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MATHEMATICS PANEL

QUARTERLY PROGRESS REPORT FOR PERIOD ENDING OCTOBER 31, 1949

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MATHEMATICS PANEL
Quarterly Progress Report
for the Period Ending October 31, 1949

A. S. Householder, Chief

Date Issued:

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MATHEMATICS PANEL

Quarterly Report to November 1, 1949

Alston S. Householder, Chief

Two of the former members of the Panel, Mrs. Hullings and Miss Martin, have transferred to the Technical Division, and a third, Mrs. Palmiter, expects to leave Oak Ridge in the near future. Replacements are in varying stages of the clearance process and termination of present employment, so that it will very likely be two or three months or longer before the Panel is back to full strength.

Work on two classified problems, to be reported elsewhere, has constituted a large fraction of the Panel's activities during the quarter under review and will continue through at least a part of the coming quarter. As a consequence certain other problems have been set aside, including the Monte Carlo estimate of collision densities in biological tissue, and the attempt to solve a certain integral equation arising in β-ray spectroscopy. It is planned to resume work on the tissue problem immediately upon termination of the classified problem.

Meanwhile renewed emphasis upon reactor problems at the Laboratory requires a re-orientation in the work of the Panel and the need for some expansion. Dr. Nelson has been giving a series of lectures, to be described below, at the Laboratory Training School to serve as a mathematical introduction to reactor theory, and members of the Panel generally are reviewing the background for problems now arising.

Final consideration is being given to the selection of a highspeed computing machine to be located in the Oak Ridge area and to be used in large part for the solution of shielding and reactor problems. The joint ORNL-NEPA committee selected last summer for an intensive review of the characteristics and availability of machines issued its report recommending the purchase of a Raytheon Computer and the Raytheon Corporation is preparing detailed estimates of cost and construction time. Meanwhile, however, word came that Edmund C. Berkeley and Associates, and the Sylvania Electric Products, Inc., were interested in constructing Mark III computers, and on October 28, Dr. Householder and Mr. P. R. Bell of ORNL, and Mr. E. E. St. John of NEPA, representing the

committee, went to Boston to talk with representatives of the two companies, and with Professor Aiken of Harvard. Selection from the alternatives of Raytheon and Mark III should be made sometime in November.

Preparation for computation of the Internal Conversion coefficients for the L-shell are being made by Dr. Rose in consultation with members of the Panel. Mrs. Dismuke's exploratory sample calculations will be described below. Since the bulk of the computations consist in the numerical integration of systems of equations a careful selection of the method is required. A method recently published by L. Fox *et al* of the National Physical Laboratory, England, appears to be ideally suited for machine solution of linear equations of this type. It is identical with a somewhat older method by Milne for equations that are not necessarily linear, but with the improvement made possible by the linearity of passing immediately to the limit of the iteration that is otherwise required for each new value of the variable. Order-of-magnitude estimates of errors due to round-off and truncation place these well within the required limits.

On October 12, Dr. Nelson presented a paper before the Mathematics Colloquium at the University of Tennessee on the subject "An Abel Integral Equation with Constant Limits of Integration."

The American Mathematical Society has accepted an invitation to hold a meeting of the Southeast Section in Oak Ridge on April 21, 22, 1950. The Committee on Arrangements for the Society includes Dr. Householder representing ORNL and Dr. Snyder, of the Health Physics Division, representing ORINS and the University of Tennessee. Tentative arrangements have been made to invite the Biometric Society to sponsor a joint session on the first day of the meetings devoted to non-statistical applications of mathematics to biology.

Publication of the proceedings of last summer's Monte Carlo conference at Los Angeles has been delayed somewhat, but it is now hoped that papers can be sent to the National Bureau of Standards for reproduction by the end of November. Dr. J. H. Curtiss of the Bureau has asked Dr. Householder to relieve him as editor-in-chief.

The detailed listing of technical problems follows.

