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CL-574 (2.2)

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PILE THEORY LECTURE NOTES

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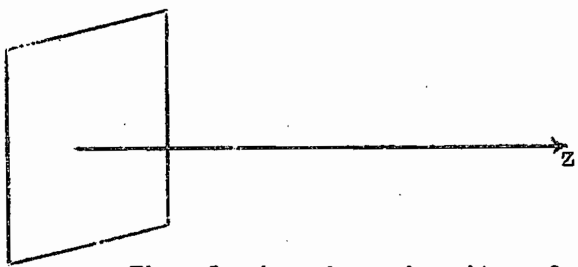
Construction of Solution of $\left(\Delta q = \frac{\partial q}{\partial \tau}\right)$

1) 1-dimension

Consider an infinite plane neutron source perpendicular to the z axis.



$q = q(E)$ = slowing down density of neutrons

The slowing down density of neutrons for a point source is:

$q = q(E)$ = slowing down density of neutrons. Since $q = q(z, \tau)$, $D \Delta q = q$. If $Ddt \equiv d\tau$ so that $\Delta q = dq/d\tau$. Also $q = q(z, \tau)$; $q = \frac{\partial q}{\partial z}$ and $q'' = \frac{\partial^2 q}{\partial z^2}$. The equation then becomes: $q'' = q$

Assume $q(z, \tau) = ce^{az + b\tau}$, where a and b are constants and c is a constant $c = c(a)$

$$q' = ace^{az + b\tau}$$

$$q'' = a^2 ce^{az + b\tau}$$

but

$$q = be^{az + b\tau}$$

$$\therefore \quad a^2 = b$$

Substituting back into the assumed solution,

$$q(z, \tau) = ce^{az + a^2}$$ [redacted]

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To connect this solution with a Fourier Series, let $a = iw$ where $i = \sqrt{-1}$ and w = parameter, therefore, $a^2 = -w^2$; the solution becomes:

$$q(z, \tau) = c(w)e^{iwz - w^2\tau} \quad ,$$

where constant c now is expressed as a function of the parameter w.

Since we are not interested in the previous history of the neutrons, let us choose $z = 0$ at the infinite plane source. Then $c(w)$ must be chosen to fit this boundary condition.

From the expression of the limiting form extending over the infinite plane of Fourier's Series, we get the Fourier Integral

$$q(z, o) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwz} \, c(w) \, dw$$

$$c(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(z, o) \, e^{-iwz} \, dz$$

where $q(z)$ is finite and continuous and single valued for all values of z or if discontinuous, must have only a finite number of maxima and minima in the neighborhood of any given point (see Byerly, "Fourier's Series and Spherical Harmonics", Chapt. II).

The general expression of the slowing down density for a point source over the infinite plane then becomes:

$$q(z, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwz - w^2\tau} \, c(w) \, dw$$

$$q(z, \tau) = \frac{1}{\sqrt{2\pi}} \int^{+\infty} e^{iwz - w^2\tau} \, dw \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(z') \, e^{-iwz'} \, dz'$$

[redacted]

where $z \rightarrow z'$ is the independent variable for which $q = q(z')$ is expressed in terms of the parameter w. Simplifying and combing:

$$q\ (z, \tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} q(z')\ dz' \int_{-\infty}^{+\infty} e^{iw(z - z') - w^2\tau}\, dw$$

which is independent of the original source.

To get $\exp\left\{-\left[w^2 - iw\ (z - z')\right]\right\}$ into the form of the Gaussian function to facilitate integration, we must complete the square of the two terms of the exponent. Add and subtract $\left[\frac{i(z - z')}{2\sqrt{\tau}}\right]^2$

$$-\left\{\left[w^2\tau - iw(z - z')\right] + \left[\frac{i(z - z')}{2\sqrt{\tau}}\right]^2 + \frac{(z - z')^2}{4\tau}\right\}$$

where $\quad i^2 = (\sqrt{-1})^2 = -1$

or $\quad -\left\{\left[w\sqrt{\tau} - \frac{i(z - z')}{2\sqrt{\tau}}\right]^2 + \frac{(z - z')^2}{4\tau}\right\}$

let $\quad A = \left[w\sqrt{\tau} - \frac{i(z - z')}{2\sqrt{\tau}}\right]^2$

then $\quad q(z, \tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(z - z')^2/4\tau}\, q(z')\ dz' \int_{-\infty}^{+\infty} e^{-A}\, dw$

