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APPROXIMATE ANALYSIS OF THE PENETRATION OF NEUTRONS THROUGH A THICK SHIELD COMPOSED OF NON-HYDROGENOUS MATERIALS

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BY: P. Morrison 7/10/52

This document consists of 30 pages.
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Contract No. W-7405, eng. 26*

ORNL Summer Shielding Session

Gale Young, Leader

Title: Approximate Analysis of the Penetration of Neutrons Through A Thick Shield Composed of Non-Hydrogenous Materials

Author: F. H. Murray

Date Issued:

DEC 14 1949

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This report is one of a series on shielding and represents part of the work of a summer group working principally at Oak Ridge National Laboratory. Participants are drawn from Nuclear Development Associates (Navy contract), NEPA, Bureau of Ships and Bureau of Aeronautics (Navy Department), AEC, RAND, General Electric, Argonne and Oak Ridge National Laboratory.

These reports are issued through ORNL on Contract No. W-7405, eng 26.

Approximate Analysis of the Penetration of Neutrons Through a Thick Shield Composed of Non-Hydrogenous Materials

By F. H. Murray

This report is chiefly concerned with extensions of the analysis of thick shields by methods related to those of ORNL 218, and of Bethe (KAPL 56). The emphasis is on the development of methods of calculation by which significant results may be obtained with a comparatively small amount of labor, at least in some idealized cases approximating actual mixtures used in shields.

In sections A and B the general equations for constant cross sections are worked out more completely than in ORNL 218, and a brief derivation is given of the asymptotic expression for the flux nv when a finite number of spherical harmonics is assumed to be sufficient.

In section C the method of moments is developed, from which one can obtain the form of the energy distribution of nv at a fixed distance from a plane source if, as in much statistical analysis, only a few terms of a Gram-Charlier series expansion is considered to be sufficient. In this method all cross sections are assumed constant, and some "absorbing" material is present. Applications can be made to mixtures not containing nuclei scattering inelastically.

In section D a generalization of the age equation is derived in the simplest extension of the more common equation for variable cross sections, which permits the derivation of asymptotic expansions for nv at large distances. The general form of the extension of the age equation leads to

a system of equations of a type studied by Birkhoff if the number of spherical harmonics is limited in advance. Here some general equations are set up; the behavior of nv at large distances from the source requires considerable further analysis if the methods of Birkhoff are applied.

In section E there is developed an approximate analysis for absorption and isotropic scattering by a simple method which has the advantage of being carried through to the end without difficulty; this leads to the diffusion equation, giving the correct diffusion length.

Section F is concerned with mixtures of nuclei for some of which the only scattering to be considered is the shadow scattering according to the Bethe-Placzek formula. Some methods of approximate analysis are developed, which lead to solutions after certain characteristic numbers and functions have been found for a form of Schroedinger equation similar to Wick's. Two approximate solutions for a mixture of water and an inelastically scattering heavy element are indicated.

A. <u>General derivation of the Boltzmann equation</u>

When a neutron is scattered by a nucleus, and has the initial value u' for $u = \ln(E_0/E)$, in the laboratory system of coordinates the probability that it will be scattered through an angle of cosine μ_0 into an infinitesimal solid angle $d\Omega$ is equal to $f(\mu_0)d\Omega$; and

$$2\pi\int_{-1}^{1} f(\mu_0)\, d\mu_0 = 1$$

If $F(z,\mu',u')$ represents nv, then the number of particles scattered from the initial infinitesimal range $du'\, d\Omega'$ into the range $d\Omega$ becomes

$$F(z,\mu',u')\, du'\, d\Omega'\, \sigma_M(u')\, f_M(u',\mu_0)\, d\Omega$$

Let Ω, Ω' be independent directions; the angle μ_0 is a function of u'-u; hence when μ_0 is held fixed, du' = du, and the expression above becomes

$$F(z,\mu',u')\, d\Omega' \sigma_M(u')\, f_M(u',\mu_0)\; du\; d\Omega \Big|_{u' = \psi(\mu_0,u)}.$$

Hence, the total probability of scattering into the range $du\,d\Omega$ per second from a particle of mass M becomes $du\, d\Omega$ times the quantity

$$I_M = \int d\Omega'\; F(z,\mu',\; u')\sigma(u')\; f_M(u',\mu_0) \Big|_{u' = \psi(\mu_0,\, u)}.$$

If $\mu_0 = Z(u'-u)$, this can be written

$$I_M = \int \sigma(u')\, F(z,\mu',\, u')\, f_M(u',Z) \frac{\partial Z}{\partial u'} \delta(Z-\mu_0)\, du'\, d\Omega'.$$

$$= \frac{1}{2\pi} \int \sigma(u')\, F(z,\mu',\, u')\, g_M(u',u)\, \delta(Z-\mu_0)\, du'\, d\Omega'.$$

$$g_M = 2\pi f_M \frac{\partial Z_M}{\partial u'}$$

Then from

$$F(z,\mu',\, u') = \frac{1}{4\pi} \sum (2n+1)\, F_n(z,\, u')\, P_n(\mu')$$

$$\delta(Z-\mu_0) = \tfrac{1}{2} \sum (2n+1)\; P_n(Z)\; P_n(\mu_0)$$

and the addition theorem for the Legendre functions, we find

$$I_M = \frac{1}{4\pi} \sum (2n+1)\, G_n^{(M)} P_n(\mu)$$

(1) $$G_n^{(M)} = \int_{u-a_M}^{u} \sigma_M(u')\, g_M(u',u)\, P_n(Z_M)\, F_n(z,u')\, du'.$$

In ordinary scattering,

$$g_M = \frac{(M+1)^2}{4M} e^{u'-u}, \qquad G_n^{(M)} = h_n^{(M)} \sigma_M F_n.$$

Generally, let

$$f_M = \frac{1}{4\pi} \sum (2r+1) \quad c_r^{(M)} P_r \ (Z_M).$$

For heavy elements, with a_M very small, $\left[a_M = 2 \ln (M+1)/M - 1) \right]$,

one may take out $\sigma_M \ F_n$ from the integrand, whence

$$G_n^{(M)} \doteq 2\pi\sigma_M F_n \int_{Z=-1}^{Z=1} f_M \frac{\partial Z}{\partial u'} P_n \ (Z_M) du' = \sigma_M F_n \ c_n^{(M)}.$$

and

$$I_M = \frac{1}{4\pi} \sum (2n+1) \sigma_M \ c_n^{(M)} F_n \ P_n \ (\mu)., \quad c_n^M = \iint \bar{f} \ P_n \ (Z) \ d\Omega$$

Then if

$$(2) \qquad \sigma C_n = (2n+1) \left[\sum_{M'} \sigma_{M'} c_n^{(M')} + \sum_{M''} \sigma_{M''} \ h_n^{(M'')} \right]$$

we have

$$\Sigma I_M = \frac{\sigma}{4\pi} \sum_n C_n \ F_n \ P_n \ (\mu)$$

and the Boltzmann equation can be written

$$(3) \qquad \left(\mu \frac{\partial}{\partial z} + \sigma\right) \ F = \frac{Q}{4\pi} + \frac{\sigma}{4\pi} \sum C_n F_n P_n \ (\mu)$$

similar to equation (2) of #218.

