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EFFECT OF THE FINITE SIZE OF THE NUCLEUS ON INTERNAL CONVERSION

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EFFECT OF THE FINITE SIZE OF THE NUCLEUS ON INTERNAL CONVERSION

N. Tralli and M. E. Rose

I Introduction

In all existing calculations of internal conversion coefficients the nucleus has been considered as a point. It was implicitly assumed that the effect of the finite size of the nucleus would be negligibly small. Fowler[1], in 1930, made a rough calculation of the effect in the case of non-radiative transitions, $J=0 \rightarrow J=0$, where J is the total angular momentum quantum number for the nucleus. However, in view of the refined calculations of internal conversion coefficients[2], a more exact investigation of the effect of the finite size of the nucleus is desirable.

The internal conversion coefficients depend on (1) the scalar and vector potentials of the radiation field, (2) the initial bound-state wave functions of the electron and (3) the continuum final-state wave functions. In the following calculations, the effect of the finite size of the nucleus on each of these is investigated separately.

II The Effective Radiation Field

The expressions for the scalar potential Φ and the vector potential $\vec{A}$ at a point $\vec{r}$ at time t due to charge density ρ' and current density $\vec{j}'$ at a nuclear point $\vec{r}'$ are

$$\Phi(r,t) = \int \frac{[\rho']}{R} \, d\tau' \tag{1}$$

and

$$\vec{A}(r,t) = \int \frac{[\vec{j}]}{R} \, d\tau' \tag{2}$$

in which $R = |\vec{r} - \vec{r}'|$, $d\tau' = r'^2 \sin\theta' d\theta' d\varphi' dr'$ is an element of the nuclear volume, and $[\rho']$ and $[\vec{j}']$ are, respectively, the nuclear charge and current density evaluated at the time $t - R/c$ (3).

For a transition in which the nuclear angular momentum quantum number goes from J' to J we assume that the charge and current densities are given by the Schroedinger expressions

$$[\rho'] = \epsilon \Psi_J^* (r') \Psi_{J'} (r') e^{-i\omega t + iKR}$$

and

$$[\vec{j}'] = -i \frac{\epsilon}{2} \left(\Psi_J^* \operatorname{grad} \Psi_{J'} - \Psi_{J'} \operatorname{grad} \Psi_J^* \right) e^{-i\omega t + iKR}$$

in which ϵ is the total nuclear charge, the Ψ's are the nuclear wave functions, $\omega = (W_{J'} - W_J)/\hbar$ where W_J is the energy associated with the angular momentum J, and $K = \omega/c$. In both of these relations, as in what follows, relativistic units with $\hbar = m = c = 1$ are used.

We shall outline the derivation of the electric multipole scalar potential and merely state the result for the vector potential since it is similar but somewhat more involved. The nuclear wave functions are assumed to be of the form

$$\Psi_J = R_J(r') Y_J^M(\theta', \varphi') \quad \text{and} \quad \Psi_{J'} = R_{J'}(r') Y_{J'}^{M'}(\theta', \varphi')$$

where R is a function of r' only and $Y_J^M(\theta', \varphi')$ is a normalized spherical harmonic. Our results will be independent of the special form of the nuclear wave functions. All that is necessary is that the Ψ_J be eigenfunctions of the

J^2 and J_z operators and of correct parity. Using the relation[4]

$$\frac{e^{iKR}}{R} = \frac{i}{K r_< r_>} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\Theta)\, \eta_l(K r_>)\, \psi_l(K r_<)$$

where
$$\eta_l(K r_>) = \sqrt{\frac{\pi K r_>}{2}}\, H^{(1)}_{l+1/2}(K r_>)$$

$$\psi_l(K r_<) = \sqrt{\frac{\pi K r_<}{2}}\, J_{l+1/2}(K r_<)$$

and the addition theorem for the Legendre polynomials

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_l^{m*}(\theta',\varphi')\, Y_l^{m}(\theta,\varphi)$$

