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ORNL SUMMER SHIELDING SESSION

Gale Young, Leader

A PROPOSED PARTICLE ATTENUATION PROBLEM

G. Goertzel
NDA

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A PROPOSED PARTICLE ATTENUATION PROBLEM

G. Goertzel, NDA

Abstract

An attempt is made to present in some detail a practical method of applying stochastic methods with quota sampling to an attenuation problem. After a discussion of background material, a specific computation set-up is given in Section 5. This is sufficiently detailed that, using Section 5 as a guide, an actual computation may be prepared.

1. The Problem

The problem considered is that of the transmission of particles through a slab of homogeneous and isotropic material. The physical arrangement considered (see Figure 1)

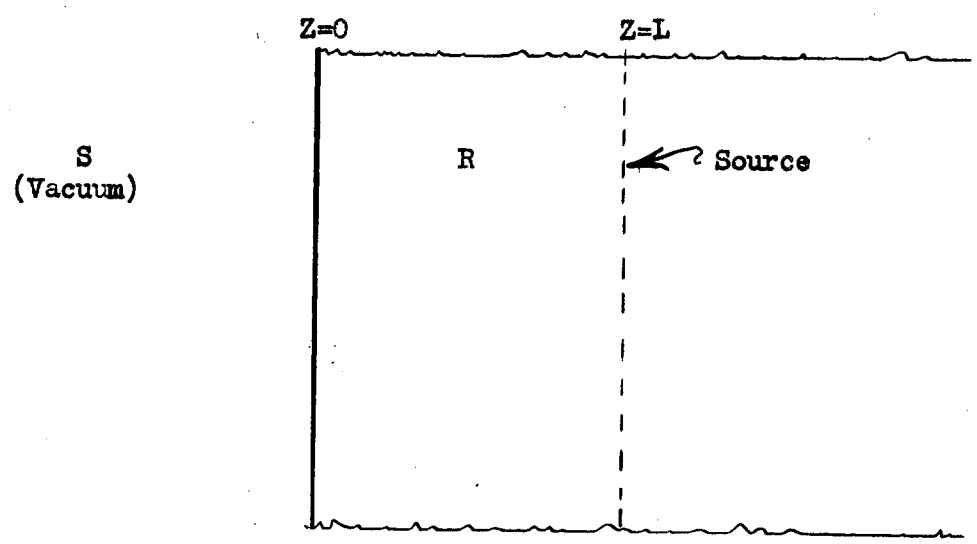

Figure 1

is a semi-infinite region extending in the direction of positive Z from Z = 0. A source of particles, invariant to displacements in its own plane, exists at Z = L.

The question to be answered is:

What is the expectation value $\langle g(E) \rangle$ of the function $g(E)$ where

$g(E)$ = a specified function of the energy E for those particles which are transmitted

= 0 for those particles which are not transmitted.

In the particular case

$g(E)$ = 1 for transmitted particles

= 0 for particles not transmitted

the expectation value of $g(E)$ is the probability that a particle is transmitted through the medium.

2. The General Method

The problem is to be solved by Monte-Carlo procedures using the quota sampling procedure discussed in a previous report.[1] The general procedure to be used, as applied to the present problem, is repeated here.

A point (E,μ,Z) in phase space just prior to a collision is denoted by α, β, etc. E is the particle energy and μ the cosine of the angle between the particle velocity and the Z-axis. The source is described by a function $S(\alpha)$ such that

$$\int_{\text{All } \alpha} S(\alpha)\, d\alpha = 1$$

and $S(\alpha)$ is the probability density for the first collision position. More detail is given in later sections.

(1) "Quota Sampling and Importance Functions in Stochastic Solution of Particle Problems" by G. Goertzel, NDA, June 21, 1949.

If further, $\Psi_n(\alpha)$ is the probability density for the n'th collision position in phase space and if R denotes that region in phase space where $0 < Z$, one has

$$\Psi_1(\alpha) = S(\alpha)$$

$$\Psi_{n+1}(\alpha) = \int_R \Psi_n(\beta)\ p(\beta,\alpha)d\beta \qquad n = 1, 2, \ldots\ldots\ldots$$

whence the function

$$\Psi(\alpha) \equiv \sum_{n=1}^{\infty} \Psi_n(\alpha)$$

satisfies the integral equation

$$\Psi(\alpha) = \int_R \Psi(\beta)\ p(\beta,\alpha)d\beta + S(\alpha).$$

In these equations, $p(\beta,\alpha)$ is the probability density for an (n+1)st collision at α for a particle whose n'th collision was at β.

In terms of the above definitions,

$$\langle g(E)\rangle = \int_S g(E) \left\{ \int_R \Psi(\beta)p(\beta,\alpha)d\beta + S(\alpha) \right\} d\alpha$$

where α denotes (E,μ,Z) and

$$\int_S d\alpha \equiv \int_{-1}^{0} d\mu \int_{-\infty}^{0} dZ \int_{0}^{\infty} dE\ .$$

The stochastic process for estimating $\langle g(E) \rangle$ without quota sampling is given below. A particle is started according to (0) and at each step the computation either ends or proceeds from (n) to (n+1).

(0): Select α_o from the distribution $S(\alpha)d\alpha$.

If α_o lies in R, i.e. if $0 \leqslant Z$, proceed to (1).

If α_o lies in S, i.e. if $Z < 0$, set $N = 0$, $g(E) = g(E_o)$ and start over.

(n), n = 1, 2, ...

Select α_n from the distribution $p(\alpha_{n-1}, \alpha)d\alpha$.

If α_n lies in R, proceed to (n+1).

If α_n lies in S, set $N = n$, $g(E) = g(E_N)$ and start over.

If α_n corresponds to an absorption process, that is, if

$$\int_{(R+S)} p(\alpha,\beta)d\beta < 1$$

set $N = n$ and $g(E_N) = 0$.

Clearly

$$\langle g(E) \rangle = \langle g(E_N) \rangle$$

whence an estimate of $\langle g(E) \rangle$ is given by the average of $\langle g(E_N) \rangle$ over all particles considered.

The modification of the above procedure caused by quota sampling requires the introduction of an importance function $Q(\alpha)$ such that

$Q(\alpha)$ is arbitrary for α in R

$Q(\alpha) = g(E)$ for α in S.

One now proceeds as follows:

(0): Select α_o from the distribution

$$\tilde{S}(\alpha)d\alpha \equiv \frac{1}{N^0} S(\alpha)\ Q(\alpha)\ d\alpha$$

$$N^0 \equiv \int_{(R+S)} S(\alpha)\ Q(\alpha)\ d\alpha$$

Compute the weight

$$W_o \equiv S(\alpha_o)/\tilde{S}(\alpha_o) = \frac{1}{Q(\alpha_o)} \int_{(R+S)} S(\alpha)\ Q(\alpha)\ d\alpha$$

If α_o lies in R, proceed to (1).

If α_o lies in S, set N = 0, and start over.

(n), n = 1, 2, ...

