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PHYSICS DIVISION

SOME METHODS FOR SOLVING MULTI-GROUP PILE EQUATIONS

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PHYSICS DIVISION

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SOME METHODS FOR SOLVING MULTI-GROUP PILE EQUATIONS

I. Introduction

The question of determining the critical multiplication constant of a chain reactor is basic to many engineering and design considerations. The physical formulation of this problem leads to extremely complex mathematical equations and has not as yet been completely carried out. An approximate, and, for many purposes, sufficiently accurate formulation of the physical problem is contained in the multi-group formalism (cf. II A below). The solution of problems in multi-group theory is still, however, mathematically complex.

It is the purpose of the present report to give in detail several methods of solving the multi-group equations for the critical multiplication constant. The neutron flux distributions easily follow. Only "one coordinate" piles (those of such a shape that the equations are separable to give a set of ordinary differential equations) are treated herein. We consider here three general methods.

Direct solution of the boundary value problem:

There exist direct analytical methods which use a straightforward approach to evaluate solutions of the group problem in closed form; a special case of this technique is the matrix method for solution of the multiregion problem.

Series expansions:

The flux distribution may be expanded in series of orthogonal functions, which, when terminated at any given term, yield approximate solutions to the problem.

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Iteration techniques:

According to the kinetic theory of reactors, any initial flux distribution will, after many neutron generations, approach the equilibrium shape. The critical multiplication constant is then simply that value which keeps the neutron level constant. By successive iteration of the group solutions, using previous values as sources, the equilibrium distributions and multiplication may be obtained.

The various methods are illustrated later in this report. We confine ourselves here to a general evaluation of the techniques.

The direct methods are undoubtedly the best approach to problems of comparative mathematical simplicity (one or two groups, or single region problem). For three or more groups, they become more complicated, and iteration techniques are advised. For two groups and more than two regions, the matrix forms are recommended in preference to other analytical techniques.

The series expansion method has the disadvantage of requiring many terms to give good flux distribution, resulting in complicated formulae in which computational errors may easily creep in. However, the use of a single term in the series gives a good first approximation of the critical multiplication, when a reasonable estimate cannot be made a priori.

The iteration method is, in general, the only way to handle problems in which properties are not constant within each region. In particular, the use of numerical methods furnish the most feasible approach to problems involving a large number of groups or complicated perturbations; the numerical operations reduce to a simple routine in which errors are unlikely to appear.

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II A. FORMULATION OF THE PROBLEM

The differential equations of multi-group theory will be written as

$$(1) \quad \nabla \cdot (D_n \nabla \phi_n) - (\alpha_n + a_n)\phi_n + \alpha_{n-1}\phi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i a_i \phi_i = 0$$

where the subscript n denotes the group index in a decreasing energy scale (thus n = 1 represents the fastest group and n = N represents the thermal (or slowest) group) and where

D_n = diffusion coefficient

ϕ_n = neutron flux

f_n = fraction of fission neutrons having energies lying in the n'th group

a_n = macroscopic absorption cross-section

α_n = slowing down cross section

k_n = multiplication constant for neutrons absorbed in the n'th group.

In general, each of these will be functions of position,

$$(2) \qquad \alpha_0 = \alpha_N = 0.$$

since neutrons can neither slow down into group 1 nor slow down from group N. ϕ_n is required to be finite and positive everywhere and to vanish on the extrapolated outer boundaries. The critical mass problem may be considered an eigenvalue problem with eigenvalue parameter λ. That is, finite solutions of Eq. (1) with prescribed functions D_n, f_n, a_n, α_n, k_n, and the above boundary conditions will exist for certain discrete values of λ. The solution corresponding to the minimum value of λ, λ_{min}, is the only one which is everywhere positive. The pile is under critical, critical, or over critical, respectively as $\lambda_{min} > 1$, $\lambda_{min} = 1$, or $\lambda_{min} < 1$. Under certain conditions the differential

[redacted]

equations are separable and equations in one coordinate result. Such a set of one coordinate equations can be handled whereas the set of partial differential equations (1) is in general quite unmanageable.

A. Cartesian Coordinates

Conditions: D_n, a_n, α_n, k_n, and f_n are functions of z only

$$\Phi_n = \varphi_n(z)\, G(x,y)$$

then equation 1 becomes

$$(3) \quad G \frac{d}{dz} D_n \frac{d}{dz} \varphi_n + \varphi_n D_n \left(\frac{\partial^2 G}{\partial x^2} \quad \frac{\partial^2 G}{\partial y^2}\right) - (\alpha_n + a_n)\, G \varphi_n + \alpha_{n-1} G \varphi_{n-1}$$

$$+ \lambda G f_n \sum_{i=1}^{N} k_i \varphi_i a_i = 0$$

or

$$(4) \quad \frac{1}{\varphi_n D_n} \left\{ \frac{d}{dz} D_n \frac{d\varphi_n}{dz} - (\alpha_n + a_n)\varphi_n + \alpha_{n-1}\varphi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i \varphi_i a_i \right\}$$

$$= -\frac{1}{G} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right)$$

Since the left side contains the independent variable z and not x or y whereas the right side does not contain z each side must equal a constant which is written as $+B^2$. B^2 will then be taken as the lowest value for which Eq. (5) has a solution satisfying the boundary conditions.

$$(5) \quad \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial^2 y^2} + B^2 G = 0$$

[redacted]

Eq. (4) becomes

$$(6)\quad \frac{d}{dz} D_n \frac{d\varphi_n}{dz} - (\alpha_n + a_n + B^2 D_n)\varphi_n + \alpha_{n-1}\varphi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i \varphi_i a_i = 0$$

B. Cylindrical Coordinates

Conditions: D_n, a_n, α_n, k_n and f_n are functions of ρ only

$$\Phi_n = \varphi_n(\rho)\, G(z)$$

Eq. (1) becomes

$$(7)\quad \frac{G}{\rho}\left(\frac{d}{d\rho} D_n \frac{d\varphi_n}{d\rho}\right) + D_n \frac{d^2G}{dz^2}\varphi_n - (\alpha_n + a_n)\varphi_n G + \alpha_{n-1}\varphi_{n-1}$$

$$+ \lambda G f_n \sum_{i=1}^{N} k_i \varphi_i a_i = 0$$

or

$$(8)\quad \frac{1}{D_n \varphi_n}\left[\frac{1}{\rho}\frac{d}{d\rho} D_n \rho \frac{d\varphi_n}{d\rho} - (\alpha_n + a_n)\varphi_n + a_{n-1}\varphi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i \varphi_i a_i\right]$$

$$= -\frac{1}{G}\frac{d^2G}{dz^2} = +B^2$$

where the buckling B^2 is determined by the lowest eigenvalue of

$$(9)\quad \frac{d^2G}{dz^2} + B^2 G = 0$$

Eq. (8) becomes

$$(10)\quad \frac{1}{\rho}\frac{d}{d\rho} D_n \rho \frac{d\varphi_n}{d\rho} - (\alpha_n + a_n + D_n B^2)\varphi_n + \alpha_{n-1}\varphi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i \varphi_i a_i = 0$$

[redacted]

C. Spherical Coordinates

Conditions: D_n, a_n, α_n, k_n and f_n are functions of r only

$$\Phi_n = \varphi_n(r)$$

Eq. (1) becomes

$$(11) \quad \frac{1}{r^2} \frac{d}{dr} \left(D_n r^2 \frac{d\varphi_n}{dr}\right) - (\alpha_n + a_n)\varphi_n + a_{n-1}\varphi_{n-1} + \lambda f_n \sum_{i=1}^{N} k_i \varphi_i a_i = 0$$

These one coordinate equations can be solved in general by the numberical integration and iteration method treated in Section II D. Other methods are applicable if the pile may be divided into regions within each of which D_n, a_n, α_n, f_n and k_n are constants.

At any interface the Φ must be continuous so that $\nabla\Phi_n$ is everywhere finite, (as required by Eg. (1).

Similarly, the normal component of $D_n \nabla \Phi_n$ is continuous at any interface. Thus in the one coordinate problems φ_n, $D_n \frac{d\varphi_n}{dr}$, $D_n \frac{d\varphi_n}{d\rho}$, $D_n \frac{d\varphi_n}{dz}$ are all continuous.

For simplicity in treating these equations, it will henceforth be assumed that:

$$f_1 = 1 \qquad f_n = 0 \text{ for } n \neq 1$$

$$a_n = 0 \text{ for } n \neq N$$

These assumptions are not necessary to the methods and the necessary extension of the work below for a more general case will usually be clear.

If we let $\sigma_n = \alpha_n + a_n$ in each region, the equations (6, 10 and 11) may be written

[redacted]

$$(12)\quad \left\{\begin{array}{l} \dfrac{1}{\rho^M}\dfrac{d}{d\rho}\rho^M\dfrac{d\varphi_1}{d\rho} - \left(\dfrac{\sigma_1}{D_1}+B^2\right)\varphi_1 + \dfrac{\lambda}{D_1}k_N\sigma_N\varphi_N = 0 \\ \qquad \text{or} \\ \nabla^2\varphi_1 - \left(\dfrac{\sigma_1}{D_1}+B^2\right)\varphi_1 + \dfrac{\lambda}{D_1}k_N\sigma_N\varphi_N = 0 \end{array}\right\} \text{ for the fastest group}$$

$$(13)\quad \left\{\begin{array}{l} \dfrac{1}{\rho^M}\dfrac{d}{d\rho}\rho^M\dfrac{d\varphi_n}{d\rho} - \left(\dfrac{\sigma_n}{D_n}+B^2\right)\varphi_n + \dfrac{\sigma_{n-1}}{D_n}\varphi_{n-1} = 0 \\ \nabla^2\varphi_n - \left(\dfrac{\sigma_n}{D_n}+B^2\right)\varphi_n + \dfrac{\sigma_{n-1}}{D_n}\varphi_{n-1} = 0 \end{array}\right\} \text{ for } n \neq 1$$

where the exponent M has the values 0, 1 and 2 for the one coordinate Cartesian, cylindrical and spherical cases respectively.

II B. Solutions in closed form

This treatment will be restricted to one coordinate symmetrical piles consisting of a core and one or more reflector regions. Further, it will be assumed that k is zero in the reflectors. The general solution of the system of coupled differential equations (eq. 12 and 13, Sect. II A) is derived first then the appropriate boundary conditions at each interface are applied. This yields a set of linear equations by means of which the unknown coefficients are eliminated, resulting in a transcendental equation which may be solved by an iterative procedure to obtain λ.