B. <u>Formulas for constant cross sections</u> (See Bethe KAPL 56.)

If F_n can be represented as a Laplace transform

$$F_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} H_n \ (s) \ e^{su} \ ds \ , \quad H_n \ (s) = \int_0^\infty e^{-su} \ F_n \ (z,u) \ du$$

one can substitute $H_n(s)\exp(su)$ in the various integrals and obtain the

final representation of F_n by integration. In the case of ordinary scattering,

$$e^{u'-u} \cdot \mathcal{J}_M \; H_n(s)\, e^{su'} = \frac{(M+1)^2}{4M} H_n(s)\, e^{su} \cdot e^{(s+1)(u'-u)}$$

$$Z_M \doteq 1 + \frac{u'-u}{\xi}, \quad \xi = 2/M$$

and from the definition of g_M,

$$h_{M,n} = \frac{(M+1)^2}{4M} \int_{Z=-1}^{Z=1} e^{(s+1)\xi\,(Z_M - 1)}\, P_n(Z_M)\, \xi_M \, dZ_M$$

$$= \frac{(M+1)^2}{4M} \cdot \frac{2\, e^{-x}}{M} \int_{-1}^{1} e^{xZ_M}\, P_n(Z_M)\, dZ_M. \qquad x = (s+1)\xi$$

From Watson, Theory of Bessel functions, p. 369, equation (3),

$$e^{xZ} = \sum_0^\infty (2n+1)\, f_n(x)\, P_n(Z)$$

$$f_n(x) = \sqrt{\frac{\pi}{2x}}\, I_{n+\frac{1}{2}}(x), \qquad I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \qquad x \gg 1.$$

Hence

$$h_{M,n} = \left(\frac{M+1}{M}\right)^2 e^{-x}\, f_n(x)$$

In particular,

$$f_0(x) = \frac{\sinh x}{x}$$

$$f_1(x) = \frac{1}{x}\left[\cosh x - \frac{\sinh x}{x}\right]$$

$$f_2(x) = \frac{1}{x}\left[\sinh x\left(1 + \frac{3}{x^2}\right) - \frac{3}{x}\cosh x\right].$$

If $\left|\frac{x}{n^2}\right| \gg 1$, $h_{Mn} \doteq 1/2x = \alpha_M/(s+1)$. This approximation is limited by the tacit assumption that F can be adequately represented by a finite number of F_n functions; a method which avoids this limitation is that of Wick (On the Space Distribution of Slow Neutrons, P. R., Mar. 1, 1949, p. 738).

Referring to (2), one finds that in the absence of inelastic scattering nuclei,

$$\sigma C_n = \left[\sum_M \sigma_M \alpha_M / (s+1)\right] (2n+1) = \frac{C\sigma}{s+1} (2n+1)$$

and from (3) one obtains the asymptotic form

$$(4) \qquad \left(\mu \frac{\partial}{\partial z} + \sigma\right) F_s = \frac{Q}{4\pi} + \frac{C\sigma}{s+1} F_s.$$

Operating on both sides by $(s+1)$ and noting that sF_s is the Fourier transform of dF/du if $F = 0$ when $u = 0$, one obtains the equation equivalent to (4)

$$(5) \qquad \left(\frac{\partial}{\partial u} + 1\right)\left(\mu \frac{\partial}{\partial z} + \sigma\right) F = \frac{1}{4\pi}\left(\frac{\partial}{\partial u} + 1\right) Q + C\sigma F.$$

Taking the Fourier transform with respect to z of both sides, $(F_p \rightarrow G)$, and replacing G by gexp(-u) one obtains

$$\frac{\partial}{\partial u}(\mu p + \sigma)\, g - C\sigma g = \frac{1}{4\pi} \frac{\partial}{\partial u}(Q e^u)$$

whence

$$g = e^{\int_0^u \frac{C\sigma\, du}{\mu p + \sigma}} \int_0^u e^{-\int_0^{u'} \frac{C\sigma\, du'}{\mu p + \sigma}} \frac{1}{4\pi(\mu p + \sigma)} \frac{d}{du'}(Q' e^{u'})\, du'$$

$$= \frac{C\sigma \bar{Q}_p}{4\pi} \frac{e^{\frac{C\sigma u}{\mu p + \sigma}}}{(\mu p + \sigma)^2} \cdot \int_0^{\varepsilon} Q_p e^{u'}\, du' = \bar{Q}_p \cdot$$

if Q is continuous in u but different from 0 only in a very small neighborhood of u = 0. It is to be remarked that these assumptions are possible if the operations above are applied to the asymptotic form of the equation (4) even if the definition of h_{Mn} requires an integral over a finite range $(u-a_M, u)$; when the Fourier transforms of the original functions F_n are introduced, all values of u are included in the domain of definition of the functions f_s.

The function g can be integrated over the unit sphere, with the result

$$G_o(p,u) = \frac{\bar{Q}\, p\, e^{-u}}{2\, p\, u}\left[e^{\frac{C\sigma u}{\sigma - p}} - e^{\frac{C\sigma u}{\sigma + p}}\right]$$

For a plane source $\bar{Q}_p$ is independent of p, and

$$(6) \qquad F_o(z,u) = \bar{Q}_p \frac{e^{-u}}{2\,u} \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \left[e^{\frac{C\sigma u}{\sigma - p}} - e^{\frac{C\sigma u}{\sigma + p}}\right] e^{pz} \frac{dp}{p}$$

The function dF_o/dz can be evaluated with the aid of the tables of Campbell and Foster, # 654.2 with the result

$$-\frac{\partial F_o}{\partial Z} = \bar{Q}_p \frac{e^{-u}}{2\,u} \sqrt{\frac{C\sigma u}{z}}\; e^{-\sigma z}\; I_1\left(2\sqrt{C\sigma u z}\right) dz$$

$$F_o = \bar{Q}_p \frac{e^{-u}\sqrt{C\sigma}}{2\sqrt{u}} \int_z^{\infty} e^{-\sigma z}\, z^{-\frac{1}{2}}\, I_1\left(2\sqrt{C\sigma u z}\right) dz$$

The method of steepest descent applied to (6) also gives the asymptotic solution which is found to be

$$(7) \qquad F_o(z,u) \sim \frac{\bar{Q}_p\, C^{\frac{1}{4}}\, e^{-u}}{4\sqrt{\pi}\,(\sigma z u)^{3/4}}\; e^{-\sigma z + 2\sqrt{C\sigma u z}}$$

The expression above, multiplied by 2z, has been derived in the analysis of a beam source; see Marshak, Review of Modern Physics, July 1947, p. 230 (161).

