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PILE THEORY LECTURE NOTES

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May 27, 1943

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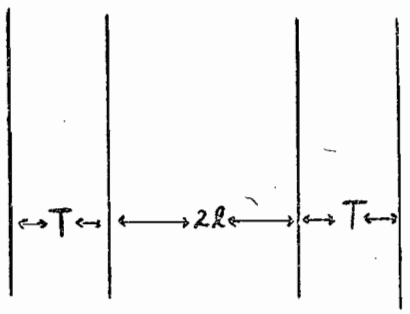

REFLECTOR THEORY

To derive the main features of reflector theory, we will start with an infinite slab pile, that is, a slab which is infinitely high of infinite breadth but has a thickness which is finite. The pile equation

$$-\Delta \varphi = \left(\frac{k-1}{M^2}\right)\varphi$$

has a solution in this case of

$$\varphi = A \cos \sqrt{-\Delta}\, x$$

Here φ is equal to nv and x runs in the direction of the finite dimension. The origin has been placed at the center of the slab and we can easily find the critical dimension of the slab by finding the point at which the cos = 0. This is when

$$x = \pm \frac{\pi}{2\sqrt{-\Delta}}$$

that is to say, that the critical thickness is $\pi/\sqrt{-\Delta}$. If now a layer of dead graphite is placed on each side of the slab, some of the neutrons which formerly escaped will be reflected back into the pile. The re-production of the pile, which ran before, will be raised and it becomes evident that our thickness is now greater than is necessary. It is possible, consequently, to decrease the dimension of the active part by placing such reflectors around the pile. To find out how much saving

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can be effected of active material, we will investigate this slab pile under a very simplified theory.

If we consider that all neutrons are slow in the pile equation, then M^2 is merely L^2: that is, the thermal diffusion area when the reflector is around the pile. The equation which governs in the active portion is the same as before and its solution is the same as before. Now, however, the boundary condition at the edges of the active pile is no longer that the neutron distribution should go to zero. This obtains at the edge of the reflector instead. At the edge of the active pile, the condition is that φ is continuous and that the flux which is $\lambda\varphi'$ is continuous. If we designate with a subscript i those quantities appropriate to the active portion, or interior, and the subscript e those appropriate to the reflector, or exterior, the boundary conditions may be written

$$\varphi_i = \varphi_e$$

$$\lambda_i \varphi_i' = \lambda_e \varphi_e' \quad .$$

Outside, that is in the reflector, a diffusion equation applies and since there is no reproduction this is the equation

$$\varphi_e'' - \varphi_e/L_e^2 = 0 \quad .$$

The solution of this equation is a hyperbolic function. If we suppose that the edge of the reflector is at a distance $\ell + T$ from the center ($\ell = \frac{1}{2}$ thickness of the active pile and T the thickness of the reflector), we may write

$$\varphi_e = B \sinh\left(\frac{\ell + T - x}{L_e}\right)$$

for the solution in the reflector on the right. It may now be seen that this does vanish at the outer boundary of the reflector. Applying the conditions at the boundary of the active portion and the reflector, we obtain

$$\varphi_i = A \cos\left(\ell\sqrt{-\Delta}\right)$$

$$= B \sinh\left(T/L_e\right)$$

$$-(\varphi'_i)_e = \sqrt{-\Delta}\, A \sin\left(\ell\sqrt{-\Delta}\right) = \frac{\lambda_e}{\lambda_i}\frac{B}{L_e}\cosh\left(\frac{T}{L_e}\right)$$

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From these the constants A and B may be eliminated and the following equation results:

$$-\left(\frac{\varphi_i}{\varphi_i'}\right) = \frac{1}{\sqrt{-\Delta}} \cot \ell \sqrt{-\Delta}$$

$$= \frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right)$$

at the boundary. This quantity is the linear extrapolation distance, that is to say, the distance beyond the end of the active pile at which the curve for φ inside comes down to zero if its extrapolation is a straight line. The equation itself may be solved for ℓ which is then the $\frac{1}{2}$ thickness of material needed for the reaction to go and gives

$$\ell = \frac{1}{\sqrt{-\Delta}} \cot^{-1}\left[\sqrt{-\Delta}\,\frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right)\right]$$

The quantity in the square brackets is very small, as may be seen as follows. The tanh (T/L_e) is less than 1. λ_i/λ_e is a small number and the $\sqrt{-\Delta}$ is so small, we only wish it were bigger, that its product times L_e is never large. Consequently, the $\cot^{-1}$ may be expanded in series. The series for small x is

$$\cot^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} + \ldots\ldots\ldots\ldots$$