in which $r_>$ is the larger and $r_<$ is the smaller of r and r', Θ is the angle between $\vec{r}$ and $\vec{r}'$, $H^{(1)}$ is the Hankel function of the first kind and J is the Bessel function, substitution in (1) gives

$$\Phi = e^{-i\omega t}\, i\, 2\pi^2 K \epsilon \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_l^m(\theta,\varphi) \int j_l(K r_<)\, \mathcal{H}_l(K r_>)\, R_J\, R_{J'}\, r'^2 dr'$$

$$\times \int d\Omega'\, Y_{J'}^{M'}\, Y_J^{M*}\, Y_l^{m*} \qquad (3)$$

in which $d\Omega' = \sin\theta' d\theta' d\varphi'$ and

$$j_l(K r_<) = \frac{J_{l+1/2}(K r_<)}{\sqrt{K r_<}} \quad \text{and} \quad \mathcal{H}_l(K r_>) = \frac{H^{(1)}_{l+1/2}(K r_>)}{\sqrt{K r_>}}$$

The integral over the three spherical harmonics is identically zero unless

$$m = M' - M$$

$$|J'-J| \leq \ell \leq |J'+J|$$

and

$$J' + J + \ell = \text{even integer.}$$

This last condition is the parity selection rule.

There are three cases to be considered:

Case I. The electron is outside the nucleus. $0 < r < R_n$ and $R_n < r < \infty$ where R_n is the nuclear radius, so that $r' = r_<$ and $r = r_>$. Considering only the lowest value of ℓ namely $\ell = |J'-J| \equiv I$ and writing $m = M' - M \equiv \mu$, expression (3) becomes

$$\Phi_1 = e^{-i\omega t}\, i\, 2\pi^2 K \epsilon\, a_1\, \mathcal{H}_I(Kr)\, Y_I^{\mu}(\theta,\varphi) \tag{4}$$

where

$$a_1 = \int_0^{R_n} \mathcal{J}_I(Kr')\, \mathcal{R}_J\, \mathcal{R}_{J'}\, r'^2 dr' \int d\Omega'\, Y_{J'}^{M'}\, Y_J^{M*}\, Y_I^{\mu *}.$$

Case II. The electron is inside the nucleus and $r > r'$. Then $r = r_>$, $r' = r_<$ and (3) becomes

$$\Phi_2 = e^{-i\omega t}\, i\, 2\pi^2 K \epsilon\, a_2\, \mathcal{H}_I(Kr)\, Y_I^{\mu}(\theta,\varphi) \tag{5}$$

where

$$a_2 = \int_0^{r} \mathcal{J}_I(Kr')\, \mathcal{R}_J\, \mathcal{R}_{J'}\, r'^2 dr' \int d\Omega'\, Y_{J'}^{M'}\, Y_J^{M*}\, Y_I^{\mu *}.$$

Case III. The electron is inside the nucleus and $r' > r$. Here $r' = r_L$ and $r = r_S$ so that (3) becomes

$$\Phi_3 = e^{-i\omega t}\, i 2\pi^2 K \epsilon\, a_3\, \mathcal{J}_I(Kr)\, Y_I^{\mu}(\theta,\varphi) \tag{6}$$

where

$$a_3 = \int_r^{R_n} \mathcal{H}_I(Kr')\, R_J\, R_{J'}\, r'^2 dr' \int d\Omega'\, Y_{J'}^{M'}\, Y_J^{M*}\, Y_I^{\mu *}.$$

Adding (4), (5) and (6) we obtain the general expression for the scalar potential at $\vec{r}$

$$\Phi = e^{-i\omega t}\, i 2\pi^2 K \epsilon \left\{ a_1 \mathcal{H}_I(Kr) + a_2 \mathcal{H}_I(Kr) + a_3 \mathcal{J}_I(Kr) \right\} Y_I^{\mu}(\theta,\varphi) \tag{7}$$

in which the range of r is $R_n < r < \infty$ in the first term in the brackets and $0 < r < R_n$ in the second and third terms.