The factor $(2z)^{-1}$ has 1/2 due to the propagation in two directions (+ z, -z) from the plane source in an infinite medium, and 1/z resulting from the isotropy of the radiation from each point of the plane.

C. <u>The double Fourier transform of F and the method of moments</u>

Returning to (3), and taking the Fourier transform with respect to z of both sides, also the Laplace transform with respect to u (already performed), and dividing both sides by σ , with $p = \sigma P$

$$H_n(p,s) = \left(\frac{1}{2\pi i}\right)^2 \iint e^{-pz-su} F_n(z,u)\,dz\,du$$

we obtain equation (2) of #218:

(7.1) $$(1+\mu P)H = \frac{Q/\sigma}{4\pi} + \frac{1}{4\pi}\sum C_n H_n P_n(\mu)$$

A continuous source distribution will be assumed, of such a character that only a small number of the quantities C_n need be assumed different from zero. As in #218 preceding equation (5), one obtains by the same algebra

$$H_n = -(Q/\sigma C_0)\frac{\Delta_{0n}}{D}$$

$$D \equiv \begin{vmatrix} a_0 C_0 - 1 & a_{01} C_1 & a_{02} C_2 \cdots \\ a_{10} C_0 & a_{11} C_1 - 1 & a_{12} C_2 \cdots \\ a_{20} C_0 & a_{21} C_1 & a_{22} C_2 - 1 \cdots \end{vmatrix}$$

where

$$a_{mn} = a_m(P) P_n(-1/P),\ m \geq n,$$

$$a_0 + P a_1 = 1,\quad nP/(2n+1)a_{n-1} + a_n + \frac{(n+1)P}{2n+1} a_{n+1} = 0$$
$$n \geq 1$$

Let

$$H_0(p,s) = \int_0^\infty e^{-pz}\, G_0(z,s)\,dz$$

$$G_0(z,s) = \int_0^\infty e^{-su}\, F_0(z,u)\,du$$

$$\left(-\frac{\partial}{\partial s}\right)^m G_0(z,s) = \int_0^\infty u^m e^{-su} F_0(z,u)\,du$$

Evidently,

(8) $$\overline{u^m} = \left[\left(-\frac{\partial}{\partial s}\right)^m G_0(z,s)\right]_{s=0} \Big/ G_0(z,0)$$

and the moments $(u^m)_{ave.}$ can be calculated from $G_0(z,s)$ which in turn is found from $H(p,s)$ and the inverse Fourier transformation; assume that $G_0(z,s)$ is defined by the pole solution ($D = 0$) and that

$$G_0(z,s) = f(s)e^{-P\sigma z} = e^{\ln f(s) - P\sigma z}$$

$$-\frac{\partial G_0}{\partial s} = \left[\sigma z \frac{\partial P}{\partial s} - f'(s)/f(s)\right] G_0$$

$$\frac{\partial^2 G_0}{\partial s^2} = \left[\sigma z \frac{\partial^2 P}{\partial s^2} - \frac{d}{ds}(f'/f)\right] G_0 + G_0 \left[\sigma z \frac{\partial P}{\partial s} - f'(s)/f(s)\right]^2$$

The average value of u becomes

(9) $$\overline{u} = \sigma z \frac{\partial P}{\partial s} - f'(s)/f(s)$$

and the standard deviation squared

(10) $$\Sigma^2 = \overline{u^2} - \overline{u}^2 = \sigma z \frac{\partial^2 P}{\partial s^2} - \frac{d}{ds}(f'/f)$$

We have evidently for the ratio of the standard deviation to the average $\overline{u}$

$$\frac{\Sigma}{\overline{u}} = \left[\frac{\sqrt{\sigma z \frac{\partial^2 P}{\partial s^2} - \frac{d}{ds}(f'/f)}}{\sigma z \frac{\partial P}{\partial s} - f'(s)/f(s)}\right]_{s=0}$$

and for very large z this ratio decreases like $1/z^{1/2}$. With some approximation, then, one can represent $F_0(z,u)$ for large z in the form

(11) $$F_0(z,u) \sim f(0)e^{-\sigma z P(0)} \cdot \frac{1}{\Sigma\sqrt{2\pi}} e^{-(u-\overline{u})^2/2\Sigma^2}$$

One can compute F_0 from G_0 by the method of steepest descent also, from

$$F_0(z,u) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{\ln f(s) - \sigma z P(s) + su}\,ds$$

In the method of steepest descent define S by assuming the derivative of the exponent to be zero: $u + f'/f - \sigma z P'(s) = 0, \quad s = S(u,z)$

$$e^{\ln f - \sigma z P(s) + su} = e^{\ln f(S) - \sigma z P(S) + Su} \times$$

$$\times e^{\frac{1}{2}(s-S)^2\left[\frac{d}{ds}\left(\frac{f'}{f}\right) - \sigma z P''(S)\right]}$$

$$F_0(z,u) = e^{\ln f(S) - \sigma z P(S) + Su} \cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-t^2 c^2} i\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Sigma} e^{\ln f(S) - \sigma z P(S) + Su}$$

From the equation defining S, expanded to first powers in S,

$$S = \frac{u - \bar{u}_0}{\sigma z P''(0) - \frac{d}{ds}(f'/f)} \qquad \bar{u}_0 = \sigma z P'(0) - (f'/f)_{s=0}$$

Hence

$$F_0(z,u) = \frac{1}{\Sigma\sqrt{2\pi}} e^{\ln f(0) - \sigma z P(0) + \frac{S^2}{2}\left[\frac{d}{ds}\left(\frac{f'}{f}\right) - \sigma z P''(0)\right]}$$

(12)

$$= \frac{f(0)}{\Sigma\sqrt{2\pi}} e^{-\sigma z P(0)} \cdot e^{(u-\bar{u}_0)^2 \Big/ 2\left[\sigma z P''(0) - \frac{d}{ds}\left(\frac{f'}{f}\right)_0\right]}$$

which is the same as (11).

In case C_0 and C_1 are the only non-zero constants to be retained, a useful method of calculating P(o) is obtained from the equation D = 0 written in the form

$$\frac{1}{a_0(P) - 1} = \frac{C_0}{1 - C_0} + \frac{C_1}{P^2}$$

The left-hand member is plotted as function of $x = 1/P^2$; the right-hand side represents a straight line in x, defined at x = 0 and x = 2, and the intersection of the curve and line defines P.

