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PILE THEORY LECTURE NOTES

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A. M. Weinberg

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Critical size of heterogeneous pile; statistical weight

Usually as a matter of economy, the lattice is variable, the best materials being put in the center. To determine the average Δ for such a structure (and hence the critical size), we consider the pile equation with variable

$$\frac{k-1}{M^2} \equiv \Delta(\underline{r}) \qquad (\underline{r}) \text{ is a vector}$$

$$(1) \qquad \nabla^2 nv + \Delta(\underline{r})\, nv = 0$$

This equation is not quite accurate but is sufficiently good if the variation of $\Delta(x)$ is not too steep. We wish to determine the average LaPlacian, $\bar{\Delta}$; i.e., the Δ of a uniform pile which has the same critical dimensions as the actual pile. For such a fictitious pile, the density $(nv)'$ satisfies

$$(2) \qquad \nabla^2 (nv) + (nv)' \bar{\Delta} = 0$$

Multiplying (1) by $(nv)'$ and (2) by (nv), integrating over the pile volume and substracting, we have

$$\bar{\Delta} \int_V (nv)(nv)'\, dV - \int_V \Delta(\underline{r})(nv)(nv)'\, dV = \int_V \left[(nv)\nabla^2(nv)' - (nv)'\nabla^2 nv\right] dV$$

The right hand side may be transformed by Green's Theorem

$$\int_V \left[nv\, \nabla^2(nv)' - (nv)'\nabla^2 nv\right] dV = \int_S (nv)\, \nabla(nv)' - (nv)'\nabla nv\Big] \cdot \underline{dS}$$

$$= 0 \quad \text{since } nv = (nv)' = 0 \text{ everywhere}$$

over the pile surface S; hence,

$$\bar{\Delta} = \frac{\int \Delta(\underline{r})\,(nv)\,(nv)'\,dV}{\int (nv)\,(nv)'\,dV}$$

If $\Delta(\underline{r})$ varies only slightly so that $(nv) \approx (nv)'$, we have

$$\bar{\Delta} = \frac{\int_V \Delta(\underline{r})(nv)^2\,dV}{\int_V (nv)^2\,dV}$$

i.e., the local $\Delta(\underline{r})$ at a point is weighted with the <u>square</u> of the neutron density times velocity in computing the critical lattice dimensions. If the pile structure is cellular, we may replace the integrals by sums and we can write

$$\bar{\Delta} = \frac{\sum_{\text{all cells}} \Delta_i (nv)_i^2}{\sum_{\text{all cells}} (nv)_i^2}$$

where $(nv)_i$ is the average nv over the i-th cell. The quantity

$$\frac{(nv)_i^2}{\sum_i (nv_i)^2}$$

which weights the Δ_i of a cell in the calculation of $\bar{\Delta}$ is called the <u>statistical weight</u>, w_i, of the i-th cell. Note that

$$\sum_i w_i = 1$$

As an example of how the statistical weight is used, let us calculate the average $\bar{\Delta}$ of a cylindrical pile of height h, radius r_c which consists of a core of height Z, radius r, $\Delta = \Delta_1$, and a shell of thicknesses $r_c - r$, $h - x$, and $\Delta = \Delta_2$.

The distribution nv is, in our approximation,

$$nv = J_0\left(\frac{2.405}{r_c} r\right) \cos\frac{\pi z}{h}$$

and so

$$\bar{\Delta} = \frac{\Delta_1 \int_0^r \int_0^{\frac{z}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz + \Delta_2 \left[\int_0^{r_c}\int_0^{\frac{h}{2}} - \int_0^r \int_0^{\frac{z}{2}}\right] r\, J_0^2\left(\frac{2.405}{r_c}\right) \cos^2\frac{\pi z}{h}\, dr\, dz}{\int_0^{r_c}\int_0^{\frac{h}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz}$$

$$\bar{\Delta} = \Delta_1 \frac{\int_0^r \int_0^{\frac{z}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz}{\int_0^{r_c}\int_0^{\frac{h}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz} + \Delta_2 \left[1 - \frac{\int_0^r \int_0^{\frac{z}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz}{\int_0^{r_c}\int_0^{\frac{h}{2}} r\, J_0^2\left(\frac{2.405}{r_c} r\right) \cos^2\frac{\pi z}{h}\, dr\, dz}\right]$$

To evaluate these averages we use the results (Jahnke-Ende, TABLES OF FUNCTIONS, p. 146)

$$\int_0^{r_c} r\, J_o^{\,2}\left(\frac{2.405\ r}{r_c}\right) dr = \frac{r^2}{2}\left\{ J_1^{\,2}\left(\frac{2.405}{r_c}\ r\right) + J_o^{\,2}\left(\frac{2.405}{r_c}\ r\right)\right\}$$

$$\int_0^{\frac{h}{2}} \cos^2 \frac{\pi z}{h}\, dz = \frac{h}{\pi}\left[\frac{\pi z}{4h} + \frac{\sin \frac{\pi z}{h}}{4}\right] = \frac{z}{4} + \frac{h}{4\pi} \sin \frac{\pi z}{h}$$