By a very analogous procedure we obtain for the vector potential

$$\left.\begin{aligned}
A_z &= e^{-i\omega t}\, \pi^2 K \epsilon \left\{ b_1 \mathcal{H}_{I-1}(Kr) + b_2 \mathcal{H}_{I-1}(Kr) + b_3 \mathcal{J}_{I-1}(Kr) \right\} Y_{I-1}^{\mu} \\
A_x + i A_y &= e^{-i\omega t}\, \pi^2 K \epsilon \left\{ c_1 \mathcal{H}_{I-1}(Kr) + c_2 \mathcal{H}_{I-1}(Kr) + c_3 \mathcal{J}_{I-1}(Kr) \right\} Y_{I-1}^{\mu+1} \\
A_x - i A_y &= e^{-i\omega t}\, \pi^2 K \epsilon \left\{ d_1 \mathcal{H}_{I-1}(Kr) + d_2 \mathcal{H}_{I-1}(Kr) + d_3 \mathcal{J}_{I-1}(Kr) \right\} Y_{I-1}^{\mu-1}
\end{aligned}\right\} \tag{8}$$

where the range of r in the terms in the brackets is the same as in the case of Φ. The condition $\operatorname{div}\vec{A} + \frac{1}{c}\dot{\Phi} = 0$ leads to the relations

$$a_1 = a_2 = a_3 = e\,\frac{I}{2(2I+1)}$$

$$b_1 = b_2 = b_3 = e\,B_I^{\mu}$$

$$c_1 = c_2 = c_3 = -e\,D_I^{\mu}$$

$$d_1 = d_2 = d_3 = e\,F_I^{\mu}$$

where e is an arbitrary constant and

$$B_I^{\mu} = \sqrt{\frac{(I+\mu)(I-\mu)}{(2I+1)(2I-1)}}\,, \quad D_I^{\mu} = \sqrt{\frac{(I-\mu)(I-\mu-1)}{(2I+1)(2I-1)}}\,, \quad F_I^{\mu} = \sqrt{\frac{(I+\mu)(I+\mu-1)}{(2I+1)(2I-1)}}$$

are the transformation coefficients for vector addition. The relations $a_1 = a_2 = a_3$, etc., are more special than is required. The general relations lead to the same results.

The potentials are therefore

$$\left.\begin{aligned}
\Phi &= e^{-i\omega t}\,\frac{ieI}{2I+1}\left\{\mathcal{H}_I(Kr) + \mathcal{J}_I(Kr)\right\} Y_I^{\mu}(\theta,\varphi) \\
A_z &= e^{-i\omega t}\,e\,B_I^{\mu}\left\{\mathcal{H}_{I-1}(Kr) + \mathcal{J}_{I-1}(Kr)\right\} Y_{I-1}^{\mu}(\theta,\varphi) \\
A_x + i\,A_y &= -\,e^{-i\omega t}\,e\,D_I^{\mu}\left\{\quad '' \qquad '' \quad\right\} Y_{I-1}^{\mu+1}(\theta,\varphi) \\
A_x - i\,A_y &= e^{-i\omega t}\,e\,F_I^{\mu}\left\{\quad '' \qquad '' \quad\right\} Y_{I-1}^{\mu-1}(\theta,\varphi)
\end{aligned}\right\} \qquad (9)$$

in which the range of r is $0 < r < \infty$ in the first term in the brackets and $0 < r < R_n$ in the second term.

Taking $e = (2I+1)\sqrt{\frac{2}{\pi}\frac{1}{I(I+1)}}$ and comparing (9) with the expressions for the electric multipole potentials for the point nucleus as given by Rose and Goertzel[2], it is seen that the effect of the finite size of the nucleus is to introduce the second term in the brackets of (9).

