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Contract No. W-35-058, Eng. 71

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THE SLOWING DOWN LENGTH IN MIXTURES

By

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THE SLOWING DOWN LENGTH IN MIXTURES

M. E. Rose

The purpose of the following is: (1) To present a general formalism based on a minimum number of assumptions which provides a very convenient starting point for all detailed calculations of the slowing down length. (2) To apply this formalism to some cases of particular interest, notably mixtures of light and heavy water. Two cases are considered: First, a mixture containing only a small amount of light water wherein the finite mass of all the non-hydrogeneous constituents is taken into account so that these results may also be applied to other cases, for example Be-H_2O mixtures. Second, mixtures which are predominantly composed of light water with small amounts of heavy water are also considered.

I. General Theory

We start with the Boltzmann equation for the collision density, $\psi(z,\mu,u)$ (the number of collisions per sec per cm^3 per logarithmic energy interval and per unit solid angle). For a plane isotropic source and a homogeneous non-absorbing slowing down medium we have

$$\lambda\mu\frac{\partial\psi}{\partial z}+\psi=\sum_k\int_{\zeta_k}^{u} d\Omega' du' f_k(u-u',\mu_0)\,\psi(z,\mu',u')c_k(u')+\frac{1}{4\pi}\delta(z)\,\delta(u) \tag{1}$$

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The notation is the customary one but is repeated here for convenience. In (1) $\mu = \cos\vartheta$ and ϑ defines the direction of motion with respect to the z-axis (normal to the plane source at z = 0) while $u = \log E_0/E$ where E_0 is the primary (source) energy. The components of the mixture are designated by the index k and the scattering kernel giving the probability that the logarithmic energy changes from 0 to u and that the scattering angle is arccos μ_0 in a collision with the kth nuclear species is (for spherically symmetric scattering in the center of gravity reference frame)

$$f_k(u,\mu_0) = \frac{(M_k+1)^2}{8\pi M_k} e^{-u} \delta(\mu_0 - \phi_k(u)), \quad u < \alpha_k$$
$$\phi_k(u) = \tfrac{1}{2}\left[(M_k+1)e^{-u/2} - (M_k-1)e^{u/2}\right] \tag{2}$$

where α_k is the maximum logarithmic energy gain[1)]

$$\alpha_k = 2\log\frac{M_k+1}{M_k-1}$$

By the conservation laws

$$f_k = 0 \qquad u > \alpha_k \tag{2a}$$

so that f_k is discontinuous at α_k. The lower limit in (1) is defined by

$$\zeta_k = 0 \qquad u < \alpha_k$$
$$\zeta_k = u - \alpha_k \qquad u > \alpha_k \tag{3}$$

and takes care of (2a). Evidently the normalization is

$$2\pi\int d\mu_0\, du'\, f_k(u-u',\mu_0) = 1 \tag{4}$$

1) While the energy decreases in each collision the logarithmic energy u must, of course, increase

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In the convection term of (1) λ is the total mean free path

$$\frac{1}{\lambda} = \sum_k N_k \sigma_k \tag{5}$$

and in the scattering term the c_k are the relative number of collisions with nuclei of type k:

$$c_k = \lambda N_k \sigma_k \tag{5a}$$

with

$$\sum_k c_k = 1$$

The primary point in the following procedure is to recognize the fact that we are interested only in the second moment of the distribution, that is, in

$$\overline{z^2} = \frac{\int z^2 \psi(z,\mu,u)\, d\mu\, dz}{\int \psi(z,\mu,u)\, d\mu\, dz} \tag{6}$$

or in the slowing down length L_s defined by

$$L_s^2 = \tfrac{1}{2}\, \overline{z^2} \tag{6a}$$

Since these quantities depend only on the initial and final energies, in fact only on u, equations involving only moments of the spatial distribution are simpler to work with than (1). These equations (see (15) below)have the further simplicity that they are simply integral equations and not integro-differential equations.

We first expand the collision density as well as the scattering kernels in Legendre series:

$$\psi(z,\mu,u) = \frac{1}{4\pi} \sum_0^\infty (2\ell + 1)\, \psi_\ell(z,u) P_\ell(\mu) \tag{7}$$

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$$f_k(u,\mu_0) = \frac{1}{4\pi} \sum_0^\infty (2\ell + 1) f_{\ell k}(u) P_\ell(\mu_0) \tag{7a}$$

$$\psi_\ell = \int P_\ell(\mu)\, \psi(z,\mu,u)\, d\Omega \tag{7b}$$

$$f_{\ell k} = \int P_\ell(\mu) f_k(u,\mu_0)\, d\Omega_0 \tag{7c}$$

$$= \frac{(M_k + 1)^2}{4M_k} e^{-u} P_\ell(\phi_k)$$

and (1) becomes

$$\frac{\lambda}{2\ell + 1}\left[\ell \frac{\partial \psi_{\ell-1}}{\partial z} + (\ell + 1)\frac{\partial \psi_{\ell+1}}{\partial z}\right] + \psi_\ell$$

$$= \sum_k \int_{\zeta_k}^{u} du'\, \psi(z,u') c_k(u') f_{\ell k}(u-u') + \delta_{\ell 0}\, \delta(z)\, \delta(u) \tag{8}$$