[redacted]

The equations in the core are

1) $\nabla^2 \varphi_1 - (\frac{\sigma_1}{D_1} \quad B^2)\varphi_1 + \frac{\lambda k_N}{D_1} \sigma_N \varphi_N = 0$ for $n = 1$

2) $\nabla^2 \varphi_n - (\frac{\sigma_n}{D_n} + B^2)\varphi_n + \frac{\sigma_{n-1}}{D_n} \varphi_{n-1} = 0$ for $n \neq 1$

Let us assume that $\nabla^2 \varphi_n = -\mu^2 \varphi_n$ in order to find the solutions of eq. 1 and 2. Substituting there results

3) $(\mu^2 + \frac{\sigma_1}{D_1} + B^2)\varphi_1 = \frac{\lambda k_N}{D_1} \sigma_N \varphi_N$ for $n = 1$

4) $(\mu^2 + \frac{\sigma_n}{D_n} + B^2)\varphi_n = \frac{\sigma_{n-1}}{D_n} \varphi_{n-1}$ for $n \neq 1$

The φ's may easily be eliminated from this system of equations by equating the product of the N right hand sides with that of the N left hand sides and dividing both sides by the product of the N φ's, resulting in an Nth degree algebraic equation in u^2.

5) $$\prod_{n=1}^{n=N} (\mu^2 + \frac{\sigma_n}{D_n} + B^2) = \lambda k_N \prod_{n=1}^{n=N} \frac{\sigma_n}{D_n}$$

The N roots of Eq. (5), μ_1, -------- μ_N, provide the general solution of eq. (1 and 2)

6) $\varphi_1 = X_1 F(\mu_1 \rho) + X_2 F(\mu_2 \rho) +$ -------- $X_N F(\mu_N \rho)$

[redacted]

7) $$\varphi_m = X_1 S_{m1} F(\mu_1 \rho) + X_2 S_{m2} F(\mu_2 \rho) + \text{-------} X_N S_{mN} F(\mu_N \rho)$$

where the function F is the solution of $\nabla^2 F + \mu^2 F = 0$ which is regular at the origin.*

The coupling coefficients S_{mn} are determined by substitution in Eq. (2) which gives

8) $$S_{mn} = \prod_{i=2}^{i=m} \frac{\sigma_{i-1}}{D_i} \frac{1}{(\mu_n^2 + \frac{\sigma_i}{D_i} + B^2)}$$

In most pile calculations eq. 5 will have one positive root which will be designated as μ_1^2 and the other roots will be negative or complex.

If all roots are negative except μ_1^2, μ_n is pure imaginary for $n \neq 1$ let $\nu_n^2 = -\mu_n^2$ for $n \neq 1$ so that

9) $$F(\mu_n x) = F(i\ \nu_n x) = G(\nu_n x); \qquad n \neq 1$$

where $G(\nu x)$ is the regular solution* of

10) $$\nabla^2 G - \nu^2 G = 0$$

In the reflectors, remembering that $k_N = 0$ and writing $\mathcal{H}_n^2 = \frac{\sigma_n}{D_n} + B^2$,

11) $$\nabla^2 \varphi_1 - \mathcal{H}_1^2 \varphi_1 = 0 \qquad \text{for } n = 1$$

12) $$\nabla^2 \varphi_n - \mathcal{H}_n^2 \varphi_n + \frac{\sigma_{n-1}}{D_n} \varphi_{n-1} = 0 \qquad \text{for } n \neq 1$$

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* For Cartesian coordinates, the assumption of symmetry indicates that only phase solutions for which $F(z) = F(-z)$ need be considered.

[redacted]

Since equation (11) contains only φ_1, its general solution can be written immediately as a linear combination of the functions $G(\mathcal{H}_1 x)$ and $H(\mathcal{H}_1 x)$ which are the regular and irregular solutions, respectively, of eq. (11).* The solutions for $n \neq 1$ will contain $G(\mathcal{H}_n x)$ and $H(\mathcal{H}_n x)$ plus multiples of the homogeneous solutions for the faster groups.

$$13)\quad \begin{cases} \varphi_1 = Y_1\, G(\mathcal{H}_1 x) + Z_1\, H(\mathcal{H}_1 x) \\ \varphi_2 = Y_2\, G(\mathcal{H}_2 x) + Z_2\, H(\mathcal{H}_2 x) + C_{2\,1}\left[Y_1\, G(\mathcal{H}_1 x) + Z_1\, H(\mathcal{H}_1 x)\right] \\ \varphi_m = Y_m\, G(\mathcal{H}_m x) + Z_m\, H(\mathcal{H}_m x) + C_{m,m-1}\left[Y_{m-1}\, G(\mathcal{H}_{m-1} x) + Z_{m-1}\, H(\mathcal{H}_m x) + \cdots C_{m,1}\,\varphi_1\right. \end{cases}$$

The coupling coefficients C_{mn} are obtained by substituting the values of the φ_n in Eq. (12) and equating the coefficients of each function $\left[Y_n\, G(\mathcal{H}_n x) + Z_n\, H(\mathcal{H}_n x)\right]$. Thus obtaining

$$14)\qquad C_{m,n} = \prod_{i=n+1}^{i=m} \frac{\sigma_{i-1}}{D_i}\,\frac{1}{(\mathcal{H}_n^2 - \mathcal{H}_n^2)} \qquad \text{where } m > n$$

In the outermost reflector, if T is the distance from the center of the core to the outer extrapolated boundary, then a boundary condition is $\varphi_n(T) = 0$ which can be satisfied only if

$$15)\qquad Y_n\, G(\mathcal{H}_n T) + Z_n(\mathcal{H}_n T) = 0 \qquad \text{for all } n.$$

or

$$16)\qquad Y_n = -Z_n\,\frac{H(\mathcal{H}_n T)}{G(\mathcal{H}_n T)}$$

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* For Cartesian coordinates, $\cosh \mathcal{H}_1 x$ and $\sinh \mathcal{H}_1 x$, respectively.

[redacted]

Since the solution φ can only be determined within a factor (depending upon the power level of the pile), it is convenient to set X_1 equal to unity. Then by using the boundary conditions (that is the continuity of φ_n and $D_n \frac{d\varphi_n}{dx}$ at each interface R_1, R_2 etc. as discussed in Section II A) (M-1)N linear equations are obtained where M is the number of regions. By eliminating the $\left[(M-1)N - 1\right]$ unknown constants of the type X_n, Y_n and Z_n a critical transcendental equation involving functions of μ results. It is possible to solve this by an iterative process by dividing the equation into a part relatively insensitive to changes in λ and a part quite sensitive to changes in λ. μ_1 is commonly the sensitive function. The critical equation may be expressed as $\frac{\mu_1 R_1 F'(\mu_1 R_1)}{F(\mu_1 R_1)}$ equal to a function of the remaining variables where F' is the derivative of F. An assumed value of μ_1* is used to obtain λ from eq. (5)** and by evaluating all the functions in the critical equation except $F(\mu_1 R_1)$ and $\mu_1 F'(\mu_1 R_1)$ a new value of μ_1 may be obtained. A new value of λ is obtained by substitution in Eq. (5) and the process is repeated until the successive values of λ converge. Trial and error, with interpolation, may also be used, of course.

The method is given in detail for a three group, two region example.

The boundary conditions at the interface R are

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* It is also possible, and may be convenient if no information is available as to a good choice of μ_1, to select a trial value of λ and solve Eq. (5) for the starting value of μ_1.

** It is necessary to find the other roots of Eq. (5) consistent with the assumed μ_1 and corresponding λ.

[redacted]

$$F(\mu_1 R) + X_2 G(\nu_2 R) + X_3 G(\nu_3 R) = Z_1 P_1$$

$$S_{21} F(\mu_1 R) + S_{22} X_2 G(\nu_2 R) + S_{23} X_3 G(\nu_3 R) = C_{21} Z_1 P_1 + Z_2 P_2$$

$$S_{31} F(\mu_1 R) + S_{32} X_2 G(\nu_2 R) + S_{33} X_3 G(\nu_3 R) = C_{31} Z_1 P_1 + C_{32} Z_2 P_2 + Z_3 P_3$$

17)

$$\mu_1 F'(\mu_1 R) + X_2 \nu_2 G'(\nu_2 R) + X_3 \nu_3 G'(\nu_3 R) = \ell_1 Z_1 \mathcal{H}_1 P_1'$$

$$S_{21} \mu_1 F'(\mu_1 R) + S_{22} X_2 \nu_2 G'(\nu_2 R) + S_{23} X_3 \nu_3 G'(\nu_3 R) = \ell_2 C_{21} Z_1 \mathcal{H}_1 P_1' + \ell_2 Z_2 \mathcal{H}_2 P_2'$$

$$S_{31} \mu_1 F'(\mu_1 R) + S_{32} X_2 \nu_2 G'(\nu_2 R) + S_{33} X_3 \nu_3 G'(\nu_3 R) = \ell_3 C_{31} Z_1 \mathcal{H}_1 P_1'$$

$$+ \ell_3 C_{32} Z_2 \mathcal{H}_2 P_2' + \ell_3 Z_3 \mathcal{H}_3 P_3'$$

where

$$\ell_i = \frac{D_i \text{ (reflector)}}{D_i \text{ (core)}}$$

$$P_i = H(\mathcal{H}_i R) - \frac{H(\mathcal{H}_i T)}{G(\mathcal{H}_i T)} G(\mathcal{H}_i R); \quad P_i' = H'(\mathcal{H}_i R) - \frac{H(\mathcal{H}_i T)}{G(\mathcal{H}_i T)} G'(\mathcal{H}_i R)$$

This system of linear equations can be solved for $\frac{\mu_1 F'(\mu_1 R)}{F(\mu_1 R)}$ by the method of determinants. If $g_i = \frac{\nu_i G'(\nu_i R)}{G(\nu_i R)}$ and

$$P_i^* = \frac{\left[H'(\mathcal{H}_i R) - \frac{H(\mathcal{H}_i T)}{G(\mathcal{H}_i T)} G'(\mathcal{H}_i R)\right]}{\left[H(\mathcal{H}_i R) - \frac{H(\mathcal{H}_i T)}{G(\mathcal{H}_i T)} G(\mathcal{H}_i R)\right]}$$

[redacted]

then the following critical equation is obtained:

$$\begin{vmatrix} +1 & 1 & 1 & 1 & 0 & 0 \\ +S_{21} & S_{22} & S_{23} & C_{21} & 1 & 0 \\ +S_{31} & S_{32} & S_{33} & C_{31} & C_{32} & 1 \\ 0 & E_2 & E_3 & \ell_1 P_1 & 0 & 0 \\ 0 & S_{21}E_2 & S_{23}E_3 & \ell_2 C_{21} P_1 & \ell_2 P_2 & 0 \\ 0 & S_{31}E_2 & S_{33}E_3 & \ell_3 C_{31} P_1 & \ell_3 C_{32} P_2 & \ell_3 P_3 \end{vmatrix}$$

