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PILE THEORY LECTURE NOTES

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M.S. for M.T. Bray 10-5-54
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B. The Pile equation and its solution

1) Simple deduction

If the medium is infinite in extent, and if n represents the density of thermal neutrons per c.c., then, from the definition of k,

$$\frac{dn}{dt} = (k-1)n,$$

where the unit of t is the time for one generation. This equation has the solution $n = n_0 e^{(k-1)t}$ which is divergent if $k > 1$, convergent if $k < 1$, stationary if $k = 1$.

Suppose now the medium is not infinite in size so that n is a function of position as well as time. Let n represent the density (per c.c.) of all neutrons; these are supposed to be either thermal or fast. In general,

$$\frac{\partial n}{\partial t} = (k-1)n - \mathcal{L}(n)$$

where $\mathcal{L}(n)$ represents the number of neutrons lost per generation per unit volume by leakage. The leakage can be considered to be made up of two parts:

a) Leakage outward, while thermal, [redacted] per unit volume per generation $= -D_t \Delta n \times \tau_t$

b) Leakage outward, while ~~thermal~~ fast, [redacted] per unit volume per generation $= -D_f \Delta n \times \tau_f$

where D_t = diffusion coefficient of thermal neutrons $= \frac{\lambda_t v_t}{3}$

D_f = diffusion coefficient of fast neutrons $= \frac{\lambda_f v_f}{3}$

τ_t = lifetime of neutrons which are slow

τ_f = lifetime of neutrons which are fast.

Evidently $\tau_t = \frac{1}{N\sigma_a v_t}$, $\tau_f = \frac{\log E_0/E_t}{\alpha} \frac{\lambda_f}{v_f}$

where α = average logarithmic energy loss per collision, E_0 = initial

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(fission) energy, E_t = thermal energy, and τ_f and τ_t are assumed to be the same for all fast neutrons, although this assumption is not essential to the argument. Then:

$$\mathcal{L}(n) = -\frac{\lambda_t}{3N\sigma_a}\Delta n - \frac{\lambda_f^2}{3\alpha}\log E_0/E_t\,\Delta n \equiv -(L^2+\tau)\Delta n$$

where $\tau \equiv \frac{\lambda_f^2}{3\alpha}\log E_0/E_t$ is the age of the fast neutrons.

Hence we may write the pile equation

$$\boxed{\frac{\partial n}{\partial t} = (k-1)n + (L^2+\tau)\Delta n}$$

which holds either for thermal or all neutrons because the equation is homogeneous, or in the steady state, $\frac{\partial n}{\partial t} = 0$, and so

$$\boxed{\Delta n = -\frac{k-1}{M^2}n}$$

where $M = \sqrt{L^2+\tau}$ is called the <u>migration length</u>.

<u>Fermi's derivation of the pile equation</u>

In the above derivation the details of the slowing down process were neglected since all the non-thermal neutrons were lumped together as "fast". An alternative derivation due to Fermi is the following:

Assume the pile is homogeneous, with macroscopic absorption cross-section $N\sigma_a$. Then the thermal neutron density, n, satisfies in the steady state,

$$\frac{\lambda}{3}\Delta n v - N\sigma_a n v + q(\tau_0) = 0 \qquad (1)$$

where $q(\tau_0)$ = number of thermals produced per c.c. per second by slowing, τ_0 = age of thermal neutrons. The slowing down density q, of neutrons destined to escape resonance capture, is

$$\Delta q = \frac{\partial q}{\partial \tau} \qquad (2)$$

with the initial condition

$$q(0) = k(N\sigma_a n v) \qquad (3)$$

since k = number of fast neutrons which escape resonance capture produced per thermal absorbed.

A solution of the slowing down equation, valid if $\frac{\Delta n v}{n v}$ is a constant, is

$$q(\tau) = k N\sigma_a n v e^{\tau\Delta} \qquad (4)$$

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where Δ, when not followed by a variable, is a <u>number</u> equal to <u>constant</u> value of $\frac{\Delta n v}{n v}$. (Note: Wigner has shown that the requirement of constant $\frac{\Delta n v}{n v}$ is unnecessary if $e^{\tau\Delta}$ is interpreted as an operator.)