We now define the generalized moments by

$$M_{n\ell}(u) = \int z^n\, \psi_\ell(z,u)\, dz \tag{9}$$

so that

$$\overline{z^2} = \frac{M_{20}(u)}{M_{00}(u)} \tag{9a}$$

From (8) we get

$$M_{n\ell} - \frac{\lambda n}{2\ell + 1}\left[\ell M_{n-1,\ell-1} + (\ell + 1) M_{n-1,\ell+1}\right] \tag{10}$$

$$= \sum_k \int_{\zeta_k}^{u} du'\, M_{n\ell}(u') c_k(u') f_{\ell k}(u-u') + \delta_{\ell 0}\, \delta_{n0}\, \delta(u)$$

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where we have used[2)]

$$\int z^n \frac{\partial \psi_\ell}{\partial z}\, dz = -n\, M_{n-1,\ell} \tag{10a}$$

since

$$(z^n \psi_\ell)_{z=\pm\infty} = 0$$

From (9a) we are interested in the case $n = \ell = 0$, $n = 2$ and $\ell = 0$. However, from (10) it is seen to be necessary to consider the cases $n = \ell = 1$ and $n = 0$, $\ell = 2$ as well. We find

$$M_{00} = \delta(u) + \sum_k \int_{\varsigma_k}^{u} du' M_{00}(u') c_k(u') f_{0k}(u-u') \tag{11a}$$

$$M_{11} - \frac{\lambda}{3}(M_{00} + 2M_{02}) = \sum_k \int_{\varsigma_k}^{u} du' M_{11}(u') c_k(u') f_{1k}(u-u') \tag{11b}$$

$$M_{20} - 2\lambda M_{11} = \sum_k \int_{\varsigma_k}^{u} du' M_{20}(u') c_k(u') f_{0k}(u-u') \tag{11c}$$

$$M_{20} = \sum_k \int_{\varsigma_k}^{u} du' M_{02}(u') f_{2k}(u-u') \tag{11d}$$

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2) Special considerations are necessary for $n = 0$ for the following reason. Because of the symmetry of the problem

$$\psi(z,\mu,u) = \psi(-z,-\mu,u)$$

so that ψ_ℓ is an even function of z if ℓ is even and an odd function if ℓ is odd. It follows that for even ℓ, ψ_ℓ are continuous at the source $z = 0$ but have a discontinuous derivative (cf. (8)). For odd ℓ there is a discontinuity in ψ_ℓ at $z = 0$ and $\partial\psi_\ell/\partial z$ has a δ-function singularity at this point. For $n = 0$ in (10a) and ℓ even the left hand side vanishes because $\partial\psi_\ell/\partial z$ is an odd function. For ℓ odd $\partial\psi_\ell/\partial z$ is an even non-singular function plus a multiple of a δ-function. The contributions of these two parts just cancel and therefore (10a) is valid in general.

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The system of moment equations is therefore a closed one. It will be noted that this result is exact, with respect to the assumptions on which (1) is based, and no neglections corresponding to diffusion theory, termination of the Legendre series (7) or any other approximative schemes are used.

The system of equation (11) can be further simplified by observing that $M_{02} = 0$. The proof is given in Appendix I. In fact, it is shown that all the moments $M_{0\ell}$ for $\ell \neq 0$ must vanish. This leads to the result that the <u>integrated collision density is isotropic</u>; that is,

$$\int \Psi(z,\mu,u)dz = \frac{1}{4\pi}\int \Psi_0(z,u)dz \qquad (12)$$

is independent of direction or of the orientation of the plane source.

Eq'ns. (11) are further simplified by recognizing that in u - space each of the M's has a δ-function singularity at the primary value $u = 0$. This follows at once from (11 a-c) and the fact that the integral of a δ-function singularity is non-singular. We, therefore, set

$$\begin{aligned} M_{00} &= K_0 + a_0\,\delta(u) \\ M_u &= K_1 + a_1\,\delta(u) \qquad (13) \\ M_{20} &= K_2 + a_2\,\delta(u) \end{aligned}$$

where K_0, K_1 and K_2 are non-singular at $u = 0$. We find

$$a_0 = 1, a_1 = \frac{1}{3}\lambda_0, \quad a_2 = \frac{2}{3}\lambda_0^2 \qquad (14)$$