PROBLEM: Tetrad analysis (yeast genetics)
ORIGIN: Dr. Seymour Pomper, Biology Division
PARTICIPATING MEMBER OF PANEL: A. S. Householder
BACKGROUND AND STATUS: Lindegren presents a set of graphs for estimating gene-to-centromere distance in yeast. No derivation is given, however, nor

any statistical evaluation of the estimates. Furthermore, the method requires the determination of the frequency of occurrence of certain combinations of characters governed by independent genes, thus necessitating a preliminary test for independence, and there is disagreement among geneticists as to the correct method of testing for independence.

Each ascus contains 4 spores. If two gene loci are under observation, then in the case of heterozygosis with respect to both characters the spores can be all of parental type, all of non-parental type, or two parental and two non-parental. In the absence of linkage, the all-parental type asci, and the all non-parental type should occur with equal frequency, while the frequency of occurrence of the other type of ascus (two parental and two non-parental type spores) depends upon the crossover probabilities of the two genes. Hence the test for independence consists in testing the equality of the two frequencies, ignoring the presence of the third.

If A_1 and A_2 denote the two characters, α_1 and α_2 the crossover propabilities, then from N asci the expected number of asci of the third type comes out to be

$$n = 2\,v\,N = N\,[\,\alpha_1\,(1-\alpha_2) + \alpha_2\,(1-\alpha_1) + \alpha_1\,\alpha_2\,/\,2\,].$$

Since two unknowns, α_1 and α_2, are involved, observations made upon a single pair of characters are not sufficient for determining the α's, but with three characters taken in pairs, the equations can be solved for the 3 α's. Lindegren's method amounts to taking the estimates

$$2\,v_{ij} = n_{ij}\,/\,N_{ij},$$

where the subscripts indicate the genes concerned, and in solving for the α's graphically. However, it is easy to solve the equations algebraically obtaining

$$3\,x_i = 1 + s\,/\,w_{jk} \qquad (i,\ j,\ k \text{ all different})$$

where

$$x_i = \alpha_i\,/\,2$$

is the so-called "distance" from gene to chromosome,

$$w_{jk} = 1 - 3\, v_{jk},$$

and

$$s^2 = w_{23}\, w_{31}\, w_{12}.$$

Unfortunately the sign of s is ambiguous, leading to two theoretically possible sets of values, and, what is worse, sampling errors may lead to a negative value of s^2. A more general approach is therefore required, partly because of these complications, but also to allow for combinations of more than 3 genes, to provide the single estimate for each distance that is simultaneously best for all observations.

We therefore return to the above expressions for v_{ij}. The estimate given above is indeed the maximum likelihood estimate made for that v independently of the others. However it is determined by the probabilities α, each of which enters other combinations. Instead of making independent maximum likelihood estimates for the v's independently, and then solving for the α's (or the x's), one should make the single maximum likelihood estimate of the α's from all available data simultaneously.

If we make the substitution

$$3\, \alpha_i = 2\, (1 - \eta_i)$$

we obtain

$$3\, v_{ij} = 1 - \eta_i\, \eta_j.$$

The probability of obtaining n_{ij} asci of the third type out of a total of N_{ij} is then

$$\binom{N_{ij}}{n_{ij}} [(2/3)(1 - \eta_i\, \eta_j)]^{n_{ij}}\, [(1/3)(1 + 2\, \eta_i\, \eta_j)]^{N_{ij} - n_{ij}}.$$

Hence the likelihood function is

$$L = \sum_{(i,j)} \{n_{ij} \log (1 - \eta_i \eta_j) + (N_{ij} - n_{ij}) \log (1 + 2 \eta_i \eta_j)\},$$

the sum being taken over all combinations (i,j). This quantity is to be maximized with respect to the η's.