The first term in ℓ we note is just equal to the old value of the critical $\frac{1}{2}$ thickness. The further terms are the saving if we define $\delta = \ell_c - \ell$ then

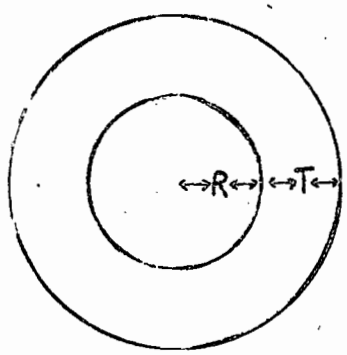
$$\delta = \frac{1}{\sqrt{-\Delta}}\left\{\left[\sqrt{-\Delta}\,\frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right)\right] - \frac{[\quad]^3}{3} + \ldots\ldots\ldots\right\}$$

$$= \frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right)\left[1 - \frac{-\Delta}{3}\left(\frac{\lambda_i}{\lambda_e} L_e \tanh \frac{T}{L_e}\right)^2 + \ldots\ldots\right]$$

The first term of this expansion is identical with the linear extrapolation. The other terms bring in the effect of the finite size of the pile, that is to say, of the curvature of the neutron distribution inside the pile.

These higher terms are very small and even under rather radical assumptions would not for a graphite plant give an effect greater than 6% in the thickness saved.

When the thickness T of the reflector is small, the tanh (T/L_e) is approximately equal to T/L_e and the linear extrapolation, or the saving, is approximately equal to $(\lambda_i/\lambda_e)T$, that is to say, for example, that in graphite piles with graphite reflectors a small reflector acts as if it were lattice.



Actually, of course, plants are never built with any infinite dimension. The shape which is most economical of active substance is a sphere. Because of the difficulty of construction, spheres are not usually used. However, the linear dimensions in three directions are approximately equal in all systems which are normally under consideration. Although the effect of the finite thickness is not great, the effect of finiteness in the other two directions is not negligible. In order to illustrate how this enters into the calculation, we will carry through the reflector theory on the same simple model but using the sphere instead of the slab. In this case the

$$\Delta\varphi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi)$$

and it is convenient in place of φ to use the $\psi = r\varphi$. The equation for the active portion becomes

$$\psi_i'' - \frac{\psi_i\,(k-1)}{L_i^2} = 0$$

and the equation in the reflector,

$$\psi_e'' - \psi_e/L_e^2 = 0 \quad .$$

Using $-\Delta = \frac{k-1}{L_i^2}$, this is an exact parallel to the case treated above.

The boundary conditions in this case are again the continuty of φ and of $\lambda\varphi'$. The first one goes over immediately into the continuity of ψ but the second one

$$\lambda_i \left(\frac{\psi_i}{r}\right)' = \lambda_e \left(\frac{\psi_e}{r}\right)'$$

It is only the same when $\lambda_i = \lambda_e$. The solutions which must be used inside replace the cos by the sin. This is necessary since otherwise there would be a singularity in φ at the center of the pile. They are

$$\psi_i \; \cancel{\varphi_i} = A \sin r\sqrt{-\Delta}$$

$$\psi_e \; \cancel{\varphi_e} = B \sinh\left(\frac{R + T - r}{L_e}\right)$$

Writing the boundary conditions then between the active region and the reflector, we obtain for the case $\lambda_i = \lambda_e$

$$A \sin R\sqrt{-\Delta} = B \sinh\left(\frac{T}{L_e}\right)$$

$$\sqrt{-\Delta}\, A \cos R\sqrt{-\Delta} = -\frac{B}{L_e} \cosh\left(\frac{T}{L_e}\right)$$

$$\frac{1}{\sqrt{-\Delta}} \tan R\sqrt{-\Delta} = -L_e \tanh\left(\frac{T}{L_e}\right)$$

The last equation may be solved for R using a $\tan^{-1}$ expansion for small angles. The $\tan^{-1}$ is

$$x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots$$

Here we are dealing with a small negative quantity and its $\tan^{-1}$ is therefore

$$\pi - x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots$$

It is, therefore, possible to write

$$R = \frac{\tan^{-1}}{\sqrt{-\Delta}} \left[-\sqrt{-\Delta}\ L_e \tanh\left(\frac{T}{L_e}\right) \right.$$

$$\frac{\pi' - \sqrt{-\Delta}\ L_e \tanh\ (T/L_e) + \frac{-\Delta\sqrt{-\Delta}}{3} L_e \tanh\left(\frac{T}{L_e}\right)^3}{\sqrt{-\Delta}}$$