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where the subscript 0 means that the quantity is evaluated at the primary energy or at $u = 0$. Then (11) becomes

$$K_0 = \sum_k \int_{\varsigma_k}^{u} K_0(u')c_k(u')f_{0k}(u-u')du' + c_k(0)f_{0k}(u) \qquad (15a)$$

$$K_1 - \frac{\lambda}{3} K_0 = \sum_k \int_{\varsigma_k}^{u} K_1(u')c_k(u')f_k(u-u')du' + \frac{\lambda_0}{3} c_k(0)f_{1k}(u) \qquad (15b)$$

$$K_2 - 2\lambda K_1 = \sum_k \int_{\varsigma_k}^{u} K_2(u')c_k(u')f_{0k}(u-u')du' + \frac{2}{3}\lambda_0^2 c_k(0)f_{0k}(u) \qquad (15c)$$

The second term on the right hand side of each of these equations represents the contribution of neutrons reaching the energy interval in question from the source energy by a single collision. The kth term in each of these sums drops out for $u > \alpha_k$ (cf. 2a).

The second moment is now given by[3)]

$$\overline{z^2} = K_2/K_0 \qquad (16)$$

The eq'ns. (15) form a general representation of the second moment problem and a convenient starting point for all the approximate procedures which have already been developed[4)].

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3) The function K_0, which gives the energy distribution of the neutrons, was the subject of a paper by G. Placzek, Phys. Rev. 69, 423, (1946).

4) See e.g. Marshak, Slowing Down of Neutrons by Elastic Collisions with Nuclei.

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Following a similar procedure one can express the higher moments of the spatial distribution in terms of K-functions[5]. However, contenting ourselves with the consideration of the second moment equations (15) we find that the primary difficulty in solving these equations arises from the fact the the K-functions are not analytic. In fact, considering for a moment the simpler case of a single element so that the sums reduce to one term, one sees that because of the source term the K-functions are discontinuous at α have discontinuous first derivative at 2α and in general discontinuous $(n-1)$st derivative at $n\alpha$. These discontinuities are simply due to the fact that $u = \alpha$ - we count neutrons which have made all numbers of collisions from 0 to while at $u = \alpha$ neutrons which have made no collisions are excluded. Again the discontinuity at $n\alpha$ is simply the contribution of neutrons which have made just $(n-1)$ collisions. For large u it is clear that these discontinuities are less and less important, occurring for $u > n\alpha$ only in the $(n-1)$st and higher derivatives. Hence the K-functions approach smooth functions asymptotically. Thus, it is a comparatively simple matter to obtain the asymptotic behavior of the K's. As is to be expected, this asymptotic behavior simply gives the Fermi age for a single element.

$$L_s^2 = \frac{1}{3}\,\frac{1}{\xi\,(1-\bar{\mu})}\int_0^u \lambda^2(u')\,du'$$

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5) For example, the calculation of $\overline{z^4} = M_{40}/M_{00}$ depends on the solution of seven coupled integral equations.

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where

$$\xi = \int_0^{\alpha} u f_0(u)\,du = 1 - \frac{(M-1)^2}{2M} \log \frac{M+1}{M-1}$$

$$\bar{\mu} = \int_0^{\alpha} f_1(u)\,du = \frac{2}{3M}$$

are the average logarithmic energy gain and the average cos of the scattering angle.

Correspondingly, for a mixture there are discontinuities of all three K-functions at $u = \alpha_k$ for each k and discontinuities in first derivatives at $\alpha_k + \alpha_j$. In fact, all points $\sum_k n_k \alpha_k$ where n_k are a set of positive integers, are points of discontinuity as is physically clear. However, one can also obtain the asymptotic behavior in this case leading to the usual Fermi age for L_s^2. In the next section we consider a case in which the Fermi age approximation is insufficient.

II. Application To Heavy Water - Light Water Mixtures

It is clear that a rigorous solution of (15) can be obtained in the limiting cases of hydrogen, for which $\alpha = \infty$ so that no point of discontinuity occurs in the region of interest, and in the trivial case of a scattering nucleus with infinite mass for which $\alpha = 0$. The latter case, of course, is trivial but the combination of the two cases, hydrogen plus infinite mass scatterer which represents H_2O quite well,

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can also be carried out for essentially the same reason that no point of discontinuity occurs for finite non-vanishing u[6]. In this case the equations reduce to first order differential equations and their solution leads to the well-known result for the slowing down length in H_2O[7].

A more general case is the one in which the mixture contains an element such as deuterium, the mass of which may not be considered infinite. In fact, in the case of D_2O mixtures the mass of the oxygen should not be taken as infinite either unless the amount of oxygen present is considerably greater than the amount of deuterium. We, therefore, consider two cases below. First, we consider the case of a small amount of H_2O in an H_2O - D_2O mixture wherein the mass of the oxygen is taken to be finite. Second, we consider a H_2O - D_2O mixture in which the former is the major constituent.

