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Contract No. W-7405, eng 26

A MONTE CARLO COMPUTATION BEING MADE FOR NEUTRON ATTENUATION IN WATER

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DECLASSIFIED Per Letter Instructions Of AEC-1-26-50

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INTRODUCTION

During the Summer Work Session on Shielding held at the Oak Ridge National Laboratory, Summer 1949, the following Monte Carlo problem was proposed. The problem was designed by H. Kahn, the Rand Corporation, and by Lewis Nelson, ORNL. This report was written by the latter.

At the end of the Work Session, the computations are well under way. They are being performed by Mr. J. P. Kelley, Mr. Buford Carter, and the Central Statistics Laboratory personnel of K-25, Oak Ridge, with the IBM installation there.

The problem was designed as a "parametric study". It is hoped that the "life histories" obtained according to the recipes given, will contain sufficient information to allow estimates of the solutions of several different problems.

A MONTE CARLO COMPUTATION BEING MADE FOR NEUTRON ATTENUATION IN WATER

1. THE PROBLEM.

A plane, monoenergetic (10 mev) source of neutrons is embedded in a three dimensionally infinite body of water. By tracing the life histories of 1200 neutrons according to the "Monte Carlo" method in connection with the sampling techniques described in Paragraph 2, an estimate is to be computed of:

(i) the attenuation by infinite slabs of several thicknesses of a plane, 10 mev source of neutrons. These estimates can be computed for various distributions in direction of the source neutrons.

(ii) other physical quantities similar to that of (i)

(iii) the "efficiency" of the various techniques used with the intention of obtaining both the experience and information on which can be based a choice of significantly improved sampling.

While the sampling is in water, later consideration is to be given to three other media, namely

1. One in which the ratio of the number of hydrogen atoms to the number of oxygen atoms is less than that of water.
2. One in which the foregoing ratio is greater than that of water.
3. A fictitious media, water in which the cross-sections are smooth monotonic functions of energy.

Very probably this will be accomplished by adjusting the life histories obtained in water. The techniques planned for this are not discussed in this report.

The cross-sections used are those given by Feshbach in ORNL-417. His estimate of the inelastic scattering cross-section of oxygen is used and is treated as absorption. The difference between his estimates of the total and inelastic oxygen cross-sections are regarded as the elastic scattering cross-section. This scattering is taken to be isotropic in the laboratory system with the energy loss given by conserving momentum. However, this can later be corrected when other estimates of the scattering law as a function of incident neutron energy warrant the effort required.

The hydrogen cross-section used is also that given by Feshbach and is treated as isotropic in the center of gravity system. The energy degradation is that required by conserving momentum.

2. THE TECHNIQUES USED.

Of the spatial coordinates, account is taken only of x, the distance measured perpendicularly from the plane of the source. The direction of a neutron, therefore, is sufficiently described by the cosine of the angle included between its path and the x-axis. For the source neutrons it is convenient to sample the angle between the neutron direction and the x-direction randomly from a set of angles equidistributed between 0 and $\pi/2$. By proper weighting, this sample can be converted into one from a more or less arbitrary distribution

of directions for the source particles, in particular either a normal or an isotropic source.

Those neutrons which are scattered by hydrogen forward with respect to their pre-collision direction are the more important ones. Those which oxygen scatters forward with respect to the axis of the material are similarly more important. Therefore the scattering angles are sampled from distributions which are accordingly prejudiced. Correction for this is accomplished by appropriately weighting the neutron after each collision.

Since the optimum distribution from which the scattering angles could be sampled is unknown, various distributions are used. Letting $n = 0, 1, \ldots, 1199$ identify the neutrons, the sampling is

RANGE OF n	DISTRIBUTION	DISTRIBUTION
	COLLISION WITH	
	HYDROGEN	OXYGEN
0 - 199	0	0
200 - 399	2	0
400 - 599	2	2
600 - 799	3	2
800 - 999	3	3
1000 -1199	4	3

The distributions 0, 2, 3, 4 are of increasing partiality to forward angles and are given below.

When a neutron's energy has fallen below 0.4 mev it is regarded as absorbed. After each neutron has made a certain number of collisions (excepting those whose energy is already below 0.4 mev) it is subjected to a game of chance in which it has the probabilities one-half of survival and death respectively. The weight of a surviving neutron is doubled, subsequent to its next collision the process is repeated, etc., until all the neutrons are extinct.

