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ON APPROXIMATE RELATIVISTIC WAVE FUNCTIONS AND INTERNAL PAIR FORMATION

J. L. JACKSON

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J. L. Jackson

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ON APPROXIMATE RELATIVISTIC WAVE FUNCTIONS AND INTERNAL PAIR FORMATION

J. L. Jackson

I. Introduction

An investigation was made of the possibility of extending the Born approximation calculation for internal pair formation[1] to take account of terms of order αZ, where α is the fine structure constant and Z is the atomic number of the nucleus. A calculation of this type would extend our knowledge of internal pair formation to higher Z as the Born approximation is valid only for $\frac{\alpha Z}{v}$ small compared to unity. The major part of the work done involved a study of approximate relativistic wave functions for electrons in a Coulomb field which, at infinity, are moving in a fixed direction. Such wave functions have a much wider area of interest than the confines of this problem. The results of this investigation were negative in the sense that none of the methods considered led to a way to perform the internal pair formation calculation. Herein, a review of the various approaches that were attempted and the reasons for their failure will be given.

In the Born approximation calculation, the effect of the Coulomb field of the nucleus upon the electron is neglected. Using Dirac plane wave functions for the electron, the probability for the transition of an electron in a negative energy state to a state of positive energy, under the perturbing action of the radiation field associated with the nuclear transition, is calculated. To carry this one step further one must improve the electronic wave functions so that they satisfy the Dirac equation for a particle moving

in a Coulomb field of charge Ze up to the first order of αZ. Three methods of obtaining such wave functions were considered. They involved:

1) Using Sommerfeld-Maue wave functions[2]

2) Using the integral equation satisfied by the wave function (obtained from the Green Function)

3) Expanding the correction to the plane wave in plane waves.

One may obtain a Green Function for either the first or second order Dirac relativistic equations. A proof that these both lead to the same integral equation is given in Appendix B.

II. Sommerfeld-Maue Functions

The Sommerfeld-Maue wave function for an electron may be written

$$\psi = \psi_0 + \frac{1}{i} \frac{\vec{\alpha} \cdot \text{grad}}{2E} \psi_0 \tag{1}$$

where

$$\psi_0 = e^{i\vec{p}\cdot\vec{r}} F(-n, 1, u)\, \Gamma(\vec{p}) \,. \tag{2}$$

Here $u = i(pr - \vec{p}\cdot\vec{r})$, $n = \alpha Z/i\nu$, and $F(-n, 1, u) = 1 - \frac{nu}{1!} \quad \frac{n(n-1)}{2!\,2!} u^2 + \ldots$

It is understood that in the second term of the right side of equation (2) the operator "grad" does not act on the factor $e^{i\vec{p}\cdot\vec{r}}$ of ψ_0, as the addition of this term would change the zeroth order part of the wave function.

This wave function satisfies the required equation

$$\left(\vec{\alpha} \cdot \frac{\text{grad}}{i} + \beta - E - \frac{Z\alpha}{r}\right) \psi = 0$$

up to the first order of αZ. In addition, however, it contains terms of higher order in αZ. One then has the choice of attempting to proceed with the wave function as it stands and perhaps, after evaluating the matrix elements, setting terms of order $(\alpha Z)^2$ and higher equal to zero or one may immediately take only that part of the wave function up to the first order of αZ and try to use that to evaluate the required matrix elements.

If one attempts to use the Sommerfeld-Maue wave function as it stands, one must perform integrals of the type:

$$\int d\tau\, e^{-i\vec{p}_I \cdot \vec{r}} \chi_\ell(r)\, Y_\ell^m(\Omega_r)\, e^{i\vec{p}_{II} \cdot \vec{r}}\, F(-n_{II}, 1, u)$$

No method of performing this integral in closed form was found. Apparently the mode of procedure would be to use the Rayleigh expansion for $e^{-i\vec{p}_I \cdot \vec{r}}$ and for the remaining part to use the expansion[3]

$$(3) \qquad e^{i\vec{p}_{II} \cdot \vec{r}} F(-n_{II}, 1, u) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(1+\lambda-n)}{2\lambda!\,\Gamma(1-n)}\, e^{-ip_{II} r}\, (2ip_{II} r)^\lambda$$

$$\times F(1+\lambda+n, 2\lambda+2, 2ip_{II} r) P_\lambda(\cos\theta)$$

where $\Gamma(\chi)$ is the gamma function of χ and θ is the angle between the vectors $\vec{p}_{II}$ and $\vec{r}$.