Case I. Small H_2O Concentration

In this case the slowing down length is obtained as a correction on the age formula for D_2O. The latter is obtained, where no H is present, by assuming a slow variation of the weighted K-function,

6) While it is possible to solve the equations (15) for a single element the solution must be carried out in each interval 0 to α, α to 2α etc. separately. Such a formal solution cannot be applied practically for the energies of interest, viz: about seven to eight slowing down intervals in deuterium.
7) LA-53, MT-17 and reference 4.

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$c_k K_n$, in the slowing down interval $u - \alpha_k$ to u, corresponding to small α_k. This means that the product $c_k K_n$ in the scattering terms are expanded in a Taylor series. Denoting all quantities referring to D_2O alone by an asterisk we obtain

$$K_0^* = \frac{1}{\xi_A^*} \tag{17a}$$

$$K_1^* = \frac{\lambda^*}{3} \frac{K_0^*}{(1-\bar{\mu}^*)A} \tag{17b}$$

$$\frac{d}{du} \xi_A^* K_2^* = 2 \lambda^* K_1^* \tag{17c}$$

where the subscript A means averaged over the constituents of the mixture, e. g.,

$$\xi_A^* = \sum_k c_k(u) \xi_k$$

and in (17) the subscript k takes on only the two values corresponding to the two constituents in D_2O. When (17c) is integrated subject to the age assumption $K_2(0) = 0$ we get, as usual,

$$2\tau = \frac{K_2^*}{K_0^*} = \frac{2}{3} \int_0^u \frac{\lambda^{*2}(u')du'}{\xi_A^*(1-\bar{\mu})_A^*} \tag{17d}$$

The effect of the H_2O present is now considered by replacing K_0, K_1 and K_2 in the scattering terms due to H by the results (17). It is to be emphasized that the usual Fermi-age assumption, valid for small slowing down interval α_k is not made for the scattering by H nuclei

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where the slowing down interval is large. In the scattering terms corresponding to the non-hydrogeneous constituents the age expansion is again made. Then one gets

$$\frac{d\,\xi^*_A K_0}{du} + c_H K_0 = A_0(u) \tag{18}$$

$$A_0 = c_H(0)e^{-u} + \int_0^u \frac{c_H(u')}{\xi^*_A(u)}\, e^{u'-u}\, du' \tag{18a}$$

Choosing the integration constant so that $K_0 = K_0^*$ for $c_H = 0$ we obtain

$$K_0 = \frac{1}{\xi^*_A}\, e^{-\int_0^u (c_H/\xi^*_A)du'} \left[1 + \int_0^u A_0(u')e^{\int_0^{u'} c_H/\xi^*_A\, du''}\right] \tag{18b}$$

Similarly

$$K_1 = \frac{1}{3}\,\frac{1}{1-\bar{\mu}^*_A}\left[\lambda K_0 + \int_0^u \frac{\lambda^*(u')K_0^*(u')}{1-\bar{\mu}^*_A(u')}\, c_H(u')e^{\frac{3}{2}(u'-u)}\, du' + \lambda_0 c_H(0)e^{-3u/2}\right] \tag{19}$$

For K_2 we obtain

$$K_2 = \frac{2}{\xi^*_A}\, e^{-\int_0^u (c_H/\xi^*_A)du'} \int_0^u A_2(u')e^{\int_0^{u'} (c_H/\xi^*_A)du''}\, du' \tag{20}$$

where

$$A_2(u) = \lambda K_1 + \int_0^u \tau(u')K^*_0(u')c_H(u')e^{u'-u}\, du' + \frac{1}{3}\lambda_0^2\, c_H(0)e^{-u} \tag{20a}$$

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and τ is defined by (17d). Eq'ns. (17) - (20) give the slowing down length from

$$L_s^2 = \frac{1}{2}\frac{K_2}{K_0} \tag{20b}$$

in terms of quadratures.

If we expand the above in powers of c_H, retaining no terms beyond first order, the result obtained is

$$\begin{aligned} L_s^2 = {} & \tau(u)\Big[1 - c_H(0)(1-e^{-u}) - \int_0^u \frac{c_H(u')}{\xi_A^*(u')}\left(e^{u'-u} - 1\right)du'\Big] \\ & + \frac{\lambda_0^2}{3} c_H(0)\left(u - 1 + e^{-u}\right) + \int_0^u du'\,\tau(u')\Big[c_H(0)(1-e^{-u}) \\ & + \int_0^{u'} \frac{c_H(u'')}{\xi_A^*(u'')} e^{u''-u'} du'' + \frac{\xi_A^*(u')}{\lambda(u')}\lambda_0 c_H(0) e^{-3u'/2} \\ & + \frac{\xi_A^*(u')}{\lambda(u')}\int_0^{u'} \frac{\lambda^*(u'')c_H(u'')}{\xi_A^*(u'')} \frac{e^{\frac{3}{2}(u''-u')}}{1-\bar{\mu}_A^*(u'')} du''\Big] \\ & + \int_0^u \frac{c_H(u')}{\xi_A^*(u')} du' \int_0^u du'\,\tau(u') + \int_0^u du' \int_0^{u'} du''\,\tau(u'')\frac{c_H(u'')}{\xi_A^*(u'')} e^{u''-u'} \end{aligned} \tag{20c}$$