18) $$-\frac{\mu_1 F'(\mu_1 R)}{F(\mu_1 R)} = \frac{\begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & S_{22} & S_{23} & C_{21} & 1 & 0 \\ 0 & S_{32} & S_{33} & C_{31} & C_{32} & 1 \\ 1 & E_2 & E_3 & \ell_1 P_1 & 0 & 0 \\ S_{21} & S_{22}E_2 & S_{23}E_3 & \ell_2 C_{21} P_1 & \ell_2 P_2 & 0 \\ S_{31} & S_{32}E_2 & S_{33}E_3 & \ell_3 C_{31} P_1 & \ell_3 C_{32} P_2 & \ell_3 P_3 \end{vmatrix}}{\text{[illegible]}}$$

In this case we assume a value μ_1' of μ_1; then

19) $$\lambda' k_N = \frac{D_1 D_2 D_3}{\sigma_1 \sigma_2 \sigma_3} (\mu_1'^2 + B^2 + \frac{\sigma_1}{D_1})(\mu_1'^2 + B^2 + \frac{\sigma_2}{D_2})(\mu_1'^2 + B^2 + \frac{\sigma_3}{D_3})$$

μ_2 and μ_3 are given by the two solutions for μ^2 of

20) $$(\mu^2 + B^2)^2 + (\mu^2 + B^2)(\frac{\sigma_1}{D_1} + \frac{\sigma_2}{D_2} + \frac{\sigma_3}{D_3} + \mu_1'^2 + B^2) + \frac{\sigma_1 \sigma_2 \sigma_3}{D_1 D_2 D_3} (\frac{\lambda k_N - 1}{\mu_1'^2 + B^2}) = 0$$

The coupling coefficients are next calculated:

[redacted]

$$S_{21} = \frac{\sigma_1}{D_2(\mu_1'^2 + \frac{\sigma_2}{D_2} + B^2)} \quad S_{22} = \frac{\sigma_1}{D_2(\mu_2 + \frac{\sigma_2}{D_2} + B^2)} \quad S_{23} = \frac{\sigma_1}{D_2(\mu_3 + \frac{\sigma_2}{D_2} + B^2)}$$

21) $$S_{31} = \frac{\sigma_1 \sigma_2}{D_2 D_3(\mu_1'^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_1^2 + \frac{\sigma_3}{D_3} + B^2)} \quad S_{32} = \frac{\sigma_1 \sigma_2}{D_2 D_3(\mu_2^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_2^2 + \frac{\sigma_3}{D_3} + B^2)}$$

$$S_{33} = \frac{\sigma_1 \sigma_2}{D_2 D_3(\mu_3^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_3^2 + \frac{\sigma_3}{D_3} + B^2)}$$

and the functionals g_1 and g_2, where

$$g_1 = \frac{\nu_1 G'(\nu_1 R)}{G(\nu_1 R)}$$

The coupling coefficients and the functionals for the reflector (these do not change with each iteration) are next calculated.

$$C_{21} = \frac{\sigma_1}{D_2} \frac{1}{(\mathcal{H}_2^2 - \mathcal{H}_1^2)}$$

22)

$$C_{31} = \frac{\sigma_1 \sigma_2}{D_2 D_3} \frac{1}{(\mathcal{H}_2^2 - \mathcal{H}_1^2)(\mathcal{H}_3^2 - \mathcal{H}_1^2)} \qquad C_{32} = \frac{\sigma_2}{D_3} \frac{1}{(\mathcal{H}_3^2 - \mathcal{H}_2^2)}$$

p_1, p_2 and p_3 are given by

[redacted]

$$23) \qquad p_1 = \frac{\varkappa_1 H'(\varkappa_1 R) - \frac{H(\varkappa_1 T)}{G(\varkappa_1 T)} G'(\varkappa_1 R)}{H(\varkappa_1 R) - \frac{H(\varkappa_1 T)}{G(\varkappa_1 T)} G(\varkappa_1 R)}$$

These are substituted in Eq. (18), and the value of $\frac{\mu_1' R F'(\mu_1' R)}{F(\mu_1' R)}$ is obtained. From this $\mu_1' R$ is found and this new value is used to repeat the process.

For the one coordinate cases the functions are given explicitly in Table I. A specific numerical example is presented in Section III B.

Table I

	Cartesian	Cylindrical	Spherical
F	$\cos(Z)$	$J_0(Z)$	$\frac{\sin(Z)}{Z} = \sqrt{\frac{\pi}{2Z}}\, J_{\frac{1}{2}}(Z)$
F'	$-\sin(Z)$	$-J_1(Z)$	$-\frac{\sin Z}{Z^2} + \frac{\cos Z}{Z} = -\sqrt{\frac{\pi}{2Z}}\, J_{3/2}(Z)$
G	$\cosh(Z)$	$I_0(Z)$	$\frac{\sinh(Z)}{Z}$
G'	$\sinh(Z)$	$I_1(Z)$	$-\frac{\sinh(Z)}{Z^2} + \frac{\cosh Z}{Z}$
H	$\sinh(Z)$	$K_0(Z)$	$\frac{\cosh(Z)}{Z}$
H'	$\cosh(Z)$	$-K_1(Z)$	$-\frac{\cosh Z}{Z^2} + \frac{\sinh Z}{Z}$

[redacted]

II C. Series Expansion Method*

This method is applicable to multigroup piles divided into regions within each of which D_n, σ_n and k_N vanishes outside the core or central region.

Since in the core each flux equation is inhomogeneous, the solutions are expressible as a particular solution plus an arbitrary constant times the homogeneous solution. To adequately represent the unknown slow neutron flux, which is a source for the fast neutron flux, an appropriate series expansion of orthogonal functions is used. In the non-central regions the analytic solutions of Section II B are used. By applying the boundary conditions at each interface the arbitrary constants are eliminated and a new expression for φ_N is obtained in terms of the same orthogonal functions. λ is obtained by means of a secular determinant which compares the coefficients of the assumed φ_N and the new φ_N derived from it.

By setting λ equal to unity the equations for the fluxes in the core are:

1) $$\nabla^2 \varphi_1 - \left(\frac{\sigma_1}{D_1} + B^2\right)\varphi_1 + \frac{k_N}{D_1}\sigma_N \varphi_N = 0$$

2) $$\nabla^2 \varphi_n - \left(\frac{\sigma_n}{D_n} + B^2\right)\varphi_n + \frac{\sigma_{n-1}}{D_n}\varphi_{n-1} = 0 \qquad \text{for } n \neq 1$$

the homogeneous solutions are obtained for

3) $$\nabla^2 \varphi_n - \left(\frac{\sigma_n}{D_n} + B^2\right)\varphi_n = 0$$

or

$$\varphi_n = G(\omega_n x)$$

* This technique is similar to that proposed by Ehrlich, Hurwitz and Stehn (GE-RE-1).

[redacted]

where $\omega_n^2 = \frac{\sigma_n}{D_n} + B^2$,

and G is the same symmetrical function as used in Section II B.

The φ_N will be represented by an appropriate series

4) $$\varphi_N = \sum_{i=0}^{\infty} a_i F(\theta_i X)$$

where F is the symmetrical solution of

5) $$\nabla^2 F + \theta_i^2 F = 0$$

and the θ_i are determined in such a manner that

6) $$\int_0^{R_1} F(\theta_i X)\, F(\theta_j X)\, d\tau = C_i\, \delta_{ij};$$

R_1 is the position of the first interface

7) $$d\tau = X^M dx$$

and M is 0, 1 or 2 when the case considered is Cartesian, cylindrical or spherical respectively.

It can be shown that the orthogonality conditions of eq. (6) are satisfied if the θ_i's are chosen in such a manner that

8) $$\left| \frac{\partial F(\theta_i X)}{\partial X} \right|_{X=R_1} = 0$$

Thus $\theta_0 R_1$, $\theta_1 R_1$, ---------- $\theta_m R_1$ are the first m successive roots of $F' = 0$.

[redacted]

For convenience a general term in eq. (4) is used to represent φ_N. Thus

9) $$\varphi_{Ni} = a_i F(\theta_i X).$$

By substituting eq. (9) into eq. (1) and adding the homogeneous solution, the solution for φ_1 is obtained.

10) $$\varphi_1 = X_1 G(\omega_1 X) + a_i F(\theta_i X) \frac{k_N \sigma_N}{D_1(\omega_1^2 + \theta_i^2)}$$

In a similar manner the solutions for the other φ_n's are determined. Thus

11) $$\varphi_2 = X_2 G(\omega_2 X) + \frac{\sigma_1 X_1 G(\omega_1 X)}{D_2(\omega_2^2 - \omega_1^2)} + \frac{k_N \sigma_N \sigma_1}{D_1 D_2} \frac{a_i F(\theta_i X)}{(\omega_1^2 + \theta_i^2)(\omega_2^2 + \theta_i^2)}$$

12) $$\varphi_3 = X_3 G(\omega_3 X) \frac{\sigma_2 X_2 G(\omega_2 X)}{D_3(\omega_3^2 - \omega_2^2)} + \frac{\sigma_1 \sigma_2 X_1 G(\omega_1 X)}{D_2 D_3(\omega_2^2 - \omega_1^2)(\omega_2^2 - \omega_1^2)}$$

$$+ \frac{k_N \sigma_N \sigma_1 \sigma_2 a_i F(\theta_i X)}{D_1 D_2 D_3(\omega_1^2 + \theta_i^2)(\omega_2^2 + \theta_i^2)(\omega_3^2 + \theta_i^2)}$$

etcetera.

The solutions in the non-central regions are the same as those used in Section II B (eq. 13) then

13) $$\varphi_1 = Y_1 G(\varkappa_1 X) + Z_1 H(\varkappa_1 X)$$

[redacted]

14) $$\varphi_n = Y_n G(\mathcal{H}_n X) + Z_n H(\mathcal{H}_n X) + C_{m,n-1}\left[Y_{n-1} G(\mathcal{H}_{n-1} X) + Z_{n-1} H(\mathcal{H}_{n-1} X)\right] + \cdots\cdots$$

in the outermost region as before $Y_n G(\mathcal{H}_n X) + Z_n H(\mathcal{H}_n X)$ is replaced by

$$Z_n\left[H(\mathcal{H}_n X) - \frac{H(\mathcal{H}_n T)}{G(\mathcal{H}_n T)}\, G(\mathcal{H}_n X)\right]$$

At each interface, for each group, from the boundary conditions requiring the continuity of φ_n and $D_n \varphi_n'$ two equations are obtained. If M is the number of regions there will be 2N(M-1) simultaneous equations from which the 2N(M - 1) unknown constants (x,y,z) can be eliminated and a solution obtained for φ_{Ni} which will be designated φ^*_{Ni}. It is to be noted that the unknowns $(x_n y_n z_n)$ will depend upon the particular a_i and θ_i used.

