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ASYMPTOTIC DISTRIBUTION OF NEUTRONS FROM A POINT SOURCE IN A MEDIUM CONTAINING HYDROGEN AND AN ISOTROPIC SCATTERER

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ASYMPTOTIC DISTRIBUTION OF NEUTRONS FROM A POINT SOURCE IN A MEDIUM CONTAINING HYDROGEN AND AN ISOTROPIC SCATTERER

T. A. Welton - G. Goertzel

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Asymptotic Distribution of Neutrons From a Point Source

In a Medium Containing Hydrogen and an Isotropic Scatterer

T. A. Welton - G. Goertzel

It has apparently been pointed out by many authors[1] that in neutron stopping calculations the effect of hydrogen can, to good approximation, be taken as a slowing of the neutrons without angular deflection. A simple justification for this assumption can be given. A neutron which suffers a small fractional energy loss in a hydrogen collision is deviated only by a small angle. A neutron which is deviated by a large angle ($>\frac{\pi}{4}$, say) loses a large amount of energy, and because of the general trend toward larger cross-sections at lower energies, stops quickly without contributing in an important way to the transmission. This assumption (in future referred to as the "straight-ahead" assumption) will be here accepted without check, although a test of its validity is not difficult, and certainly important.

Greuling[2] has pointed out that there exists a family of energy dependences for the hydrogen cross-section which lead to analytical solutions of the "straight-ahead" problem in terms of the confluent hypergeometric function. Young[3] has given a useful summary of the available knowledge on this point.

If hydrogen is combined with material which scatters isotropically (approximately), with or without true inelastic scattering, we obtain an interesting shielding material. We may expect that the "straight-ahead" approximation will here be much better than in the case of pure hydrogen, since any small hydrogen scattering will be masked by isotropic deflections . It is then to be hoped

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1) Wigner C-137, Mon P-283

2) Mon P-172

3) Mon P-293

that a simple treatment of such a material can be given by combining the "straight-ahead" approximation for hydrogen with some simple treatment of the isotropic processes. As a first step in such a program, G. Young[4] has proposed the treatment of a material composed of hydrogen plus an isotropic, elastic scatterer, whose cross-section is independent of energy. Under these conditions, the processes of degradation by hydrogen and diffusion through the material are independent in the following sense. Let $S(E,\ell)dE$ be the number of neutrons in the energy range dE, at a distance ℓ from a point source of neutrons in pure hydrogen of the same atomic concentration as prevailing in the actual shield material. Now consider a problem in which a plane, isotropic source emits neutrons at a steady rate into a medium consisting of the isotropically scattering nuclei, with the hydrogen removed. Let $P(x,\ell)d\ell$ be the density of these neutrons at a distance x from the source which have travelled a total path-length ℓ in the range $d\ell$ from the source. It is then clear that, because of the independence of scattering cross-section of energy, the resultant density of neutrons in space and energy at a distance r from the source is given by:

$$\psi(x, E) = \int_0^\infty d\ell \; P(x,\ell)\, S(E,\ell) \qquad (1)$$

The function P is of course zero for $\ell < x$.

The function P can only be calculated with difficulty, but its Laplace transform with respect to ℓ can be simply calculated in many situations. Imagine that a capture cross-section α is added to the isotropic scattering cross-section σ_s, and a unit plane isotropic source is present at $x = 0$. Neutrons which arrive at x with a path length ℓ will be attenuated by a

4) Discussion with F. H. Murray

factor $e^{-\alpha \ell}$. Therefore, if we define:

$$Q(x,\alpha) = \int_0^\infty d\ell \, e^{-\alpha \ell} \, P(x,\ell)$$

then $Q(x,\alpha)$ is just the neutron density at x which would exist in a medium with scattering cross-section σ_s, capture cross-section α, and a unit plane isotropic source at $x = 0$.

Murray[5] has proposed an ingenious method for calculating $P(x,\ell)$. Consider an isotropic plane source (in the pure scattering material) which emits a single neutron per unit area with unit velocity, at time zero. The density of neutrons, $N(x, t)$, at time t and distance x is just $P(x,t)$. This must be true to within a constant factor since time of flight and path length are directly connected (are in fact equal, with unit velocity). To show that the factor is unity, we note that the time integral of $N(x, t)$ is just the total density at x which would result if one neutron were emitted per second per unit area. It is obvious from the definition of $P(x,\ell)$ that its integral over all ℓ must be just the total density at x, with the same steady source. Therefore $P(x,\ell) = N(x,\ell)$. The equivalence to the previous recipe is obvious as soon as one attempts to calculate the time dependent problem by the method of Laplace transformation.

We now calculate $S(E,\ell)$ and $P(x,\ell)$. The function $S(E,\ell)$ satisfies:

$$\frac{dS(E,\ell)}{d\ell} + \sigma_H(E)\, S(E,\ell) = \int_E^\infty \frac{dE'}{E'} \sigma_H(E')\, S(E',\ell) + \delta(E-E_0)\, \delta(\ell) \qquad (2)$$

Here $\sigma_H(E)$ is the hydrogen cross-section. The source term on the right gives

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5) CL-FHM-1

a unit point source at $\ell = 0$ with energy E_o. We write:

$$S(E,\ell) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega\ell}\, T(E,\omega) \tag{3}$$

Equation (2) becomes)

$$\left[\sigma_H(E) + i\omega\right] T(E,\omega) = \int_E^{\infty} \frac{dE'}{E'} \sigma_H(E')\, T(E',\omega) + \delta(E-E_o) \tag{4}$$

The solution of (4) is easily shown to be:

$$T(E,\omega) = \frac{\delta(E-E_o)}{\sigma_H(E_o) + i\omega} + \frac{1(E_o - E)\,\sigma_H(E_o)}{E_o\left[\sigma_H(E_o)+i\omega\right]\left[\sigma_H(E)+i\omega\right]}\, e^{\int_E^{E_o} \frac{dE'}{E'} \frac{\sigma_H(E')}{\sigma_H(E')+i\omega}} \tag{5}$$

where $1(E_o-E)$ is the usual unit step function.