In this treatment it is not presupposed that spores in any given ascus can be tested for more than two of the characters. If they can be tested for three characters, then a slightly improved method is available, since then the distinguishable asci fall into 5 classes. An ascus may be of the third type with respect to no pair of genes; or with respect to each of two pairs; or with respect to every pair. These probabilities can be expressed as

$$P_0 = \{1 + 2 (\eta_1 + \eta_2 + \eta_3) + 2 (\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2)\} / 9,$$

$$P_1 = (1 + 2 \eta_2 \eta_3) / 3 - P_0,$$

$$P_2 = (1 + 2 \eta_3 \eta_1) / 3 - P_0,$$

$$P_3 = (1 + 2 \eta_1 \eta_2) / 3 - P_0,$$

$$P_4 = 1 - P_0 - P_1 - P_2 - P_3 .$$

The likelihood function is then

$$L = n_0 \log P_0 + n_1 \log P_1 + n_2 \log P_2 + n_3 \log P_3 + n_4 \log P_4,$$

where the n's represent the number of asci found in the various classes. The best estimates for the η's are then those which maximize this function.

Procedures are being worked out for carrying out the solution for the maximum in each of the two cases discussed.

PROBLEM: Fitting exponential decay curves
PARTICIPATING MEMBER OF PANEL: A. S. Householder
ORIGIN AND BACKGROUND: Continuation. See previous Quarterly Reports.
STATUS: The report on Prony's method appeared during the past quarter as ORNL 455. Previously the endeavor has been to develop effective methods for

solving the least-square equations. The equations for the maximum likelihood fit are now being examined. If y_i are the measurements, and

$$\eta_i \equiv y_i / (1 - \mathcal{E}_i)$$

are the best estimates, these must satisfy

$$f_0 \equiv a_0 \eta_0 - a_1 \eta_1 + a_2 \eta_2 - \cdots \pm \eta_p \equiv 0$$

$$f_N \equiv a_0 \eta_N - a_1 \eta_{N+1} + a_2 \eta_{N+2} - \cdots \pm \eta_{N+p} \equiv 0,$$

while the equations for minimizing the likelihood function have the form

$$\lambda_0 \eta_0 + \lambda_1 \eta_1 + \cdots + \lambda_N \eta_N \equiv 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_0 \eta_{p-1} + \lambda_1 \eta_p + \cdots + \lambda_N \eta_{N+p-1} \equiv 0,$$

where the λ's are Lagrange multipliers, and

$$\mathcal{E}_0 \equiv a_0 \lambda_0,$$

$$\mathcal{E}_1 \equiv a_0 \lambda_1 - a_1 \lambda_0,$$

$$\mathcal{E}_2 \equiv a_0 \lambda_2 - a_1 \lambda_1 + a_2 \lambda_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathcal{E}_{N+p} \equiv (-1)^p \lambda_N.$$

The equations $f \equiv 0$ are easily expressible in matrix form as

$$H a \equiv 0,$$

where H is the matrix of the η's, and this is in turn equivalent to a matrix equation

$$A' H = 0,$$

where each column of the matrix A consists of the elements of the vector a preceded and followed by zeros. Since

$$\lambda' H = 0,$$

it follows that the vector λ with N+1 components λ_r is expressible in terms of a vector b with $N-p+1$ components:

$$\lambda = A b,$$

so that the number of unknowns is reduced by p.

A direct solution in the general case seems out of the question, and an iterative solution is sought. In the special case $p = 1$, $\alpha = \alpha_1$ is the single positive root of

$$\frac{\alpha\ [1 + 2\alpha + 3\alpha^2 + \ldots + (N+1)\ \alpha^N]}{(N+1) + N\alpha + \ldots + \alpha^N} = \frac{y_1\ 2y_2 + \cdots + (N+1)\ y_{N+1}}{(N+1)\ y_0 + Ny_1 + \cdots + y_N}$$

PROBLEM: Effect of intra-class correlation upon test of significance of difference.