This solution for φ^*_{Ni} will be in terms of the functions $G(\omega_n X)$ and the particular $F(\theta_i X)$. Since $G(\omega_n X) = F(\sqrt{-1}\,\omega_n X)$ as seen from eq. (3,5), $G(\omega_n X)$ can be expanded in terms of $F(\theta_i X)$ or

15) $$G(\omega_n X) = S_n \sum_{i=0}^{\infty} b_{ni}\, F(\theta_i X)$$

then the solutions for φ^*_{Ni} and φ^*_N are

16) $$\varphi^*_{Ni} = a_i \sum_{j=0}^{\infty} A_{ij}\, F(\theta_j X)$$

17) $$\varphi^*_N = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i\, A_{ij}\, F(\theta_j X)$$

since λ was set equal to unity originally

[redacted]

18) $$\frac{1}{\lambda}\varphi_N = \varphi_N^*$$

each side of eq. (18) is multiplied by $C_k\, F(\theta_k X)$ and integrated over the core where C_k is the normalizing factor of eq. (6).

19) $$\frac{1}{\lambda}\int_0^{R_1} C_k\, F(\theta_k X)\,\varphi_N d\tau = \int_0^{R_1} C_k\, F(\theta_k X)\,\varphi_N^*\, d\tau$$

20) $$\frac{1}{\lambda}\int_0^{R_1} C_k F(\theta_k X) \sum_{i=0}^{\infty} a_i\, F(\theta_i X) d\tau = \int_0^{R_1} C_k\, F(\theta_k X) \sum_{i=0} \sum_{j=0} a_i\, A_{ij}\, F(\theta_j X) d\tau$$

21) $$\frac{1}{\lambda} a_k = \sum_{i=0}^{\infty} a_i\, A_{ik}$$

This results in an infinite set of linear homogeneous equations in the unknowns a_i. There exists a non-trivial solution only when the determinant of the coefficients vanishes. Thus there is a secular determinant that determines λ. Since in any practical case the non-diagonal coefficients $A_{i,k}$ are significantly smaller than the diagonal coefficient A_{ii} (where $i < k$) and the diagonal coefficients $A_{ii} \longrightarrow 0$ as $i \longrightarrow \infty$, the approximations to λ determined by finite determinants converge to the true λ as the degree of the determinant increases.

As an example the three group, two region case is set up.

Let

$$\ell_i = \frac{D_i(\text{reflector})}{D_i(\text{core})} \qquad P_i = H(\mathcal{H}_i R) - \frac{H(\mathcal{H}_i T)}{G(\mathcal{H}_i T)}\, G(\mathcal{H}_i R)$$

[redacted]

$F_i = F(\theta_i R)$; $G_i = G(\omega_i R)$; $P_i' = \varkappa_i \left[H'(\varkappa_i R) - \frac{H(\varkappa_i T)}{G(\varkappa_i T)} G'(\varkappa_i R) \right]$

$G_i' = \omega_i G'(\omega_i R)$

Core Reflector

$$\eta_{1i} = \frac{k_3 \sigma_3}{D_1(\omega_i^2 + \theta_i^2)} \; ; \; \alpha_1 = \frac{\sigma_1}{D_2(\omega_2^2 - \omega_1^2)} \; ; \qquad \gamma_1 = \frac{\sigma_1}{D_2(\varkappa_2^2 - \varkappa_1^2)}$$

$$\eta_{2i} = \frac{\sigma_1 \eta_{1i}}{D_2(\omega_2^2 + \theta_i^2)} \; ; \; \alpha_2 = \frac{\sigma_2}{D_3(\omega_3^2 - \omega_2^2)} \; ; \qquad \gamma_2 = \frac{\sigma_2}{D_3(\varkappa_3^2 - \varkappa_2^2)}$$

$$\eta_{3i} = \frac{\sigma_2 \eta_{2i}}{D_3(\omega_3^2 + \theta_i^2)} \; ; \; \alpha_3 = \frac{\sigma_2 \alpha_1}{D_3(\omega_3^2 - \omega_1^2)} \; ; \qquad \gamma_3 = \frac{\sigma_2 \gamma_1}{D_3(\varkappa_3^2 - \varkappa_1^2)}$$

Then the boundary conditions at the interface take the form

$$X_1 G_1 + a_i \eta_{1i} F_i = Z_1 P_1$$

$$X_1 G_1' + 0 = \ell_1 Z_1 P_1'$$

$$X_2 G_2 + \alpha_1 X_1 G_1 + a_i \eta_{2\,1} F_i = Z_2 P_2 + \gamma_1 P_1 Z_1$$

22)

$$X_2 G_2' + \alpha_1 X_1 G_1' + 0 = \ell_2 \left[Z_2 P_2' + Z_1 \gamma_1 P_1' \right]$$

$$X_3 G_3 + \alpha_2 X_2 G_2 + \alpha_3 X_3 G_3 + a_i \eta_{3i} F_i = Z_3 P_3 + \gamma_2 Z_2 P_2 + \gamma_3 Z_1 P_1$$

$$X_3 G_3' + \alpha_2 X_2 G_2 + \alpha_3 X_3 G_3' + 0 = \ell_3 \left[Z_3 P_3' + \gamma_2 Z_2 P_2' + \gamma_3 Z_1 P_1' \right]$$

[redacted]

Since from eq. (8) $\left.\dfrac{\partial F(\theta_i X)}{\partial X}\right|_{X=R} = 0$

By solving these equations in pairs for X_i and Z_i the solutions are obtained in terms of the variables a_i, F_i, η_{1i}, η_{2i} and η_{3i}. Thus

$$X_1 = C_{11}\eta_{1i} a_i F_i \quad ; \quad Z_1 = E_{11}\eta_{1i} a_i F_i$$

23) $$X_2 = (C_{22}\eta_{2i} + C_{21}\eta_{1i}) a_i F_i \quad ; \quad Z_2 = (E_{22}\eta_{2i} + E_{21}\eta_{1i}) a_i F_i$$

$$X_3 = (C_{33}\eta_{3i} + C_{32}\eta_{2i} + C_{31}\eta_{1i}) a_i F_i ; \quad Z_3 = (E_{33}\eta_{3i} + E_{32}\eta_{2i} + E_{31}\eta_{1i}) a_i F_i$$

where the C and E coefficients are given by

$$C_{11} = \frac{\ell_1 P_1'}{-\ell_1 G_1 P_1'' + G_1'' P_1} \qquad E_{11} = \frac{G_1''}{-\ell_1 G_1 P_1'' + F_1'' P_1}$$

$$C_{22} = \frac{\ell_2 P_2''}{-\ell_2 G_2 P_2'' + G_2'' P_2} \qquad E_{22} = \frac{G_2''}{-\ell_2 G_2 P_2'' + G_2'' P_2}$$

$$C_{33} \quad \frac{\ell_3 P_3''}{-\ell_3 G_3 P_3'' + G_3'' P_3} \qquad E_{33} = \frac{G_3''}{-\ell_3 G_3 P_3'' + G_3'' P_3}$$

$$C_{21} \quad \frac{\ell_2 P_2''(\alpha_1 G_1 C_{11} - \gamma_1 P_1 E_{11}) + P_2(\ell_2 \gamma_1 P_1' E_{11} - \alpha_1 G_1'' C_{11})}{-\ell_2 G_2 P_2'' + G_2'' P_2}$$

[redacted]

$$E_{21} = \frac{G_2'(\alpha_1 G_1 C_{11} - \gamma_1 P_1 E_{11}) + G_2(\ell_2 \gamma_1 P_1' E_{11} - \alpha_1 G_1' C_{11})}{-\ell_2 G_2 P_2' + G_2' P_2}$$

$$C_{32} = \frac{\ell_3 P_3'(\alpha_2 G_2 C_{22} - \gamma_2 P_2 E_{22}) + P_3(\ell_3 \gamma_2 P_2' E_{22} - \alpha_2 G_2' C_{22})}{-\ell_3 G_3 P_3' + G_3' P_3}$$

$$E_{32} = \frac{G_3'(\alpha_2 G_2 C_{22} - \gamma_2 P_2 E_{22}) + G_3(\ell_3 \gamma_2 P_2' E_{22} - \alpha_2 G_2' C_{22})}{-\ell_3 G_3 P_3' + G_3' P_3}$$

$$C_{31} = \frac{\ell_3 P_3'(\alpha_2 G_2 C_{21} + \alpha_3 G_1 C_{11} - \gamma_2 P_2 E_{21} - \gamma_3 P_1 E_{11}) + P_3(\ell_3 \gamma_2 P_2' E_{21} + \ell_3 \gamma_3 P_1' E_{11} - \alpha_2 G_2' C_{21} - \alpha_3 G_1' C_{11})}{-\ell_3 G_3 P_3' + G_3' P_3}$$

$$E_{31} = \frac{G_3'(\alpha_2 G_2 C_{21} + \alpha_3 G_1 C_{11} - \gamma_2 P_2 E_{21} - \gamma_3 P_1 E_{11}) + G_3(\ell_3 \gamma_2 P_2' E_{21} + \ell_3 \gamma_3 P_1' E_{11} - \alpha_2 G_2' C_{21} - \alpha_3 G_1' C_{11})}{-\ell_3 G_3 P_3' + G_3' P_3}$$

Thus from eq. (22, 23) the solution for φ^*_{3i} is

24) $$\varphi^*_{3i} = (C_{33}\eta_{3i} + C_{32}\eta_{2i} + C_{31}\eta_{1i})a_i F(\theta_i R)\, G(\omega_3 X) + \alpha_2(C_{22}\eta_{2i} + C_{21}\eta_{1i})a_i$$

$$F(\theta_i R)G(\omega_2 X) + \alpha_3(C_{11}\eta_{1i})a_i F(\theta_i R)G(\omega_1 X) + \eta_{3i} a_i F(\theta_i X)$$

substituting the expansion of eq. (15) into eq. (24) there results

[redacted]

$$= \Big[(C_{33}\eta_{3i} + C_{32}\eta_{2i} + C_{31}\eta_{1i}) S_3 \Big(\sum_{j=0}^{\infty} b_{3j} F(\theta_j X) + \alpha_2 (C_{22}\eta_{2i} + C_{21}\eta_{1i})$$

$$S_2 \Big(\sum_{j=0}^{\infty} b_{2j} F(\theta_j X) + \alpha_3 (C_{11}\eta_{1i}) S_1 \Big(\sum_{j=0}^{\infty} b_{1j} F(\theta_j X) \Big) \Big] a_i F(\theta_i R) + \eta_{3i} a_i F(\theta_i X)$$

26) $$A_{ij} = \Big[(C_{33}\eta_{3i} + C_{32}\eta_{2i} + C_{31}\eta_{1i}) S_3 b_{3j} + \alpha_2 S_2 b_{2j} (C_{22}\eta_{2i} + C_{21}\eta_{1i})$$

$$+ \alpha_3 S_1 b_{1j} (C_{11}\eta_{1i}) \Big] F(\theta_i R) + \eta_{3i} \delta_{ij}$$

The secular determinant is in the form

$$\left.\begin{array}{llll} A_{00} - \frac{1}{\lambda} & A_{10} & A_{20} & A_{30} \text{ ---------- } \\ A_{01} & A_{11} - \frac{1}{\lambda} & A_{21} & A_{31} \text{ ---------- } \\ A_{02} & A_{12} & A_{22} - \frac{1}{\lambda} & A_{32} \text{ ---------- } \\ A_{03} & A_{13} & A_{23} & A_{33} - \frac{1}{\lambda} \text{ -------- } \end{array}\right\} = 0$$

Thus the various approximations to $\frac{1}{\lambda}$ are

27) $$\frac{1}{\lambda} = A_{00}$$

28) $$\frac{1}{\lambda} = \frac{A_{00} + A_{11} + \sqrt{(A_{00} + A_{11})^2 - 4(A_{00}A_{11} - A_{01}A_{10})}}{2}$$

etera

[redacted]

Constants for the expansion in eq. (15) and the equation for determining θ_i for the three one coordinate geometries are given in Table II. The other functions are obtained from Table I (Section II B).