Select α_n from the distribution

$$\tilde{p}(\alpha_{n-1},\alpha)d\alpha \equiv \frac{1}{N^*} p(\alpha_{n-1},\alpha)\ Q(\alpha)\ d\alpha$$

$$N^* \equiv \int_{(R+S)} p(\alpha_{n-1},\alpha)\ Q(\alpha)\ d\alpha$$

Compute the weight

$$W_n \equiv W_{n-1}\ p(\alpha_{n-1},\alpha_n)/\tilde{p}(\alpha_{n-1},\alpha_n) =$$

$$= W_{n-1} \frac{1}{Q(\alpha_n)} \int_{(R+S)} p(\alpha_{n-1},\alpha)\ Q(\alpha)\ d\alpha$$

If α_n lies in R, proceed to (n+1).

If α_n lies in S, set N = n and start over.

Clearly

$$\langle g(E) \rangle = \langle W_N \, g(E_N) \rangle = \langle W_N \, Q(\alpha_N) \rangle$$

whence an estimate of $\langle g(E) \rangle$ is given by the average of $W_N Q(\alpha_N)$ over all particles considered.

As was demonstrated in reference (1), it is of interest to consider what happens if $Q(\alpha)$ satisfies

$$Q(\alpha) = g(E), \quad \alpha \text{ in } S$$

$$= \int_{(R+S)} p(\alpha,\beta) \, Q(\beta) \, d\beta, \quad \alpha \text{ in } R.$$

In this case, one has

$$W_o = \frac{1}{Q(\alpha_o)} \int_{(R+S)} S(\alpha) \, Q(\alpha) \, d\alpha$$

$$W_n = W_{n-1} \, Q(\alpha_{n-1})/Q(\alpha_n).$$

Thus it follows

$$W_n Q(\alpha_n) = \int_{(R+S)} S(\alpha) \, Q(\alpha) \, d\alpha$$

and one has the zero-variance estimate

$$\langle g(E) \rangle = \int_{(R+S)} S(\alpha)\ Q(\alpha)\ d\alpha\ .$$

3. Discussion of General Method

One should note that a variance is defined as positive or zero, so that the statement that the ideal $Q(\alpha)$ gives zero variance implies that a neighboring $Q(\alpha)$ will give small variance and further that the variance will depend on the square of the departure of $Q(\alpha)$ from the ideal.

Thus a reasonable estimate of $Q(\alpha)$ should improve the variance of the computation greatly. I think one should go to great trouble to select and use a satisfactory $Q(\alpha)$. This philosophy is reflected in the details described in Section 5.

The reason that use of an importance function is significant is that in the stochastic experiment one no longer requires that a particle is or is not transmitted, but rather permits all particles to be transmitted. Thus in the non-quota sampling procedure one had

$$\langle g(E) \rangle = \langle g(E_N) \rangle$$

with most of the $g(E_N)$ zero. That is, one had $\langle W_N g(E_N) \rangle$ where for most experiments W_N was zero and for a very few W_N was unity. The quota sampling has the effect of permitting the W_N to take a continuum of values so that the $W_N g(E_N)$ may have a smaller spread.

The efficiency of the quota sampling will thus be partly indicated by the spread in values of $W_N g(E_N)$. It is clear, however, that if all W_N lie in the range $0 < W_n < 1$, the variance is less with quota sampling than without quota sampling.

4. The Importance Functions

The first task to be considered is the selection of $Q(\alpha)$. Since α denotes the position just prior to a collision, unless anisotropic scattering is very important one may neglect the dependence of $Q(\alpha)$ on the velocity direction, μ. Further, an exponential dependence of Q on Z would seem to express our present state of knowledge adequately.

We thus take for Q the form*

$$Q(\alpha) = G\, g(E)\, e^{-\tilde{K} w(E) Z} \qquad 0 < Z$$

$$= g(E) \qquad Z < 0$$

where G is a constant between 1/2 and 1 and is necessary to fit the $Q(\alpha)$ on both sides of the plane $Z \equiv 0$. $w(E)$ is the reciprocal mean free path of the medium for particles of energy E and $\tilde{K}$ is a number which in general should depend on E. We will take $\tilde{K}$ independent of E and such that $0 \leqslant \tilde{K} \leqslant 1$. Clearly, the larger $\tilde{K}$, the stronger is the effect of $Q(\alpha)$.

In the detailed treatment of a collision, one does not in general construct $p(\alpha,\beta)$ and use this. One rather considers a collision in stages, say as follows:

0. Position just prior to collision is (E_o, μ_o, Z_o).
1. Choose collision type σ; one may have various kinds of collisions of which the following will be considered:
   a) Inelastic and isotropic.
   b) Elastic and isotropic in the center of gravity coordinates for collision with any element but H.
   c) Collision of type b) but with H.
   d) Absorption.

* This choice of Q is not adequate if $g(E)$ varies rapidly with energy, especially if $g(E)$ decreases drastically for large E.

2. Select E and μ, energy and angle after collision.

3. Select Z, knowing Z_0, E, μ.

Clearly, for step 1, one needs the relative importance of collisions of types a, b, c, d. The importance of type d is zero, so this poses no problem. To obtain the importance of types a, b, c, one needs

$$q_\sigma(\alpha) = \int p_\sigma(\alpha,\beta)Q(\beta)d\beta$$

where p_σ is the probability density for going to β if a collision of type σ is made at α. Similar quantities are needed in step 2.

We will not attempt to use an exact value for $q_\sigma(\alpha)$, etc. since this is based on our approximate $Q(\beta)$, so that the work would not be justified. Nonetheless, a large amount of effort will have to be expended in evaluating and using the importance functions appropriate to the various steps.

In Section 5, a detailed procedure is suggested. The various considerations leading to these choices are outlined in Section 6. Of course, the choices of Section 5 are not unique -- they are merely ones that seems practical to me.

Before beginning Section 5, a few remarks are in order. First, the handling of type a collisions (inelastic and isotropic) is the most difficult for several reasons, one of which is the greater detail needed in presentation of the physical data. Thus, the treatment of these in Section 5 is based on a particular representation of the data and should be modified if this representation is inadequate or inconvenient.

Second, since the quota sampling procedure does not affect the expectation value of the final answer, it will be clearly advantageous, wherever tables are needed for the quota sampling, to use fairly coarse tabular intervals. In this

case, however, it is necessary to use care that the weights computed in each choice are consistent with the sampling procedure used.

5. Detailed Computation Procedure

5.1 Physical Data

For the particular computation considered, one needs specify $g(E)$ and L. One needs also specify, for the source neutrons, the energy and angle distribution $s(E,\mu)$. In addition, one needs a table, as a function of energy, of the quantities

$w(E)$	The reciprocal mean free path at energy E
$\alpha_a(E)$	The probability that a collision is of type a
$\alpha_{b1}(E)$	The probability that a collision is of type b with a nucleus of mass M_1
$\alpha_c(E)$	The probability that a collision is of type c
$\alpha_d(E)$	The probability that a collision is an absorption

$$\alpha_a + \alpha_{b1} + \alpha_{b2} + \cdots + \alpha_c + \alpha_d = 1$$

We call this TABLE A.