The performance of the angular integration leaves one with a double sum where the range of one of the parameters of summation is infinite. The coefficients of these sums involve the linear coefficients of vector addition and radial integrals. The radial integrals apparently do not converge term by term.

If, on the other hand, one attempts to work only with that part of the Sommerfeld-Maue wave function that is at most of first order in αZ, one obtains for the wave function

$$(4) \qquad \psi = L_1(u)\, e^{i\vec{p}\cdot\vec{r}}\, \Gamma(\vec{p}) + \frac{e^{i\vec{p}\cdot\vec{r}}}{2Ei}\, \Gamma(\vec{p})\, \vec{\alpha}\cdot \text{grad}\, L_1(u)$$

where

$$L_1(u) = 1 - n \int_0^u \frac{e^x - 1}{x}\, dx$$

This wave function, however, has shortcomings. Although it satisfies the Dirac equation to the first order of αZ, it does not remain finite over all space. It may readily be seen that the above function diverges logarithmically as u approaches infinity. This difficulty may be circumverted by adding terms of higher order in αZ so that it remains finite for large u. A way to do this without returning to the original Sommerfeld-Maue wave function and its complications would be to multiply the divergent terms by $e^{i\alpha Zu}$. Again, however, no way was found to evaluate the matrix elements.

III. Integral Equation

Another attempt to obtain an improved wave function was made using the Green function of the Dirac equation. This may be obtained most simply by multiplying the Dirac equation

$$(\vec{\alpha} \cdot \frac{\text{grad}}{i} + \beta - E)\psi = -V\psi$$

by the adjoint operator $\frac{\vec{\alpha} \cdot \text{grad}}{i} + \beta + E$. One then obtains

$$(5) \qquad (\nabla^2 + p^2)\psi = (\frac{\vec{\alpha} \cdot \text{grad}}{i} + \beta + E)V\psi$$

where we have let $p^2 = E^2 - 1$. The Green function of the operator $\nabla^2 + p^2$ is well known and may be written as

$$(6) \qquad G(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi} \frac{e^{ip|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|} .$$

ψ must therefore satisfy the integral equation

$$(7) \quad \psi = e^{i\vec{p} \cdot \vec{r}} \Gamma(p) - \int d\tau' \, e^{ip|\vec{r} - \vec{r}\,'|} (\vec{\alpha} \cdot \text{grad}'/i + \beta + E)V(r')\psi(r')/|r - r'|$$

where grad' is the operator gradient with respect to the primed coordinate. If one then sets $V = -\alpha Z/r'$, one may obtain the first-order approximation in αZ by setting the ψ that occurs in the integral equal to $e^{i\vec{p} \cdot \vec{r}\,'} \Gamma(\vec{p})$. The resulting approximation wave function is

$$(8) \qquad \psi = e^{i\vec{p}\cdot\vec{r}}\,\Gamma(p) +$$

$$\int \frac{d\tau'}{4\pi}\,\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\left\{\frac{Z\alpha}{i}\,\frac{\vec{\alpha}\cdot\vec{r}'}{r'^3} - \frac{2\alpha Z_E}{r'}\right\} e^{i\vec{p}\cdot\vec{r}'}\,\Gamma(\vec{p})$$

In evaluating the integral over the second term in the braces, after expanding $e^{ip|\vec{r}-\vec{r}'|}/|\vec{r}-\vec{r}'|$ and $e^{i\vec{p}\cdot\vec{r}'}$ and performing the angular integration, one is left with radial integrals of the following form

$$H_{\ell+\frac{1}{2}}(pr)\int_{r'=0}^{r} J_{\ell+\frac{1}{2}}(pr')\,J_{\ell+\frac{1}{2}}(pr') +$$

$$J_{\ell+\frac{1}{2}}(pr)\int_{r'=r}^{\infty} dr'\;H_{\ell+\frac{1}{2}}(pr')\,J_{\ell+\frac{1}{2}}(pr')$$

As the second term of this expression is divergent one apparently cannot make use of this approach for improving the wave function.