Finally, each life history is "integrated over its first collision. This process is given in detail in Paragraph 10.

3. NOTATION.

The notation used is

n: index for identifying neutrons

i: index for identifying collisions ($i = 0$, indicates the neutron source)

x_i: position of the i-th collision

ρ_i: distance along flight path between i-th and (i + 1)-st collisions

α_i: neutron energy between collisions i and (i + 1)

μ_i: measured total cross-section for energy α_i

λ_i: angle between ρ_i and x-axis

γ_i: $\cos \lambda_i$

Δx_i: $\rho_i \gamma_i$

Θ_i: angle between ρ_{i-1} and ρ_i (the scattering angle of the i-th collision)

φ_i: azimuthal angle of the i-th collision with respect to ρ_{i-1} as pole

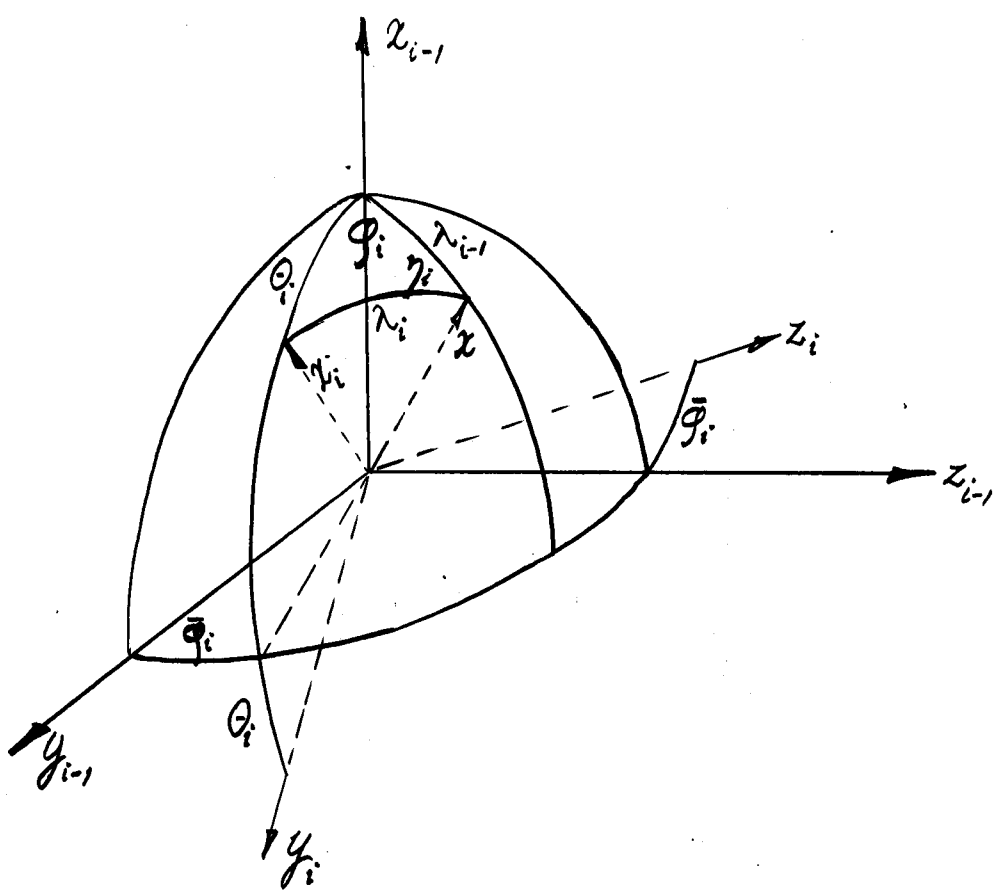
η_i: azimuthal angle of the i-th collision with respect to the x-axis as pole

$\xi^{(k)}$: a random variable

$F_k(t)$: the probability function of $\xi^{(k)}$

$f_k(t)$: the frequency function or probability density of $\xi^{(k)}$

Some of the angles are indicated in the diagram in which x_{i-1} and x_i are the directions of ρ_{i-1} and ρ_i respectively.