D. A generalization of the age equation

If it can be assumed that the quantities h_{nM} may be evaluated in terms of F_n and its first derivative F_n' with sufficient accuracy, and if the cross sections are functions of u for which only the first derivative need be considered, then it is found that

$$h_{0M} = 1 - \xi_M \frac{(\sigma_M F_0)'}{\sigma_M F_0} = 1 - \xi_M \frac{\sigma_M'}{\sigma_M} - \xi_M \frac{F_0'}{F_0}$$

$$h_{1M} = \frac{\xi_M}{3}\left[1 + \frac{(\sigma_M F_1)'}{\sigma_M F_1}\right] = \frac{\xi_M}{3}\left[1 + \frac{\sigma_M'}{\sigma_M} + \frac{F_1'}{F_1}\right]$$

Then we have

$$\sigma C_0 = d_0 + \Sigma \sigma_M - \Sigma \xi_M \sigma_M' - \sigma y F_0'/F_0, \quad \sigma y = \Sigma \sigma_M \xi_M$$

$$\sigma C_1 = 3d_1 + \Sigma \sigma_M \xi_M + \Sigma \xi_M \sigma_M' + \sigma y F_1'/F_1$$

or,

$$C_0 = \overline{C_0} - y F_0'/F_0$$

(13) $$C_1 = \overline{C_1} + y F_1'/F_1$$

From the equations obtained from the Fourier transformation with respect to z one obtains immediately

$$G_0 = a_0 Q p/\sigma + a_{00}(\overline{C}_0 G_0 - y G_0') + a_{01}(\overline{C}_1 G_1 + y G_1')$$

$$G_1 = a_1 Q p/\sigma + a_{10}(\overline{C}_0 G_0 - y G_0') + a_{11}(\overline{C}_1 G_1 + y G_1')$$

Let $\sigma' dt = du/y$; these equations may be transformed into the system of equations

$$\left(\frac{d}{\sigma dt} + 1 - \overline{C}_0\right) G_0 + P G_1 = Q p/\sigma$$

$$-P G_0 + \left(\frac{d}{\sigma dt} + \overline{C}_1 + P a_0/a_1\right) G_1 = 0$$

$$P = p/\sigma, \quad a_1 = \frac{1 - a_0}{P}$$

Now σ is variable, and it is more convenient to write these

$$
(14)\qquad
\begin{aligned}
&\left(\frac{d}{dt}+\sigma-\sigma\bar{c}_0\right)G_0+pG_1=Qp\\
&-pG_0+\left(\frac{d}{dt}+\sigma\bar{c}_1+pa_0/a_1\right)G_1=0
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{d}{dt}+A\right)G_0+pG_1=Qp\\
&-pG_0+\left(\frac{d}{dt}+B\right)G_1=0
\end{aligned}
$$

$$
(15)\qquad \left[\left(\frac{d}{dt}+B\right)\left(\frac{d}{dt}+A\right)+p^2\right]G_0=\left(\frac{d}{dt}+B\right)Qp
$$

Hence the equation of the second order for G_0,

$$
(16)\qquad \left[\frac{d^2}{dt^2}+(A+B)\frac{d}{dt}+p^2+AB+\frac{dA}{dt}\right]G_0=\left(\frac{d}{dt}+B\right)Qp
$$

Since $y\,dt = du/\sigma$ and y is the average value of σ which may be assumed a small quantity, this equation is approximately

$$
\left[\frac{\sigma^2 d^2}{du^2}+\frac{A+B}{y}\sigma\frac{d}{du}+\frac{1}{y^2}\left(p^2+AB+\frac{dA}{dt}\right)\right]G_0=\frac{1}{y}\left(\frac{d}{dt}+B\right)Qp
$$

which is of the type studied by Birkhoff for which approximate solutions may be obtained easily, $1/y$ being a large quantity. For a first approximation one treats the equation in G_0 as if all coefficients were constants.

Let $g_1(t)$, $g_2(t)$ be solutions of the homogeneous equation; then one can write the solution of the non-homogeneous equation in the form

$$
G_0=\alpha g_1(t)+\beta g_2(t)+\int_0^t\left[g_1(w)g_2(t)-g_2(w)g_1(t)\right]\frac{\left(\frac{d}{dw}+\beta(w)\right)Qp(w)}{g_1(w)g_2'(w)-g_2(w)g_1'(w)}\,dw
$$

Let S_i be a root of the equation

$$
S^2+(A+B)S+p^2+A'+AB=0
$$

$$
(17)\qquad
\begin{aligned}
S_1&=-\frac{A+B}{2}+\sqrt{\left(\frac{A+B}{2}\right)^2-(p^2+A'+AB)}\\
S_2&=-\frac{A+B}{2}-\sqrt{\left(\frac{A+B}{2}\right)^2-(p^2+A'+AB)}
\end{aligned}
$$

Approximately,

$$g_1 = e^{\int S_1 dW}, \quad g_2 = e^{\int S_2 dW}$$

and

$$g_1 g_2' - g_2 g_1' = (S_2 - S_1)\, e^{\int_0^t (A+B)\, dW}$$

$$= -2\sqrt{\left(\frac{A+B}{2}\right)^2 - (p^2 + A' + AB)}\; e^{\int_0^t (A+B)\, dW}$$

Finally,

$$G_0 = \alpha e^{\int S_1 dW} + \beta e^{\int S_2 dW} + \int_0^t \left[e^{\int_0^t S_1(w)\, dw}\, e^{\int_0^w S_2(w')\, dw'} - e^{\int_0^w S_1(w')\, dw'}\, e^{\int_0^t S_2(w')\, dw'} \right]$$

$$\times\, e^{-\int_0^w (S_1 + S_2)\, dw'}\, \frac{\left(\frac{d}{dw} + B(w)\right) Q\, p(w)\, dw}{2\sqrt{\left(\frac{A+B}{2}\right)^2 - (p^2 + A' + AB)}}$$

(18)

$$= \alpha e^{\int_0^t S_1 dw} + \beta e^{\int_0^t S_2 dw} + 1/2 \int_0^t \frac{\left[e^{\int_w^t S_1(w')\, dw'} - e^{\int_w^t S_2(w')\, dw'} \right] \left(\frac{d}{dw} + B(w)\right) Q\, p(w)\, dw}{\sqrt{\left(\frac{A+B}{2}\right)^2 - (p^2 + A' + AB)}}$$

If Q is a function different from zero only within a small interval ($0 < u < \varepsilon$) one obtains by an integration by parts

$$G_0 = \alpha e^{\int S_1 dW} + \beta e^{\int S_2 dW} + \frac{\bar{Q}}{2}\left[(B + S_1)\, e^{\int_0^t S_1 dw} - (B + S_2)\, e^{\int_0^t S_2 dw} \right] \left[\left(\frac{A+B}{2}\right)^2 - (p^2 + A' + AB) \right]^{-\frac{1}{2}}$$

Let

$$\alpha + \frac{\bar{Q}}{2}(B + S_1) = 0, \qquad \beta = 0$$

(19)

$$G_0 = \frac{-\bar{Q}(B + S_2)}{2\sqrt{\left(\frac{A+B}{2}\right)^2 - (p^2 + A' + AB)}}\; e^{\int_0^t S_2 dw}$$

From the analysis for constant cross sections, it is necessary to consider solutions which may give $S = 0$, or S_2 if $A + B < 0$; and $S_2 = 0$ requires that $p^2 + AB + A' = 0$.