ORIGIN: Dr. W. L. Russell (and others), Biology Division

PARTICIPATING MEMBER OF PANEL: A. S. Householder

BACKGROUND AND STATUS: In order to judge whether a factor has or does not have a particular effect it sometimes happens that the samples taken from the two populations, where the factor is and where it is not present, consist of sub-samples within each of which additional factors may tend to uniformize the effect in question. In one extreme case these additional factors may cause all members of a given sub-sample to be alike, in which case the examination of a single individual from each sub-sample would provide all the information available, and the size of the sample is simply the number of sub-samples. In the other extreme case the additional factors are negligible, and the segrega-

tion into sub-samples may be ignored.

The distributions involved are of the so-called "contagious" type of which various examples have been considered by Feller and others, and characterized by the fact that out of a family of populations, one selects a population at random and then from it a sample at random. If the effect in question is the presence or absence of a specified character, the populations are distinguished by the relative frequency p of occurrence of the character within that population, while the parameter p has a certain cumulative distribution $F(p)$ among the populations.

Now suppose that $F_1(p)$ represents the distribution over the family of populations in which the factor is present, and $F_2(p)$ represents the distribution over the family of populations in which the factor is absent. The null hypothesis is that $F_1(p) \equiv F_2(p)$, and the admissible alternative is that either $F_1(p) \geq F_2(p)$ for all p or else that $F_1(p) \leq F_2(p)$ for all p. If nothing is known concerning the form of the functions $F(p)$ then a non-parametric test must be invoked. Sharper tests would, of course, be possible in cases where one could suppose that the functions F are of a specified form $F(p, \alpha)$ depending upon a parameter α, in which case, if

$$F_i\ (p) \equiv F\ (p, \alpha_i),$$

the null hypothesis would be simply $\alpha_1 \equiv \alpha_2$.

PROBLEMS: Miscellaneous statistical calculations
ORIGIN: Biology Division
PARTICIPATING MEMBERS OF PANEL: A. S. Householder, Rosemary Crook, Lewis Nelson, R. R. Coveyou
BACKGROUND AND STATUS: Several short calculations of standard deviations, chi-square, least squaring, etc. have been performed from time to time for Dr. Whittinghill, Dr. Henry Kohn, and other members of the Biology Division, besides brief consultations on various statistical questions.

PROBLEMS: Calculations of cloud chamber tracks
ORIGIN: Mr. C. D. Moak, Physics Division
PARTICIPATING MEMBER OF PANEL: K. M. Martin
STATUS: Work completed.

PROBLEM: A Monte Carlo shielding experiment
ORIGIN: The summer work session on shielding
PARTICIPATING MEMBER OF PANEL: Lewis Nelson
BACKGROUND: See last Quarterly Report.
STATUS: A major part of the computation has been construction of numerical tables for use in tracing the life histories. These tables have been completed and checked. The tracing of life histories has begun.

PROBLEM: A short course in mathematics
ORIGIN: Dr. F. C. VonderLage
PARTICIPATING MEMBER OF PANEL: Lewis Nelson
BACKGROUND AND STATUS: It was thought necessary to offer a short review in mathematics for the benefit of those who are to enroll in the Laboratory's training course in "pile physics". Between 20 and 30 students have attended this class which meets on Tuesdays and Thursdays, 8 to 9:30. Five meetings have been held. Perhaps five more will be held.

The aim is to review some of the important ideas of the calculus, such as the derivative with its mean value theorem; the integral as the limit of a sum; the gradient, its geometric meaning and how to compute it in rectangular, spherical and cylindrical coordinates. In addition an effort is being made to discuss differential equations, both ordinary and partial. This is an attempt to convey the idea of an eigen-value problem and of a boundary value problem.

Time permits neither rigor nor detail. Nevertheless, it is hoped that identification of some of the basic mathematical concepts which are needed in elementary pile physics will:

(1) eliminate dread of them in those students who understood them insufficiently well

(2) provide a basis for each student to satisfy independently his own needs in using these ideas when the time for it arrives.