4. THE RANDOM VARIABLES.

Since several different random variables are used, a table of them and their associated functions is set here for convenience. For brevity we write

$$E(x) \qquad \int_0^x e(\alpha)\, d\alpha$$

$$e(\alpha) \qquad \frac{2}{(2\pi)^{1/2}} \quad \exp(-\alpha^2/2)$$

k	$\xi^{(k)}$	$F_k(t)$	$f_k(t)$	range of t
0	$\xi^{(0)}$	t	1	$(0, 1)$
1	$\xi^{(1)}$	$E(t)$	$e(t)$	$(0, \infty)$
2	$1-\frac{\xi^{(1)}}{2},\ 0 \le \xi^{(1)} \le 2$	$\frac{E(2)-E[2(1-t)]}{E(2)}$	$\frac{2}{E(2)}\, e[2(1-t)]$	$(0, 1)$
3	$1-\frac{\xi^{(1)}}{3},\ 0 \le \xi^{(1)} \le 3$	$\frac{E(3)-E[3(1-t)]}{E(3)}$	$\frac{3}{E(3)}\, e[3(1-t)]$	$(0, 1)$
4	$1-\frac{\xi^{(1)}}{4},\ 0 \le \xi^{(1)} \le 4$	$\frac{E(4)-E[4(1-t)]}{E(4)}$	$\frac{4}{E4}\, e[4(1-t)]$	$(0, 1)$
5	$\cos\varphi,\ 0 \le \varphi \le \pi$	$1-\frac{1}{\pi}\arccos t$	$\frac{1}{\pi}\,\frac{1}{(1-t^2)^{1/2}}$	$(-1, 1)$
6	$-\log \xi^{(0)}$	$1-e^{-t}$	e^{-t}	$(0, \infty)$

5.0 THE INITIAL ANGLE.

The starting angles are chosen as follows

$$\gamma_0^{(n)} = \cos(0.45 \quad n) \qquad 0 \leq n \leq 199$$

$$= \cos\left[0.45 \quad (n - 200)\right] \qquad 200 \leq n \leq 399$$

$$= \cos\left[0.45 \ (n - 400)\right] \qquad 400 \leq n \leq 599$$

$$= \cos\left[0.45 \quad (n - 600)\right] \qquad 600 \leq n \leq 799$$

$$= \cos\left[0.45 \quad (n - 800)\right] \qquad 800 \leq n \leq 999$$

$$= \cos\left[0.45 \ (n - 1000)\right] \qquad 1000 \leq n \leq 1199$$

5.1 DETERMINATION OF γ_i AND α_i AFTER COLLISION WITH HYDROGEN.

Choose ξ_i	when n is in
$\xi_i^{(0)}$	(0,199)
$\xi_i^{(2)}$	(200,399)
$\xi_i^{(2)}$	(400,599)
$\xi_i^{(3)}$	(600,799)

$\xi_i^{(3)}$ (800,999)

$\xi_i^{(4)}$ (1000,1199)

$k = 0, 2, 3, 4$

$$\cos \Theta_i = (\xi_i^{(k)})^{1/2} \qquad \sin \Theta_i = (1 - \xi_i^{(k)})^{1/2}$$

$$\cos \varphi_i = \xi_i^{(5)}$$

$$\gamma_i = \gamma_{i-1} \cos \Theta_i + \sin \lambda_{i-1} \sin \Theta_i \cos \varphi_i$$

$$\sin \lambda_i = (1 - \gamma_i^2)^{1/2}$$

$$\alpha_i = \xi_i^{(k)} \alpha_{i-1} .$$

To correct for the choice of the scattering angle from an incorrect distribution, the weight of each neutron is multiplied by

$$A_i^{(k)} = \frac{1}{f_k(\xi_i^{(k)})} = \frac{E(k)}{k\, e(\xi_i^{(1)})} .$$

5.2 DETERMINATION OF γ_i AND α_i AFTER COLLISION WITH OXYGEN.

Choose ξ_i	when n is in
$\xi_i^{(0)}$	(0, 199)
$\xi_i^{(0)}$	(200,399)
$\xi_i^{(2)}$	(400,599)
$\xi_i^{(2)}$	(600,799)
$\xi_i^{(3)}$	(800,999)
$\xi_i^{(3)}$	(1000,1199)

$k = 0, 2, 3$

$$\gamma_i = 2 \xi_i^{(k)} - 1$$

$$\sin \lambda_i = 2 \left[\xi_i^{(k)} (1 - \xi_i^{(k)} \right]^{1/2}$$

$$\cos \eta_i = \xi_i^{(5)}$$

$$\cos \Theta_i = \gamma_{i-1} \gamma_i + \sin \lambda_i \cos \eta_i$$

$$\alpha_i = \frac{2}{17} \left(\frac{15}{2} + \cos \Theta_i \right) \alpha_{i-1} .$$