Finally, the inverse Fourier transformation of G_0 gives F_0:

(20)

$$F_0(z, u) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{-\bar{Q}(B + S_2)\, e^{\int_0^t S_2 dw + pz}}{2\sqrt{\left(\frac{A+B}{2}\right)^2 - (p^2 + AB + A')}}\, dp.$$

A considerably more complete treatment of the general age equation for a limited number of spherical harmonics might be developed with the methods of Birkhoff (On the asymptotic character of the solutions of certain linear differential equations containing a parameter, Trans. Amer. Math. Soc., vol. 9, pp. 219-231 (1908)). The preliminary analysis for such a treatment is presented below; the application to a system of equations instead of to a single equation offers no new difficulties.

One can write $\sigma_m F_n = f(p,u)$, and expanding f in a Taylor series,

$$\int_{u-a}^{u}\left[\sum \frac{f^{(m)}}{m!}(u'-u)^m\right] e^{u'-u} P_n(Z)\,du'$$

$$= \sum \frac{f^{(m)}\xi^{m+1}}{m!}\int_{-1}^{1}(Z-1)^m e^{\xi(Z-1)} P_n(Z)\,dZ$$

$$= \sum_m \frac{f^{(m)}\xi^{m+1}}{m!}\frac{d^{(m)}}{d\xi^m}\int_{-1}^{1} e^{\xi(Z-1)} P_n(Z)\,dZ$$

$$= 2\sum_m \frac{f^{(m)}\xi^{m+1}}{m!}\frac{d^m\left[\bar{f}_n(\xi)e^{-\xi}\right]}{d\xi^m}$$

In the particular case f = exp(su), the integral is

$$2\xi\sum\frac{(s\xi)^m}{m!}\left[\bar{f}_n(\xi)e^{-\xi}\right]^{(m)} = 2\xi\bar{f}_n\left[\xi(1+s)\right]e^{-\xi(1+s)}$$

Hence in the elastic scattering, obtains

$$G_n^{(M)} = \left[\xi_M\frac{(M+1)^2}{2M}\right]\sum(\sigma_M F_n)^m\frac{\xi_M^m}{m!}\frac{d^m}{d\xi^m}\left[\bar{f}_n(\xi_M)e^{-\xi_M}\right]$$

$$\bar{f}_n(x) = \sqrt{\frac{\pi}{2x}}\,I_{n+\frac{1}{2}}(x)$$

$$\xi_M\frac{(M+1)^2}{2M} = \frac{(M+1)^2}{M^2}$$

As before, $h_{n,M}^{(\xi_M)} = \frac{(M+1)^2}{M^2}\bar{f}_n(\xi_M)e^{-\xi_M}$

$$G_n^{(M)} = \sum_m \frac{\xi_M^m}{m!}\frac{d^m(\sigma_M F_n)}{du^m}\frac{d^m h_{n,M}(\xi_M)}{d\xi_M^m}$$

(21)

If several nuclei are present, let $\xi_M = b_M \xi$; the last equation can be written

$$G_n^{(M)} = \sum_m \xi^m \frac{d^m(\sigma_M F_n)}{du^m} \frac{b^m}{m!} \frac{d^m h_{nM}(b_M \xi)}{d(b_M \xi)^m}$$

$$= \sum_m \xi^m \frac{d^m(\sigma_M F_n)}{du^m} \frac{1}{m!} \frac{d^m h_{nM}(b_M \xi)}{d\xi^m}$$

Let

$$t = u/\xi$$

Then

$$G_n^{(M)} = \sum_m \left[\frac{1}{m!} \frac{d^m h_{nM}(b_M \xi)}{d\xi^m}\right] \frac{d^m(\sigma_M F_n)}{dt^m}$$

This can be written

$$G_n^{(M)} = \sum_m A_{nm}^{(M)} \frac{d^m}{dt^m}(\sigma_M F_n)$$

$$\xi = \Sigma \sigma_M \xi_M / \Sigma \sigma_M$$

One can write this also

(22) $$G_n^{(M)} = \sigma_M \Sigma_m B_{nm}^{(M)} \frac{d^m F_n}{dt^m}$$

Consider the solutions of the type

$$F_n = \bar{F}_n e^{\int s\,dt}$$

$$\frac{dF_n}{dt} = s\bar{F}_n e^{\int s\,dt} + \bar{F}' e^{\int s\,dt}$$

Since

$$\int_0^t s\,dt = \frac{1}{\xi}\int_0^u s\,du'$$

in general, the variations of the exponent with t predominate so that approximately,

$$\frac{d^m F_n}{dt^m} = s^m F_n$$

hence

$$G_n^{(M)} = \left[\sum B_{nM}^{(M)} s^m\right] \sigma_M F_n = \bar{h}_{nM} \sigma_M F_n$$

The function S is still to be determined. Considering S as a parameter, independent of z, one can solve the equation D = 0, where D is defined in terms of P,S in the same manner as for the case of constant coefficients. In particular, P is defined as function of S near S = 0. We now define S as a function of P by the inverse relation from D = 0, taking the root near S = 0, or which may be made 0 for suitable values of P, with u arbitrary. Then S becomes a well defined function of (p,u). Also, for constant coefficients, from (5) of #218, or from the double Fourier transform (section C),

$$H_n = -(Q/\sigma C_0)\,\Delta_{0n}/D$$

$$G_n = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} H_n e^{su}\,du = -(Q/\sigma C_0)\frac{\bar{\Delta}_{0n}}{\left(\frac{\partial D}{\partial s}\right)_{D=0}} e^{s_1 u}$$

The quantities F_n considered above are defined similarly to this G_n, and with the change exp(su)→exp($\int$sdt) we find immediately.

$$(23) \qquad G_n = -\left[(Q/\sigma C_0)\frac{\bar{\Delta}_{0n}}{\frac{\partial D}{\partial s}}\right]_{D=0} e^{\int_0^t s\,dt}$$

With another Fourier inverse transformation one obtains

$$(24) \qquad F_n(z,u) = \frac{-1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} (Q/\sigma C_0)\left[\frac{\bar{\Delta}_{0n}}{\frac{\partial D}{\partial s}}\right]_{D=0} e^{t\left(\frac{1}{t}\int_0^t s\,dt'\right)+pz}\,dp$$

This may be evaluated with the aid of a saddle-point method.