PROBLEM: Numerical integration
ORIGIN: Dr. S. M. Dancoff, Physics Division
PARTICIPATING MEMBERS OF PANEL: Donna Palmiter, Ann Forbes
STATUS: Work completed.

PROBLEM: Miscellaneous interpolating, graphing, etc.
ORIGIN: Dr. M. E. Rose, Physics Division
PARTICIPATING MEMBERS OF PANEL: Donna Palmiter, Ann Forbes, R. R. Coveyou, M. K. Hullings

BACKGROUND: Preliminary to computation of the Internal Conversion Coefficients for the K-shell and L-shell on the Mark III, many miscellaneous hand computations were required. These were mainly interpolations and preparation of graphs.
STATUS: Work completed.

PROBLEM: Monte Carlo computations
ORIGIN: Dr. Lewis Nelson, Math Panel
PARTICIPATING MEMBERS OF PANEL: Donna Palmiter, Ann Forbes
BACKGROUND: The Monte Carlo shielding experiment (mentioned separately) required some hand computation in preparation of the necessary curves and tables.
STATUS: Work completed.

PROBLEM: Sample solutions of an eigen-value problem
ORIGIN: Dr. M. E. Rose
PARTICIPATING MEMBER OF PANEL: Nancy Dismuke
BACKGROUND: This is for the purpose of testing the method of Dr. M. E. Rose (ORNL 481) for determining the eigen-value, W, for the pair of first order differential equations (wave equations)

$$
\begin{aligned}
dF/dx &= (2\nu/x)\ F + x\ [\,1 - W + V(x)]\ G \\
dG/dx &= x\,[\,1 + W - V(x)\,]\ F - (2\,\nu/x)\ G
\end{aligned}
\tag{1}
$$

where

$$V(x) = (-2\,\alpha\,Z/x^2)\ \varphi_z\ (x)$$

F/G is known near the origin and for large values of x.
STATUS: Briefly the method is to integrate from 0 to some point x_ℓ, referred to below as integration region I and from large x (large enough x that the asymtotic value for F/G is valid) inward to x_ℓ, integration region II. For the correct eigen-value F, G, $\frac{1}{F}\frac{dF}{dx}$, $\frac{1}{G}\frac{dG}{dx}$ are continuous at x_ℓ (actually of course they will be continuous for any point x). Since the initial conditions in both regions I and II fix only the ratio F/G, F and G can be made continuous

at x_f by multiplying by the proper constant. By calculating the logarithmic derivatives it can be seen that they will be continuous if the ratio

$$(2)\ R(W) = [(F/G)_{x_f} \text{ on curve of Region I}]/[(F/G)_{x_f} \text{ on curve of Region II}]$$

is equal to 1. The method is to integrate equations (1) in region I and in region II for two values of W and to calculate the ratio given by (2) for each W. With these two values of $R(W)$ it should be possible to extrapolate or interpolate to find a third value of W which will make $R(W)$ nearer to 1. This value of W is then used to integrate equations (1) and again the ratio $R(W)$ is calculated. If this third value of W is not sufficiently close to the eigenvalue, a fourth can be found, and so on.

The real problem in testing Rose's method is integrating equations (1). The numerical integration is being done with the help of Miss Phyllis Johnson of Y-12 on the IBM equipment there. The integration procedure followed is due to L. F. Hausman and M. Schwarzchild, *Rev. of Scientific Instruments 18,* 877 (1947).