IV. Expansion in Plane Waves

As the matrix element is readily integrated when the initial and final states are plane waves (as is the case in the Born approximation), a convenient way to express the correction term to the plane wave, ψ', would be as an expansion in terms of plane waves. One would therefore like to write

$$\psi^{(s)} = e^{i \vec{p}_o \cdot \vec{r}} \Gamma^{(s)}(\vec{p}_o) + \psi' \tag{9}$$

where $$\psi' = \sum_{i=1}^{4} \int d^3\vec{k}\, b_i(\vec{k})\, e^{i \vec{k} \cdot \vec{r}} \Gamma^{(s)}(\vec{k})$$

and $\Gamma^{(s)}(\vec{p}_o)$ is the s-th Dirac plane wave spinor. The sum is over the four Dirac spinors of an electron with momentum $(\vec{k})$. If one then inserts this $\psi^{(s)}$ into the Dirac first order equation and neglects the term $V\psi'$, one obtains

$$(\vec{\alpha} \cdot \frac{\text{grad}}{i} + \beta - E_o)\psi' = - V e^{i \vec{p}_o \cdot \vec{r}} \Gamma^{(s)}(\vec{p}_o) \tag{10}$$

Inserting the expression for ψ' from equation (9) and putting $V = -\alpha Z/n$ the resulting equation is:

$$\sum_{i=1}^{u} \int d^3\vec{k}\, b^i(\vec{k}) \left[E^i - E_o\right] e^{i \vec{k} \cdot \vec{r}} \Gamma^i(\vec{k}) \tag{11}$$

$$= + \frac{Z\alpha}{r} e^{i \vec{p}_o \cdot \vec{r}} \Gamma^{(s)}(\vec{p}_o)$$

Here $E^i = + \sqrt{k^2 + 1}$ for $i = 1, 2$

$= - \sqrt{k^2 + 1}$ $\quad i = 3, 4$

Introducing the notation $\Gamma_j^{(i)}(\vec{k})$ for the j-th component of the i-th Dirac plane wave spinor and making use of the Fourier Transform Theorem one obtains the following equation for the $b^i(\vec{k})$

$$(12) \qquad \sum_{i=1}^{4} b^i(\vec{k}) \left[E^i - E_o \right] \Gamma_\ell^{(i)}(\vec{k}) = \int \frac{d^3\vec{r}}{(2\pi)^3} \, e^{-i\vec{k}\cdot\vec{r}} \, e^{i\vec{p}_o\cdot\vec{r}} \, \frac{Z\alpha}{r} \, \Gamma_\ell^{(s)}(\vec{p}_o)$$

Evaluating the integral on the right side of equation (12) and making use of the orthogonality properties of the $\Gamma_\ell^{(i)}(\vec{k})$ one obtains

$$(13) \qquad b^j(\vec{k}) = -\frac{Z\alpha}{2\pi^2} \sum_{\ell=1}^{4} \frac{\bar{\Gamma}_\ell^{(j)}(\vec{k}) \, \Gamma_\ell^{(s)}(\vec{p}_o)}{(E_j - E_o) \left| \vec{p}_o - \vec{k} \right|^2}$$

One may then write for the m-th component of ψ'

$$(14) \qquad \psi'_m = -\frac{Z\alpha}{2\pi^2} \int d^3\vec{k} \sum_{\ell=1}^{4} \frac{\Gamma_\ell^{(s)}(\vec{p}_o)}{\left| \vec{p}_o - \vec{k} \right|^2} \sum_{j=1}^{4} \frac{\bar{\Gamma}_\ell^{(j)}(\vec{k}) \, \Gamma_m^{(j)}(\vec{k})}{E_j - E_o}$$

Following a procedure identical to that of appendix B (see equations B-6 and B-9) and letting $E^2 = k^2 + 1$, the result for ψ' is

$$(15) \qquad \psi' = -\frac{Z\alpha}{2\pi^2} \int d^3\vec{k} \, e^{i\vec{k}\cdot\vec{x}} \, \frac{(\vec{\alpha}\cdot\vec{k} + \beta + E_o)}{(k^2 - p_o^2) \left| \vec{p}_o - \vec{k} \right|^2}$$

This correction to the wave function apparently is infinite due to the singularity at $\vec{k} = \vec{p}_o$.

