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PILE THEORY LECTURE NOTES

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IV. Diffusion and slowing down in a reacting medium

If $k > 1$, the chain reaction will be just self-sustaining provided the leakage is not too large. The leakage depends on the surface to volume ratio of the pile; for a given geometric shape, the leakage therefore becomes smaller as the size of the structure increases. After the structure reaches a certain critical size, the leakage will be sufficiently small to allow the reaction to be self maintaining.

To calculate the critical size, in the case of say, a sphere, we start with pile equation with $k > 1$

$$\Delta nv + \frac{k-1}{M^2} nv = 0 \qquad \left(\frac{k-1}{M^2} = -\Delta \right)$$

or, for a sphere,

$$\frac{d^2(nv)}{dr^2} + \frac{2}{r}\frac{dnv}{dr} \; \frac{k-1}{M^2} nv = 0$$

There are two solutions which satisfy the boundary condition $nv = 0$ at $r = R_c$. First there is the trivial solution

$$nv = 0$$

which is the only stationary solution which satisfies the boundary condition if the curvature of the distribution (i. e., $k - 1/M^2$) is too small or too large to bring the neutron density to zero at the boundary. The second solution,

$$nv = A \frac{\sin\sqrt{\frac{(k-1)}{M^2}}\, r}{r\sqrt{\frac{(k-1)}{M^2}}}$$

will be a non trivial solution ($A \neq 0$) only if at the boundary $r = R_c$, $nv = 0$. This will be the case if and only if

$$\sqrt{\frac{k-1}{M^2}}\, R_c = \pi$$

$$\boxed{R_c = \frac{\pi M}{\sqrt{k-1}} \quad \frac{\pi}{\sqrt{[illegible]}}}$$

This is the formula for the critical radius of a chain reacting sphere.

For a chain reacting cylinder (radius r_c, height h), the pile equation may be written

$$\frac{\partial^2 (nv)}{\partial r^2} + \frac{1}{r} \frac{\partial\, nv}{\partial r} + \frac{\partial^2 nv}{\partial z^2} + \frac{k-1}{M^2}\, nv = 0$$

The solution which vanishes at the boundary of the cylinder (i.e. at $r = r_c$, $z = \pm h/2$) is of the form

$$nv = J_o\left(\frac{2.405\, r}{r_c}\right) \cos \frac{\pi z}{h}$$

where, in order for nv to satisfy the differential equation, we must have by substituting into the pile equation

$$\left(\frac{2.405}{r_c}\right)^2 J_o'' + \left(\frac{2.405}{r_c}\right) \frac{1}{r} J_o' - \frac{\pi^2}{h^2} J_o + \frac{k-1}{M^2} J_o = 0$$

or

$$J_o'' + \frac{1}{\frac{2.405\, r}{r_c}} J_o' + \left(\frac{r_c}{2.405}\right)^2 \left(\frac{k-1}{M^2} - \frac{\pi^2}{h^2}\right) J_o = 0$$

Comparing this with Bessel's equation

$$J_o'' + 1/x\, J_o' + J_o = 0 \,,$$

we see that the original expression for nv is a non-trivial solution of the pile equation provided

$$\left(\frac{r_c}{2.405}\right)^2 \left(\frac{k-1}{M^2} - \frac{\pi^2}{h^2}\right) = 1$$

i. e.,

$$\boxed{\frac{k-1}{M^2} = -\Delta = \frac{\pi^2}{h^2} + \left(\frac{2.405}{r_c}\right)^2}$$

PROBLEM 1: Show that the critical length, L, of a reacting cube is given by

$$-\Delta = \frac{3\pi^2}{L^2}$$

PROBLEM 2: Show that for a rectangular parallelepiped of sides a, b, c, the critical dimensions are related by

$$-\Delta = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} .$$

PROBLEM 3: Show that if the length of each edge of a reacting cube is changed by a certain per cent, the required $-\Delta$ of the structure is changed by twice this per cent.

PROBLEM 4: Find the critical radius for a cylinder for which $k = 1.04$, $M^2 = 650$ cm^2, $h = 7$ meters. If h is reduced to 6.5 meters and r and M^2 remain the same, what k is required to maintain the reaction?

Optimum dimensions for a cylinder

For a cylinder the smallest volume which will chain react can be calculated as a straightforward maximum-minimum problem. Thus, in general, the volume V is

$$V = \pi r_c^2 h$$

and the critical condition is

$$\frac{\pi^2}{h^2} + \overline{\frac{2.405}{r_c^2}}^{\,2} = -\Delta ,$$

or

$$r_c^2 = -\frac{\overline{2.405}^2 h^2}{\pi^2 + h^2 \Delta}$$

Hence

$$V = -\frac{\overline{2.405}^2 \pi h^3}{\pi^2 + h^2 \Delta}$$

To determine minimum volume, we differentiate and put $\frac{dV}{dh} = 0$:

$$\frac{dV}{dh} = \frac{-3(\pi^2 + h^2\Delta)\overline{2.405}^2 \pi h^2 + \overline{2.405}^2 \pi h^3 . 2h\Delta}{(\pi^2 + h^2\Delta)^2} = 0$$

$$\therefore \quad -3(\pi^2 + h^2\Delta) + 2h^2 \Delta = 0$$

$$h^2 = \frac{-3\pi^2}{\Delta} \quad \text{at minimum volume}$$

$$r_c^2 = -\frac{3}{2}\frac{\overline{2.405}^2}{\Delta} \quad \text{at minimum volume}$$

from which we get the minimum volume:

$$\boxed{V = 148.3\ (-\ \Delta)^{-3/2}} \quad \text{for optimum cylinder.}$$

Note that critical volume is proportional to the 3/2 power of $1/\Delta$. This follows directly from dimensional arguments, since Δ has the dimensions $(\text{length})^{-2}$.

PROBLEM: Show that volume of a chain reacting cube is $(3\pi^2)^{3/2}(-\Delta)^{-3/2}$, of a chain reacting sphere is

$$4/3 \cdot \pi \cdot \pi^{3/2}(-\Delta)^{-3/2} .$$