Further, for the inelastic process, one needs specify the energy distribution after the collision. It will be assumed here that this, or our knowledge of this, can adequately be represented by a set of quantities $\gamma_{\sigma\tau}$ such that if a neutron has energy E before a collision, such that

$$E_\sigma < E \leqslant E_{\sigma+1}$$

it will have one of the energies E_τ after the collision with the probability of a specific E_τ given by $\gamma_{\sigma\tau}$. Thus $\sum_{(\tau)} \gamma_{\sigma\tau} = 1$. Of course, this representation is useful only if the variables σ and τ need not take on very many values.

In addition a logarithmic and exponential table will be needed. Also a table of square roots (or squares) and of sines and cosines.

5.2 Quota Sampling Tables

A choice is made for G and $\tilde{K}$. Once these are selected, the following tables are prepared.

QI. $F(\mu, \lambda)$

If $\mathcal{E}(X)$ denotes the function

$$\mathcal{E}(X) = 0 \qquad X < 0$$
$$= 1 \qquad 0 < X$$

we define

$$F(\mu, \lambda) \equiv \int_{-1}^{\mu} f(\mu', \lambda)\, d\mu' \Big/ \int_{-1}^{1} f(\mu', \mu)\, d\mu'$$

$$F(\lambda) = 1/2 \int_{-1}^{1} f(\mu', \lambda)\, d\mu'$$

$$f(\mu, \lambda) \equiv J\, e^{-\tilde{K}\lambda} + \mathcal{E}(-\mu)(1 - M)\, e^{\lambda/\mu}$$

$$J \equiv \frac{G}{1 + \tilde{K}\mu}$$

$F(\mu, \lambda)$ may be expressed in terms of exponential integrals.

We will need to tabulate, for various values μ_σ of μ and λ_τ of λ the following quantities:

$$\mu_\sigma;\ \lambda_\tau;\ F_{\sigma\tau} = F\left(\mu_\sigma, \frac{\lambda_\tau + \lambda_{\tau+1}}{2}\right) \ ; \ \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}} = \frac{\mu_{\sigma+1} - \mu_\sigma}{F_{\sigma+1,\tau} - F_{\sigma\tau}} .$$

Herein, $\sigma = 0, 1, \ldots\ldots, (\Sigma - 1), \Sigma$

$$\mu_0 = -1 \qquad F_{0\tau} = 0$$
$$\mu_\Sigma = 1 \qquad F_{\Sigma\tau} = 1$$

$\sum$ will be approximately ten. It seems to me that no more than some forty values of τ will be necessary.

QII. $H(E,x)$

$$H(E,x) = \int_0^E dE' \; g(E') \; e^{-w(E')x}$$

We tabulate:

$$E_\sigma; \; x_\tau \; ; \; H_{\sigma\tau} \equiv H\left(E_\sigma, \frac{x_\tau + x_{\tau+1}}{2}\right); \; \frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}} = \frac{E_{\sigma+1} - E_\sigma}{H_{\sigma+1,\tau} - H_{\sigma\tau}} \; .$$

It again seems likely that no more than some forty values of τ should be needed. For σ, one needs specifically consider $w(E)$ and $g(E)$ as functions of E, so that no estimates for the number of values of σ will be given here.

QIII. $G_{\sigma\tau\nu}$

$$G_{\sigma\tau\nu} = \frac{1}{K_{\sigma\nu}} \sum_{\tau'=1}^{\tau-1} \gamma_{\sigma\tau'} F\left[w(E_{\tau'}), \; Z_\nu\right] g(E_{\tau'})$$

$$G_{\sigma o\nu} = 0$$

$$K_{\sigma\nu} \equiv \sum_\tau \gamma_{\sigma\tau} F\left[w(E_\tau), \; Z_\nu\right] g(E_\tau)$$

We tabulate thus:

$$E_\sigma; \; E_{\tau+1}; \; Z_\nu \; ; \; \gamma_{\sigma\tau} \; ; \; \Delta G_{\sigma\tau\nu} \equiv G_{\sigma,\tau+1,\nu} - G_{\sigma\tau\nu} \; .$$

As was stated before, this particular method of handling inelastic collisions will be feasible only if both σ and τ take on at most a few values.

QIV. $K_{\sigma\nu}$

We tabulate: E_σ, Z_ν, $K_{\sigma\nu}$

QV. F_τ

We tabulate: λ_τ, $F_\tau = F\left(\frac{\lambda_\tau + \lambda_{\tau+1}}{2}\right)$ (c.f. QI)

5.3 The Computation Procedure

In the following, "select ξ" will mean select ξ at random from a population equidistributed in the interval (o, 1). The initial or (0) step in following a particle proceeds as follows:

(0): 1) Set $E_o = E_s$, $\mu_o = \mu_s$.*

2) Look up $w(E_o)$ and $g(E_o)$ in Table A.

If $0 \leqslant \mu_o$

3a) Select ξ

4a) Compute $s = \frac{-1}{1 + K\mu_o} \log \xi$

5a) Look up $e^{K\mu_o s}$ and compute.

$W_o = e^{K\mu_o s}/(1 + K\mu_o s)$

$Z_o = L + \mu_o s/w(E_o)$

6a) Proceed with step (1)

If $\mu_o < 0$

3b) Look up $e^{-Kw(E_o)L}$ and $e^{w(E_o)L/\mu_o}$ and compute

$$D = e^{-Kw(E_o)L} - \left(1 - \frac{1}{J}\right) e^{w(E_o)L/\mu_o}$$

4b) Select ξ and compute

$$\rho = D\xi + e^{w(E_o)L/\mu_o} - e^{-Kw(E_o)L}$$

*We do not consider here other forms of source than the monochromatic, unidirectional one with energy E_s and angle μ_s. For other sources the necessary modifications are not difficult.

If $0 < \rho$

5ba) Compute $W_N g(E_N) = J_o D\, g(E_o)$, $(N = 0)$

6ba) Start a new particle.

If $\rho < 0$

5bb) $s = \frac{-1}{1 + \hat{K}\mu_o} \log(1 - e^{\hat{K}w(E_o)L} D\xi)$

6bb) $Z_o = L + \mu_o s/w(E_o)$; look up $e^{\hat{K}w(E_o)Z_o}$

$$W_o = \frac{D}{1 + \hat{K}\mu_o} e^{\hat{K}w(E_o)Z_o}$$

7bb) Proceed with (1).

All steps subsequent to the (0) step follow the same procedure, as given below for the (n) step. We know E_{n-1}, μ_{n-1}, Z_{n-1}, W_{n-1}.