In the foregoing we have omitted the effect of "first" collisions in the case of the non-hydrogeneous constituents although these have

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been included for the hydrogen collisions. These effects could be included in a straightforward manner which in the case of D_2O alone, may be described as follows: The equations obtained by making the Taylor expansions in the slowing down integrals for D and O are considered to apply for $u > \alpha_D$ only and not for all u. To obtain the initial conditions $K_n(\alpha_D +)$ the integral equations for $u < \alpha_D$ are solved. This can be done directly if the slowing down in O is neglected so that effectively one deals with a single element. Then in the interval 0 to α_D the eq'n (15) become differential equations, of first order for K_0 and K_2 and of second order with constant coefficients for K_1. From (15) we get $K_n(0)$ and the discontinuities $K_n(\alpha_D +) - K_n(\alpha_D -)$. A similar procedure is followed in the case that a small amount of H_2O is present. Fortunately, on the basis of the age calculations for D_2O it seems advantageous to dispense with these additional complications. Such effects are considered in the case of small D_2O concentration treated below.

Case 2. Small D_2O Concentration

We now treat the mass of the O nucleus as infinite since the comparison of the slowing down and average scattering in O is now to be made with that due to H nuclei which are present in much greater quantity than the D nuclei. With infinite mass constituents in the mixture Eq's. (15) must be modified. If j designates the infinite mass constituent we have

$$f_{\ell j}(u-u') = \delta(u-u')\ \delta_{\ell 0} \qquad (21)$$

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The substitution (13) now gives

$$a_0 = \frac{1}{1 - c_j(0)}$$

$$a_1 = \frac{1}{3} \frac{\lambda_0}{1 - c_j(0)} \tag{22}$$

$$a_2 = \frac{2}{3} \frac{\lambda_0^2}{(1-c_j(0))^2}$$

and the moment equations

$$(1-c_j)K_0 = \sum_{k \neq j} \left[\int_{\zeta_k}^{u} du' K_0(u') f_{0k}(u-u') c_k(u') + a_0 c_k(0) f_{0k}(u) \right] \tag{23a}$$

$$K_1 - \frac{\lambda}{3} K_0 = \sum_{k \neq j} \left[\int_{\zeta_k}^{u} du' K_1(u') f_{1k}(u-u') c_k(u') + a_1 c_k(0) f_{1k}(u) \right] \tag{23b}$$

$$(1-c_j)K_2 - 2\lambda K_1 = \sum_{k \neq j} \left[\int_{\zeta_k}^{u} du' K_2(u') f_{0k}(u-u') c_k(u') + a_2 c_k(0) f_{0k}(u) \right] \tag{23c}$$

where here and in the following the a's are given by (22).

When the elements of finite mass are hydrogen plus a second element (deuterium) whose effective concentration are c_H and c_2 respectively the Eq. (23) becomes, for $u > \alpha$

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$$(c_H + c_2)K_0 = \int_0^u K_0(u')f_{0H}(u-u')c_H(u')du' + \int_{u-\alpha}^u K_0(u')f_{02}(u-u')c_2(u')du'$$

$$+ a_0 c_H(0) f_{0H}(u) \qquad (23a)$$

$$K_1 - \frac{\lambda}{3}K_0 = \int_0^u K_1(u')f_{1H}(u-u')c_H(u')du' + \int_{u-\alpha}^u K_1(u')f_{12}(u-u')c_2(u')du'$$

$$+ a_1 c_H(0) f_{1H}(u) \qquad (23b)$$

$$(c_H + c_2)K_2 - 2\lambda K_1 = \int_0^u K_2(u')f_{0H}(u-u')c_H(u')du' + \int_{u-\alpha}^u K_2(u')f_{02}(u-u')c_2(u')$$

$$+ a_2 c_H(0) f_{0H}(u) \qquad (23c)$$

where α refers to the non-hydrogeneous element of finite mass (otherwise designated by subscript 2). For $u < \alpha$ the only changes are the addition of another source term $c_2(0)f_{\ell 2}(u)$ ($\ell = 0$ for K_0 and K_2, $\ell = 1$ for K_1) corresponding to collisions with element 2 in which the primary energy is the source energy, and the replacement of the lower limit in the scattering integral for element 2 by the value zero.