This latter integral is of the form of

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{a}$$

$$A = x^2 = \left[w\sqrt{\tau} - \frac{i(z - z')}{2\sqrt{\tau}}\right]^2 ;\ x = \left[w\sqrt{\tau} - \frac{i(z - z')}{2\sqrt{\tau}}\right]$$

$$2x\,dx = 2\left[w\sqrt{\tau} - \frac{i(z - z')}{2\sqrt{\tau}}\right]\sqrt{\tau}\, dw$$

$$dx = \sqrt{\tau}\, dw$$

where $a = 1$, ...

$$\therefore \quad \frac{1}{\sqrt{\tau}}\int_{-\infty}^{\infty} e^{-A}\, dw = \frac{\sqrt{\pi}}{\sqrt{\tau}}$$

$$q(z, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} e^{-(z - z')^2/4} \quad q(z')\, dz'$$

$q(z')$ = δ function which equals 0 at the source and equals 1 for all values of z' away from the source. The slowing down density for a point source then becomes:

$$q(z, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-\left[(z - z')^2/4\tau\right]}$$

for 1-dimension
$$\boxed{q(z, \tau) = \frac{Q}{2\sqrt{\pi\tau}} e^{-\left[(z - z')^2/4\tau\right]}}$$

where there are Q point sources per $cm.^2$.

2) Calculation of the slowing down density for 3-dimensions

$$q = q(x, y, z, \tau)$$

$c = c(\vec{w})$ where $\vec{w}$ is a vector parameter in 3 dimensional space.

$$c(\vec{w}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} q(\vec{r}, o)\, e^{-i\vec{w}\cdot\vec{r}'}\, d\vec{w}$$

and therefore

$$c(\vec{\omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} q(\vec{r}, 0)\, e^{-i\vec{\omega}\vec{r}}\, d\omega$$

because:

$$c(\vec{\omega}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(\vec{x}, 0, 0, 0)\, e^{-i\omega_x x}\, d\omega_x \frac{1}{\sqrt{2\pi}} \int_{-\infty} q(0, \vec{y}, 0, 0)\, e^{-i\omega_y y}\, d\omega_y \left[c(\omega_z)\right]$$

$$\left[c(\omega_z)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(0, 0, \vec{z}, 0)\, e^{-i\omega_z z}\, d\omega_z$$

and $q(\vec{x}, 0, 0, 0)\, q(0, \vec{y}, 0, 0)\, q(0, 0, \vec{z}, 0) = q(\vec{r}, 0)$

$$\vec{\omega} = \vec{i}\omega_x + \vec{j}\omega_y + \vec{k}\omega_z$$
$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$$

and since $q(\vec{r}, o) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} e^{iw\vec{r}} c(w)dw$

$$q(\vec{r},\tau) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} e^{iw\vec{r} - w^2\tau} dw \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} q(\vec{r}')e^{-iw\vec{r}'} d\vec{r}'$$

where $\vec{r} \longrightarrow \vec{r}'$ is the independent variable for which $q = q(\vec{r}')$ is expressed in terms of the parameter w.

Simplifying and combining

$$q(\vec{r},\tau) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{+\infty} q(\vec{r}')\, d\vec{r}' \int_{-\infty}^{+\infty} e^{iw(\vec{r}-\vec{r}') - w^2\tau} dw$$

which is independent of the original source. Now to get the

$$\left\{ - \left[w^2\tau - iw(\vec{r} - \vec{r}') \right] \right\}$$

into the form of the Gaussian function to facilitate integration, we must complete the square of the two terms of the exponent as in the 1-dimensional case. Then

$$q(\vec{r}, \tau) = \left(\frac{1}{2\sqrt{\pi\tau}}\right)^3 \int_{-\infty}^{+\infty} e^{-(r-r')^2/4\tau}\, q(\vec{r}')\, d\vec{r}'$$

and since $q(\vec{r}') = \delta$ function which equals 0 at the source and equals 1 for all vaues of $\vec{r}'$ away from the source, the slowing down density becomes for a point source

$$q(\vec{r}, \tau) = \left(\frac{1}{2\sqrt{\pi\tau}}\right)^3 e^{-(\vec{r}-\vec{r}')/4\tau}$$

and where there are Q point sources per $cm.^3$

$$\boxed{q(\vec{r}, \tau) = \frac{Q}{(2\sqrt{\pi\tau})^3}\, e^{-(r-r')^2/4\tau}}$$