(n): $n = 1, 2, \ldots\ldots$

1) Look up in Table A the quantities*

$w(E_{n-1})$, $\alpha_a(E_{n-1})$, $\alpha_{b1}(E_{n-1})$, . . . , $\alpha_c(E_{n-1})$

2) Look up in Table QIV

$K_{\sigma\nu}$ such that $E_\sigma < E_{n-1} \leqslant E_{\sigma+1}$

$Z_\nu < Z_{n-1} \leqslant Z_{\nu+1}$

$q_a = K_{\sigma\nu}$

3) Look up in Table QV

F_τ such that $\lambda_\tau < Z_{n-1} W(E_{n-1}) \leqslant \lambda_{\tau+1}$

$q_b = g(E_{n-1}) F_\tau$.

4) Using Table QII

a) If $\mu_o > 0$ determine σ, τ, such that

$E_\sigma < E_{n-1} \leqslant E_{\sigma+1}$ $\qquad$ $x_\tau < \hat{K}Z_{n-1} \leqslant x_{\tau+1}$

* This is partly a repetition of Step 11 in (n - 1).

b) If $\mu_o < 0$ determine σ, τ, t, such that

$$E_\sigma < E_{n-1} \leqslant E_{\sigma+1} \qquad\qquad x_t < -Z_{n-1}/\mu_{n-1} \leqslant x_{t+1}$$

$$x_t < KZ_{n-1} \leqslant x_{t+1}$$

Then compute

$$H_{m\tau} = H_{\sigma\tau} + \frac{\Delta H_{\sigma\tau}}{\Delta E_{\sigma\tau}} (E_{n-1} - E)$$

<u>If $\mu_{n-1} < 0$</u>

$$H_{mt} = H_{\sigma t} + \frac{\Delta H_{\sigma t}}{\Delta E_{\sigma t}} (E_{n-1} - E_\sigma).$$

$$q_c = \frac{1}{E_{n-1}} \left\{ J_{n-1} H_{m\tau} + \mathcal{E}(-\mu_{n-1})(1 - J_{n-1}) H_{mt} \right\}$$

5) Compute $N = q_a \alpha_a + q_b(\alpha_{b1} + \alpha_{b2} + \alpha_{b3} + \ldots) + q_c\alpha_c$

Then select ξ and determine

case a) $N\xi < q_a\alpha_a$

case b1) $q_a\alpha_a \leqslant N\xi < q_a\alpha_a + q_b\alpha_{b1}$

case b2) $q_a\alpha_a + q_b\alpha_{b1} \leqslant N\xi < q_a\alpha_a + q_b(\alpha_{b1} + \alpha_{b2})$

. . . . . .

case c) $N - q_c\alpha_c \leqslant N\xi < N$

<u>Case a</u>

6a) $W_1 = \frac{N}{q_a} W_{n-1}$

7a) Using Table QIII, select ξ and find

E_τ, $\gamma_{\sigma\tau}$, $\Delta G_{\sigma\tau\nu}$ such that

$$E_\sigma < E_{n-1} \leqslant E_{\sigma+1}; \; Z_\nu < Z_{n-1} \leqslant Z_{\nu+1}$$

$$G_{\sigma\tau\nu} \leqslant \xi < G_{\sigma,\tau+1,\nu}$$

$$E_n = E_\tau$$

$$W_2 = W_1 \frac{\gamma_{\sigma\tau}}{\Delta G_{\sigma\tau\nu}}$$

8a) Look up $W(E_n)$ in Table A.

9a) Using Table QI, select ξ and find μ_σ, $\frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}}$, $F_{\sigma\tau}$, such that

$$F_{\sigma\tau} \leqslant \xi F_{\sigma+1,\tau} \; ; \; \lambda_\tau < W(E_n) Z_{n-1} \leqslant \lambda_{\tau+1}$$

10a) Compute

$$\mu_n = \mu_\sigma + (\xi - F_\sigma) \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}}$$

$$W_3 = 1/2 \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}} W_2$$

<u>Case bi</u>

6bi) $w_1 = \frac{N}{q_b} W_{n-1}$

7bi) Using Table QI, select ξ_1 and find μ_σ, $\frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}}$, $F_{\sigma\tau}$ such that

$$F_{\sigma\tau} \leqslant \xi_1 < F_{\sigma+1,\tau}; \lambda_\tau < w(E_{n-1}) Z_{n-1} \leqslant \lambda_{\tau+1}$$

8bi) Compute

$$\mu_n = \mu_\sigma + (\xi_1 - F_{\sigma\tau}) \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}}$$

9bi) Select ξ_2, look up $\cos\pi\xi_2$ and compute

$$\omega = \mu_{n-1}\mu_n + \sqrt{1 - \mu_{n-1}^2}\sqrt{1 - \mu_n^2} \cos\pi\xi_2$$

10bi) Compute

$$E_n = E_{n-1}\left(\frac{\omega + \sqrt{\omega^2 + M_1^2 - 1}}{M_1 + 1}\right)^2$$

$$w_3 = \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}} \frac{\left(\omega + \sqrt{\omega^2 + M_1^2 - 1}\right)^2}{2M_1\sqrt{\omega^2 + M_1^2 - 1}} w_1$$

Case c

6c) $W_1 = \frac{N}{q_c} W_{n-1}$

$0 \leqslant \mu_{n-1}$

7ca) Using Table QII, with $H_{m\tau}$ as found in 4, select ξ_1 and find $H_{\sigma\tau}$, $\frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}}$, E_σ, such that

$$H_{\sigma\tau} < \xi_1 H_{m\tau} < H_{\sigma+1,\tau}; \quad x_\tau < \widehat{KZ}_{n-1} \leqslant x_{\tau+1}$$

8ca) Compute $E_n = E_\sigma + \frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}} (H_{m\tau}\xi_1 - H_\sigma)$

9ca) Compute $W_3 = W_1 \frac{1}{E_{n-1}} H_{m\tau} \frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}}$

$\mu_{n-1} < 0;\ 1 - J_{n-1} \geqslant 0$

7cb) With $H_{m\tau}$ and H_{mt} as found in 4, select ξ_1 and compute

$$\rho = J_{n-1} H_{m\tau} - \xi_1 \left\{ J_{n-1} H_{m\tau} + \left(1 - J_{n-1}\right) H_{mt} \right\}$$

8cba) $\rho > 0$

Select ξ_2 and, using Table QII, find

$\frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}}$, $H_{\sigma\tau}$, E_σ, such that

$$H_{\sigma\tau} \leqslant H_{m\tau}\xi_2 < H_{\sigma+1,\tau}; \quad x_\tau < \widehat{KZ}_{n-1} \leqslant x_{\tau+1}$$

Compute $E_n = E_\sigma + \frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}} \left(H_{m\tau}\xi_2 - H_{\sigma\tau}\right)$

Look up in Table QII

$\frac{\Delta E_\sigma}{\Delta H_{\sigma t}}$ such that $x_t < \widehat{KZ}_{n-1} \leqslant x_{t+1}$

