# PILE THEORY LECTURE NOTES

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A. M. Weinberg

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## V. Control of Medium

A chain reacting pile which is in a steady state has an effective multiplication constant of exactly unity. If a foreign substance which absorbs neutrons is introduced into the medium, the multiplication constant will be reduced and the chain reaction will stop unless the size of the pile is increased. The fundamental problem of the theory of control is to calculate the change in k caused by a control rod of given size, shape, and absorbing material.

It is evident that the increased absorption of a control rod is only part of the reason that a rod decreases k, especially if the rod is large. For a sizeable rod the neutron distribution is perturbed by the presence of the rod (Fig. 1). The increased curva-

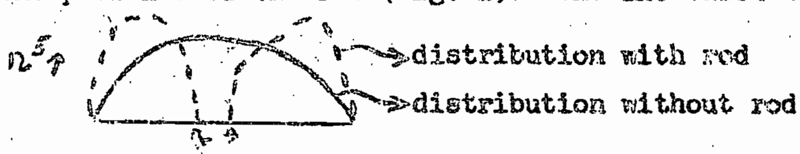



Fig. 1

ture in the neutron distribution increases the leakage outward (as well as increasing the leakage into the rod); for usual sized rods this increased outward leakage is about as effective as the absorption in (or leakage into) the rod in reducing the multiplication factor.

### Spherical Control "rod"

The simplest control rod is a black sphere of radius $r_o$ at the center of a spherical pile of radius R. We shall suppose that the control sphere is sufficiently large so that the neutron density extrapolates to zero at a distance $\lambda/\sqrt{3}$ inside the surface of the rod; this $\lambda/\sqrt{3}$ correction may be taken into account by redefining $r_o$ to be the physical radius of the sphere decreased by $\lambda/\sqrt{3}$. We shall suppose at first that the control sphere is black to all neutrons, fast and slow, and that the fast distribution is affected by the presence of the control sphere in exactly the same way as the slow distribution. This will be roughly true for very large rods; for smaller rods it is not at all the case, and we shall have to make a more extensive calculation later to take the fact that the rod is transparent to fast neutrons into account.

We now pose the following problem: Suppose the pile is just critical with radius R, multiplication constant $k_0$, and LaPlacian $\Delta_0$. If the control sphere of effective radius $r_0$ (including $\lambda / \sqrt{3}$ correction) is placed in the center of the pile, what (higher) value of k and $\Delta$ must the pile have in order to just maintain the chain reaction?

The thermal neutron distribution without a control is

$$(1) \qquad (nv)_0 = \frac{\sin \sqrt{\frac{k_0 - 1}{M^2}}\, r}{r} \;;$$

with a control sphere it is

$$(2) \qquad (nv)_1 = \frac{\sin \sqrt{\frac{k - 1}{M^2}}\,(r - r_0)}{r}$$

since $(nv)_1$ is a solution of the pile equation which vanishes on the surface of the control sphere. If the control is black to all neutrons, (2) also describes the spatial distribution of fast neutrons (except for a factor $e^{\tau\Delta}$) and we could find the value of k required to make R the critical radius from the condition $(nv)_1 = 0$ at $r = R$; i.e.,

$$(R - r_0)\sqrt{-\Delta} = \sqrt{\frac{k - 1}{M^2}}\,(R - r_0) = \pi$$

or

$$\sqrt{-\Delta} = \frac{\pi}{R - r_0} = \frac{\pi}{R}\,\frac{1}{1 - \frac{r_0}{R}}$$

But, since $R = \pi / \sqrt{-\Delta_0}$,

$$\sqrt{-\Delta} = \sqrt{-\Delta_0}\;\frac{1}{1 - r_0/R}$$

or

$$\frac{k - 1}{k_0 - 1} = \frac{1}{\left(1 - \frac{r_0}{R}\right)^2}$$

Since $r_0/R << 1$, we can expand the denominator, and we obtain

$$(3) \qquad \frac{k - 1}{k_0 - 1} = 1 + \frac{2r_0}{R} \;,$$

which gives the fractional increase in k-1 required by the introduction of a spherical control which is black to both thermal and fast neutrons. If $k = k_o$, we find from (2) the interesting result that a completely black sphere at the center of a reacting spherical pile increases the critical radius by an amount exactly equal to the effective radius of the control sphere.