TABLE II

General	Cartesian	Cylindrical	Spherical
S_n	$\frac{2\omega_n}{R}\sinh(\omega_n R)$	$\frac{2}{R}\omega_n I_1(\omega_n R)$	$\frac{2\omega_n}{R}\sqrt{\frac{\pi}{2\omega_n R}}\, I_{3/2}(\omega_n R)$
b_{ni}	$\frac{(-1)^i}{\omega_n^2+\theta_i^2}$	$\frac{1}{(\omega_n^2+\theta_i^2)\left\lvert J_o(\theta_i R)\right\rvert}$	$\frac{1}{(\omega_n^2+\theta_i^2)\sqrt{\frac{\pi}{\theta_i R}}\, J_{1/2}(\theta_i R)}$
$\left.\frac{\partial F(\theta_i X)}{\partial X}\right\rvert_{r=R} = 0$	$\sin(\theta_i R) = 0$	$J_1(\theta_i R) = 0$	$\sqrt{\frac{\pi}{2\theta_i R}}\, J_{3/2}(\theta_i R) = 0$

[redacted]

II D. Neutron Flux Iterations*

This method is applicable in the one coordinate cases even when d, σ, and k are functions of the one coordinate ρ. In principle a neutron flux distribution for a particular group (usually the thermal group) is arbitrarily assumed. This distribution is used as a source to generate the flux distribution of the next less energetic group (the slowest group distribution is, of course, used to generate the fastest group distribution). This new distribution is then regarded as a source which similarly generates the flux distribution of the next lower group. By repeating this process of using each new distribution as a source for the next lower group the successive approximations of the flux distribution for any particular group will approach the same form. In the limit, each successive approximation to a particular distribution differs only by a constant multiplicative factor which is simply related to the effective multiplication constant.

If D, σ and k_N are functions of ρ, we must write eq. (12 and 13) Section II A in a more general form thusly:

$$1)\quad \frac{1}{\rho^M}\frac{d}{d\rho}\left(\rho^M D_1 \frac{d\varphi_1}{dp}\right) - (\sigma_1 + D_1 B^2)\varphi_1 + \lambda k_N \sigma_N \varphi_N = 0$$

$$2)\quad \frac{1}{\rho^M}\frac{d}{d\rho}\left(\rho^M D_n \frac{d\varphi_n}{d\rho}\right) - (\sigma_n + D_n B^2)\varphi_n + \sigma_{n-1}\varphi_{n-1} = 0 \qquad \text{for } n \neq 1$$

So as to write each of these second order equations as two first order equations we let

$$3)\qquad I_n = \rho^M D_n \frac{d\varphi_n}{d\rho}$$

---

* Cf. Mon P-360.

[redacted]

then eq. (1 and 2) become

$$4)\qquad \frac{dI_n}{d\rho} = \rho^M(\sigma_n + D_n B^2)\varphi_n - \rho^M \sigma_{n-1}\varphi_{n-1} \qquad \text{for } n \neq 1$$

$$5)\qquad \frac{dI_1}{d\rho} = \rho^M(\sigma_1 + D_1 B^2)\varphi_1 - \rho^M \lambda k_N \sigma_N \varphi_N$$

The iteration may in simple cases be performed by the use of analytic functions and analytic solutions of the differential equations. For complicated situations (more than 2 regions or three groups) it is more convenient to do the problem numerically.

For this numerical approximation method eq. (3, 4, 5) are replaced by appropriate difference equations. Thus utilizing the definition of a derivative

6)
$$6)\qquad \left(\frac{d\varphi}{dx}\right)_\rho = \lim_{h \to 0} \frac{\varphi(\rho + \frac{1}{2}h) - \varphi(\rho - \frac{1}{2}h)}{h}$$

a small lattice spacing h will be used so that

$$7)\qquad \left(\frac{d\varphi}{dx}\right)_\rho \doteq \frac{\varphi(\rho + \frac{1}{2}h) - \varphi(\rho - \frac{1}{2}h)}{h}$$

this is approximate since h is finite. Thus eq. (3, 4, 5) become

$$8)\qquad \frac{\varphi_n(\rho + \frac{1}{2}h) - \varphi_n(\rho - \frac{1}{2}h)}{h} = \frac{I_n(\rho)}{\rho^M D_n(\rho)}$$

[redacted]

9) $$\frac{I_n(\rho + h) - I_n(\rho)}{h} \doteq \left[\rho^M(\sigma_n + D_n B^2)\varphi_n - \rho^M \sigma_{n-1}\varphi_{n-1}\right]_{\rho = \rho + \frac{1}{2}h} \quad \text{for } n \neq 1$$

10) $$\frac{I_1(\rho + h) - I_1(\rho)}{h} \doteq \left[\rho^M(\sigma_1 + D_1 B^2)\varphi_1 - \rho^M \lambda k_N \varphi_N\right]_{\rho = \rho + \frac{1}{2}h}$$

By rewriting there are obtained the difference equations to be solved.

Definition of I
the differential.

11) $$\varphi_n(\rho + \tfrac{1}{2}h) = \varphi_n(\rho - \tfrac{1}{2}h) + \frac{hI_n(\rho)}{\rho^M D_n(\rho)}$$

12) $$I_n(\rho + h) = I_n(\rho) + h\left[\rho^M(\sigma_n + D_n B^2)\varphi_n - \rho^M \sigma_{n-1}\varphi_{n-1}\right]_{\rho = \rho + \frac{1}{2}h} \quad \text{for } n \neq 1$$

13) $$I_1(\rho + h) = I_1(\rho) + h\left[\rho^M(\sigma_1 + D_1 B^2)\varphi_1 - \rho^M \lambda k_N \sigma_N \varphi_N\right]_{\rho = \rho + \frac{1}{2}h}$$

Equations (12 and 13) are inhomogeneous in form because of the presence of the term in φ_{n-1}. Therefore, the complete solution is expressible as any particular solution I_n^P plus an undetermined constant α times the homogeneous solution I_n^H. Likewise there will be a particular and a homogeneous solution of φ_n.

[redacted]

Thus

$$14)\quad I_n^P(\rho+h) = I_n^P(\rho) + h\left[\rho^M(\sigma_n + D_n B^2)\varphi_n^P - \rho^M \sigma_{n-1}\varphi_{n-1}\right]_{\rho=\rho+\frac{1}{2}h} \qquad \text{for } n \neq 1$$

$$15)\quad I_n^H(\rho+h) = I_n^H(\rho) + h\left[\rho^M(\sigma_n + D_n B^2)\varphi_n^H\right]_{\rho=\rho+\frac{1}{2}h}$$

$$16)\quad \varphi_n^P(\rho+\tfrac{1}{2}h) = \varphi_n^P(\rho-\tfrac{1}{2}h) + \frac{hI_n^P(\rho)}{\rho^M D_n(\rho)}$$

$$17)\quad \varphi_n^H(\rho+\tfrac{1}{2}h) = \varphi_n^H(\rho-\tfrac{1}{2}h) + \frac{hI_n^H(\rho)}{\rho^M D_n(\rho)}$$

$$18)\quad I_n = I_n^P + \alpha I_n^H$$

$$19)\quad \varphi_n = \varphi_n^P + \alpha\varphi_n^H$$

The constant α is determined in each iteration by the boundary condition that $\varphi_n(T) = 0$ at the edge of the pile (T) and from eq. (19). Thus

$$20)\quad \alpha = -\frac{\varphi_n^P(T)}{\varphi_n^H(T)}$$

[redacted]

Analogous to Eq. (14) for the fast group the following equation is used:

$$21)\qquad I_1^P(\rho + h) = I_1^P(\rho) + h\left[\rho^M(\sigma_1 + D_1B^2)\varphi_1^P - \rho^M k_N \sigma_N \varphi_N\right]_{\rho = \rho + \frac{1}{2}h}$$

where λ is set equal to unity so that the ratio of any φ_n to the next iterated value of φ_n will approach λ in the limit that these two functions for a given φ_n become identical in form.

To solve the difference equations values for φ_N are assumed; the process converges faster the more accurate the assumed form of φ_N is. From eq. 3 and the symmetry of the pile $I_n^P(0)$ and $I_n^H(0)$ are always equal to zero. Next values for $\varphi_1^H(0 + \frac{1}{2}h)$ and $\varphi_1^P(0 + \frac{1}{2}h)$ are arbitrarily assumed. If h was chosen so that T/h is an integer than I_1^P and I_1^H can be evaluated at $\rho = s\,h$ (where $s = 0, 1, 2, 3, \text{---------}, T/h$) by eq. (15, 21) and in between φ_1^P and φ_1^H can be evaluated at $\rho = (s + \frac{1}{2})h$ from eq. (16, 17). To determine $\varphi_n'(T)$ a quadratic interpolation can be used, that is assuming that

$$22)\qquad \varphi_n = a\,x^2 + bx + c$$

and let $x = 0$ when $\rho = T - \frac{1}{2}h$ then

$$23)\qquad \varphi_n(T - \tfrac{1}{2}h) = c$$

$$24)\qquad \varphi_n(T - 3/2h) = ah^2 - bh + c$$

$$25)\qquad \varphi_n(T + \tfrac{1}{2}h) = ah^2 + bh + c$$

by solving eq. (22, 23, 24, 25) for $\varphi_n(T)$ there results

[redacted]

26) $\varphi_n(T) = .375\,\varphi_n(T + \frac{1}{2}h) + .75\,\varphi_n(T - \frac{1}{2}h) - .125\,\varphi_n(T - 3/2h)$

Upon solving for $\varphi_1^H(T)$ and $\varphi_1^P(T)$ a value for x is obtained which yields by eq. (19, 20) φ_1^0 the new flux distribution. These values for φ_1^0 are used as a source function in an entirely analogeous manner to obtain φ_2'; however, eq. (14) is used instead of eq. (21) to obtain I_2^P.