8cbb) $\underline{\rho < 0}$

Select ξ_2 and, using Table QII, find

$$\frac{\Delta E_\sigma}{\Delta H_{\sigma t}}, \; H_{\sigma t}, \; E_\sigma, \text{ such that}$$

$$H_{\sigma t} \leqslant H_{mt}\, \xi_2 < H_{\sigma+1,t}; \; x_t < -\frac{Z_{n-1}}{\mu_{n-1}} \leqslant x_{t+1}$$

Compute

$$E_n = E_\sigma + \frac{\Delta E_\sigma}{\Delta H_{\sigma t}} (H_{mt}\, \xi_2 - H_{\sigma t})$$

Look up in Table QII

$$\frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}} \quad \text{such that } x_\tau < KZ_{n-1} \leqslant x_{\tau+1}$$

9cb) Compute

$$w_3 = w_1 \frac{1}{E_{n-1}} \frac{J_{n-1} H_m + (1 - J_{n-1}) H_{mt}}{J_{n-1} \dfrac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} + (1 - J_{n-1}) \dfrac{\Delta H_{\sigma t}}{\Delta E_\sigma}}$$

$\underline{\mu_{n-1} \quad 0; \; 1 - J_{n-1} \quad 0}$

7cc) With $H_{m\tau}$ and H_{mt} as found in 4 and using Table QII, select ξ_1 and find

$$\frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}}, \; \frac{\Delta E_\sigma}{\Delta H_{\sigma t}}, \; E_\sigma, \; H_{\sigma\tau}, \text{ such that}$$

$$H_{\sigma\tau} < \xi_1 H_m \leqslant H_{\sigma+1,\tau} \qquad x_\tau < KZ_{n-1} \leqslant x_{\tau+1}$$

$$x_t < -Z_{n-1}/\mu_{n-1} < x_{t+1}$$

8cc) Select ξ_2 and compute

$$\rho = J_{n-1} \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} \xi_2 - \left\{ \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} J_{n-1} + (1 - J_{n-1}) \frac{\Delta H_{\sigma t}}{\Delta E_\sigma} \right\}$$

$\underline{\text{If } \rho \leqslant 0}$ compute

$$E_n = E_\sigma + \frac{\Delta E_\sigma}{\Delta H_{\sigma\tau}} (\xi_1 H_{m\tau} - H_{\sigma\tau})$$

and continue.

If $\rho > 0$, start again at 7cc.

9cc) Same as 9cb.

10c) Compute $\omega = \sqrt{E_n/E_{n-1}}$. Select ξ. Look up $\cos\pi\xi$. Compute

$$\mu_n = \mu_{n-1}\,\omega + \sqrt{1 - \mu_{n-1}^2}\,\sqrt{1 - \omega^2}\,\cos\pi\xi .$$

11) Look up $w(E_n)$ and $g(E_n)$ in Table A.

If $\mu_n \geqslant 0$

12a) Select ξ.

13a) Compute $s = \dfrac{1}{1 + \tilde{K}\mu_n} \log \xi$.

14a) Look up $e^{\tilde{K}\mu_n s}$

15a) Compute

$$W_n = w_3 \frac{e^{\tilde{K}\mu_n s}}{1 + \tilde{K}\mu_n} ,$$

$$Z_n = Z_{n-1} + \mu_n s/w(E_n).$$

16a) Proceed with $(n + 1)$.

If $\mu_n \leqslant 0$

12b) Look up $e^{-\tilde{K}w(E_n)Z_{n-1}}$ and $e^{w(E_n)Z_{n-1}/\mu_n}$ and compute

$$D = e^{-\tilde{K}w(E_n)Z_{n-1}} - \left(1 - \frac{1}{J}\right) e^{w(E_n)Z_{n-1}/\mu_n}$$

13b) Select ξ and compute

$$\rho = D\xi + e^{w(E_n)Z_{n-1}/\mu_n} - e^{-w(E_n)Z_{n-1}}$$

If $\rho \geqslant 0$

14ba) Compute

$$W_n\, g(E_n) = w_3\, J_n D\, g(E_n) \qquad (N = n)$$

15ba) Start a new particle

If $\rho < 0$

14bb) $$s = -\frac{1}{1 + \tilde{K}\mu_n} \log\left(1 - e^{\tilde{K}w(E_n)Z_{n-1}} D\xi\right)$$

15bb) $$Z_n = Z_{n-1} + \mu_n s/w(E_n)$$

Look up $e^{\tilde{K}w(E_n)Z_n}$ and compute

$$W_n = w_3 \frac{D}{1 + \tilde{K}\mu_n} e^{\tilde{K}w(E_n)Z_n}$$

16bb) Proceed with $(n + 1)$

Note that the thing determined by all this computation is the value of $W_n g(E_n)$ averaged over all particles treated.

6. The Why's and Wherefore's

The importance function Q has been previously selected as

$$Q = G\, g(E)\, e^{-\tilde{K}w(E)Z}\, \mathcal{E}(Z) + g(E)\, \mathcal{E}(-Z)$$

where

$$\mathcal{E}(Z) = 1, \qquad Z > 0$$
$$= 0, \qquad Z < 0.$$

In examining the prescription of Section 5, we will state the probability functions p as determined by the physics and will also state the $\tilde{p}$ as determined by the prescriptions in Section 5. The weights (w) of the particles must be multiplied, at each random selection, by the corresponding value of $p/\tilde{p}$. This will be compared with the instructions given above.

Further, the reasons for the choice of the various $\tilde{p}$ used in Section 5 will be given. In many cases it was necessary to select a $\tilde{p}$ which was relatively easy to compute, rather than one which might have been desirable for other reasons. Since the process of determining the $\tilde{p}$ proceeds best in the reverse order from that of Section 5, we discuss Section 5 in reverse here.

6.1 <u>Steps 11 through 16</u>

We are concerned with determining Z_n, already knowing E_n and μ_n and, of course, Z_{n-1}. One may write

$$Z_n = Z_{n-1} + \left[\mu_n / w(E_n)\right] s$$

and determine s. Clearly

$$p(s) = e^{-s}$$

$$\begin{aligned} Q(s) &= G\, g(E_n)\, e^{-\hat{K}w(E_n)Z_n}\, \mathcal{E}(Z_n) \\ &\quad + g(E_n)\, \mathcal{E}(-Z_n) \\ &= G\, g(E_n)\, e^{-\hat{K}w(E_n)Z_{n-1}}\, e^{-\hat{K}\mu_n s}\, \mathcal{E}\left[Z_{n-1}w(E_n) + \mu_n s\right] \\ &\quad + g(E_n)\, \mathcal{E}(-Z_{n-1}w(E_n) - \mu_n s). \end{aligned}$$

On the basis of the general principles of the quota-sampling procedure, one wants

$$\tilde{p}(s) = p(s)\, Q(s) \Big/ \int_0^\infty p(s)\, Q(s)\, ds.$$

Steps 12 - 15 in Section 5(n) correspond to a choice of s at random according to the probability density $\tilde{p}(s)$. Then, one should have

$$\frac{W_n}{W_3} = \frac{p(s)}{\tilde{p}(s)} = \frac{\int_0^\infty p(s)\, Q(s)\, ds}{Q(s)}$$

That this statement is true for 15a, 14ba, and 15bb is easily verified.