Putting (4) into (1) we obtain

$$\frac{\lambda}{3}\Delta n v - N\sigma_a n v + k N\sigma_a n v e^{\tau\Delta} = 0$$

or

$$\Delta n v - \kappa^2 n v + k\kappa^2 n v e^{\tau\Delta} = 0$$

Since we assume $\frac{\Delta n v}{n v} = \Delta$ (a constant) we may write the "characteristic equation of the pile"

$$\boxed{k = (1 - L^2\Delta)e^{-\tau\Delta}} \quad , L^2 = 1/\kappa^2 = \text{"pile diffusion length"}$$

which relates the multiplication factor and the Laplacian or curvature of of the neutron distribution, Δ. The characteristic equation has, for a given k, an infinite number of roots Δ_i, each of which corresponds to a harmonic in the distribution near a perturbing source or near a boundary. Since $\Delta \equiv \frac{\Delta n v}{n v}$, we obtain from the characteristic equation the pile equation on expanding the exponential:

$$\boxed{\Delta n v = -\frac{k-1}{M^2} n v}$$

which is the same as before if v is constant. The pile equation is the fundamental equation of macroscopic pile theory.

Formally, the pile equation is identical with the steady state diffusion equation with absorption. The main difference is that k may be $\gtrless$ 1; i.e. the Laplacian, or the curvature of the thermal neutron density,

$$\Delta = \frac{\Delta n v}{n v} = \frac{-(k-1)}{M^2} .$$

is positive or negative depending on value of k.

<u>Calculation of Migration area, M^2</u>

The migration area $M^2 = L^2 + \tau$ is calculated as follows:

1) τ : The "age" is given by

$$\tau = \frac{3}{\alpha(1 - \frac{2}{3A})} \int_{E_0}^{E} \lambda^2(E) \frac{dE}{E}$$

where λ = m.f.p., A = atomic wt. of scatterer, α = slowing down parameter (= .158 for carbon). τ is best determined experimentally ($4\tau = r_0^2$ where r_0 = range of fission neutrons,) because energy of fission neutrons is not known very accurately, nor is λ (E) known too well. τ for thermals in graphite $\sim$ 380 cm^2.

Problem: Suppose volume ratio of metal to graphite is V_0/V_1. What is value of τ in such a lattice?

2) Diffusion length, $\frac{1}{L^2} = 3N^2\sigma\sigma_a$ (neglecting absorption and forward scattering.)

Since σ and σ_a are constants of whole medium, σ_i of each region must be weighted with probability that neutrons will be in that region, i.e.,

$$\sigma = \frac{\sigma_0 \bar{n}_0 V_0 + \sigma_1 \bar{n}_1 V_1}{\bar{n}_0 V_0 + \bar{n}_1 V_1} \approx \sigma_1$$

where V_0 = volume of metal/cell
V_1 = volume of moderator/cell
$V_0 \ll V_1$
$\bar{n}_0$, $\bar{n}_1$ = average density of neutrons in two media.

$$\sigma_a = \frac{\sigma_{a0}\bar{n}_0 V_0 + \sigma_{a1}\bar{n}_1 V_1}{\bar{n}_0 V_0 + \bar{n}_1 V_1} = \frac{[\sigma_{a0}\bar{n}_0 V_0 + \sigma_{a1}\bar{n}_1 V_1]}{\sigma_{a1}[\bar{n}_0 V_0 + \bar{n}_1 V_1]}\sigma_{a1}$$

$$\sigma_a \approx \frac{\text{total absorbed}}{\text{absorbed in moderator}}\sigma_{a1} \approx \frac{1}{1-f}\sigma_{a1}$$

$$\boxed{\therefore L^2 \approx \frac{1-f}{3N^2\sigma_1\sigma_{a1}} = \frac{(1-f)}{\kappa_1^2} = (1-f)L_1^2}$$

$\frac{1}{\kappa_1} = L_1$ = diffusion length of moderator.

Since $(1-f) \sim 12\%$, $L^2 \sim 2500$ cm^2 (for graphite), $L^2 \sim 300$, $M^2 = L^2 + \tau \sim 700$.