E. Absorption and scattering, with isotropic scattering after each collision

We consider an infinite medium with macroscopic cross section for scattering σ_s and with total cross section $\sigma = \sigma_s + \sigma_a$. It is assumed that scattering is isotropic in the laboratory system to a sufficient approximation; if ξ is the average value of the change in log E for one scattering collision, the energy distribution after one collision is approximately*

$$f_1(u)\,du = \frac{1}{2\xi}\,du,\quad o \leq u \leq 2\xi, \quad = 0, \Big/ \begin{matrix} u < o \\ u > 2\xi \end{matrix}$$

and after n collisions

$$(25)\quad f_n(u)\,du, \quad f_n(u) = \int_o^u f_{n-1}(u')f_1(u-u')\,du', \quad o \leq u \leq 2\xi$$

hence

$$(26)\quad \psi_n(s) = \int_o^\infty f_n(u)\,e^{-su}\,du, \quad \psi_1(s) = e^{-\xi s}\,\frac{\sinh \xi s}{\xi s}$$

$$\psi_n(s) = e^{-n\xi s}\left(\frac{\sinh \xi s}{\xi s}\right)^n$$

One can obtain the distribution in space and in energy after n collisions by considering the (n-1)th distribution in space and in energy as the source function for the distribution after one more collision; the energy distribution results from that of the (n-1)st distribution according to the combination equation (25). Assuming that the numerical values of the cross sections do not change appreciably in any interval $(0 < u < 2\xi)$, let

*(A very similar problem was treated by Ericksson, On the Stationary Energy And Spacial Distribution of Neutrons in a Medium of Infinite Size, Archiv for Mat., As och Fysik, Band 34 B, N:o2.)

$F_n(z, \mu, u)$ be the flux nv after n collisions, while F_n^o represents F integrated over the unit sphere. One can employ the equation similar to the Boltzmann equation

$$\mu \frac{\partial F_n}{\partial z} + \sigma F_n = \frac{1}{4\pi} \sigma_s \bar{F}^o_{n-1} , \quad \mu \frac{\partial F_o}{\partial z} + \sigma F_o = \frac{1}{4\pi} \delta(u)\delta(z)$$

if $\bar{F}^o_{n-1}$ indicates F^o_{n-1} with f_{n-1} replaced by f_n. Let

$$G_n = \int_{-\infty}^{\infty} F_n \quad e^{-pz}\, dz$$

and assume

$$G_n = g_n\,(p,\mu,u) f_n(u)$$

The equation above becomes

$$(27) \qquad (\mu p + \sigma)\, g_n = \frac{1}{4\pi} \quad \sigma_s\, g^o{}_{n-1} , \quad (\mu p + \sigma)\, g_1 = \frac{\sigma s}{4\pi} \quad g^o{}_o$$

where $g^o_n = 2\pi \int_{-1}^{1} g_n\,(p,\mu,u)\, du$, $\quad g^o{}_o = a_o(p/\sigma)\,/\sigma$

From (27) dividing both sides by $(\mu\, p + \sigma)$ and integrating over the unit sphere,

where $\quad g^o_n = \frac{\sigma s}{\sigma} \quad a_o\,(p/\sigma)\, g^o_{n-1}$,

$$a_o\,(P) = \frac{1}{2P} \log \frac{1+P}{1-P} = \frac{1}{P} \tanh^{-1} P$$

Let $\quad \alpha = \log \left\{ \frac{\sigma_s}{\sigma} \quad a_o\,(p/\sigma) \right\}$

We have $\quad g^o_1 = \ell^{\alpha} g^o_o$

$$g^o_n = \ell^{n\alpha} g^o_o$$

and for the total flux

$$G^o = g_o^o \sum_1^\infty \ell^{n\alpha} f_n(u)$$

The approximate value of f_n for large n may be obtained as follows: Since

$$\psi_n(s) = \int_0^\infty f_n(u)\, e^{-su}\, du,$$

$$-\left.\frac{d\psi_n}{ds}\right|_{s=o} = \int_0^\infty f_n(u) \cdot u\, du = \bar{u}_n\, \psi_n(0)$$

$$\left.\frac{d^2\psi_n}{ds^2}\right|_{s=o} = \int_0^\infty f_n(u)\, u^2\, du = \overline{u_n^2}\, \psi_n(0)$$

$$\bar{u}_n = -\left.\frac{d\psi_n}{ds}\right|_{s=o} = -\frac{d}{ds}\left[\ell^{-ns\xi}\left(\frac{\sin \xi s}{s\xi}\right)^n\right]_{s=o} = n\xi$$

Also,

$$\overline{u_n^2} = (n\xi)^2 + \frac{n}{3}\xi^2$$

and the standard deviation results from the identity.

Hence $\sigma^2 = \frac{n}{3}\xi^2 = \overline{u_n^2} - \bar{u}_n^2$

$$f_n(u) \cong \frac{\ell^{-(u-n\ )^2\ 2\left(\frac{n\ ^2}{3}\right)}}{\sqrt{2\pi}\ \sqrt{\frac{n}{3}\ \xi^2}}$$

and the total flux becomes

$$G^o = \frac{g_o^o}{\xi\sqrt{\frac{2}{3}\pi}} \sum_n n^{-\frac{1}{2}} \ell^{\,n\alpha - (u-n\xi)^2 / \frac{2}{3} n \xi^2}$$

When n is large, the sum can be replaced by an integral which may be evaluated with the aid of Campbell's tables, #807 (Collected Works of G. A. Campbell).