6.2 Steps 6bi through 10bi

The importance of a particle at E_n, μ_n, Z_{n-1} is given by the

$$\int_0^\infty p(s)\ Q(s)\ ds$$

of Section 6.1, i.e. by

$$q(E_n,\mu_n) = g(E_n) \left\{ J_n \exp(-Kw(E_n)Z_{n-1}) + (1 - J_n) \exp(Z_{n-1}w(E_n)/\mu_n)\ \mathcal{E}(-\mu_n) \right\}$$

Thus, one should use

$$\tilde{p}(E,\mu) = p(E,\mu)\ q(E,\mu) \qquad \text{(normalized)}$$

to select E_n, μ_n. Now, for type bi collisions*

$$p(E,\mu) = \frac{(M_i + 1)^2}{4\pi E_{n-1} M_i} \left\{ 1 - \mu_{n-1}^2 - \omega^2 - \mu^2 + 2\mu\mu_{n-1}\omega \right\}^{-1/2}$$

$$\omega = \frac{M_i + 1}{2} \sqrt{\frac{E}{E_{n-1}}} - \frac{M_i - 1}{2} \sqrt{\frac{E_{n-1}}{E}} .$$

It is clear that the use of $p(E,\mu)\ q(E,\mu)$ for $\tilde{p}$ will involve great effort. On the other hand, except for H one expects no M_i less than 8, and usually higher. Thus E will often not change much from E_{n-1}, so that the major emphasis is on the angular change. Also, the average of $p(E,\mu)$ over all E is independent of μ. Thus, if one selects μ from

$$\tilde{p}(\mu) = q(E_{n-1},\mu) \qquad \text{(normalized)}$$

* See Appendix I.

this modification should not materially reduce the efficiency of the quota sampling. Of course, this choice of $\tilde{p}(\mu)$ corresponds to quota sampling as if the collision were elastic and isotropic in the laboratory coordinates.

The choice of μ_n from this distribution is effectively carried out in (n), 7bi and 8bi. One should note that what actually is used in 7bi and 8bi is not

$$q(E_{n-1},\mu) \Big/ \int_{-1}^{1} q(E_{n-1},\mu)\,d\mu$$

but

$$\frac{\int_{\mu_\sigma}^{\mu_{\sigma+1}} q(E_{n-1},\mu)\,d\mu \Big/ (\mu_{\sigma+1} - \mu_\sigma)}{\int_{-1}^{1} q(E_{n-1},\mu)\,d\mu} = \frac{\Delta F_{\sigma\tau}}{\Delta\mu_\sigma}$$

for $\mu_\sigma < \mu \leqslant \mu_{\sigma+1}$. This is a consequence of using a coarse tabular interval.

Once μ_n is chosen, we still need E_n. Thus, we select ω according to 9bi, so that*

$$p(\omega|\mu) = \frac{1}{\pi}\left[1 - \mu_n^2 - \mu_{n-1}^2 - \omega^2 + 2\mu_n\mu_{n-1}\,\omega\right]^{-1/2}$$

and compute, according to 10bi,

$$E_n = E_{n-1}\left(\frac{\omega + \sqrt{\omega^2 + M_1^2 - 1}}{M_1 + 1}\right)^2 .$$

These steps assure that E_n is consistent with μ_n. Now then

$$\tilde{p}(\omega,\mu) = \frac{\Delta F_{\sigma\tau}}{\Delta\mu_\sigma}\, p(\omega|\mu)$$

---

* See Appendix I, $p(\mu\,|\,\omega)$, since an interchange of ω and μ is permissible there.

$$\tilde{p}(E,\mu) = \tilde{p}(\omega,\mu)\,\frac{d\omega}{dE} =$$

$$= \frac{\Delta F_{\sigma\tau}}{\Delta\mu_\sigma}\, p(\omega|\mu)\, \frac{(M_i+1)^2}{2E_{n-1}} \left(\frac{\sqrt{\omega^2 + M_i^2 - 1}}{\left[\omega + \sqrt{\omega^2 + M_i^2 - 1}\right]^2} \right)$$

Thus, as in 10bi

$$\frac{w_3}{w_1} = \frac{p(E,\mu)}{\tilde{p}(E,\mu)} = \frac{1}{2M_i}\, \frac{\left[\omega + \sqrt{\omega^2 + M_i^2 - 1}\right]^2}{\sqrt{\omega^2 + M_i^2 - 1}}\, \frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}} .$$

6.3 <u>Steps 7a through 10a</u>

Type <u>a</u> collisions are inelastic and isotropic, so that

$$p(E,\mu) = f(E)\, g(\mu) \quad \text{where } g(\mu) = 1/2$$

expresses the isotropic nature of the collision. Furthermore, we have assumed that the possible final energies E form a discrete set, E_τ, such that if

$$E_\sigma < E_{n-1} < E_{\sigma+1}$$

the probability that $E_n = E$ is $\gamma_{\sigma\tau}$. Thus we have

$$p(E_\tau,\mu) = 1/2\, \gamma_{\sigma\tau}$$

and using $q(E,\mu)$ as defined in Section 6.2, should use

$$\tilde{p}(E_\tau,\mu) = \frac{\frac{1}{2}\gamma_{\sigma\tau}\, q(E_\tau,\mu)}{\frac{1}{2}\sum_\tau \gamma_{\sigma\tau} \int_{-1}^{1} q(E_\tau,\mu)\,d\mu} .$$

A selection from this distribution can conveniently be effectuated; this is carried out in (n), 7a - 10a. The procedure is based on the equation

$$\tilde{p}(E_\tau, \mu) = \tilde{p}(\mu | E_\tau) \int_{-1}^{1} p(E_\tau, \mu) d\mu$$

where
$$\tilde{p}(\mu | E_\tau) = \frac{\tilde{p}(E_\tau, \mu)}{\int_{-1}^{1} \tilde{p}(E_\tau, \mu) d\mu} = \frac{q(E_\tau, \mu)}{\int_{-1}^{1} q(E_\tau, \mu) d\mu} .$$

Thus, we first select E_n from the discrete distribution

$$\tilde{p}(E_\tau) = \int_{-1}^{1} \tilde{p}(E_\tau, \mu) d\mu$$

and then select μ_n from $\tilde{p}(\mu | E_\tau)$.

Step 7a is concerned with selecting E_n and steps 9a and 10a concern the selection of μ_n. The remarks in Section 6.2 on the coarse tabular interval used in selecting μ_n from $q(E,\mu)$ hold here also. In fact the μ_n choice is carried out here in the same manner as in Section 6.2.