The potentials for magnetic multipole radiation are obtained from the potentials for electric multipole radiation by means of the duality principle, $\vec{E}^m = \vec{H}^e$ and $\vec{H}^m = -\vec{E}^e$. Choosing $\Phi^m = 0$, $\vec{E}^m = -\frac{1}{c}\dot{\vec{A}}^m = iK\vec{A}^m$ so that $\vec{A}^m = -\frac{i}{K}\vec{H}^e$. The relations[5]

$$iH_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(A_x + iA_y) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(A_x - iA_y)$$

$$\pm i(H_x \pm iH_y) = \left(\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y}\right)A_z - \frac{\partial}{\partial z}(A_x \pm A_y)$$

then yield

$$\left.\begin{aligned}
A_x &= -e^{-i\omega t}\frac{e\mu}{2I+1}\left\{\mathcal{H}_I(Kr) + \mathcal{J}_I(Kr)\right\}Y_I^{\mu}(\theta,\varphi) \\
A_x + iA_y &= e^{-i\omega t}\frac{e\sqrt{(I-\mu)(I+\mu+1)}}{2I+1}\left\{\quad '' \qquad '' \quad\right\}Y_I^{\mu+1}(\theta,\varphi) \\
A_x - iA_y &= e^{-i\omega t}\frac{e\sqrt{(I+\mu)(I-\mu+1)}}{2I+1}\left\{\quad '' \qquad '' \quad\right\}Y_I^{\mu-1}(\theta,\varphi)
\end{aligned}\right\} \quad (10)$$

as the expressions for the magnetic multipole vector potential.

Since the range of r in $\mathcal{J}(Kr)$ in (9) and (10) is $0 < r < R_n$, for large r the electric and magnetic multipole potentials for the finite nucleus reduce to those for the point nucleus. Consequently, the expressions for the Poynting flux and the number of gamma rays emitted per unit time are the same as for the point nucleus.

In the non-relativistic calculations of internal conversion coefficients only the most singular term of the Hankel function is retained. In this approximation, therefore, the effect of the finite size of the nucleus is zero. In general, a measure of the effect of the finite size of the nucleus on the potentials is the value of the ratio

$$\frac{\mathcal{J}_I(KR_n)}{\mathcal{H}_I(KR_n)} \simeq \frac{i\,(I-1/2)}{\{\Gamma(I+3/2)\}^2}\left(\tfrac{K}{2}R_n\right)^{2I+1}$$

For $I=1$, this is of order $i\,(KR_n)^3/9\pi < i\,K^3\ 3\times10^{-5}$. Since the ratio decreases with increasing I, we may say that the effect of the finite size of the nucleus on the multipole potentials is negligibly small for all I and for all gamma rays of interest.

III The Continuum Wave Functions

In the case of a central field one obtains the Dirac radial equations

$$\left.\begin{aligned} G' + \frac{K}{r}G - (W+1-V)F &= 0 \\ F' - \frac{K}{r}F + (W-1-V)G &= 0 \end{aligned}\right\} \qquad (11)$$

in which $G = rg$ and $F = rf$ where g and f are the Dirac radial functions, W is the energy inclusive of rest energy, $V = -\alpha Z/r$ is the potential energy, the primes denote differentiation with respect to r, and

$$K = -(j + 1/2) \qquad \text{for} \quad j = \ell + 1/2$$
$$K = j + 1/2 \qquad \text{"} \quad j = \ell - 1/2$$

where j is the total angular momentum of the electron and ℓ corresponds to the orbital angular momentum in the non-relativistic limit.