The method of approximation adopted in the solution of these equations and the details of solution are described in appendix II. Briefly, the essential part of the procedure is to treat the non-hydrogeneous collisions by an age procedure while treating the hydrogen

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collision terms rigorously. The result is

$$L_s^2 = \frac{e^{\alpha}}{2B_0} \chi(u) \left[\frac{G_2(\alpha)}{\chi(\alpha)} + \left(\frac{F_2}{F_1\chi}\right)_{\alpha} - \left(\frac{F_2}{F_1\chi}\right)_{u} - \int_{\alpha}^{u} \left(\frac{F_2}{\chi F_1 Z}\right)_{u'} du' \right] \tag{24}$$

see Eq's. (2.4) and (2.6) of the appendix II. Auxiliary quantities are defined or given by Eq's. (2.3a, b), (2.4a), (2.5a) (2.8), (3.1), (3.3) and (3.5).

In the limit of zero deuterium or D_2O concentration (24) reduces to the well-known formula for H_2O as it should.

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APPENDIX I

Each moment $M_{0\ell}$ for $\ell \neq 0$ satisfies the integral equation

$$M_{0\ell} = \sum_k \int_{\zeta_k}^{u} M_{0\ell}(u')c_k(u')f_{\ell k}(u-u')du' \qquad (1.1)$$

Since $f_{\ell k}(u-u') = 0$ for $u' < \zeta_k = u - \alpha_k$, the lower limit on each integral can be replaced by zero. Therefore we can write

$$M_{0\ell} = \int_0^u M_{0\ell}(u')H(u,u')du' \qquad (1.2)$$

where

$$H = \sum_k c_k(u')f_{\ell k}(u-u') \qquad (1.3)$$

Since both H and M_0 are bounded in the region of integration we have

$$M_{0\ell} < H_m (M_{0\ell})_m u \qquad (1.4)$$

where the subscript m indicates maximum value in the interval 0 to u. Using the right side of (1.4) in (1.2) we have

$$M_{0\ell} < \frac{u^2}{2!} H_m^2 (M_{0\ell})_m$$

and by iteration

$$M_{0\ell} < \lim_n \frac{u^n}{n!} H_m^n (M_{0\ell})_m \qquad (1.5)$$

For finite u the limit is zero and therefore all $M_{0\ell}$ ($\ell \neq 0$) vanish identically.

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APPENDIX II

We first consider the Eq. (23c') and for the non-hydrogeneous element the age-expansion is made. In first approximation, where only one term of the expansion is taken, we would find

$$G_2 - 2\lambda K_1 = e^{-u}\left[\int_{\alpha}^{u} G_2(u')e^{u'}du' + B_2\right] \tag{2.1}$$

where

$$G_2 = c_1 K_2 \tag{2.1a}$$

$$B_2 = a_2 c_H(0) + \int_0^{\alpha} c_1(u')K_2(u')e^{u'}du' \tag{2.1b}$$

We have taken cognizance of the fact that in the hydrogen scattering integral, taken from 0 to $u > \alpha$, the moment function K_2 which appears in the integrand must be defined from $u < \alpha$ in an analytically different manner than for $u > \alpha$. This gives rise to the second term in the constant B_2. The evaluation of this term is considered below.

For deuterium the next term in the age-expansion is important and including this term we find instead of (2.1)

$$G_2 - 2\lambda K_1 = e^{-u}\left[\int_{\alpha}^{u} G_2(u')e^{u'}du' + B_2\right] - \xi_2 \frac{dc_2K_2}{du} \tag{2.2}$$

where ξ_2 is the average logarithmic energy gain in element 2 (deuterium). Differentiation of (2.2) would lead to a second order differential equation with non-constant coefficients. To avoid this

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situation result (2.1) is used in the last term of (2.2) and we obtain after differentiation of (2.2) the first order equation

$$\frac{dG_2}{du} + \left(1 + \frac{d}{du} \log F_1 - \frac{1}{F_1}\right) G_2 = \frac{1}{F_1}\left(F_2 + \frac{dF_2}{du}\right) \tag{2.3}$$

where

$$F_1 = 1 + \xi_2 \frac{d}{du} \frac{c_2}{c_1} \tag{2.3a}$$

is a known function of u and

$$F_2 = -2\lambda K_1 \left(1 - \xi_2 \frac{c_2}{c_H}\right) + 2\xi_2 \frac{c_2}{c_H} \frac{d}{du} \lambda K_1 \tag{2.3b}$$

is determined below (see Eq. (2.8)).