It follows that, for our detailed method of choice,

$$\tilde{p}(E_n, \mu_n) = \tilde{p}(\mu_n | E_n)\ \tilde{p}(E_n)$$

$$\tilde{p}(E_n) = G_{\sigma, \tau+1, \nu} - G_{\sigma, \tau, \nu} \qquad \text{for } E_n = E_\tau$$

$$\frac{p(E_n)}{\tilde{p}(E_n)} = \frac{\gamma_{\sigma\tau}}{\Delta G_{\sigma\tau\nu}} \qquad \text{for } E_n = E_\tau .$$

This gives a factor in the weight of w_2/w_1 as in 7a. Further (note σ and τ have new meanings here, not as above)

$$\tilde{p}(\mu_n | E_n) = \frac{\Delta F_{\sigma\tau}}{\Delta \mu_\sigma} \qquad \text{for } \mu_\sigma < \mu_n < \mu_{\sigma+1}$$

whereas $\qquad p(\mu_n) = 1/2.$

Thus

$$\frac{w_3}{w_2} = \frac{1}{2}\frac{\Delta\mu_\sigma}{\Delta F_{\sigma\tau}}.$$

6.4 Steps 7c through 10c

We are now considering H collisions. Here the argument differs in extreme from Section 6.2. We select the energy in an H collision as if the process were straight ahead. This is plausible, since in general a high energy particle is more important than a low energy one, and the particles losing least energy in a collision are those most forward. Further in an H collision the angular distribution is markedly forward.

Thus, rather than use

$$\frac{p(E,\mu)\ q(E,\mu)}{\int dE \int d\mu\ p(E,\mu)\ q(E,\mu)}$$

to select E_n and μ_n, we select E_n from

$$\mathcal{P}(E) = \frac{q(E,\mu_{n-1})\int p(E,\mu)\ d\mu}{\int\left[q(E',\mu_{n-1})\int d\mu\ p(E',\mu)\right] dE'}$$

and then select μ_n from

$$p(\mu|E_n)$$

that is, from the actual probability density for μ_n if E_n is known.

Now, from Appendix I it is apparent that, with $M = 1$,

$$\int p(E,\mu)d\mu = \frac{1}{\pi E_{n-1}}\int\frac{d\mu}{(1-\mu_{n-1}^2-\omega^2-\mu^2+2\mu\mu_o\omega)^{1/2}} \quad (E < E_{n-1})$$

$$= \frac{1}{E_{n-1}}$$

so that

$$\tilde{p}(E) = q(E,\mu_{n-1}) \Big/ \int_0^{E_{n-1}} q(E',\mu_{n-1})dE' \ .$$

This function cannot be tabulated directly as it depends on the four variables, E_{n-1}, Z, μ_{n-1}, E. However, if we define $H(E,x)$ as in Section 5 (see discussion of Table QII) we have

$$\int_0^E \tilde{p}(E')dE' =$$

$$= \frac{J_{n-1}\, H(E, \tau Z_{n-1}) + \mathcal{E}(-\mu_{n-1})\,(1 - J_{n-1})\, H\left(E, \frac{-Z_{n-1}}{\mu_{n-1}}\right)}{J_{n-1}\, H(E_{n-1}, \tau Z_{n-1}) + \mathcal{E}(-\mu_{n-1})\,(1 - J_{n-1})\, H\left(E_{n-1}, \frac{-Z_{n-1}}{\mu_{n-1}}\right)}$$

The procedure of (n), steps 7ca - 9ca, 7cb - 9cb, and 7cc - 9cc, are concerned with selecting E_n from $\tilde{p}(E)$, including, of course, the necessary modification to permit the use of a coarse set of tabular intervals for $H(E,x)$. Thus for 7ca - 9ca, where $\mu_{n-1} \geqslant 0$ the procedure of 8ca gives

$$\tilde{p}(E) = \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} \frac{1}{H_{m\tau}} \qquad \text{for } E_\sigma < E < E_{\sigma+1}$$

and, since

$$p(E) = 1/E_{n-1}$$

one has w_3/w_1 as in 9ca.

For 7cb - 9cb, the distribution used is

$$\tilde{p}(E) = \frac{J_{n-1}\, H_{\sigma\tau} + (1 - J_{n-1})\, H_{\sigma t}}{J_{n-1}\, H_{m\tau} + (1 - J_{n-1})\, H_{mt}}$$

for $E_\sigma < E < E_{\sigma+1}$ and τ and t are so selected as to correspond to Z_{n-1} and Z_{n-1}/μ_{n-1}.

Since here $1 - J_{n-1}$ is, by hypothesis, positive, one has to select E from a sum of two distribution. This is carried out by deciding on one or the other with one game of chance and then selecting E with a second game of chance. Thus, which of the two distributions (the t or τ) is decided in 7cb by the sign of ρ. For these results, one has the next steps in 8cba or 8cbb.

Now clearly

$$\tilde{p}(E) = \tilde{p}(\rho > 0)\, \tilde{p}(E|\rho > 0) + \tilde{p}(\rho < 0)\, \tilde{p}(E|\rho < 0).$$

From 7cb

$$\tilde{p}(\rho > 0) = 1 - \tilde{p}(\rho < 0) = J_{n-1} H_{m\tau} \Big/ \left[J_{n-1} H_{m\tau} + (1 - J_{n-1}) H_{mt}\right];$$

from 8cba

$$\tilde{p}(E|\rho > 0) = \frac{1}{H_{m\tau}} \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} \qquad E_\sigma < E < E_{\sigma+1};$$

and from 8cbb

$$\tilde{p}(E|\rho < 0) = \frac{1}{H_{mt}} \frac{\Delta H_{\sigma t}}{\Delta E_\sigma} \qquad E_\sigma < E < E_{\sigma+1}$$

so that

$$\tilde{p}(E) = \frac{J_{n-1} \dfrac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} + (1 - J_{n-1}) \dfrac{\Delta H_{\sigma t}}{\Delta E_\sigma}}{J_{n-1} H_{m\tau} + (1 - J_{n-1}) H_{mt}}$$

whence, with $p(E) = 1/E_{n-1}$, the value of w_3/w_1 specified in 9cb is easily seen to be $p(E)/\tilde{p}(E)$.