Again the neutron's weight is multiplied by

$$A_i^{(k)} = \frac{1}{f_k(\xi_i^{(k)})} = \frac{E(k)}{k\, e\, (\xi_i^{(1)})} .$$

6. DETERMINATION OF ρ_i.

$$\rho_i = \frac{1}{\mu_i} \xi_i^{(6)} .$$

7. DETERMINATION OF THE KIND OF COLLISION.

σ_O = oxygen cross-section, total

σ_E = oxygen cross-section, elastic

σ_I = oxygen cross-section, inelastic

σ_H = hydrogen cross-section, (total = elastic) .

We define

$$K(\alpha) \equiv \frac{2\ \sigma_h\ (\alpha)}{2\ \sigma_h(\alpha) + \sigma_E(\alpha)} \qquad Q(\alpha) \equiv \frac{2\ \sigma_h(\alpha)\ + \sigma_E\ (\alpha)}{2\ \sigma_h(\alpha)\ + \sigma_O(\alpha)} \quad .$$

The inelastic scattering is regarded as absorption. However, absorption of a sample neutron reduces the efficiency of the computations. Therefore the collision types are sampled from a distribution which allows no absorption. The neutron is then weighted to account for use of the incorrect distribution of collision types.

Choose $\xi_i^{(0)}$ and if:

$\xi_i^{(0)} \leq K(\alpha_i)$, the collision is with hydrogen

$\xi_i^{(0)} > K(\alpha_i)$, the collision is elastic with oxygen.

In both cases the neutron's weight is multiplied by $Q(\alpha_i)$.

8. DEATH OF A NEUTRON.

When a neutron's energy becomes less than 0.4 mev it is regarded as extinct. After a neutron has undergone a certain number of collisions it is given a probability 1/2 of survival after each collision. The number of collisions per neutron which will be used will depend on the time per collision which the computations require. Perhaps so many as ten collisions per neutron is feasible, possibly only five or six.

9. THE WEIGHT.

After each collision a neutron's weight is

$$W_i = A_i^{(k)} \; Q(\alpha_i) \; \delta_i \; W_{i-1}$$

where $A_i^{(k)}$ accounts for the choosing the scattering angle from a fictitious distribution; where $Q(\alpha_i)$ accounts for the method's allowing only elastic scattering by oxygen ($Q = 1$ for $\alpha \leq 6$ mev); and where δ_i is 1 or 2 according as the neutron was <u>not</u> subjected to game of chance for survival or was so subjected and survived the game.

Another factor, $S^{(n)}$, is included in case the directional distribution of the source is to be converted from that used to another one.

9.1 SCATTERING BY HYDROGEN.

If we let Θ^* denote the scattering angle which actually occurs in the case of a collision with hydrogen and let

$$\zeta^* \equiv \cos \Theta^*$$

the associated probability function and density are

$$P^* (\zeta^* \leq U) \equiv F^* (U) = U^2 \qquad f^*(U) = 2U \qquad 0 < U < 1.$$

In the computation, page 8, a scattering angle Θ, whose cosine we now abbreviate by ζ, was computed by

$$\zeta = (\xi^{(k)})^{1/2} .$$

If we let the probability function and density of ζ be respectively

$$P(\zeta \leq v) \equiv F(v), \quad f(v)$$

it follows that

$$F(v) = F_k(v^2) \qquad f(v) = 2v \; f_k(v^2).$$

Therefore the weight which should be given a neutron after a collision with hydrogen is

$$A^{(k)} = \frac{f^*(\zeta)}{f(\zeta)} = \frac{1}{f_k(\zeta^2)} = \frac{1}{f_k(\xi^{(k)})} = \frac{E(k)}{k\, e(\xi^{(1)})} .$$

The following table gives an indication of the probability distributions from which the cosine of the angle of scattering by hydrogen is drawn. The case $k = 0$ is that of the actual scattering hence the corresponding weight is unity.

$$P(v < \zeta) \text{ or } P(\Theta < \Theta), \quad v = \cos \Theta, \quad \zeta = \cos \Theta$$

Θ/π	v	k			
		0	2	3	4
1/6	/2	0.25	0.401	0.548	0.683
1/4	/2	0.5	0.715	0.869	0.955
1/3	1/2	0.75	0.908	0.978	0.997