By repeating this process we eventually have a solution for φ_N^0 and the first approximation to λ will be

27) $$\frac{\int k_N\,\sigma_N\,\varphi_N\,d\tau}{\int k_N\,\sigma_N\,\varphi_N^0\,d\tau}$$

where the volume element $d\tau$ is given by

28) $$d\tau = \rho^{M-1}\,d\rho$$

When the process is repeated, the values of φ_n^H and I_n^H are used unchanged. To make the calculations less cumbersome it is useful to set the new value of $\varphi_n^{0P}(0 + \frac{1}{2}h) = \dfrac{\varphi_n(0 + \frac{1}{2}h)}{\lambda}$ since this tends to make the correction term α vanish. Successive approximations of λ are obtained by eq. (27) except that successive iterated values of φ_N are substituted for φ_N and φ_N^0 respectively.

MODIFICATIONS FOR PILE REGIONS

To apply this method to a pile divided into regions within each of which D_M, k_N and σ_N are constants it is advisable to use different lattice spacing values so that the width of each region is an integral number times the

[redacted]

h used in that region. The modification comes in crossing an interface between two regions. The boundary conditions require that I_n and φ_n be continuous in crossing an interface. Thus if R_1 is the coordinate of the interface separating regions 1 and 2 which have the lattice spacings h_1 and h_2 respectively, the value of $I_n(R_2)$ is determined by the difference equations in region 2 and $\varphi_n^P(R_2)$ is determined by the interpolation eq. (26) as applied to region 2. Now $\varphi_n^P(R_2 + \frac{1}{2}h_3)$ must be so chosen that when $\varphi_n(R_2)$ is obtained by backwards interpolation in region 3 it will have the same value as $\varphi_n(R_2)$ obtained from region 2. Thus

$$29)\quad \varphi_n^P(R_2) = .375\,\varphi_n^P(R_2 - \tfrac{1}{2}h_3) + .75\,\varphi_n^P(R_2 + \tfrac{1}{2}h_3) - .125\,\varphi_n^P(R_2 + 3/2h_3)$$

$$30)\quad \varphi_n^P(R_2 + \tfrac{1}{2}h_3) = \varphi_n^P(R_2 - \tfrac{1}{2}h_3) + A\, I_n^P(R_2) \quad ; \quad A = \frac{h}{\rho D_n(\rho)} \text{ at } R_2$$

$$31)\quad \varphi_n^P(R_2 + 3/2h_3) = \varphi_n^P(R_2 + \tfrac{1}{2}h_3) + B\, I_n^P(R_2 + h_3) \quad ; \quad B = \frac{h}{\rho D_n(\rho)} \text{ at } R_2 + h_3$$

$$32)\quad I_n^P(R_2 + h) = I_n^P(R_2) + C\,\varphi_n^P(R_2 + \tfrac{1}{2}h) - E\,\varphi_{n-1}(R_2 + \tfrac{1}{2}h_3)$$

$C = h\rho(\sigma_n + D_n B^2)$ at $R_2 + \frac{1}{2}h_3$

$E = h\rho\sigma$ at $R_2 + \frac{1}{2}h_3$

where the coefficients A, B, C and E are obtained from eq. (16, 14, 21). Upon solving for $\varphi_n^P(R_2 + \frac{1}{2}h_3)$ there results

$$33)\quad \varphi_n^P(R_2 + \tfrac{1}{2}h_3) \quad \frac{\varphi_n^P(R_2) + I_n^P(R_2)\left[.375\,A + .125\,B\right] - .125\,B\,E\,\varphi_{n-1}(R_2 + \frac{1}{2}h_3)}{1 - .125\,B\,C}$$

Similarly the result for $\varphi_n^H(R_2 + \frac{1}{2}h_3)$ is

[redacted]

$$34)\quad \varphi_n^H(R_2 + \tfrac{1}{2}h_3) = \frac{\varphi_n^H(R_2) + \Gamma_n^H(R_2)\left[.375A + .125B\right]}{1 - .125\,B\,C}$$

since $E = 0$ in the homogeneous case.

Upon repeated iterations as in the general case λ is approximated by

$$35)\quad \lambda \doteq \frac{\int \varphi_n^{\ell}\, d\tau}{\int \varphi_n^{\ell+1}\, d\tau}$$

where it is assumed for simplicity that $k_N \neq 0$ only in one region and where φ_n^{ℓ} and $\varphi_n^{\ell+1}$ are successive iterations of any φ_n. A slightly better approximation to λ is

$$36)\quad \frac{\int \left[\varphi_n^{\ell}\right]^2 d\tau}{\int \varphi_n^{\ell+1}\,\varphi_n^{\ell}\, d\tau}$$

but as the forms of the same φ_n become identical eq. (35) is an adequate approximation. In carrying out the integrations in eq. (27, 35, 36) any form of numerical integration may be used. Although Simpson's Rule is superior to the Trapezoidal Rule the latter is adequate when the iterations of any φ_n closely approach a common form.

[redacted]

III A. Specific Example

In this section a specific example of a three group, two region cylindrical pile is computed using each of the three methods described in Section II.

The constants of the pile are:

For the core $R_1 = 14.51$ cm $B^2 = .0016098^*$

$D_1 = 1.793$ $\sigma_1 = .038198$

$D_2 = .8866$ $\sigma_2 = .049977$

$D_3 = .2840$ $\sigma_3 = .078583$

For the reflector $R_1 = 14.51$ cm $R_2 = 46.04$ cm $B^2 = .0018881^*$

$D_1 = .945$ $\sigma_1 = .025933$

$D_2 = .5303$ $\sigma_2 = .0091811$

$D_3 = .5367$ $\sigma_3 = .0012406$

*The B^2 are most commonly obtained from Eq. (9) of Section III A by considering the function G to be a symmetric function Z which vanishes at a point H + L above the center line. H is the half height of the cylinder, and L the "extrapolation" length of the medium. Hence,

$$B^2 = \left[\frac{\pi}{2(H + L)}\right]^2 .$$

Core and reflector may have different values for L, and, hence, different B^2.

L can be obtained as an average over energy (CF-Mon P-284), or, in case the Z direction is also reflected, from previous calculations of reflector savings.

[redacted]

III B.

In the analytic method it is best to determine the functions in the reflector first since these do not change with each iteration.

$$\mathcal{H}_1^2 = \frac{\sigma_1}{D_1} + B^2 = .029330 \qquad \mathcal{H}_1 = .17126$$

$$\mathcal{H}_2^2 = \frac{\sigma_2}{D_2} + B^2 = .019201 \qquad \mathcal{H}_2 = .13857$$

$$\mathcal{H}_3^2 = \frac{\sigma_3}{D_3} + B^2 = .0041997 \qquad \mathcal{H}_3 = .064805$$

$$C_{21} = \frac{\sigma_1}{D_2} \frac{1}{(\mathcal{H}_2^2 - \mathcal{H}_1^2)} = -4.8279$$

$$C_{31} = \frac{\sigma_1 \sigma_2}{D_2 D_3} \frac{1}{(\mathcal{H}_2^2 - \mathcal{H}_1^2)(\mathcal{H}_3^2 - \mathcal{H}_1^2)} = 3.2865$$

$$C_{32} = \frac{\sigma_2}{D_3} \frac{1}{(\mathcal{H}_3^2 - \mathcal{H}_2^2)} = -1.1404$$

According to Table I (Section II B) G, G', H, H' are replaced by the functions I_0, I_1, K_0 and $-K_1$ respectively

	$\mathcal{H}_1 R_1$	$\mathcal{H}_2 R_1$	$\mathcal{H}_3 R_1$	$\mathcal{H}_1 T$	$\mathcal{H}_2 T$	$\mathcal{H}_3 T$
Argument	2.4850	2.0107	.94032	7.8848	6.3798	2.9836
I_0	3.2523	2.2967	1.2336	383.90	95.178	4.8164
I_1	2.4827	1.6066	.52407			
K_0	.063466	.11241	.45885	.00016552	.00082606	.03540
K_1	.075283	.13791	.66717			

[redacted]

$$p_1 = \frac{-\mathcal{H}_1\left[K_1(\mathcal{H}_1R_1) + \frac{K_o(\mathcal{H}_1T)}{I_o(\mathcal{H}_1T)}\ I\ (\mathcal{H}_1R_1)\right]}{K_o(\mathcal{H}_1R_1) - \frac{K_o(\mathcal{H}_1T)}{I_o(\mathcal{H}_2T)}\ I_o(\mathcal{H}_1R_1)} = -.20315$$

$p_2 = -.17005$

$p_3 = -.096682$

$$\ell_1 = \frac{D_1(\text{reflector})}{D_1(\text{core})} = .52705$$

$\ell_2 = .59813$ $\qquad$ $\ell_3 = 1.8898$

In the determinant the following are used:

$\ell_1 p_1 = -.10707$ $\qquad$ $\ell_2 C_{21} p_1 = .58864$ $\qquad$ $\ell_3 C_{31} p_1 = -1.2617$

$\ell_2 p_2 = -.10171$ $\qquad$ $\ell_3 C_{32} p_2 = .36648$ $\qquad$ $\ell_3 p_3 = -.18271$

In the core it is necessary to assume a trial value for μ_1.