When $1 - J_{n-1} < 0$, the device used above is no longer applicable. The procedure used in this case, in steps 7cc - 9cc, consists of selecting E as follows:

Select E from the distribution

$$\tilde{p}_\sigma(E) = \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} \frac{1}{H_{m\tau}} \qquad \text{for } E_\sigma < E < E_{\sigma+1} \cdot$$

Keep this E if, and only if

$$0 \geqslant \rho = J_{n-1} \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} \xi_2 - \left[\frac{\Delta H_\sigma}{\Delta E_\sigma} J_{n-1} + (1 - J_{n-1}) \frac{\Delta H_{\sigma t}}{\Delta E_\sigma}\right]$$

where ξ_2 is a random variable equidistributed in the interval (o, 1). Thus

$$\tilde{p}(E) = \frac{\tilde{p}_o(E)\ \tilde{p}(\rho < 0|E)}{\int dE\ \tilde{p}_o(E)\ \tilde{p}(\rho < 0|E)} \quad ;$$

but

$$\tilde{p}(\rho < 0|E) = \frac{\frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} J_{n-1} + (1 - J_{n-1}) \frac{\Delta H_{\sigma t}}{\Delta E_\sigma}}{J_{n-1} \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma}}$$

$$\int dE\ \tilde{p}_o(E)\ \tilde{p}(\rho < 0|E) = \frac{J_{n-1} H_{m\tau} + (1 - J_{n-1}) H_{mt}}{J_{n-1} H_{m\tau}} \quad .$$

From this we have

$$\tilde{p}(E) = \frac{J_{n-1} \frac{\Delta H_{\sigma\tau}}{\Delta E_\sigma} + (1 - J_{n-1}) \frac{\Delta H_{\sigma t}}{\Delta E_\sigma}}{J_{n-1} H_{m\tau} + (1 - J_{n-1}) H_{mt}} \quad .$$

This is the result desired, so that w_3/w_1 is correctly the same in 9cc as in 9cb.

6.5 Steps 1 through 6

The considerations of Sections 6.2 - 6.4 have given us, to the degree of approximation used therein, an estimate of the importance of types a, b, and c collisions. The scheme used in 6.2 gives equal importance to all type b collisions, independent of the mass M_i involved.

Thus, from 6.3 the importance of a type a collision is

$$q_a = \frac{1}{2} \sum_{\tau} \gamma_{\sigma\tau} \int_{-1}^{1} q(E_\tau, \mu) d\mu = K_{\sigma\nu}$$

where

$$E_\sigma < E_{n-1} < E_{\sigma+1} \text{ and } Z_\nu < Z_{n-1} < Z_{\nu+1} .$$

Similarly, from 6.2 one has for the importance of a type b collision, consistent with the modifications of 6.2,

$$q_b = \int_{-1}^{1} q(E_{n-1}, \mu) d\mu = g(E_{n-1}) F_\tau \qquad \text{for } \lambda_\tau < Z_{n-1}\, w(E_{n-1}) < \lambda_{\tau+1} .$$

Further, from 6.4,

$$q_c = \frac{1}{E_{n-1}} \int_0^{E_{n-1}} q(E', \mu_{n-1}) dE'$$

$$= \frac{1}{E_{n-1}} \left[J_{n-1} H_{m\tau} + \mathcal{E}(-\mu_{n-1}) (1 - J_{n-1}) H_{mt} \right].$$

The validity of the choices of step 5 and the w_1/w_{n-1} of 6a, 6b, and 6c are clearly consistent with the above.

7. Conclusion

I hope that following the above procedure will enable the estimation of $\langle g(E) \rangle$ for a particular case to an accuracy of some 10%, say with somewhere around 100 particles. An experiment carried out in this way will be extremely informative. Of course, one will study in some detail the distribution of the values of $W_n g(E_n)$ for the various particles to discover the efficiency of the technique and the error in $\langle g(E) \rangle$.

Furthermore, while no mention of the possibility has been given above, it is clear that more than one problem can be solved at a time or, in other words, given the solution of one problem, one can change the physical parameters and obtain the answer to the problem thus defined by merely recomputing the weights. This is not much work compared with the total effort involved in a computation.

Clearly, the limitation to be imposed on this procedure is that as the modified problem differs more from the unmodified, the variance of the answer increases, provided the original quota sampling procedure was reasonably efficient.

APPENDIX I

Collisions Elastic in the Center of Gravity Coordinate System

Assume a neutron having a velocity initially of $\vec{V}_o$ collides with a nucleus of mass M. We wish to find the probability density for the energy and direction cosine, E and μ, of the neutron after the collision. We have, in the center of gravity system before the collision, for the neutron velocity,

$$\frac{M}{M+1}\vec{V}_o$$

whence after the collision

$$\frac{M}{M+1} V_o \vec{\xi}$$

where $\vec{\xi}$ is a unit vector selected at random from an isotropic population. Hence, converting back to the laboratory reference frame,

$$\vec{V} = \frac{1}{M+1}\vec{V}_o + \frac{M}{M+1} V_o \vec{\xi} \quad .$$

Now let $\qquad \lambda = \frac{1}{V_o}\vec{\xi}\,\vec{V}_o$

so that $\qquad -1 < \lambda < 1 \qquad p(\lambda) = \frac{1}{2}$

and let $\qquad \kappa^2 = \frac{V^2}{V_o^2} = \frac{E}{E_o}$

so that $\qquad \kappa^2 = \left(\frac{1}{M+1}\right)^2 \left\{M^2 + 1 + 2M\lambda\right\}$

and $\qquad \frac{M-1}{M+1} < \kappa < 1, \; p(\kappa^2) = \frac{(M+1)^2}{4M} = E_o\, p(E).$

Further, if $\qquad \omega = \frac{1}{V_o V}\vec{V}\,\vec{V}$

$$\omega = \frac{M+1}{2}\kappa - \frac{M-1}{2}\frac{1}{\kappa}$$

and $$\mu = \mu_o \omega + \sqrt{1 - \mu_o^2} \quad \sqrt{1 - \omega^2} \qquad \cos \zeta$$

where $$p(\zeta) = \frac{1}{\pi} \qquad 0 < \zeta < \pi \, .$$

Thus* $$p(\mu \mid \omega) = -p(\zeta) \frac{d\zeta}{d\mu} = \frac{1}{\pi} \left[1 - \mu_o^2 - \omega^2 - \mu^2 + 2\mu\mu_o\omega\right]^{-1/2}$$

and $$p(E,\mu) = p(E)\ p(\mu|\omega) = \frac{(M + 1)^2}{4ME_o\pi} \left[1 - \mu_o^2 - \omega^2 - \mu^2 + 2\mu\mu_o\omega\right]^{-1/2}$$

---

* $p(\mu \mid \omega)$ means probability density of μ if ω.

APPENDIX II

Abstract Report on a Small Monte Carlo Computation

A semi-infinite homogeneous and isotropic plane slab was studied. A source of particles was placed in this slab sixteen mean free paths from the boundary of the slab. Isotropic scattering was assumed, and no energy changes were considered. The particles leaving the source were directed toward and normal to the boundary of the slab.

Fifty particles were considered. Importance sampling was used throughout. The computation was carried out on a hand machine in somewhat less than two man weeks. The results obtained are given below, as a function of probability of survival in a collision. Thus, a parametric study in terms of change in composition has been carried out. It is hoped, at some future date, to find the effect of varying the scattering law from the isotropic case.

Probability of Survival in a Collision	Probability of Penetration	Percent Probable Error
0	0.113×10^{-6} (e^{-16})	0%
.3	0.260×10^{-6}	18%
.4	0.335×10^{-6}	14%
.5	0.478×10^{-6}	10%
.6	0.863×10^{-6}	11%
.7	2.48×10^{-6}	21%