If the sphere is transparent to fast neutrons and is so small that its presence does not appreciably change the fast neutron distribution, the effect of the rod would of course be reduced. To estimate the reduction in effectiveness, we may calculate the actual increased leakage caused by the presence of the rod directly.

In the absence of the rod, the per cent thermal leakage is

$$L_{t_o} = \frac{\text{Thermal leakage}}{\text{Thermal absorption}} = \frac{-\frac{4\pi\lambda}{3} R^2 \frac{d}{dr} (nv)_o\big|_R}{4\pi N \sigma_a \int_o^R r^2 (nv)_o \, dr} = \frac{L^2}{M^2}(k_o - 1)$$

as may be verified by direct integration of (1). In the presence of the control to compute the corresponding quantity, we use the distribution (2):

$$L_t = \left(\frac{\text{Thermal leakage}}{\text{Thermal absorption}}\right)_{\text{with control}} = \frac{-\frac{4\pi\lambda}{3}\left(R^2\frac{d}{dr}(nv)_1\big|_R - r_o^2\frac{d}{dr}(nv)_1\big|_{r_o}\right)}{4\pi N \sigma_a \int_{r_o}^R r^2 (nv)_1 \, dr}$$

$$= \frac{L^2}{M^2}\left(1 + \frac{2r_o}{R}\right)(k_o - 1)$$

The per cent leakage while fast is given by $L_f = \frac{\tau}{M^2}(k_o - 1)$

in both cases since the total leakage is $k_o - 1$ without the control, and we have assumed that the fast distribution is unaffected by the rod. We have, therefore

$$(k_o - 1) = L_{t_o} + L_f = k_o - 1$$

$$k - 1 = L_t + L_f = \left[\frac{\tau}{M^2} + \frac{L^2}{M^2}\left(1 + 2\frac{r_o}{R}\right)\right]\left[k_o - 1\right] = (k_o - 1) + \frac{L^2}{M^2}\frac{2r_o}{R}(k_o - 1)$$

or

$$\frac{k-1}{k_o - 1} = 1 + 2\frac{L^2}{M^2}\frac{r_o}{R}.$$

Since $L^2/M^2 \sim \frac{1}{2}$, this result suggests that a rod black to only thermal neutrons is only $\frac{1}{2}$ as effective as a rod which is black to all neutrons. This result is correct only if the control can be assumed to have no affect on the fast distribution; actually the effectiveness lies between the all black and thermal values.

The effectiveness of a spherical control can be estimated from the following numerical values. Since the m.f.p. in the pile is about 2.5 cm., a rod of actual radius 4.45 cm. has an effective radius $r_o$ of 3.0 cm. If the pile has a critical radius $R \sim 300$ cm., the change in $k - 1$ is only 2% for an all black sphere, and somewhat more than 1% for a thermally black rod. If $k - 1 \sim .05$, the overall change is only .001 in the first case, .0005 in the second.

## Cylindrical Control Rod

For a cylindrical rod the theory is much the same as for a spherical rod, but the analytic details are somewhat more complicated. We consider an all black infinitely long cylindrical rod of effective radius $r_o$ at the center of an infinitely long pile of radius R. The thermal distribution without the rod is

$$(4) \qquad (nv)_o = J_o\left(\sqrt{\frac{k_o - 1}{M^2}}\, r\right)$$

while with the control rod

$$(5) \qquad (nv)_1 = Y_o\left(\sqrt{\frac{k-1}{M^2}}\, r_o\right) J_o\left(\sqrt{\frac{k-1}{M^2}}\, r\right) - Y_o\left(\sqrt{\frac{k-1}{M^2}}\, r\right) J_o\left(\sqrt{\frac{k-1}{M^2}}\, r_o\right)$$

since $(nv)_1$ is a solution of the pile equation which satisfies the boundary condition $(nv)_1 = 0$ at $r_o$. To determine k with the rod in place, we have