The solution of (2.3) subject to the initial condition applied at $u = \alpha$ is readily found to be

$$G_2 = \chi(u)\left[\frac{G_2(\alpha)}{\chi(\alpha)} + \left(\frac{F_2}{F_1 \chi}\right)_\alpha - \left(\frac{F_2}{F_1 \chi}\right)_u - \int_\alpha^u \left(\frac{F_2}{\lambda F_1^2}\right)_{u'} du'\right] \tag{2.4}$$

arguments being indicated by a subscript where convenient. In (2.4) we use the abbreviation

$$\chi(u) = \frac{1}{F_1(u)} \exp\left[-u + \int_0^u \frac{du'}{F_1(u')}\right] \tag{2.4a}$$

which is a known function. In (2.4) the argument α actually means $\alpha + \varepsilon$ where ε is infinitesimal but positive. It should be added that (2.4) is to be used only for large u. In addition to G_2 it is necessary to find $G_0 = c_H K_0$ for $u > \alpha$. This can be done quite easily if the variation of K_0 over the slowing down interval α is

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neglected as in (2.1) above. Although this approximation is not necessary, in the sense that the same procedure as led to (2.4) may be applied, it seems a reasonable one and will be made for both K_0 and K_1. Then we find for G_0

$$G_0 = e^{-u}\left[B_0 + \int_{\alpha}^{u} G_0(u')e^{u'}\,du'\right] \tag{2.5}$$

where

$$B_0 = a_0 c_H(0) + \int_0^{\alpha} G_0(u')e^{u'}\,du' \tag{2.5a}$$

is a constant. The asymptotic solution of (2.5) is the constant

$$G_0 = B_0 e^{-\alpha} \tag{2.6}$$

It is primarily for this reason that assumption just discussed was made.

Proceeding in the same fashion we find

$$K_1(1-\bar{\mu}_2 c_2) - \frac{\lambda}{3c_H} G_0 = e^{-3u/2}\left[B_1 + \int_{\alpha}^{u} K_1(u')c_H(u')e^{3u'/2}\,du'\right] \tag{2.7}$$

where

$$\bar{\mu}_2 = 2/3M_2$$

$$B_1 = a_1 c_H(0) + \int_0^{\alpha} K_1(u')c_H(u')e^{3u'/2}\,du' \tag{2.7a}$$

The solution of (2.7) is

$$K_1 = e^{-\varphi}\left[K_1(\alpha) + \int_{\alpha}^{u} e^{\varphi(u')}P(u')\,du'\right]$$

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where

$$\varphi = \log\frac{1-\bar{\mu}_2 c_2(u)}{1-\bar{\mu}_2 c_2(0)} + \int_0^u \frac{\frac{3}{2} - c_H - \frac{3}{2}\bar{\mu}_2 c_2}{1 - \bar{\mu}_2 c_2}\, du' \tag{2.8a}$$

$$P = G_0 \frac{\frac{1}{3}\frac{d\lambda_H}{du} + \frac{1}{2}\lambda_H}{1 - \bar{n}_2 c_2} \tag{2.8b}$$

and λ_H is the mean free path in hydrogen of the same density; i.e., $\lambda_H = 1/N_H\sigma_H$ where N_H = number of H atoms per cm^3 and σ_H is the miscroscopic cross-section for H. In (2.8b) G_0 is given by (2.6). This result for K_1 is to be used in (2.3b) to find F_2.

To complete the solution of the problem we need $K_n(\alpha+)$ and $K_n(u)$ for $u < \alpha$. Once the latter are found the initial values $K_n(\alpha+)$ are found from

$$K_n(\alpha+) = K_n(\alpha-) + \Delta_n \tag{2.9}$$

where the discontinuities $_n$ are obtained directly from the integral equations (23'):

$$\Delta_0 = -\frac{(M_2-1)^2}{4M_2}\left(\frac{c_H}{c_H + c_2}\right)_0 \left(\frac{1}{c_H + c_2}\right)_\alpha \tag{2.9a}$$

$$\Delta_1 = \frac{(M_2-1)^2}{12M_2}\left(\frac{c_H}{c_H + c_2}\right)_0 \left(\lambda_0 - \frac{\lambda_\alpha}{(c_H + c_2)_\alpha}\right) \tag{2.9b}$$

$$\Delta_2 = \frac{(M_2-1)^2}{6M_2}\left(\frac{c_H}{c_H + c_2}\right)_0 \frac{1}{(c_H + c_2)_\alpha} \times\left[\lambda_\alpha\left(\lambda_0 - \frac{\lambda_\alpha}{(c_H + c_2)_\alpha}\right) - \frac{\lambda_0^2}{(c_H + c_2)_0}\right] \tag{2.9c}$$

For $u < \alpha$ we have

$$(c_H + c_2)K_0 = e^{-u}\left[a_0 \gamma(0) + \int_0^u \gamma(u')K_0(u')e^{u'}du' \right] \qquad (3.0)$$

$$\gamma(u) = c_H(u) + \frac{(M_2+1)^2}{4M_2} c_2(u)$$

The solution of (3.0) is obtained by differentiation and is

$$K_0 = \frac{a_0\gamma(0)}{c_H + c_2} \exp\left[\frac{(M_2-1)^2}{4M_2} \int_0^u \frac{c_2}{c_H + c_2} du'\right] \qquad (3.1)$$

Comparison with (2.5) and (2.5a) shows that K_0 or G_0 is now completely defined.