<u>Experimental determination of Δ and k.</u> (Theory of exponential experiment)

The fundamental experiment (called the <u>exponential</u> or <u>intermediate</u> experiment) is the measurement of Δ, and from this, k. The theory is identical with the theory of the σ pile, at least in the approximation to which the equation

$$\Delta n v = -\frac{k-1}{M^2} n v$$

is valid. A source of fast neutrons, (Ra-Be) is placed at the bottom of a pile of lattice. The size of the pile must be large compared to the volume of a lattice cell, in order for homogeneous theory to be applicable. If the source is placed in x = y = o plane, the thermal neutron distribution is (neglecting end effects)

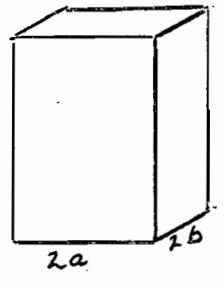
$$n v = \sum_{\mu,\nu} A_{\mu\nu} \cos\frac{\mu\pi x}{2a} \cos\frac{\nu\pi x}{2b}\, e^{-\frac{z}{c_{\mu\nu}}} \qquad \mu, \nu = \text{odd integers}$$

where, in order for equation to be satisfied

$$\Delta = -\frac{k-1}{M^2} = -\frac{\mu^2\pi^2}{4a^2} - \frac{\nu^2\pi^2}{4b^2} + \frac{1}{c_{\mu\nu}^2}$$

Again, for 1,1 harmonic, and a = b,

$$\boxed{\Delta = \frac{1}{c^2} - \frac{\pi^2}{2a^2}}$$



The intermediate experiment consists of a measurement of the relaxation length c (suitably corrected for end effects). From this Δ is determined by use of the above equation.

From the observed Cd ratio (C.R)(corrected for Cd absorption) we determined (1-f) as follows:

$\frac{\lambda n v}{f} = A \times C.R.$ where A is found from previous standardization experiments.

$$1-f = \frac{\text{thermals absorbed in moderator per cm}^3}{\text{thermals produced/cm}^3} = \frac{N\sigma_a n v}{f} = \frac{N\sigma_a}{\lambda}\frac{\lambda n v}{f}$$

$$\therefore \boxed{1-f = \frac{\pi^2}{3} A \times C \cdot R.}$$

From 1-f and π we determine $L^2 (= (1-f)/L_1^2)$ and hence M^2. The multiplication factor is thus

$$\boxed{k-1 = -M^2 \Delta}$$

However, because of the great uncertainties in the values of τ, the quantity k-1 is less accurate than Δ.

Technique of exponential experiment

Certain details of the actual experimental technique may be mentioned here.

Placement of sources: We can eliminate certain harmonics by placing sources with sufficient symmetry. For example, we can eliminate (1,3) and (3,1) harmonics by placing four identical point sources at points $(\frac{a}{2},0)$, $(0,-\frac{a}{2})$, $(-\frac{a}{2},0)$, $(0,-\frac{a}{2})$. (Exercise: Prove this statement.)

Natural neutrons: Because of spontaneous fission there will always be "natural" neutrons present in the pile. The distribution of these will be determined by the geometry of the pile, and their intensity must be subtracted from the observed intensities in determining n. The importance of the natural neutrons increases as the critical position is reached. (See Wheeler & Ibser, C-71.)

Shielding of pile: The $\lambda/\sqrt{3}$ correction is applicable strictly only if the flow at a surface is unidirectional; this will not be the case if any appreciable wall reflection is present. In order to minimize the effect of wall reflection, sides of pile are draped with Cd sheets. This still does not eliminate reflection of fast neutrons.

Least squares fitting of exponential: Usually the exponential is fitted by a least squares technique in order to determine the best relaxation length.

Correction to observed thermal utilization for finite size of pile

The observed $(1-f)_o$ is different than the $(1-f)$ in an infinite lattice because some of the thermal neutrons leak out. Since the fraction of thermal neutrons which leak out per unit volume is

$$-L^2\Delta,$$

we may write

$$(1-f)_{obs.} = \frac{\text{actual moderator thermal neutron density} \times N_1\sigma_{a1}}{\text{thermal neutrons produced/c.c.}}$$

$$= \frac{\text{thermal density if no leakage} \times (1+L^2\Delta) \times N_1\sigma_{av}}{\text{thermal neutrons produced/c.c.}}$$

$$= (1-f)(1+L^2\Delta)$$

$$\therefore (1-f) = (1-f)_o(1-L^2\Delta) \text{ since } L^2\Delta \ll 1.$$