9.2 SCATTERING BY OXYGEN.

The actual probability function and probability density obeyed by a neutron are respectively

$$P^*(\gamma \leq u) \equiv F^*(u) = \frac{1}{2}(1+u) \qquad f^*(u) = \frac{1}{2} \qquad 0 < u < 1.$$

Since the probability function and density of $\xi^{(k)}$ are

$$P(\xi^{(k)} \leq t) = F_k(t), \qquad f_k(t)$$

and in Paragraph 5.2

$$\gamma = 2\,\xi^{(k)} - 1$$

it follows that for $-1 < u < 1$

$$P(\gamma \leq u) = F_k\left(\frac{1+u}{2}\right) \qquad f_\gamma(u) = \frac{1}{2}\, f_k\left(\frac{1+u}{2}\right).$$

Therefore the weight to be given a neutron because of the partiality of distribution from which an oxygen scattering is drawn is

$$A^{(k)} = \frac{f^*(\gamma)}{f_\gamma(\gamma)} = \frac{1/2}{1/2\, f_k(\xi^{(k)})} = \frac{E(k)}{k\, e(\xi^{(1)})} .$$

A few values of $P(\gamma > u)$, the probability function from which the γ are sampled, are listed in the following table.

$P(\gamma > u)$ or $P(\Theta \leq \Theta)$, $u = \cos \Theta$, $\gamma = \cos \Theta$

Θ/π	u	k			
		0	2	3	4
1/6	$\sqrt{3}/2$	0.067	0.112	0.160	0.211
1/4	$\sqrt{2}/2$	0.146	0.241	0.341	0.442
1/3	1/2	0.25	0.401	0.548	0.683
1/2	0	0.5	0.715	0.869	0.955
2/3	-1/2	0.75	0.908	0.978	0.997
3/4	$-\sqrt{2}/2$	0.854	0.956	0.992	0.999
5/6	$-\sqrt{3}/2$	0.933	0.983	0.998	1.000

10. INTEGRATION OVER THE FIRST COLLISION.

Let us consider the path described by a certain neutron. Since $\gamma_0 = \cos \lambda_0$ was taken positive and since the source also produced a neutron with $\lambda_0' = \pi - \lambda_0$ the path of this neutron is taken as the mirror image in the plane $x = 0$ of the neutron with initial angle λ_0. Thus we consider the mirror image of each trajectory and suppose we thereby have doubled the number of life histories actually recorded.

Let us now compute from the life histories obtained an estimate of the particle current across the plane $x = a$. We will do this by computing an estimate of the current due to particles which cross the plane between their i-th and (i + 1)-st collisions. The sum (over i) of these estimates is an estimate of the total current. For this purpose we fix i and consider a particular neutron immediately after its i-th collision.

We define

$$Z_i^{(n)} = \sum_{j=1}^{i-1} \rho_j^{(n)} \gamma_j^{(n)}$$

and denote by Z_i the random variable which is the difference of the coordinates (on the x-axis) of the i-th and of the first collisions of a neutron.

Also we denote by $\mathcal{E}(a)$ the discrete random variable

$$\mathcal{E}(a) = -1 \quad \text{when a neutron crossed the plane } x = a \text{ from right to left}$$

$$= 0 \quad \text{when a neutron does not cross the plane } x = a$$

$$= 1 \quad \text{when a neutron crosses the plane } x = a \text{ from left to right.}$$

We can now write

$$P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E}, \gamma_o = \gamma_o^{(n)}, Z_i = z_i^{(n)}, \gamma_i = \gamma_i^{(n)} \right\} =$$

$$= P_i(\gamma_o^{(n)}, Z_i^{(n)}, \gamma_i^{(n)})\, P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E} \,/\, \gamma_o = \gamma_o^{(n)}, Z_i = Z_i^{(n)}, \gamma_i = \gamma_i^{(n)} \right\}$$

where $\mathcal{E}(a)$ refers to crossing the plane $x = a$ between the i-th and (i + 1)-st collisions. Integrating both sides of this equation over all $\gamma_o^{(n)}$, $Z_i^{(n)}$ and $\gamma_i^{(n)}$ gives the probability that between its i-th and (i + 1)-st collisions a neutron does not cross the plane $x = a$, crosses from left to right, crosses from right to left according as $\mathcal{E} = 0, 1, -1$,

$$P_i \left[\mathcal{E}(a) = \mathcal{E} \right] \equiv P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E} \right\} =$$

$$= \int P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E} \,|\, \gamma_0 = \gamma_o^{(n)}, Z_i = Z_i^{(n)}, \gamma_i = z_i^{(n)} \right\} \, d\, P_i\, (\gamma_o^{(n)}, Z_i^{(n)}, \gamma_i^{(n)}).$$