The regular solutions of (11), normalized per unit energy interval, are[6]

$$\left\{ \begin{matrix} F \\ G \end{matrix} \right\} = \sqrt{1 \mp W}\,(2pr)^{\gamma} \frac{e^{\pi\alpha Z W/2p}\,|\Gamma(\gamma + i\alpha Z W/p)|}{2(\pi p)^{1/2}\,\Gamma(2\gamma+1)}$$

$$\times \left\{ e^{-ipr+i\eta(\gamma + i\alpha Z W/p)}\, F(\gamma+1+i\alpha Z W/p,\, 2\gamma+1,\, 2ipr) \mp \text{c.c.} \right\} \qquad (12)$$

in which

$$e^{2i\eta} = -\frac{K - i\alpha Z/p}{\gamma + i\alpha Z W/p} \qquad \text{and} \qquad \gamma = \sqrt{K^2 - \alpha^2 Z^2},$$

p is the momentum of the electron and c.c. represents complex conjugate. The asymptotic behavior of (12) is given by

$$F_\infty = -\sqrt{\frac{W-1}{\pi p}}\,\sin(pr + \delta) \qquad \text{and} \qquad G_\infty = \sqrt{\frac{W+1}{\pi p}}\,\cos(pr + \delta)$$

where

$$\delta = \frac{\alpha Z W}{p}\ln 2pr - \arg \Gamma(\gamma + i\alpha Z W/p) + \eta - \frac{\pi\gamma}{2}$$

The irregular solutions of (11) with the same asymptotic behavior and relative phase may be obtained from the regular solutions by replacing γ by $-\gamma$. They are

$$\left\{ \begin{matrix} \bar{F} \\ \bar{G} \end{matrix} \right\} = \sqrt{1 \mp W} \, (2pr)^{-\gamma} \frac{e^{\pi\alpha ZW/2p} \left| \Gamma(-\gamma + i\alpha ZW/p) \right|}{2(\pi p)^{1/2} \, \Gamma(-2\gamma+1)}$$

$$\times \left\{ e^{-ipr+i\eta'} (-\gamma + i\alpha ZW/p) F(-\gamma+1+i\alpha ZW/p, -2\gamma+1; 2ipr) \mp c.c. \right\} \tag{13}$$

in which

$$e^{2i\eta'} = e^{2i\eta(-\gamma)} = - \frac{K - i\alpha Z/p}{-\gamma + i\alpha ZW/p}$$

Their asymptotic behavior is given by

$$\bar{F}_\infty = -\sqrt{\frac{W-1}{\pi p}} \sin(pr + \delta') \quad \text{and} \quad \bar{G}_\infty = \sqrt{\frac{W+1}{\pi p}} \cos(pr + \delta')$$

where

$$\delta' = \delta(-\gamma) = \frac{\alpha ZW}{p} \ln 2pr - \arg \Gamma(-\gamma + i\alpha ZW/p) + \eta' + \frac{\pi\gamma}{2}$$

The solutions in the case of the finite nucleus may be expressed in terms of (12) and (13).[7]

$$\mathcal{F} = F + H\bar{F} \quad \text{and} \quad \mathcal{G} = G + H\bar{G} \tag{14}$$

in which

$$H = \left(\frac{FG_i - GF_i}{F_i\bar{G} - G_i\bar{F}}\right)_{r=R_n} = (2pR_n)^{2\gamma}\,\frac{N(\gamma)}{N(-\gamma)}\,\mathcal{J}\,\frac{\cos\eta}{\cos\eta'}$$

where the F_i and G_i are the solutions inside the nucleus, and

$$\frac{N(\gamma)}{N(-\gamma)} = \frac{|\Gamma(\gamma + i\alpha ZW/p)|}{|\Gamma(-\gamma + i\alpha ZW/p)|}\;\frac{|\Gamma(-2\gamma+1)|}{|\Gamma(2\gamma+1)|}$$

$$\mathcal{J} = \frac{\tan\eta\,\cos(\omega-\beta) + \sin(\omega-\beta)}{\tan\eta'\,\cos(\omega+\beta) + \sin(\omega+\beta)}$$

in which

$$\tan\eta = \frac{(KW+\gamma)\,\alpha Z/p}{\gamma^2 - K\gamma + \left(\frac{\alpha Z}{p}\right)^2 W(W+1)}$$

$$\tan\eta' = \frac{(KW-\gamma)\,\alpha Z/p}{\gamma^2 + K\gamma + \left(\frac{\alpha Z}{p}\right)^2 W(W+1)}$$

$$\tan\beta = -\frac{\alpha ZW}{\gamma p}, \qquad \tan\omega = \sqrt{\frac{W+1}{W-1}}\left(\frac{F_i}{G_i}\right)_{r=R_n}$$