Let $\mu_1' = .08$

then $\lambda' k_3 = \frac{D_1\ D_2\ D_3}{\sigma_1\ \sigma_2\ \sigma_3}\left(\mu_1^2 + B^2 + \frac{\sigma_1}{D_1}\right)\left(\mu_1^2 + B^2 + \frac{\sigma_2}{D_2}\right)\left(\mu_1^2 + B^2 + \frac{\sigma_3}{D_3}\right) = 1.61698$

μ_2 and μ_3 are obtained from eq. 20 Section II B

thus

$$(\mu^2 + B^2)^2 + (\mu^2 + B^2)\left(\frac{\sigma_1}{D_1} + \frac{\sigma_2}{D_2} + \frac{\sigma_3}{D_3} + \mu_1^2 + B^2\right) + \frac{\sigma_1\ \sigma_2\ \sigma_3}{D_1\ D_2\ D_3}\ \frac{\lambda k_3 - 1}{(\mu_1^2 + B^2)} = 0$$

or $\mu_2^2 = -.26785$; $\mu_3^2 = -.097750$

$\nu_2 = \sqrt{-\mu_2^2} = .51754$ $\qquad$ $\nu_3 = \sqrt{-\mu_3^2} = .31265$

[redacted]

$$S_{21} = \frac{\sigma_1}{D_2(\mu_1^2 + \frac{\sigma_2}{D_2} + B^2)} = .66920$$

$$S_{22} = \frac{\sigma_1}{D_2\ (\mu_2^2 + \frac{\sigma_2}{D_2} + B^2)} = -.20528$$

$$S_{23} = \frac{\sigma_1}{D_2(\mu_3^2 + \frac{\sigma_2}{D_2} + B^2)} = -1.0833$$

$$S_{31} \quad \frac{\sigma_1\ \sigma_1}{D_2D_3(\mu_1^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_1^2 + \frac{\sigma_3}{D_3} + B^2)} = .41363$$

$$S_{32} = \frac{\sigma_1\ \sigma_2}{D_2D_3(\mu_2^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_2^2 + \frac{\sigma_3}{D_3} + B^2)} = -3.4538$$

$$S_{33} = \frac{\sigma_1\ \sigma_2}{D_2D_3(\mu_3^2 + \frac{\sigma_2}{D_2} + B^2)(\mu_3^2 + \frac{\sigma_3}{D_3} + B^2)} = -1.0558$$

The functions g_1 and g_2 are next calculated.

In this case $g_1 = \dfrac{\nu_1\ I_o(\nu_1 R_1)}{I_o(\nu_1 R_1)}$

[redacted]

ν_i	$\nu_i R_1$	$I_o(\nu_i R)$	$I_1(\nu_i R)$	g_i
.51754	7.5095	$e^{\nu_2 R_1}$ .148218	$e^{\nu_2 R_1}$ .137964	.48174
.31265	4.53655	18.0531	15.9120	.27557

Substituting these values in the determinants of eq. 18.

$$\frac{\mu_1 J_1(\mu_1 R_1)}{J_o(\mu_1 R_1)} = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ .66920 & -.20528 & -1.0833 & -4.8279 & 1 & 0 \\ .41363 & -3.4538 & -1.0558 & 3.2865 & -1.1404 & 1 \\ 0 & .48174 & .27557 & -.10707 & 0 & 0 \\ 0 & -.098892 & -.29852 & .58664 & -.10171 & 0 \\ 0 & -1.6638 & -.29095 & -1.2617 & .36648 & -.18271 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -.20528 & -1.0833 & -4.8279 & 1 & 0 \\ 0 & -3.4538 & -1.0558 & 3.2865 & -1.1404 & 1 \\ 1 & .48174 & .27557 & -.10707 & 0 & 0 \\ .66920 & .098892 & -.29852 & .58664 & -.10171 & 0 \\ .41363 & -1.6638 & -.29095 & -1.2617 & .36648 & -.18271 \end{vmatrix}}$$

$$\frac{\mu_1 R_1 J_1(\mu_1 R_1)}{J_o(\mu_1 R_1)} = \frac{14.51(.071519)}{1.6339} = .63513$$

From which $\mu_1'' = .071917$, this is used as the next trial value of μ_1 and the process repeated as shown in the following table.

[redacted]

Trial No.	1	2	3	4	5	6
μ_1	.08	.071917	.073354	.073105	.073167	.073152
λk_3	1.61698	1.51314	1.53058	1.52747	1.52823	1.52807
ν_2	.51754	.51825	.51812	.51816	.51815	
ν_3	.31265	.30949	.31004	.30992	.30994	
ε_2	.48174	.48244	.48232	.48236	.48235	
ε_3	.27557	.27237	.27293	.27280	.27282	
S_{21}	.66920	.68221	.67996	.68036	.68026	
S_{22}	- .20528	- .20457	- .20470	- .20466	- .20467	
S_{23}	-1.0833	-1.1396	-1.1295	-1.1317	-1.1312	
S_{31}	.41363	.42350	.42179	.42210	.42203	
S_{32}	-3.4538	-3.7000	-3.6535	-3.6677	-3.6641	
S_{33}	-1.0558	-1.0988	-1.0910	-1.0927	-1.0924	
$\frac{\mu_1 R_1 J_1(\mu_1 R_1)}{J_o(\mu_1 R_1)}$	.63513	.66547	.66012	.66145	.66114	
Next μ_1	.071917	.073354	.073105	.073167	.073152	

[redacted]

III C. Series Expansion Method.

This calculation is more tractable when broken up into the determination of the functions and constants associated with:

1) The reflector region
2) The homogeneous core solution
3) The combinations of 1 and 2
4) The expansion of the particular core solution
5) The final solution of A's and λk

$$\mathcal{H}_1^2 = \frac{\sigma_1}{D_1} + B^2 = .029330 \qquad \mathcal{H}_1 = .17126$$

$$\mathcal{H}_2^2 = \frac{\sigma_2}{D_2} + B^2 = .019201 \qquad \mathcal{H}_2 = .13857$$

$$\mathcal{H}_3^2 = \frac{\sigma_3}{D_3} + B^2 = .0041997 \qquad \mathcal{H}_3 = .064805$$

According to Table I (Section II B) G, G', H, H' are replaced by the functions I_o, I_1, K_o and $-K_1$ respectively

	$\mathcal{H}_1R$	$\mathcal{H}_2R$	$\mathcal{H}_3R$	$\mathcal{H}_1T$	$\mathcal{H}_2T$	$\mathcal{H}_3T$
Argument	2.4850	2.0107	.94032	7.8848	6.3798	2.9836
I_o	3.2523	2.2967	1.2336	383.90	95.178	4.8164
I_1	2.4827	1.6066	.52407			
K_o	.063466	.11241	.45885	.00016552	.00082606	.035405
K_1	.075283	.13791	.66717			

[redacted]

$$P_1 = K_o(\mathcal{H}_1 R) - \frac{K_o(\mathcal{H}_1 T)}{I_o(\mathcal{H}_1 T)} I_o(\mathcal{H}_1 R) = .063465$$

$$P_2 = .11239 \qquad P_3 = .44978$$

$$P_1' = -\mathcal{H}_1\left[K_1(\mathcal{H}_1 R) + \frac{K_o(\mathcal{H}_1 T)}{I_o(\mathcal{H}_1 T)} I_1(\mathcal{H}_1 R)\right. = -.012893$$

$$P_2' = -.019112 \qquad P_3' = -.043486$$

$$\gamma_1 = \frac{\sigma_1}{D_2(\mathcal{H}_2^2 - \mathcal{H}_1^2)} = -4.8279$$

$$\gamma_2 = \frac{\sigma_2}{D_3(\mathcal{H}_3^2 - \mathcal{H}_2^2)} = -1.1404$$

$$\gamma_3 = \frac{\sigma_2 \gamma_1}{D_3(\mathcal{H}_3^2 - \mathcal{H}_1^2)} = 3.2865$$

$$\omega_1^2 = \frac{\sigma_1}{D_1} + B^2 = .022914 \qquad \omega_1 = .15137$$

$$\omega_2^2 = \frac{\sigma_2}{D_2} + B^2 = .057979 \qquad \omega_2 = .24079$$

$$\omega_3^2 = \frac{\sigma_3}{D_3} + B^2 = .27831 \qquad \omega_3 = .52755$$

[redacted]

In this case G_i and G_i' are replaced by $I_o(\ _iR)$ and $_iI_1(\ _iR)$

ω_1	$\omega_1 R$	$I_o(\omega_1 R)$	$I_1(\omega_1 R)$	$\omega_1 I_1(\omega_1 R)$
.15137	2.1964	2.6223	1.9078	.28878
.24079	3.4939	7.3405	6.1717	1.4861
.52755	7.6548	309.75	288.75	152.33

$$\alpha_1 = \frac{\sigma_1}{D_2(\omega_2^2 - \omega_1^2)} = 1.2287$$

$$\alpha_2 = \frac{\sigma_2}{D_3(\omega_3^2 - \omega_2^2)} = .79869$$

$$\alpha_3 = \frac{\sigma_2 \alpha_1}{D_3(\omega_2^2 - \omega_1^2)} = .84661$$

$$S_1 = \frac{2}{R}\omega_1 I_1(\omega_1 R) = .039804$$

$S_2 = .20484$ $\qquad$ $S_3 = 20.997$

$$\ell_1 = \frac{D_1(\text{reflector})}{D_1(\text{core})} = .52705 \qquad \ell_2 = .59813 \qquad \ell_3 = 1.8898$$

[redacted]

$C_{11} = -.18799$	$E_{11} = 7.9891$
$C_{22} = -.045555$	$E_{22} = 5.9222$
$C_{33} = -.00087453$	$E_{33} = 1.6210$
$C_{21} = .079174$	$E_{21} = 21.562$
$C_{32} = .00099637$	$E_{32} = 1.7798$
$C_{31} = -.000041509$	$E_{31} = 2.5150$

The function $F(\theta_i R)$ becomes in this case $J_o(\theta_i R)$

$\therefore$ the $\theta_i R$ are given by the roots of $J_1(\theta_i R) = 0$ with $R = 14.51$ the first three values of θ, θ^2, $J_o(\theta R)$ are as follows:

$\theta_o = 0$	$\theta_o^2 = 0$	$J_o(\theta_o R) = 1$
$\theta_1 = .26407$	$\theta_1^2 = .069733$	$J_o(\theta_1 R) = -.40276$
$\theta_2 = .48350$	$\theta_2^2 = .23377$	$J_o(\theta_2 R) = .30012$

The coefficients η_{ni} are given by:

$$\eta_{1i} = \frac{k_3 \sigma_3}{D_1(\omega_1^2 + \theta_i^2)} \qquad \eta_{2i} = \frac{\sigma_1 \eta_{1i}}{D_2(\omega_2^2 + \theta_i^2)} \qquad \eta_{3i} = \frac{\sigma_2 \eta_{2i}}{D_3(\omega_3^2 + \theta_i^2)}$$

$\eta_{10} = 1.9127$	$\eta_{20} = 1.4213$	$\eta_{30} = .89869$
$\eta_{11} = .47306$	$\eta_{21} = .15959$	$\eta_{31} = .080689$
$\eta_{12} = .17075$	$\eta_{22} = .025215$	$\eta_{32} = .0086650$

The coefficients b_{ni} are given by:

$$b_{ni} = \frac{1}{(\omega_n^2 + \theta_i^2)\, J_o(\theta_i R)}$$

[redacted]

$b_{10} = 43.641$	$b_{20} = 17.248$	$B\ b_{30} = 3.5931$
$b_{11} = -26.799$	$b_{21} = -19.441$	$b_{31} = -7.1338$
$b_{12} = 12.981$	$b_{22} = 11.421$	$b_{32} = 6.5068$

The coefficients A_{ij} are given by:

$$A_{ij} = \left[(C_{33}\eta_{3i} + C_{32}\eta_{2i} + C_{31}\eta_{1i})S_3 b_{3j} + (C_{22}\eta_{2i} + C_{21}\eta_{1i})\alpha_2 S_2 b_{2j} + C_{11}\eta_{1i}\alpha_3 S_1 b_{1j}\right]$$

$$F(\theta_i R) + \eta_{3i}\delta_{ij}$$

$A_{oo} = .65607$	$A_{10} = .016279$	$A_{20} = -.0034545$
$A_{01} = -.033509$	$A_{11} = .091160$	$A_{21} = -.0035784$
$A_{02} = .079944$	$A_{12} = -.010834$	$A_{22} = .011817$

The successive approximations to λk using first, second and third degree determinants are:

$(\lambda k)' = 1.5242$

$(\lambda k)'' = 1.5265$

$(\lambda k)''' = 1.5278$

[redacted]

III D. Numerical Iteration Example

The application of a numerical iteration method as given in Section II D to the specific example treated in the two previous sections is here demonstrated.