Following the suggestion of R. Coveyou the following transformations were made where $r = x^2 / 2$

Region I	$F = r^{-\gamma} F$ $G = r^{-\gamma} G$	$\gamma = \sqrt{\nu^2 - \alpha^2 Z^2}$	To remove singularities in the coefficients at the origin
Region II	$f = e^{\lambda x} F$ $g = e^{\lambda x} G$	$\lambda = \sqrt{1 - W^2}$	To prevent increase of the functions over many powers of 10; hence to increase accuracy of the numerical integrations

$F(x_{n+1})$ and $G(x_{n+1})$ are expanded in Taylor series about x_n. $F'(x_n)$, $G'(x_n)$ were obtained from the differential equations (1). Higher derivatives in the series are approximated by central differences through the fourth. The difference equations thus obtained are (since the form is the same for both regions, we simply write F and G):

$$
(3) \quad \begin{aligned} F(x_{n+1}) &= \{1 + a_{11}(x_n)\, h\}\; F(x_n) + h\, a_{12}\, G(x_n) + D_1\,(x_n) \\ G(x_{n+1}) &= h\, a_{12}\, G(x_n) + \{1 + a_{22}\,(x_n)\}\; G(x_n) + D_2\,(x_n). \end{aligned}
$$

The a's are the coefficients of (1); D_1 and D_2 are the second, third and fourth derivative terms of the Taylor's series (approximated by central differences through the fourth); and h is the difference interval.

$$
D_1(x_n) = (1/12)F(x_{n+2}) + (1/3)F(x_{n+1}) - F(x_n) + (2/3)F(x_{n-1}) - (1/12)F(x_{n-2})
$$

D_2 is the same with F replaced by G.

(a) For the case being tested $Z = 92$, $\nu = +1$. $V(x)$ had been previously determined. For region I integration is from $x = .024$ to $x = 5.620$; for region II integration is from $x = 10.820$ to $x = 2.620$. The interval widths, h_n, used were determined by the values of x for which $V(x)$ was recorded. h_n ranges from .004 to .2 .

(b) The coefficient of $F(x_n)$ and $G(x_n)$ in equation (3) are calculated before the so called "integration" is begun. An integration run is then made, that is equation (3) is calculated over the entire integration interval. For the first integration run D_1 and D_2 are taken as zero. For the other runs D_1 and D_2 are calculated from the preceding integration run.

(c) As soon as approximations to F and G are available a correction run can be made, i.e., D_1 and D_2 can be calculated from the preceding integration run for use in the succeeding one. This procedure is continued until the difference between two successive integration runs is small. An explicit estimate of the accuracy obtainable has not been made. However, the fifth difference of the integrated functions will shed some light on this question, as will an estimate of the round-off error.

In the calculation 8 digits are carried for F, G, and D and for the coefficients of equation (3). If F or G had increased beyond the 8 place capacity allowed, then from that point on $F/10^k$ and $G/10^k$ would have been integrated. This did not occur in this problem, rather F and G decreased, so that accuracy greater than 4 or 5 significant figures can not be expected.

PROBLEM: Change of weight with change of dependent variable in curve fitting by the method of least squares.

ORIGIN: Within the Panel

PARTICIPATING MEMBER OF PANEL: R. R. Coveyou

BACKGROUND AND STATUS: In a current problem, the following situation arose: We were given a set

$$(1) \qquad \sigma(Z_1), \ldots, \sigma(Z_n) \equiv \sigma_1, \sigma_2, \ldots, \sigma_n$$

of experimentally determined values of a function $\sigma(Z)$, together with a set

$$(2) \qquad W(Z_1), \ldots, W(Z_n)$$

of statistical weights of these determinations. Also, physical theory predicts a functional form

$$(3) \qquad \sigma \equiv \sigma(\alpha, \beta, \gamma, \delta; Z),$$

where α, β, γ, δ are parameters whose determination is one of the chief objects of the experiment. The determination of the parameters α, β, γ, δ is, of course, to be done by the method of least squares. For reasons of computational convenience, it was decided that the fit should be made, not to σ, but to another function which we shall denote by $\lambda(\sigma)$. Then we compute

$$(4) \qquad \lambda_1 \equiv \lambda(\sigma_1) = \lambda[\sigma(Z_1)], \lambda_2, \ldots, \lambda_n,$$