At the nuclear radius we have

$$\left(\frac{H\bar{G}}{G}\right)_{R_n} = -\mathcal{J}\,\frac{\cos\beta - \sin\beta\tan\eta'}{\cos\beta + \sin\beta\tan\eta}$$

with a similar expression for $\left(H\bar{F}/F\right)$

We assume the extreme case of the shell model for the nucleus in order to determine $(F_i/G_i)_{R_n}$. In the table below are given values of the ratio $(H\bar{G}/G)_{R_n}$ for $Z = 20, 90$, $W = 1.1, 5$ and $K = \pm 1, \pm 5$

$K =$	-1	+ 1	- 5	+ 5
$Z = 20$, $W = 5$	- 0.0012	- 0.093	0.00015	- 1.53
" $W = 1.1$	- 0.028	- 0.95	- 0.00010	- 16.5
$Z = 90$, $W = 5$	0.251	1.005	0.0022	0.085
" $W = 1.1$	- 0.065	- 0.395	- 0.133	- 1.15

In the case of $K = -1$ this ratio has also been calculated assuming a parabolic charge distribution in r^2 in the nucleus and the ratio of charge density at the edge of the nucleus to its density at the center as given by Feenberg[8]. The results are

$Z = 20$, $W = 5$	0.0011
" $W = 1.1$	-0.0001
$Z = 90$, $W = 5$	0.379
" $W = 1.1$	-0.0915

The ratio $(H\bar{F}/F)$ behaves in a similar manner. In those cases where the ratio is small, the effect of the finite size of the nucleus is small. In the other cases it will be necessary to investigate the rate at which the ratios decrease with increasing r before the effect of the finite size of the nucleus

may be determined.*

IV The Bound-State Wave Functions

The wave functions for the K-shell are determined by means of a perturbation calculation. For $r > R_n$ the potential is Coulombian $V = -\frac{\alpha Z}{r}$. Assuming the extreme case of a shell model for the nucleus, the potential for $r < R_n$ is given by

$$V = -\frac{\alpha Z}{R_n} = -\frac{\alpha Z}{r} - \alpha Z\left(\frac{1}{R_n} - \frac{1}{r}\right)$$

where

$$V' = -\alpha Z\left(\frac{1}{R_n} - \frac{1}{r}\right) \tag{15}$$

is the perturbation.

The unperturbed solutions of the Dirac radial equations are the Coulomb functions

$$\begin{Bmatrix} G_K \\ F_K \end{Bmatrix} = \pm\frac{\sqrt{\Gamma(2\gamma_K+n'+1)}}{\Gamma(2\gamma_K+1)\sqrt{n'!}}\sqrt{\frac{1\pm E_K}{2\alpha Z(\alpha Z-\lambda K)}}\;\lambda^{3/2}(2\lambda r)^{\gamma_K}e^{-\lambda r}$$

$$\times\left\{\mp n' F(-n'+1, 2\gamma_K+1; 2\lambda r) + \left(\frac{\alpha Z}{\lambda} - K\right)F(-n', 2\gamma_K+1; 2\lambda r)\right\}$$

in which E is the energy inclusive of the rest energy, $n' = 0, 1, 2, 3, \ldots$, $\lambda = \sqrt{1-E^2}$ and the F's are confluent hypergeometric functions which terminate. For small r the solutions reduce to

$$\begin{Bmatrix} G_K \\ F_K \end{Bmatrix} = \pm\frac{\sqrt{\Gamma(2\gamma_K+n'+1)}}{\Gamma(2\gamma_K+1)\sqrt{n'!}}\sqrt{\frac{1\pm E_K}{2\alpha Z(\alpha Z-\lambda K)}}\;2^{\gamma_K}\lambda^{\gamma_K+3/2}\left(\mp n' - K + \frac{\alpha Z}{\lambda}\right)r^{\gamma_K}. \tag{16}$$

* The termination of the summer visit of N. Tralli prevented the inclusion of this investigation in this report.