$$(nv)_1 = 0 \quad \text{at } r = R, \text{ from which}$$

$$(6) \qquad \frac{Y_o\left(\sqrt{\frac{k-1}{M^2}}\, r_o\right)}{J_o\left(\sqrt{\frac{k-1}{M^2}}\, r_o\right)} = \frac{Y_o\left(\sqrt{\frac{k-1}{M^2}}\, R\right)}{J_o\left(\sqrt{\frac{k-1}{M^2}}\, R\right)}$$

which is a transcendental equation whose first root gives the value of $\frac{k-1}{M^2}$ with a rod. If $r_o = 0$, $Y_o\left(\sqrt{(k-1)/M^2}\, r_o\right) \to \infty$ and this equation reduces to the usual

$$J_o\left(\sqrt{\frac{k_o - 1}{M^2}}\, R\right) = 0$$

or

$$\sqrt{\frac{k_o - 1}{M^2}} = \frac{2.405}{R}$$

which is the usual formula for the critical radius of an infinitely long cylindrical pile.

In the general case, we may write

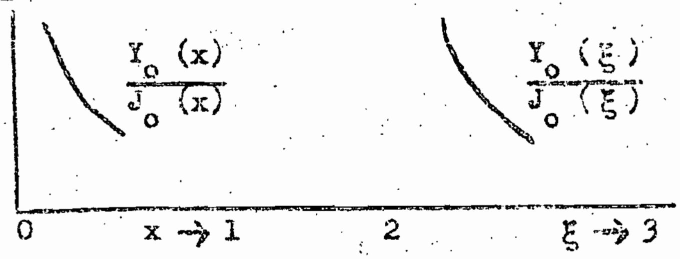
$$\sqrt{\frac{k - 1}{M^2}}\, r_o = x$$

$$\sqrt{\frac{k - 1}{M^2}}\, R = \frac{R}{r_o}\, x \equiv \xi$$

and the transcendental equation becomes

$$\frac{Y_o(x)}{J_o(x)} = \frac{Y_o(\xi)}{J_o(\xi)} \quad . \tag{7}$$

This equation can be solved most readily by plotting the functions on the left and on the right hand side on the same coordinate system and reading off on the abscissa the values of $\xi$ and $x$ which make the two functions equal




Each pair of values $x$, $\xi$ corresponds to a value of $R/r_o$ given by $\frac{R}{r_o} = \frac{\xi}{x}$ ; conversely, $\xi$ is a function of $r/R_o$, $\xi\ (r/R_o)$. The value of $k - 1$ with the control rod is given by

$$\sqrt{\frac{k - 1}{M^2}} = \frac{\xi\,(r/R_o)}{R}$$

and so

$$\frac{k-1}{k_o-1} = \frac{\xi^2(r/R_o)}{(2.405)^2} = 1 + 2\,\frac{\xi\,(r/R_o) - 2.405}{2.405} \tag{8}$$

if the rod is all black, or

$$\frac{k-1}{k_o-1} = 1 + 2\,\frac{L^2}{M^2}\;\frac{\xi\,(r/R_o) - 2.405}{2.405}$$

if the rod effects only the slow neutron distribution. Actually for rods of the size ordinarily considered ($r/R_o \sim .005$), the effectiveness lies midway between these two values.

In the following table the roots of equation (6) are tabulated along with the corresponding $\frac{k-1}{k_o-1}$ for rod black to all neutrons:

| $x$ | $\xi$ | $r_o/R$ | $\frac{k-1}{k_o-1}$ | $1 + \frac{2r_o}{R}$ |
|---|---|---|---|---|
| 0 | 2.405 | 0 | 1 | 1 |
| .005 | 2.682 | .00186 | 1.243 | 1.0037 |
| .01 | 2.721 | .0037 | 1.278 | 1.0074 |
| .02 | 2.771 | .0072 | 1.327 | 1.0144 |
| .05 | 2.865 | .0175 | 1.418 | 1.0350 |

The last column gives the percentage change in $k - 1$ for a control sphere in a spherical pile having the same ratio $r_o/R$ as the cylindrical rod. It is immediately evident that a cylindrical rod is much more effective than a spherical one of the same corresponding radius.