Conclusion

It would seem that one can not treat a pure Coulomb field as a small perturbation acting on a plane wave and get non-divergent answers. The only approach considered which did not lead to divergencies was the one involving the Sommerfeld-Maue wave functions where the Coulomb field is not part of the perturbation but the term involving $\alpha Z/r$ is part of the zeroth order equation. A possible method of procedure might be to use a screened Coulomb field. This would certainly eliminate the divergencies that occur in obtaining approximate wave functions but would undoubtedly further complicate the calculation of the internal pair formation.

I wish to acknowledge my indebtedness to Dr. M. E. Rose for suggesting this study and for his valuable guidance while the work was in progress.

APPENDIX A - Notation

Following the notation of M. E. Rose[1], Internal Pair Formation one writes for the ratio of the number of pairs per second with the electron and positron travelling in the solid angles $d\Omega_-$ and $d\Omega_+$ respectively and the positron energy between W_+ and $W_+ + dW_+$ to the number of quanta per second

$$\text{(A-1)} \qquad d\gamma_\ell (W_+; \omega_+ \varphi_+ ; \omega_- \varphi_-) = \frac{\alpha k}{32\pi^3} \; p_+ \, p_- \sum_{s_+ s_-} \left| (\psi_- | V + \vec{\alpha} \cdot \vec{A} | \psi_+) \right|^2 d\Omega_+ \, d\Omega_- \; dW_+$$

where $\vec{\alpha}$ is the Dirac matrix vector, p_+ and p_- are the positron and electron momenta respectively, k is the energy of the gamma ray, and ψ_+ and ψ_- the wave functions of the positron and electron respectively. The radiation field is represented by the V and $\vec{A}$ in the gauge where

a) electric 2^ℓ pole

$$A_H = - C_\ell \frac{(2\ell + 1)}{\sqrt{2}} \sqrt{\frac{(\ell - m)(\ell - m - 1)}{(2\ell + 1)(2\ell - 1)}} \; Y_{\ell -1}^{m+1} \; \chi_{\ell -1}(kr)$$

$$A_{-1} = C_\ell \frac{(2\ell + 1)}{\sqrt{2}} \sqrt{\frac{(\ell + m)(\ell + m - 1}{(2\ell + 1)(2\ell + 2)}} \; Y_{\ell -1}^{m+1} \; \chi_{\ell -1}(kr)$$

$$\text{(A-2)} \qquad A_o = C_\ell \frac{(2\ell + 1)}{\sqrt{2}} \sqrt{\frac{(\ell + m)(\ell - m)}{(2\ell + 1)(2\ell - 1)}} \; Y_{\ell -1}^{m} \; \chi_{\ell -1}(kr)$$

$$V = i \, C_\ell \, \ell \; Y_\ell^m \; \chi_\ell (kr)$$

b) magnetic 2^{ℓ} pole

$$A_{+1} = C_{\ell} \sqrt{(\ell - m)(\ell + m + 1)/2} \; Y_{\ell}^{m+1} \; \chi_{\ell}(kr)$$

$$A_{-1} = C_{\ell} \sqrt{(\ell + m)(\ell - m + 1)/2} \; Y_{\ell}^{m-1} \chi_{\ell}(kr)$$

$$A_{o} = \frac{C_{\ell}}{\sqrt{2}} m \quad Y_{\ell}^{m} \chi_{\ell}(kr)$$

$$V = 0$$

Here $C_{\ell} = \sqrt{\dfrac{2}{\pi \ell (\ell + 1)}}$,

$$A_{+1} = \frac{1}{\sqrt{2}} (A_x + iA_y), \quad A_{-1} = \frac{1}{\sqrt{2}}(A_x - iA_y), A_o = A_z \quad ,$$

and

$$\chi_{\ell}(x) = \sqrt{\pi/2X} \; H_{\ell + \frac{1}{2}}(x)$$

where $H_{\ell+\frac{1}{2}}(x)$ is the Hankel Function of the first kind. The normalization is $1/\pi^2 k$ quanta/sec.