The perturbed functions are

$$\Psi^G = \Psi_0^G + \Psi_1^G \qquad \text{and} \qquad \Psi^F = \Psi_0^F + \Psi_1^F$$

and the perturbed energy is

$$E = E_0 + E_1$$

in which the unperturbed wave functions are the K-shell Coulomb wave functions

$$\Psi_0^G = \sqrt{1+\gamma}\ \frac{(2\alpha Z)^{\gamma+1/2}}{\sqrt{2\,\Gamma(2\gamma+1)}}\ r^{\gamma}$$

$$\Psi_0^F = -\sqrt{1-\gamma}\ \frac{(2\alpha Z)^{\gamma+1/2}}{\sqrt{2\,\Gamma(2\gamma+1)}}\ r^{\gamma}$$

and

$$E_0 = \gamma = \sqrt{1-\alpha^2 Z^2}$$

The perturbation energy is given by

$$E_1 = \int_0^{R_n} V'\left(\Psi_0^{G\,2} + \Psi_0^{F\,2}\right) dr = \frac{\alpha^2 Z^2}{\gamma\,\Gamma(2\gamma+2)}\ (2\alpha Z R_n)^{2\gamma} \tag{17}$$

For $Z = 90$, $E_1 \simeq 0.028$ $\left(E_1/E_0 \simeq 0.04\right)$ and is smaller for smaller Z.

The perturbation to the wave function is given by

$$\Psi_1^G = \sum_n{}' \frac{V_{nK}'}{E_K - E_n}\, G_n \qquad \text{and} \qquad \Psi_1^F = \sum_n{}' \frac{V_{nK}'}{E_K - E_n}\, F_n$$

From the orthogonality relations for the spherical harmonics, V'_{nK} vanishes unless both wave functions belong to the same $K\ (=-1)$. Writing $\Psi_{0,-1}$ as the wave function belonging to $n'=0$, $K=-1$ and $\Psi_{n',-1}$ as the wave function belonging to n', $K=-1$, then

$$\Psi_1^G = \sum_{n'}{}' \frac{V'_{n',0}}{\gamma - E_{n'}} \Psi^G_{n',-1}$$

in which

$$V'_{n',0} = \int_0^{R_n} V'\left(\Psi^G_{n',-1}\,\Psi^G_{0,-1} + \Psi^F_{n',-1}\,\Psi^F_{0,-1}\right) dr$$

$$= \frac{(2\alpha Z)^{\gamma+3/2}\,R_n^{2\gamma}}{\gamma\,\Gamma(2\gamma+2)}\;\frac{\sqrt{\Gamma(2\gamma+n'+1)}}{\sqrt{n'!}}\;\frac{2^{\gamma-3}\,(\lambda_{n'})^{\gamma+3/2}}{\sqrt{\alpha Z(\alpha Z+\lambda_{n'})}}$$

$$\times\left\{\sqrt{\frac{1+\gamma}{1-E_{n'}}}\left[(1-n')\lambda_{n'}+\alpha Z\right] + \sqrt{\frac{1-\gamma}{1+E_{n'}}}\left[(1+n')\lambda_{n'}+\alpha Z\right]\right\} \qquad (18)$$

The ratio of the first term of Ψ_1^G to the unperturbed wave function is of the order of -6×10^{-5} for $Z = 20$ and -4×10^{-4} for $Z = 90$ at the nuclear radius. Thus the first term of Ψ_1^G constitutes a negligible correction to the unperturbed wave function Ψ_0^G. The contribution of successive terms is decidedly less. Similarly, the perturbation term Ψ_1^F is a negligible correction to the unperturbed wave function Ψ_0^F. Consequently, the effect of the finite size of the nucleus on the K-shell wave functions is negligible.

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