To obtain K_1 for $u < \alpha$ we note that the pertinent integral equation

$$K_1 - \frac{\lambda}{3}K_0 = \int_0^u K_1(u')\left[\left(c_H(u) + c_2(u')\frac{(M_2+1)^3}{8M_2}\right)e^{-\frac{3}{2}(u-u')} - c_2(u')\frac{(M_2+1)^2(M_2-1)}{8M_2}e^{-\frac{1}{2}(u-u')}\right]du'$$

$$+ a_1\left[\left(c_H(0) + c_2(0)\frac{(M_2+1)^3}{8M_2}\right)e^{-3u/2} - c_2(0)\frac{(M_2+1)^2(M_2-1)}{8M_2}e^{-u/2}\right] \qquad (3.2)$$

can be put in the form of a linear second order differential equation.[8] Unfortunately, for a mixture the coefficients are not constant. However, if we set $c_2 = 0$ as is consistent with the approximation here considered, the resulting first order equation gives

$$K_1 = \frac{1}{3} e^{-\frac{3u}{2} + \int_0^u c_1 du} \left\{ \lambda_0 \frac{1 + c_H(0)}{c_H(0)} + \int_0^u e^{\frac{3u'}{2} - \int_0^{u'} c_H du''} \left[\frac{d}{du}(\lambda K_0) + \frac{3}{2}\lambda K_0 \right] du' \right\} \tag{3.3}$$

or alternatively

$$K_1 = \frac{1}{3}\lambda K_0 + \frac{1}{3} e^{-\frac{3u}{2} + \int_0^u c_H du} \left\{ \lambda_0 \left(\frac{1 + c_H}{c_H} \right)_0 - \frac{\lambda_0 a_0 \gamma(0)}{(c_H + c_2)_0} + \int_0^u du' c_H(u') \lambda(u') K_0(u') \right.$$

$$\left. \times e^{\frac{3u'}{2} - \int_0^{u'} c_H du''} \right\} \tag{3.3a}$$

where K_0 is given by (3.1)

From this result B_1 is determined (cf. 2.7a). Also from (3.3) and (2.9b) one obtains $K_1(\alpha +)$ so that $K_1(u)$ for $u > \alpha$ is now determined, see (2.8).

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8) It may be mentioned that for equations of the type considered the reduction to a differential equation depends only on the fact the kernel is of the form of a sum of terms each of which is a product of function of u and a function of u'. In general the order of the differential equation is equal to the number of terms required to express the kernel in this form.

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There remains only the determination of $K_2(u)$ for $u < \alpha$. The integral equation

$$(c_H+c_2)K_2-2\lambda K_1 = e^{-u}\left[\int_0^u K_2(u')\,\gamma(u')e^{u'}\,du' + a_2\gamma(0)\right] \qquad (3.4)$$

gives the differential equation

$$\frac{d}{du}(c_H+c_2)K_2-\frac{(M_2-1)^2}{4M_2}c_2K_2 = 2\left[\frac{d\lambda K_1}{du}+\lambda K_1\right] \qquad (3.4a)$$

and obtaining the intial condition from (3.4) we find the result

$$K_2 = \frac{2\lambda K_1}{c_H+c_2} + 2K_0\left[\frac{\lambda_0^2}{3}\,\frac{1}{(c_H+c_2)_0} + \int_0^u \frac{\lambda K_1}{(c_H+c_2)K_0}\left(1-\frac{(M_2-1)^2}{4M_2}\,\frac{c_2}{c_H+c_2}\right)du'\right] \qquad (3.5)$$

where K_0 and K_1 are given by (3.1) and (3.3).

The limiting case of pure H_2O is obtained by setting $\alpha = 0$ and $c_2 = 0$. Then $B_0 = 1$ and $G_0 = 1$ while $K_1(\alpha)=K_1(0)=\frac{1}{3}\lambda_0(1+1/c_H)_0$.

Then also

$$K_1 = \frac{1}{3}e^{-\varphi}\left[\lambda_0+\frac{\lambda}{c_H}e^{\varphi}+\int_0^u \lambda\, e^{\varphi(u')}\,du'\right]$$

$$\varphi = \int_0^u \left(\tfrac{3}{2}-c_H\right)du'$$

$$G_2(\alpha) = G_2(0) = \frac{2}{3}\,\frac{\lambda_0^2}{c_H(0)}\,(2+c_H(0))$$

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With the aid of these results it is easy to verify that the above formulae reproduce the well-known water formula for the slowing down length.

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