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PILE THEORY LECTURE NOTES

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May 13, 1943

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D. Mathematical note on Bessel functions[1]

Since the actual geometry of the pile is usually cylindrical, it will be convenient for future reference to collect here some properties of Bessel functions.

The equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \alpha^2 u = 0$$

which describes, for example, diffusion in a circular cylinder of infinite length with production proportional to the density, is a special case of "Bessel's equation". A solution of this equation which is regular at the origin is

$$J_o(\alpha r) = 1 - \left(\frac{\alpha r}{2}\right)^2 + \frac{\left(\frac{\alpha r}{2}\right)^4}{2!\cdot 2!} - \frac{\left(\frac{\alpha r}{2}\right)^6}{3!\cdot 3!} + \cdots$$

which is a Bessel function of the first kind and order 0. The Bessel function of the first kind can be generalized to higher orders by recursion formulas, the only one of which we shall need is

$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x)$$

or

$$J_o'(x) = -J_1(x) ,$$

from which we deduce the series

$$J_1(x) = \frac{x}{2} - \frac{2(x/2)^3}{2!\cdot 2!} + \frac{3(x/2)^5}{3!\cdot 3!} - \cdots$$

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[1] See, e.g., Whittaker and Watson, A COURSE OF MODERN ANALYSIS.

The $J_o(x)$ function resembles a damped cosine wave, $J_1(x)$ resembles a damped sine. Thus, the equation

$$J_o(x) = 0$$

has infinitely many roots which asymptotically are separated by π. (Compare with $\cos x = 0$)

We shall need the following relations between $J_o(x)$ and $J_1(x)$:

$$\frac{d}{dx} J_o(x) = - J_1(x) \qquad \int xJ_o(x)dx = xJ_1(x)$$

A second solution of Bessel's equation which is regular at infinity but has a singularity at the origin is

$$Y_o(x) = \frac{2}{\pi} J_o(x) \left[\log(\tfrac{1}{2}x) + 0.5772\right] + (\tfrac{1}{2}x)^2 - \frac{(1 + \frac{1}{2})(\frac{1}{2}x)^4}{2!^2} + \ldots$$

or, more generally,

$$xY_n'(x) - nY_n(x) = - xY_{n+1}(x) \quad ,$$

and in particular

$$Y_o'(x) = - Y_1(x)$$

The functions $Y_n(x)$ are useful in problems involving hollow cylinders. The Y_o and Y_1 satisfy the relations

$$\frac{d}{dx} Y_o(x) = - Y_1(x) \quad , \qquad \int xY_o(x)dx = Y_1(x)$$

If we substitute ix for x in $J_o(x)$, we obtain the Bessel functions of imaginary argument. For example,

$$I_o(x) \equiv J_o(ix) = 1 + \left(\frac{x}{2}\right)^2 + \frac{\left(\frac{x}{2}\right)^4}{2! \cdot 2!} \quad \ldots$$

which satisfies the differential equation,

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - u = 0$$

$I_o(x)$ and $I_1(x)$ satisfy the relation

$$\frac{dI_o(x)}{dx} = I_1(x) \quad , \qquad \int xI_o(x)dx = xI_1(x)$$

The Bessel functions of imaginary argument are analagous to the hyperbolic functions. $I_n(x)$ becomes infinite as $e^x/\sqrt{x}$ as $x \to \infty$. The analogue of $Y_o(x)$, i.e., the Bessel function of imaginary argument which is infinite at the origin, is denoted by $K_o(x)$. It is a second linearly independent solution of Bessel's equation with imaginary argument. $K_n(x)$ is infinite at the origin and approaches zero as $e^{-x}/\sqrt{x}$ as $x \to \infty$.

<u>Expansions in Bessel functions</u>

Suppose α_n and α_m are two roots of the equation

$$J_o(\alpha) = 0 \quad .$$

Then

$$\int_o^1 xJ_o(\alpha_n x)\, J_o(\alpha_m x) = 0 \qquad \text{if } n \neq m$$

$$= \tfrac{1}{2} J_1^2(\alpha_n) \qquad \text{if } n = m$$

This may be proved from the differential equation for J_o:

$$\left|\begin{array}{l} \dfrac{d^2J_o(\alpha_n x)}{dx^2} + \dfrac{1}{x}\dfrac{dJ_o(\alpha_n x)}{dx} + \alpha_n^2 J_o(\alpha_n x) = 0 \\[2ex] \dfrac{d^2J_o(\alpha_m x)}{dx^2} + \dfrac{1}{x}\dfrac{dJ_o(\alpha_m x)}{dx} + \alpha_m^2 J_o(\alpha_m x) = 0 \end{array}\right.$$

Multiply first equation by $xJ_o(\alpha_m x)$, second by $-xJ_o(\alpha_n x)$, add, and integrate from $x = 0$ to $x = 1$. Result is

$$\int_0^1 x\left[J_o(\alpha_m x)\frac{d^2J_o(\alpha_n x)}{dx^2} - J_o(\alpha_n x)\frac{d^2J_o(\alpha_m x)}{dx^2}\right]dx$$

$$+\int_0^1\left(J_o(\alpha_m x)\frac{dJ_o(\alpha_n x)}{dx} - J_o(\alpha_n x)\frac{dJ_o(\alpha_m x)}{dx}\right)dx$$

$$= (\alpha_m^2 - \alpha_n^2)\int_0^1 xJ_o(\alpha_n x)\,J_o(\alpha_m x)\,dx$$

Integrating by parts, and using fact that $J_o(\alpha_n) = J_o(\alpha_m) = 0$,

we obtain

$$(\alpha_m^2 - \alpha_n^2)\int_0^1 xJ_o(\alpha_n x)\,J_o(\alpha_m x)\,dx = 0$$

or, if $\alpha_n \neq \alpha_m$,

$$\int_0^1 xJ_o(\alpha_n x)\,J_o(\alpha_m x)\,dx = 0 \quad .$$

<u>EXERCISE</u>: Prove, from the differential equation, that

$$\int_0^1 x\,J_o^2(\alpha_n x)\,dx = \tfrac{1}{2}\left[J_1^2(\alpha x) + J_o^2(\alpha x)\right]_0^1$$

The orthogonality property of the Bessel functions enables us to write, formally,

$$f(x) = \sum a_n\,J_o(\alpha_n x) \qquad 0 \leq x \leq 1$$

where, by orthogonality,

$$a_n = \frac{\int_0^1 xf(x)J_o(\alpha_n x)\,dx}{\int_0^1 x\,J_o^2(\alpha_n x)\,dx}$$

The possibility of such an expansion depends on further properties of the set of $J_o(\alpha_m x)$ (completeness) which we shall not consider here. (cf. Whittaker & Watson, A COURSE OF MODERN ANALYSIS, Chap. XVII)