Comparing

$$G^o \cong \sqrt{\frac{3}{2\pi}}\ \frac{g_o^o}{\xi}\ e^{\frac{3u}{\xi}} \int_0^\infty e^{-\left(\frac{3}{2}-\alpha\right)n - \frac{3u^2}{2\xi^2}\frac{1}{n}} \frac{dn}{\sqrt{n}}$$

with the integral

$$\frac{e^{-\sqrt{\sigma p}}}{\sqrt{p}} = \int_0^\infty e^{-pg - \frac{\sigma}{4g}} \frac{dg}{\sqrt{\pi g}}$$

we obtain

$$G^o = \sqrt{\frac{3}{2}}\ \frac{g_o^o}{\xi}\ e^{\frac{3u}{\xi}}\ \frac{e^{-\frac{2u}{\xi}\sqrt{\frac{3}{2}\left(\frac{3}{2}-\alpha\right)}}}{\sqrt{\frac{3}{2}-\alpha}}$$

$$= \frac{g_o^o}{\xi}\ \frac{e^{\frac{3u}{\xi}\left[1-\sqrt{1-\frac{2}{3}\alpha}\right]}}{\sqrt{1-\frac{2}{3}\alpha}}$$

With $g_o^o = e^{\alpha}/\sigma_s$, we find for nv,

$$F^o = \frac{1}{\xi\sigma_s}\ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{pz+\alpha+\frac{3u}{\xi}\left[1-\sqrt{1-\frac{2}{3}\alpha}\right]} \frac{dp}{\sqrt{1-\frac{2}{3}\alpha}}$$

$$= \frac{1}{\xi\sigma_s}\ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{pz+\frac{4}{3}\alpha+\frac{3u}{\xi}\left[1-\sqrt{1-\frac{2}{3}\alpha}\right]} dp.$$

If this is evaluated by the method of steepest descent, p is determined from the equation ($\alpha \ll 1$)

$$\frac{\partial}{\partial p}\left[pz + \frac{4}{3}\alpha + \frac{3u}{\xi}\left[1 - \sqrt{1 - \frac{2}{3}\alpha}\right]\right] = 0$$

$$z + \frac{4}{3}\left(\frac{1}{a_o}\frac{\partial a_o}{\partial P}\right) + \frac{3u}{\xi\sqrt{1 - \frac{2}{3}\alpha}}\left(\frac{1}{3}\right)\left(\frac{1}{a_o}\frac{\partial a_o}{\partial P}\right) = 0$$

$$-z\sigma P = \left(\frac{4}{3} + \frac{u}{\xi\sqrt{1 - 2\alpha/3}}\right)\frac{P}{a_o}\frac{\partial a_o}{\partial P}, \quad P = \frac{p}{\sigma} < 0.$$

Substituting in the exponent in the integrand, with

$$1 - \sqrt{1 - \frac{2}{3}\alpha} = \alpha/3$$

we have

$$e^{\frac{4}{3}\alpha + \frac{u}{\xi}\alpha} = \left(\frac{4}{3} + \frac{u}{\xi}\right)\left(\frac{P}{a_o}\frac{\partial a_o}{\partial P}\right)$$

$$= e^{\left(\frac{4}{3} + \frac{u}{\xi}\right)}\left[\alpha - \frac{P}{a_o}\frac{\partial a_o}{\partial P}\right]$$

Near $\alpha = 0$

$$\alpha = A_o(P - P_o) - \frac{A'_o}{2}(P - P_o)^2,$$

$$-z\sigma = -\left(\frac{4}{3} - \frac{u}{\xi}\right)A \qquad A = \frac{1}{a_o}\frac{da_o}{dP} \quad P > 0$$

F. Application of a method of Wick* in case the scattering from a heavy element can be represented by the diffraction formula $\left[RJ_1(kR \sin\theta)/\sin\theta\right]^2$.

*G. C. Wick, On the Space Distribution of Slow Neutrons, P. R., Vol. 75, p. 745 (1949)

The formulas to be given apply only to the case of constant cross sections; with total cross section assumed unity, the double Fourier transform with respect to the space variable of equation (3) yields

$$(28)\quad (1+\mu p)K = \frac{1}{4\pi} + \frac{1}{4\pi}\sum C_n K_n P_n(\mu)$$

Multiplying each side by $P_n(\mu)$ and integrating over the unit sphere gives

$$K_0 + pK_1 = Q + C_0 K_0$$

$$(29)\quad \frac{np}{2n+1}K_{n-1} + K_n + \frac{(n+1)p}{2n+1}K_{n+1} = C_n K_n/(2n+1)$$
$$= K_n\left[\sum\left(\frac{\sigma_{M'}}{\sigma}\right)c_n^{(M')} + \sum\left(\frac{\sigma_{M''}}{\sigma}\right)h_n^{(M'')}\right]$$

Formulas for $h_n^{(M'')}$ were derived in B; it remains to derive approximate formulas for the quantities

$$c_n^{(M')} = \frac{1}{\pi}\iint P_n(Z)\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2 d\Omega \qquad k = \sqrt{2mE}/\hbar$$

which is the same as evaluating K_n in equation (1) with

$$\sigma_M = \pi R^2, \quad g_M = 2\pi\frac{\partial Z}{\partial u'}\left[\left(\frac{1}{\pi}\right)\left(\frac{J_1(kR\sin\theta)}{\sin\theta}\right)^2\right]$$

Here it is assumed that $K_n(p,u')$ retains approximately the same value as u' varies over the range $u-a_M \leq u' \leq u$, which is justified if M is very large.

Now if the product kR is large, one can write for small angles

$$\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2 \doteq \frac{(kR)^2}{4\pi}e^{-\left[(kR)^2/4\right]\sin^2\theta}$$
$$\doteq \frac{(kR)^2}{4\pi}e^{-\frac{(kR)^2}{2}}\, e^{+\frac{(kR)^2}{2}\cos\theta}$$
$$= \frac{kR}{4\sqrt{\pi}}e^{-\frac{(kR)^2}{2}}\sum(2m+1)I_{m+\frac{1}{2}}\left(\frac{k^2R^2}{2}\right)P_m(Z)$$

from which

$$(30)\quad C_n^{(M')} \doteq \sqrt{\pi}\, kR\, e^{-\frac{(kR)^2}{2}} I_{n+\frac{1}{2}}\left(\frac{k^2R^2}{2}\right) \cong e^{-\frac{(n+\frac{1}{2})^2}{k^2R^2}}, \quad \frac{2n}{k^2R^2} << 1.$$

Hence in case a large number of equations (29) have to be solved, writing $y_n = 1-C_n/(2n+1)$, one obtains Wick's equation

$$\frac{d^2K_n}{dw^2} + \frac{1}{w}\frac{dK_n}{dw} + (2 - 2p^{-1}y_n)K_n = 0$$

with p determined from the boundary conditions K_n bounded, $\lim_{n\to\infty} K_n = 0$.