Since the $\gamma_0^{(n)}$, $Z_i^{(n)}$ and $\gamma_i^{(n)}$ in our file of life histories are sampled from

$$P^* (\gamma_o^{(n)}, Z_i^{(n)}, \gamma_i^{(n)}) = P (\gamma_o^{(n)}, Z_i^{(n)}, \gamma_i^{(n)}) / W_i^{(n)}$$

an estimate of the integral on the right is given by a sum over n. That is

$$\frac{1}{2N} \sum_{(n)} P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E}, \gamma_o = \gamma_o^{(n)}, Z_i = Z_i^{(n)}, \gamma_i = \gamma_i^{(n)} \right\} W_i^{(n)}$$

is our estimate of

$$P_i \left[\mathcal{E}(a) = \mathcal{E} \right] .$$

The particle current across the plane $x = a$ is therefore estimated by

$$\sum_{(i)} \left\{ P_i \left[\mathcal{E}(a) = 1 \right] - P_i \left[\mathcal{E}(a) = -1 \right] \right\} .$$

10.1 COMPUTATION OF $P_i^{(n)}$.

We adopt the abbreviation

$$P_i^{(n)} \left[\mathcal{E}(a) = \mathcal{E} \right] \equiv P_i \left\{ \rho_o < \infty, \rho_i < \infty, \mathcal{E}(a) = \mathcal{E} \,\middle|\, \gamma_o = z_o^{(n)}, Z_i = Z_i^{(n)}, \gamma_i = \gamma_i^{(n)} \right\} .$$

The various possibilities are tabulated. In this table the numbers in parentheses in the fourth and fifth columns refer to the equation number which gives the corresponding expression for $P_i^{(n)}$.

RANGE OF			$P_i^{(n)}$	
Z_i	γ_i	γ_0	$\mathcal{E}(a) = 1$	$\mathcal{E}(a) = -1$
$(-a, a)$	$(0, 1)$	$(0, 1)$	(1)	0
	$(0, 1)$	$(-1,0)$	0	0
	$(-1,0)$	$(0, 1)$	0	0
	$(-1,0)$	$(-1,0)$	(2)	0
(a, ∞)	$(0, 1)$	$(0, 1)$	0	0
	$(0\ 1)$	$(-1,0)$	0	0
	$(-1,0)$	$(0,1)$	0	(3)
	$(-1,0)$	$(-1,0)$	(2)	0
$(-a,-\infty)$	$(0, 1)$	$(0,1)$	(1)	0
	$(0, 1)$	$(-1,0)$	0	(4)
	$(-1,0)$	$(0, 1)$	0	0
	$(-1,0)$	$(-1,0)$	0	0

In equations (1) through (4) the signs have been so chosen that the γ_o, Z_i and γ_i are the actual values which occur in the life histories.

(1) $\quad Z_i < a, \quad \gamma_i > 0, \quad \gamma_o > 0, \quad \mathcal{E}(a) = 1$

$$P_i^{(n)} = \int_0^{a-Z_i} \frac{\mu_o}{\gamma_o} \exp\left(-\frac{\mu_o}{\gamma_o} x\right) \exp\left\{-\frac{\mu_i}{\gamma_i}(a - x - Z_i)\right\} dx =$$

$$= \frac{\mu_o}{\gamma_o}\left(\frac{\mu_o}{\gamma_o} - \frac{\mu_i}{\gamma_i}\right)^{-1}\left[\exp\left\{-\frac{\mu_i}{\gamma_i}(a - Z_i)\right\} - \exp\left\{-\frac{\mu_o}{\gamma_o})a - Z_i)\right\}\right]$$

(2) $\quad -a < Z_i, \quad \gamma_i < 0, \quad \gamma_o < 0, \quad \mathcal{E}(a) = 1$

$$P_i^{(n)} = \int_0^{-\infty} -\frac{\mu_o}{\gamma_o} \exp\left(\frac{\mu_o}{\gamma_o} x \exp\left\{\frac{\mu_i}{\gamma_i}(a - x + Z_i)\right\}\right. dx =$$

$$= \frac{\mu_o}{\gamma_o}\left(\frac{\mu_o}{\gamma_o} - \frac{\mu_i}{\gamma_i}\right)^{-1} \exp\left\{\frac{\mu_i}{\gamma_i}(a + Z_i)\right\}$$