It is first necessary to choose appropriate lattice spacings; here the core will be divided into ten steps and the reflector into six. The constants used are as given below:

	Core	Reflector
σ_1	.0381977	.0259330
σ_2	.0499775	.00918109
σ_3	.0785833	.00124064
D_1	1.793	.945
D_2	.8866	.5303
D_3	.284	.5367 Thermal
B^2	.00160981	.00188809
h	1.451	5.255
ρ	$0 \longrightarrow 14.51$	$14.51 \longrightarrow 46.04$

The procedure is fully explained in Section II D and can be applied immediately by replacing ρ^M by ρ since this is a one coordinate cylindrical case.

Computation sheets for the first iteration are shown in tables (III $\longrightarrow$ VIII).

From these the first approximation for k is 1.5286, successive approximations for k are 1.5336 and 1.5314. These values apparently approaches a limit not far from the values obtained by the other methods which were 1.5281 for the analytic method and 1.5278 for the series expansion method.

TABLE III. φ_1 CORE

ρ	$h\rho(\sigma_1 + B_1^2 D_1)$	$\frac{h}{\rho D_1}$	φ_1^H	I_1^H	φ_3	$h\rho\sigma_3$	φ_1^P	I_1^P	φ'_1
0				0				0	
.7255	.0432492		1.0		1.0	.0827244	1.0		1.55828
1.451		.557724		.0432492				- .0394752	
2.1765	.129748		1.02412		1.0	.248173	.977984		1.54973
2.902		.278862		.176127				- .160757	
3.6275	.216246		1.073235		1.0	.413622	.933155		1.53232
4.353		.185908		.408210				- .372588	
5.0785	.302745		1.14912		1.0	.579071	.863888		1.50542
5.804		.139431		.756100				- .690121	
6.5295	.389243		1.25454		1.0	.744520	.767664		1.46805
7.255		.111545		1.24442				-1.13583	
7.8905	.470376		1.39335		1.0	.909969	.640968		1.41885
8.706		.0929540		1.89982				-1.74430	
9.4315	.562240		1.56995		1.0	1.07542	.478828		1.35530
10.157		.0796749		2.78251				-2.55050	
10.8825	.648738		1.79165		1.0	1.24087	.275617		1.27586
11.608		.0697155		3.94482				-3.61257	
12.3335	.735237		2.06667		1.0	1.40632	.0237649		1.17754
13.059		.0619693		5.46431				-5.00142	
13.7875	.821735		2.40529		1.0	1.57176	- .286170		1.05665
14.51		.0557724	2.60324	7.44082			- .467306	-6.80834	.98602
			2.82028				- .665887		

1.45

TABLE IV. φ_1 REFLECTOR

ρ	$h\rho(\sigma_2 + B_2^2 D_1)$	$\frac{h}{\rho D_1}$	φ_1^H	I_1^H	φ_1^P	I_1^P	φ_1'
14.51		.383243	2.60324	7.44082	- .467306	- 6.80834	.98602
17.1375	2.49615		4.31290		- 1.84739		.56041
19.765		.281348		18.20647		- 11.41970	
22.3925	3.26156		9.43526		- 5.06030		.20720
25.02		.222256		48.98014		- 27.92417	
27.6475	4.02697		20.32139		- 11.26661		.07838
30.275		.183678		130.81377		- 73.29447	
32.9025	4.79238		44.34900		- 24.72919		.02990
35.53		.156511		343.35103		- 191.80615	
38.1575	5.55779		98.08721		- 54.74896		.01101
40.785		.136345		888.49914		- 496.08937	
43.4125	6.32320		219.22963		-122.38827		.00289
46.04		.120783	337.40329	2274.73194	-188.36495	-1269.97488	0
			493.97858		-275.77965		

122.38827
153.39137
275.77964

$$\varphi_1' = \varphi_1^P + \frac{188.36495}{337.40329}\,\varphi_1^H = \varphi_1^P + .558278\,\varphi_1^H$$

1.2

4.8

TABLE V. φ_2 CORE

ρ	$h\rho(\sigma_2+B_1^2D_2)$	$\frac{h}{\rho D_2}$	φ_2^H	I_2^H	φ_1'	$h\rho\sigma_1$	φ_2^P	I_2^P	φ_2'
0				0				0	
.7255	.0541137		1		1.55828	.0402108	1		1.06889
1.451		1.12790		.0541137				- .00854599	
2.1765	.162341		1.06103		1.54973	.120632	.990361		1.06345
2.902		.563952		.226362				- .0347168	
3.6275	.270569		1.18869		1.53232	.201054	.970782		1.05267
4.353		.375968		.547985				- .0801324	
5.0785	.378796		1.39471		1.50542	.281475	.940655		1.03674
5.804		.281976		1.07630				- .147554	
6.5295	.487024		1.698820		1.46805	.361897	.899048		1.01604
7.255		.225581		1.90336				- .240979	
7.8905	.595251		2.12756		1.41885	.442318	.844688		.991255
8.706		.187984		3.16979				- .365761	
9.4315	.703479		2.72343		1.35530	.522740	.775931		.963547
10.157		.161129		5.08567				- .528379	
10.8825	.811706		3.54288		1.27586	.603161	.690794		.934861
11.608		.140988		7.96145				- .737206	
12.3335	.919934		4.66535		1.17754	.683583	.586857		.908251
13.059		.125323		12.2533				-1.00228	
13.7875	1.02816		6.20097		1.05665	.764005	.461248		.888430
14.51		.112790	7.18085	18.6289	.98602		.389067	-1.33533	.883752
			8.30212				.310636		

$$\phi_2' = \phi_2^P + \alpha\phi_2^H$$

TABLE VI. φ_2 REFLECTOR

ρ	$h\rho(\sigma_2+B_2^2D_2)$	$\frac{h}{\rho D_2}$	φ_2^H	I_2^H	φ_1'	$h\rho\sigma_1$	φ_2^P	I_2^P	φ_2'
14.51		.682942	7.18085	18.6289	.98602		.389067	- 1.33533	.88375
17.1375	.916997		13.9192		.56041	2.33546	- .125859		.83302
19.765		.501365		31.3928				- 2.75956	
22.3925	1.19818		29.6585		.20720	3.05161	-1.50941		.53375
25.02		.396063		66.9290				- 5.20040	
27.6475	1.47937		56.1666		.07838	3.76775	-3.56910		.30019
30.275		.327316		150.020				-10.77574	
32.9025	1.76055		105.271		.02990	4.48389	-7.09617		.15589
35.53		.278905		335.355				-23.40297	
38.1575	2.04174		198.803		.01101	5.20003	-13.62338		.07206
40.785		.242969		741.259				-51.27563	
43.4125	2.32292		378.906		.00289	5.91617	-26.08177		.02087
46.04		.215236	532.290	1621.43	0		-36.66918	-111.8786	0
			727.896				-50.16207		

$$\varphi_2' = \varphi_2^P + .0688895\varphi_2^H$$

TABLE VII. φ_3 CORE

ρ	$h\rho(\sigma_3+B_1^2D_3)$	$\frac{h}{\rho D_3}$	φ_3^H	I_3^H	φ_2^0	$h\rho\sigma_2$	φ_3^P	I_3^P	φ_3^0
0				0				0	
.7255	.0832061		1		1.06889	.0526113	.65		.665714
1.451		3.52113		.0832061				- .00215174	
2.1765	.249618		1.29298		1.06345	.157834	.642423		.662741
2.902		1.760565		.405957				- .00963995	
3.6275	.416031		2.00769		1.05267	.263057	.625451		.657000
4.353		1.17371		1.24122				- .0263452	
5.0785	.582443		3.46452		1.03674	.368279	.594529		.648971
5.804		.8802825		3.25911				- .0618755	
6.5295	.748855		6.33346		1.01604	.473502	.540061		.63958
7.255		.704226		8.00195				- .138545	
7.8905	.915268		11.9686		.991255	.578724	.442494		.630576
8.706		.586855		18.9564				- .307207	
9.4315	1.08168		23.0933		.963547	.683947	.262208		.625102
10.157		.503019		43.9360				- .682597	
10.8825	1.24809		45.1939		.934861	.789170	- .0811513		.629037
11.608		.440141		100.342				-1.52165	
12.3335	1.41450		89.3585		.908251	.894392	- .750892		.653311
13.059		.391237		226.740				-3.39612	
13.7875	1.58092		178.068		.888430	.999615	-2.07958		.718627
14.51		.352113	256.267	508.251	.883752		-3.24547	-7.57186	.781577
			357.030				-4.74573		

TABLE VIII. φ_3 REFLECTOR

ρ	$h\rho(\sigma_3 + B_2^2 D_3)$	$\frac{h}{\rho D_3}$	φ_3^H	I_3^H	φ_2'	$h\rho\sigma_2$	φ_3^P	I_3^P	φ_3'
14.51		.674798	256.267	508.251	.883752		- 3.24547	- 7.57186	.781577
17.1375	.202989		421.652		.833027	.826826	- 5.74527		.88068
19.765		.495387		593.842				- 9.42685	
22.3925	.265232		715.834		.53375	1.08036	-10.41521		.83359
25.02		.391340		783.704				-12.76594	
27.6475	.327476		1022.53		.30019	1.33390	-15.41103		.657273
30.275		.323413		1118.56				-18.21311	
32.9025	.389720		1384.29		.15589	1.58743	-21.30139		.451704
35.53		.275579		1658.05				-26.76215	
38.1575	.451964		1841.21		.07206	1.84097	-28.67648		.256774
40.785		.240072		2490.21				-39.85555	
43.4125	.514208		2439.04		.02087	2.09451	-38.24468		.083031
46.04		.225541	2830.46	3744.38	0		-44.47859	-59.56498	0
			3283.55				-51.67903		

$$\varphi_3' = \varphi_3^P + .01571426\,\varphi_3^H$$