APPENDIX B - Equivalence of Integral Equation Obtained for the First and Second Order Dirac Equations

The integral equation obtained for a solution of the Dirac equation of second order in Section III was

$$\text{(B-1)} \qquad \psi = e^{-ip\cdot\vec{r}}\,\Gamma(\vec{p}) - \int \frac{d\tau'}{4\pi}\,\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\left(\frac{\vec{\alpha}\cdot \text{grad}'}{i} + \beta + E\right) V(\vec{r}')\,\psi(\vec{r}')$$

In order to be able to compare the above integral equation with the one which will be derived from the Dirac first order equation, it should be noticed that the Green Function

$$-\frac{e^{ip|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \quad \text{may be replaced by}$$

$$-\frac{\cos p|\vec{r}-\vec{r}'|}{4\pi|\vec{r}-\vec{r}'|} \quad \text{as their difference}$$

$$\frac{\sin p|\vec{r}-\vec{r}'|}{4\pi|\vec{r}-\vec{r}'|} \quad \text{is a solution of the}$$

homogeneous equation. Making use of

$$\text{(B-2)} \qquad 0 = \int d\tau'\,\nabla'\cdot\left(\vec{\alpha}\,\frac{\cos p|\vec{r}-\vec{r}'|}{|\vec{r}-\vec{r}'|}\,V(r')\,\psi(r')\right)$$

for a well behaved $\psi(r')$. We may then write

$$0 = \int d\tau' \; \frac{\cos p|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|} \; \vec{\alpha} \cdot \text{grad}' \, V(\vec{r}') \, \psi(\vec{r}')$$

$$+ \int d\tau' \; V(\vec{r}') \, \psi(\vec{r}') \, \vec{\alpha} \cdot \text{grad}' \; \frac{\cos p|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|}$$

One may therefore rewrite equation (B-1).

$$\text{(B-3)} \quad \psi(\vec{r}) = e^{i\vec{p}\cdot\vec{r}} \, \Gamma(\vec{p}) - \int \frac{d\tau'}{4\pi} \; \frac{\cos p|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|} \left\{ \beta + E \right\} V(\vec{r}') \, \psi(\vec{r}')$$

$$- \frac{1}{4\pi i} \int d\tau' \; V(\vec{r}') \, \psi(\vec{r}') \, \vec{\alpha} \cdot (\vec{r}' - \vec{r}) \left\{ \frac{p \sin p|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|^2} + \frac{\cos p|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|^3} \right.$$

It will be shown that the Green Function of the Dirac first order equation leads exactly to equation (B-3).

The Green Function of the Dirac first order equation is a four by four matrix which satisfies the equation

$$\text{(B-4)} \qquad \sum_{j=1}^{4} \left(\vec{\alpha} \cdot \frac{\text{grad}}{i} + \beta - E_o \right)_{\ell j} \quad G(r, r', j, k)$$

$$= \delta(\vec{r} - \vec{r}') \, \delta_{\ell k}$$

If one writes

$$\text{(B-5)} \quad \delta(\vec{r} - \vec{r}')\,\delta_{\ell k} = \sum_{i=1}^{4} \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\, \Gamma_{\ell}^{(i)}(\vec{p})\, \overline{\Gamma_{k}^{(i)}}(\vec{p})$$

where $\Gamma_{\ell}^{(i)}(\vec{p})$ is the ℓ-th component of the i-th Dirac plane wave spinor and $\overline{\Gamma_{\ell}^{(i)}}(\vec{p})$ is its complex conjugate, one sees that the Green Function may be written as

$$\text{(B-6)} \quad G(\vec{r}, \vec{r}'; j, k) = \sum_{i=1}^{4} \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}}{E_i - E_o}\, \Gamma_{j}^{(i)}(\vec{p})\, \overline{\Gamma_{k}^{(i)}}(\vec{p})$$

Multiplying the numerator and the denominator of the right side of the preceding equation by $E_i + E_o$ one obtains

$$\text{(B-7)} \quad G(\vec{r}, \vec{r}'; j, k) = E_o \sum_{i=1}^{4} \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}}{E_i^{\,2} - E_o^{\,2}}\, \Gamma_{j}^{(i)}(\vec{p})\, \overline{\Gamma_{k}^{(i)}}(\vec{p})$$

$$+ \sum_{i=1}^{4} \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}}{E_i^{\,2} - E_o^{\,2}}\, E_i\, \Gamma_{j}^{(i)}(\vec{p})\, \overline{\Gamma_{k}^{(i)}}(\vec{p})$$