Consequently the integral in the denominator is

$$\frac{hr_c}{8}\, J_1^{\,2}\,(2.405)$$

and so

$$\bar{\bar{\Delta}} = w_1\, \Delta_1 + (1 - w_1)\, \Delta_2$$

where

$$w_1\left(\frac{r}{r_c}, \frac{z}{h}\right) = \left(\frac{r}{r_c}\right)^2 \left[1 + \frac{J_o^{\,2}(2.405\ r/r_c)}{J_1^{\,2}\,(2.405)}\right]\left[\frac{z}{h} + \frac{1}{\pi} \sin \frac{\pi z}{h}\right]$$

If the core is of the same length as the pile, the weighting factor is

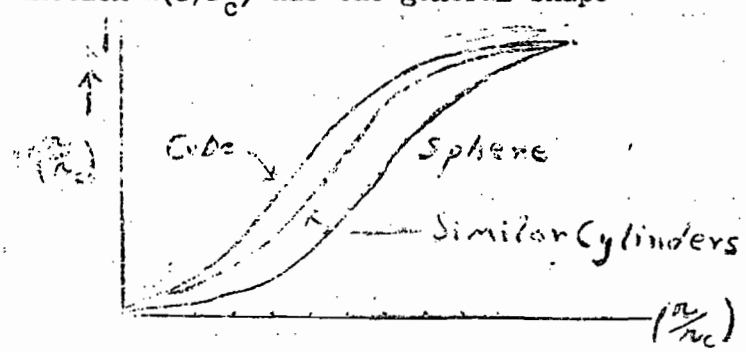
$$w_1\left(\frac{r}{r_c}\right) = \left(\frac{r}{r_c}\right)^2 \left[1 + \frac{J_o^{\,2}(2.405\ r/r_c)}{J_1^{\,2}\,(2.405)}\right]$$

EXERCISE: Calculate the statistical weight w(r) for the following geometries:

a) A sphere of radius r within a sphere of radius r_c

b) A cube of edge r_c.

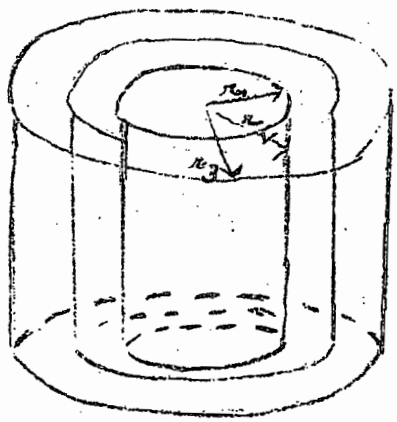
The statistical weights are greatest for the central cells, least for the outer cells. The function $w(r/r_c)$ has the general shape



From such a curve we can find the average $\bar{\Delta}$ for a cored structure, since

$$\bar{\Delta} = w\left(\frac{r_1}{r_c}\right)\Delta_1 + \left[w\left(\frac{r_2}{r_c}\right) - w\left(\frac{r_1}{r_c}\right)\right]\Delta_2 + \left[w\left(\frac{r_3}{r_c}\right) - w\left(\frac{r_2}{r_c}\right)\right]\Delta_3 + \ldots$$

where r_1, r_2, r_3 .... are the radii of the cores (see Fig. 2).



Pile Averages

In estimating the difficulties of design for a given pile, the ratio of maximum to average activity is often needed (since the design must be made to take care of the worst points in the pile). These averages are, for a cylindrical pile of height h, radius r_c, as follows:

Pile activity at a point, $nv = J_o\left(\frac{2.405\ r}{r_c}\right)\cos\frac{\pi Z}{h}$

Total pile activity $= 4\pi \int_0^{r_c}\int_0^{h/z} rJ_o(2.405\ r/r_c)\cos\frac{\pi Z}{h}\ drdZ$

Total pile activity $= 4h\ \frac{r_c^2\ J_1(2.405)}{2.405}$

Average pile activity $\frac{4\ J_1(2.405)}{2.405\ \pi}$

$$\frac{\text{Center point activity/per unit volume}}{\text{Average pile activity}} = \frac{2.405\,\pi}{4J_1(2.405)} = 3.64$$

$$\frac{\text{Average activity of central rod}}{\text{Average pile activity}} = \frac{\dfrac{2}{h}\displaystyle\int_0^{h/2} \cos\frac{\pi Z}{h}\,dZ}{\dfrac{4J_1(2.405)}{2.405\,\pi}}$$

$$= \frac{2}{\pi} \cdot \frac{2.405\,\pi}{4J_1(2.405)}$$

$$= \frac{2.405}{2J_1(2.405)} = 2.32 \quad .$$

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