In order to find the amount of saving, we will have to subtract this quantity from the critical radius of the sphere with no reflector which is

$$R_c = \frac{\pi'}{\sqrt{-\Delta}}$$

The rest is that the saving

$$\delta = R_c - R = L_e \tanh\left(\frac{T}{L_e}\right) \left[1 - \frac{-\Delta}{3} \left\{ L_e \tanh\left(\frac{T}{L_e}\right)^2 + \ldots\ldots \right\} \right]$$

exactly as before. This formal similarity with the previous case is, however, not general and it is somewhat misleading. The first term is not the extrapolation distance for φ but that for ψ, and in the case where $\lambda_i \neq \lambda_e$, the second boundary condition becomes

$$A\lambda_i \left\{ R\sqrt{-\Delta}\ \cos R\sqrt{-\Delta} - \sin R\sqrt{-\Delta} \right\} = -B\lambda_e \left\{ \frac{R}{L_e} \cosh\left(\frac{T}{L_e}\right) + \sinh\left(\frac{T}{L_e}\right) \right\}$$

which with a little algebra can be shown to result in

$$R = \frac{\tan^{-1}}{\sqrt{-\Delta}} \left[-\sqrt{-\Delta}\ \frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right) \frac{1}{\left\{ 1 + \left(1 - \frac{\lambda_i}{\lambda_e}\right) \frac{L_e}{R} \tanh\left(\frac{T}{L_e}\right) \right\}} \right]$$

which gives for δ

$$\frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right) \left\{ \frac{1}{1 + \left(1 - \frac{\lambda_i}{\lambda_e}\right) \frac{L_e}{R} \tanh\left(\frac{T}{L_e}\right)} \right\} \left\{ 1 - \frac{-\Delta}{3} \left[\frac{\lambda_i}{\lambda_e} L_e \tanh\left(\frac{T}{L_e}\right) \frac{1}{1 + \left(1 - \frac{\lambda_i}{\lambda_e}\right) \frac{L_e}{R} \tanh\left(\frac{T}{L_e}\right)} \right]^2 + \ldots \right\}$$

It now appears that only in the case when $\lambda_i = \lambda_e$ is there no correction

for finiteness in the leading term. The amount of the correction for finiteness in this term depends on $1 - \lambda_i/\lambda_e$. The formula which we have written involves R itself which is unknown, and in fact is essentially what we are trying to compute. However, it involves R only in a small correction and if R_c is used here in place of R, the error will not be great. The δ computed this way will be substantially accurate, and it may be improved quickly by successive approximation, substituting the new R in the formula at each stage.

On the basis of the general type of theory we have discussed above, it is possible to get an idea of the relative effectiveness of two different reflectors. In the formula for saving, if we consider only the leading term and omit the corrections for finiteness, those factors which depend on the material of the reflector are

$$\frac{L_e}{\lambda_e} \tanh\left(\frac{T}{L_e}\right)$$

and the ratio of the amounts saved by use of two different reflectors can be found in the ratio

$$\frac{\dfrac{L_{e_1}}{\lambda_{e_1}} \tanh \dfrac{T}{L_{e_1}}}{\dfrac{L_{e_2}}{\lambda_{e_2}} \tanh \dfrac{T}{L_{e_2}}}$$

For thick reflectors, the value of the tanh will be approximately 1 and L_e/λ_e is a measure of the goodness of the reflector. In its present form, this says that the longer the relaxation distance and the shorter the thermal free path, the better the reflector is. This may also be stated in another simple form. Since

$$L_e = \frac{1}{\sqrt{3\sigma\sigma_a}}$$

and

$$\lambda_{e_2} = \frac{1}{\sigma} \quad ,$$

$$\frac{L_e}{\lambda_e} = \sqrt{\frac{\sigma}{3\sigma_a}}$$

that is to say, it is proportional to the square root of the number of collisions made before capture in the material of the reflector. A

better reflector is one which turns back into the pile a larger fraction of the neutrons which escape from the active edge. That is, it is made of material which has a high albedo. From this consideration one would see that a high albedo goes with a large number of collisions before absorption.

Consider for a moment one of the deficiencies of the theory as developed above. We have considered that all neutrons are born thermal, actually neutrons are born fast and have to slow down. While they are slowing down, they are not absorbed so rapidly and the effect of the fast neutrons then is to increase the number of collisions before absorption. Consequently, the reflector will be expected to be somewhat better than we have calculated. By approximately calculating where the neutrons become thermal and solving the diffusion equation for thermal neutrons with this source function, a more accurate theory may be developed. The slowing functions have been calculated by several people, and in one case their application to reflector theory has been carried through numerically by Fermi. The case which he did is that of a semi-infinite graphite pile with a graphite reflector. He found only a slight change.

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