(3) $\quad a < Z_i, \quad \gamma_i < 0, \quad \gamma_o > 0, \quad \mathcal{E}(a) = -1$

$$P_i^{(n)} = \int_0^{\infty} \frac{\mu_o}{\gamma_o} \exp\left(-\frac{\mu_o}{\gamma_o} x\right) \exp\left\{-\frac{\mu_i}{\gamma_i}(a - x - Z_i)\right\} dx =$$

$$= \frac{\mu_o}{\gamma_o}\left(\frac{\mu_o}{\gamma_o} - \frac{\mu_i}{\gamma_i}\right)^{-1} \exp\left\{-\frac{\mu_i}{\gamma_i}(a - Z_i)\right\}$$

(4) $\quad Z_i < -a, \quad \gamma_i > 0, \quad \gamma_o < 0, \quad \mathcal{E}(a) = -1$

$$P_i^{(n)} = \int_0^{a+Z_i} -\frac{\mu_o}{\gamma_o} \exp\left(\frac{\mu_o}{\gamma_o} x\right) \exp\left\{\frac{\mu_i}{\gamma_i}(a - x + Z_i)\right\} dx$$

$$= \frac{\mu_o}{\gamma_o}\left(\frac{\mu_o}{\gamma_o} - \frac{\mu_i}{\gamma_i}\right)^{-1} \left[\exp\left\{\frac{\mu_i}{\gamma_i}(a + Z_i)\right\} - \exp\left\{\frac{\mu_o}{\gamma_o}(a + z_i)\right\}\right]$$

11. VARIOUS QUANTITIES TO BE ESTIMATED.

In this section are listed several quantities capable of estimation from the data obtained in the preceding sections. Let us define

$\psi(a, \alpha, \gamma) \equiv$ number of particles per unit volume, per unit α, per unit γ at (a, α, γ) in phase space.

11.1 PARTICLE CURRENT.

Definition: $\iint \psi(a, \alpha, \gamma) \; \gamma \, v \, d\alpha \, d\gamma$

Estimate: $\frac{1}{2N} \sum_{i,n} W_i^{(n)} \; \mathcal{E} \; P_i^{(n)}$.

11.2 PARTICLE DENSITY.

Definition: $\iint \psi(a, \alpha, \gamma) \; d\alpha \, d\gamma$

Estimate: $\frac{1}{2N} \sum_{(i,n)} W_i^{(n)} \; \mathcal{E} \; P_i^{(n)} \; \frac{1}{\gamma_i^{(n)} \; v_i^{(n)}}$.

11.3 **ENERGY DENSITY.**

Definition: $\iint \psi(a, \alpha, \gamma)\ \alpha\, d\alpha\, d\gamma$

Estimate: $\frac{1}{2N} \sum_{(i,n)} W_i^{(n)} \mathcal{E}\, P_i^{(n)} \frac{\alpha_i^{(n)}}{\gamma_i^{(n)}\ v_i^{(n)}}$.

11.4 **PARTICLE COLLISION DENSITY.**

Definition: $\iint \mu\, \psi(a, \alpha, \gamma)\ d\alpha\, d\gamma$

Estimate: $\frac{1}{2N} \sum_{(i,n)} W_i^{(n)} \mathcal{E}\, P_i^{(n)} \frac{\mu_i^{(n)}}{\gamma_i^{(n)}\ v_i^{(n)}}$.

11.5 **RATE OF PARTICLE COLLISION.**

Definition: $\iint v\, \mu\, \psi(a, \alpha, \gamma)\ d\alpha\, d\gamma$

Estimate: $\frac{1}{2N} \sum_{(i,n)} W_i^{(n)} \mathcal{E}\, P_i^{(n)} \frac{\mu_i^{(n)}}{\gamma_i^{(n)}}$.

11.6 **PARTICLE FLUX.**

Definition: $\iint v\ \psi(a, \alpha, \gamma)\ d\alpha\, d\gamma$

Estimate: $\frac{1}{2N} \sum_{(i,n)} W_i^{(n)} \mathcal{E}\, P_i^{(n)} \frac{1}{\gamma_i^{(n)}}$.