and treat these as the experimental data. The problem then arises, what are the statistical weights which should be attached to the λ's? Let us denote these new weights by

$$(5) \qquad W_1^*, \ldots, W_n^*.$$

If we adopt the criterion that the contribution of a single point to R, the weighted sum of squared residuals shall be the same, which ever way the computation is made, we have

$$(6) \qquad W_i(\bar{\sigma}_i - \sigma_i)^2 \equiv W_i^*(\bar{\lambda}_i - \lambda_i)^2,$$

where the barred quantities are the values of σ(or λ) calculated from the formula (3) for the best choice of the parameters α, β, γ, δ. Then we clearly have

$$W_i^* = W_i\,(\overline{\sigma_i} - \sigma_i)^2 / (\overline{\lambda_i} - \lambda_i)^2 . \tag{7}$$

Now, if we make the assumption that $\overline{\sigma_i}$ and σ_i; $\overline{\lambda_i}$ and λ_i do not differ very much, we can write

$$W_i^* = W_i / (\partial\lambda / \partial\sigma)^2_{\sigma = \sigma_i} \tag{8}$$

which is the usual formula. However, in our problem, this condition is violated; though $\overline{\sigma_i}$ and σ_i are reasonably close together, the transformation from σ to λ is such that $\overline{\lambda_i}$ and λ_i may be quite widely separated. Hence, a more accurate formula was sought. The formulas below were derived and applied, with gratifying improvement in the results.

Now, if we have reasonable estimates σ_{ei} for $\overline{\sigma_i}$, λ_{ei} for $\overline{\lambda_i}$, and if we require that a small change in any λ_{ei} shall have the same effect on R as the corresponding change in σ_{ei}, we have, if R_i is the contribution of the i-th point to R,

$$dR_i = d\,[\,W_i^*\,(\lambda_{ei} + \delta\lambda_{ei} - \lambda_i)^2\,] = d\,[\,W_i\,(\sigma_{ei} + \delta\sigma_{ei} - \sigma_i)^2\,], \tag{9}$$

where $\delta\lambda_{ei}$, $\delta\sigma_{ei}$ are corresponding small changes in λ_{ei}, σ_{ei}.
Then we have

$$2\,W_i^*\,(\lambda_{ei} + \delta\lambda_{ei} - \lambda_i)\,\delta\lambda_{ei} = 2\,W_i\,(\sigma_{ei} + \delta\sigma_{ei} - \sigma_i)\,\delta\sigma_{ei},$$

or

$$W_i^*\,(\lambda_{ei} - \lambda_i)\,\delta\lambda_{ei} = W_i\,(\sigma_{ei} - \sigma_i)\,\delta\sigma_{ei}; \tag{10}$$

$$W_i^* = W_i\,[(\sigma_{ei} - \sigma_i)/(\lambda_{ei} - \lambda_i)]\,(\delta\sigma_{ei} / \delta\sigma_{ei}) \tag{11}$$

$$W_i^* \cong W_i\,[(\sigma_{ei} - \sigma_i)/(\lambda_{ei} - \lambda_i)]\,(d\sigma/d\lambda)_{\lambda = \lambda_{ei}} , \tag{12}$$

In spite of the fact that these weights are not constant, but depend upon the

estimated σ's and λ's, hence must be recomputed at each iteration, the total effect is that of a saving of labor (of course, for our particular problem), while the improvement in accuracy in the use of (12) over the use of (8) is quite appreciable, and well worth the extra labor.

(13) $$R = \int_{\sigma_i}^{\bar{\sigma}_i} d\,R\,,$$

and, if we carry out this integration, using (12) and (9), the result is equivalent to using (7) to determine the weights.

Hence, we arrive at the prescription:

(14) In calculating corrections, (12) is the correct weight formula.

(15) In calculating R, (7) is the correct weight formula.