The asymptotic expansion of the Bessel functions may be derived from the formula

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(1/2)} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} e^{zt}\, dt$$

(Watson, Theory of Bessel Functions, p. 79) for large n, with the ratio n/z constant,

$$I_{n+\frac{1}{2}}(z) \sim \frac{t_0^{n+1}}{\sqrt{2\pi z}} \frac{e^{z\sqrt{1+\alpha^2}}}{\sqrt{1-\alpha t_0}} \qquad \alpha = n/z, \quad t_0 = \sqrt{1+\alpha^2} - \alpha.$$

Employing the expansions of all Bessel functions involved, one obtains for each $\alpha << 1$,

$$\frac{C_n}{2n+1} = \sum_{M''} \left(\frac{\sigma_{M''}}{\sigma}\right)\left(\frac{M''+1}{M''}\right)^2 \frac{e^{-\frac{1}{2}(n+\frac{1}{2})^2/X_{M''}}}{2X_{M''}} + \sum_{M'} \left(\frac{\sigma_{M'}}{\sigma}\right) e^{-\left(\frac{n+\frac{1}{2}}{kR}\right)^2}$$

When this expression is introduced on the right of (28), it is seen that for $n + \frac{1}{2} \angle\angle kR$, the contribution of the diffraction scattering is nearly constant; if G can be represented by a small number of terms, the diffraction scattering terms on both sides of the equation cancel, and one is left only inelastic scattering, considered as absorption, and normal collisions with other scattering nuclei. This problem can be solved by other methods, giving the ordinary diffusion length for the total energy, with total cross section obtained by omitting the diffraction scattering terms. The condition $n \angle kR$, or that a small number of spherical harmonics will be sufficient, requires that the ratio (inelastic scattering cross section) divided by (total cross section of normal scatters) be not too large. If σ' is the total cross section minus the diffraction scattering cross section, the argument used above does not apply when the value of p from the diffusion equation $a_0(p/\sigma') = \sigma'/\sigma_s$ is so close to σ' that functions K_n $(n + \frac{1}{2} > kR)$ are not negligibly small compared to those for which this inequality does not hold, since K_n is proportional to $a_n(p/\sigma')$.

For lead at 10 Mev the value of kR is nearly 6; in those cases in which a large number of spherical harmonics need be considered, for the fission spectrum with inelastic scatters present, the characteristic numbers and functions of the differential equation need detailed examination.

The case of variable cross sections

Here it is convenient to start with equation (3) and definitions (1) and (2); each component of the expression $\sigma C_n F_n$ is represented by an integral over σF_n which is a function of u', multiplied by a function of u-u'; taking the double Fourier transform of both sides of (3) one obtains

on the right

$$(31)\quad T_{p,s}(\sigma C_n 7_n) = (2n+1)\left[\sum C_n^{(M')} T_{p,s}(\sigma_{M'} 7_n) + \sum T_{p,s}(\sigma_{M''} 7_n)\, T_s\left\{q P_n(Z_{M''})\right\}\right]$$

where

$$T_s\left\{q P_n(Z_{M''})\right\} \cong \left(\frac{M+1}{M}\right)^2 e^{-\frac{1}{2}(n+\frac{1}{2})^2} / X_{M''} / 2X_{M''}$$

Instead of (29) we obtain the system (p = -k)

$$(32)\quad -\frac{nk}{2n+1} T_{p,s}(7_{n-1}) + T_{p,s}(\sigma 7_n) - \frac{(n+1)k}{2n+1} T_{p,s}(7_{n+1}) = \frac{1}{2n+1} T_{p,s}(\sigma C_n 7_n).$$

For small u,

$$\sigma = \sigma(0) + u\,\sigma'(0),$$

hence

$$T_{p,s}(\sigma 7_n) = \int_0^\infty e^{-su}(\sigma_0 + u\sigma_0')\, T_p(7_n)\, du$$

$$= \sigma_0\left[T_{ps}(7_n) - \frac{\sigma'}{\sigma_0}\frac{d}{ds} T_{ps}(7_n)\right]$$

If the terms of (31) are expanded in the same manner, the right-hand side of (32) becomes ($P = k/\sigma_0$)

$$-\frac{nP}{2n+1} K_{n-1} + K_n - \frac{(n+1)P}{2n+1} K_{n+1} = \left[\sum C_n^{(M')}\left(\frac{\sigma_{M'}^{\circ}}{\sigma_0}\right) + \sum\left(\frac{\sigma_{M''}^{\circ}}{\sigma_0}\right) T_s\left(q P_n(Z_{M''})\right)\right] K_n$$

$$+\left[\left(\frac{\sigma_0'}{\sigma_0}\right) - \sum C_n^{(M')}\left(\frac{\sigma_{M'}'}{\sigma_0}\right) - \sum\left(\frac{\sigma_{M''}'}{\sigma_0}\right) T_s\left\{q P_n(Z'')\right\}\right] \times \frac{dK_n}{ds}$$

Even in the case of hydrogen the expression for $T_s(gP_n(Z_1)$ contains a factor $1/x_1$; if all these terms are neglected in first approximation, for x_M^{-1} small, there remains

$$T_{ps}(\sigma c_n \mathcal{F}_n) = (2n+1) \sum c_n^{(M')} T_{ps}(\sigma_{M'} \mathcal{F}_n).$$

Now $c_n^{(M')}$ is independent of $x_{M'}$; hence one can immediately take the inverse Laplace transform of this term, and of the right member of (3), also in (32), to give the system corresponding to (32) with coefficients depending on u, for small u. In this approximate system the normal scatterers appear as absorbers. It results that when $T_p(F_n) = K_n$, $p/\sigma = -k$, one obtains as before

$$\frac{d^2 K_n}{dw^2} + \frac{1}{w}\frac{dK_n}{dw} + (2 - 2k^{-1} y_n) K_n = 0$$

For small u, one can expand $(\sigma_{M'}/\sigma)$ in powers of u and apply a perturbation calculation. Without this calculation, a particular approximate solution of the problem is obtained by determining k as function of u over the entire range of u, from a solution of the eigenvalue problem for the preceding equation, or from a solution of the determinant equation $D(k,u) = 0$, where D is defined under (7.1). With the employment of the Fourier transform with respect to the space variable alone, this method is similar to that described in ORNL 218, where k is determined from $D(k,u) = 0$.

Methods of calculating perturbations undoubtedly vary considerably with the mixture in a given shield, and these will be considered later.

It is clear that the scattering is overestimated (for a given energy loss) if in the expression $T_{ps}(gP_n(Z))$ the exponential function is replaced by unity. One can then proceed as in the last section of ORNL 218, solving first for the functions $F_n \exp(u)$. This has the effect of replacing $s+1$ by s in the definitions of $x_{M''}$. Multiplying through by the parameter s has the

effect everywhere of replacing the transforms $T_{ps}(f)$ by $sT_{ps}(f) = T_{ps}(df/du)$, except in the last terms of (31). One can then take the inverse Laplace transforms of both sides of the equations, to obtain equations involving only the Fourier transform with respect to the space variable z. For a mixture of a heavy element and water, the hydrogen terms on the right cancel with terms on the left, as in the case of no diffraction scattering, and in principle the solution of the equations for quantities $d/du(F_n \exp(u))$ can be completed, if σ_H is of the form $b\exp(u)$. This approach corresponds to forward scattering in collisions with hydrogen nuclei; the characteristic number k and the characteristic function is the same as if a collision with a hydrogen nucleus were equivalent to absorption.