Equation (B-7) may be put into convenient form by making use of the orthogonality of the $\Gamma_{j}^{(i)}$ and by applying the following theorem:

$$\text{(B-8)} \qquad \sum_{i=1}^{4} E_i \, \Gamma_j^{(i)}(\vec{p}) \, \overline{\Gamma_k^{(i)}}(\vec{p}) = (\vec{\alpha} \cdot \vec{p} + \beta)_{jk}$$

This may be demonstrated most simply by observing that the result of operating with the matrix on the left side of equation (B-8) or any of the four Dirac plane wave spinors of a particle of momentum $\vec{p}$ is the same as that of operating with $\vec{\alpha} \cdot \vec{p} + \beta$, i.e.

$$\text{(B-9)} \qquad \sum_{k=1}^{4} \left(\sum_{i=1}^{4} E_i \, \Gamma_j^{(i)}(\vec{p}) \, \overline{\Gamma_k^{(i)}}(\vec{p}) \right) \Gamma_k^{(s)}(p) =$$

$$= \sum_{i=1}^{4} E_i \, \Gamma_j^{(i)}(\vec{p}) \, \delta_{is}$$

$$= E_s \, \Gamma_j^{(s)}(\vec{p}) \qquad (s = 1, ..4) \qquad (j = 1, ..4)$$

As any spinor may be expressed as a linear combination of the four Dirac plane wave spinors for a particle of momentum of momentum $\vec{p}$, this proves (B-8). Therefore we may write

$$\text{(B-10)} \qquad G(\vec{r}, \vec{r}\,'; j, k) = \int \frac{d^3\vec{p}}{(2\pi)^3} \, \frac{(\vec{\alpha} \cdot \vec{p} + \beta + E_o)_{jk} \, e^{i\vec{p} \cdot (\vec{r} - \vec{r}\,')}}{p^2 - p_o^2}$$

where we have used $E^2 = p^2 + 1$ and $E_o^2 = p_o^2 + 1$.

Making use of the following integrals

$$\text{(B-11)} \qquad \int \frac{d^3\underset{\rightarrow}{p} \quad e^{i\underset{\rightarrow}{p}\cdot(\underset{\rightarrow}{r}-\underset{\rightarrow}{r}')}}{(2\pi)^3 \left(p^2 - p_o^2\right)} = \frac{\cos p_o |r - r'|}{4\pi |r - r'|}$$

and

$$\text{(B-12)} \qquad \int \frac{d^3\underset{\rightarrow}{p} \; \underset{\rightarrow}{\alpha}\cdot\underset{\rightarrow}{p} \quad e^{i\underset{\rightarrow}{p}\cdot(\underset{\rightarrow}{r}-\underset{\rightarrow}{r}')}}{(2\pi)^3 \left(p^2 - p_o^2\right)} =$$

$$\frac{i\underset{\rightarrow}{\alpha}\cdot(\underset{\rightarrow}{r}-\underset{\rightarrow}{r}')}{4\pi|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|^2}\left\{\frac{\cos p_o|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|}{|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|} + p_o \sin p_o|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|\right\}$$

one finally obtains, upon suppressing the indices j and ℓ

$$G(\underset{\rightarrow}{r}, \underset{\rightarrow}{r}') = \frac{(\beta + E_o)\cos p_o|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|}{4\pi|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|}$$

$$+ \frac{i\underset{\rightarrow}{\alpha}\cdot(\underset{\rightarrow}{r}-\underset{\rightarrow}{r}')}{4\pi|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|^2}\left\{\frac{\cos p_o|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|}{|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|} + p_o \sin p_o|\underset{\rightarrow}{r}-\underset{\rightarrow}{r}'|\right\}$$

as the Green Function of the operator $\frac{\vec{\alpha} \cdot \text{grad}}{i} + \beta - E_o$. Using this Green Function one then obtains an integral equation for the solution of the equation

$$\left(\frac{\vec{\alpha} \cdot \text{grad}}{i} + \beta - E_o\right) \psi = - V \psi$$

which is identical with equation (B-3).

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