It follows directly from the previously mentioned observation of Greuling that a simple expression can be obtained for $T(E,\omega)$ if σ_H varies as a power of E:

$$\sigma_H(E) = AE^{-1/n} \tag{6}$$

Here A and n are arbitrary constants, with n restricted to be a positive number (not necessarily an integer), if σ_H is to decrease with increasing energy. We take σ_H as the integration variable in (5):

$$-n\,\frac{d\sigma_H}{\sigma_H} = \frac{dE}{E}$$

$$\int_E^{E_o} \frac{dE'}{E'} \frac{\sigma_H(E')}{\sigma_H(E')+i\omega} = n\int_{\sigma_H(E_o)}^{\sigma_H(E)} \frac{d\sigma_H}{\sigma_H+i\omega} = \log\left[\frac{\sigma_H(E)+i\omega}{\sigma_H(E_o)+i\omega}\right]^n \tag{7}$$

We then have

$$T(E,\omega) = \frac{\delta(E-E_o)}{\sigma_H(E_o)+i\omega} + \frac{1(E_o-E)\sigma_H(E_o)}{E_o} \frac{\sigma_H(E)+i\omega}{\sigma_H(E_o)+i\omega}^{n-1} \tag{8}$$

The Fourier inversion (3) can then be obtained in terms of the confluent hypergeometric function, which has extremely convenient expansions in powers of x and of $\frac{1}{x}$. For our purposes, however, we shall restrict our attention to the cases $n = 1$ or 2, since these yield reasonable $\left(\frac{1}{E} \text{ or } \frac{1}{\sqrt{E}}\right)$ representations of σ_H and simplify the further analysis considerably. For $n = 1\left(\frac{1}{E}\right)$:

$$T(E,\omega) = \frac{\delta(E-E_o)}{\sigma_H(E_o)+i\omega} + \frac{1(E_o-E)\sigma_H(E_o)}{E_o} \frac{1}{\left[\sigma_H(E_o) + i\omega\right]^2}$$

$$S(E,\ell) = 1(\ell)\left\{\delta(E-E_o) - \frac{1(E_o-E)\sigma_H(E_o)}{E_o} \frac{\partial}{\partial\sigma_H(E_o)}\right\} e^{-\sigma_H(E_o)\ell} \tag{9}$$

For $n = 2\left(\frac{1}{\sqrt{E}}\right)$:

$$T(E,\omega) = \frac{\delta(E-E_o)}{\sigma_H(E_o)+i\omega} + \frac{1(E_o-E)\ \sigma_H(E_o)}{E_o\left[\sigma_H(E_o)+i\omega\right]^3}\left[\sigma_H(E)+i\omega\right]$$

$$= \frac{\delta(E-E_o)}{\sigma_H(E_o)+i\omega} + \frac{1(E_o-E)\sigma_H(E_o)}{E_o}\left\{\frac{1}{\left[\sigma_H(E_o+i\omega\right]^2} + \frac{\sigma_H(E)-\sigma_H(E_o)}{\left[\sigma_H(E_o)+i\omega\right]^3}\right\}$$

$$S(E,\ell) = 1(\ell)\left\{\delta(E-E_o) - \frac{1(E_o-E)\sigma_H(E_o)}{E_o} \frac{\partial}{\partial\sigma_H(E_o)} + \frac{1(E_o-E)\sigma_H(E_o)\left[\sigma_H(E)-\sigma_H(E_o)\right]}{E_o} \frac{\delta^2}{\delta\sigma_H^2(E_o)}\right\} e^{-\sigma_H(E_o)\ell} \tag{10}$$

Now, referring to equation (1), we have for the final energy and space density of neutrons at distance x from the source:

$$\psi(x,E) = \int_0^\infty d\ell\, P(x,\ell) S(E,\ell)$$

$$= F\left(E, E_0 \frac{\partial}{\partial \sigma_H(E_0)}\right) \int_0^\infty d\ell\, P(x,\ell)\, e^{-\sigma_H(E_0)\ell}$$

$$= F\left(E, E_0 \frac{\partial}{\partial \sigma_H(E_0)}\right) Q(x\sigma_H(E_0) \tag{11}$$

where F is the differential operator appearing in (9) or (10), and Q is the Laplace transform of P, previously defined.

Our only remaining problem is to determine Q. From the arguments given earlier, $Q(x,\alpha)$ is related to the solution of the following plane transport problem.

$$\mu \frac{\partial N(x,\mu\alpha)}{\partial x} + \sigma_s N + \alpha N = \frac{\sigma_s}{2} \int_{-1}^{1} d\mu' N(x,\mu'\alpha) + \tfrac{1}{2}\delta(x) \tag{12}$$

In order to find Q, we integrate N over μ to obtain a density in the plane problem.

That is:

$$Q(x,\alpha) = \int_{-1}^{1} d\mu\, N(x,\mu,\alpha) \tag{13}$$

This is the standard problem of a plane isotropic source in a homogeneous medium which has isotropic elastic scattering cross-section σ_s and capture cross-section α. We sketch the solution. Write:

$$N(x,\mu,\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{R(k,\alpha)}{\sigma_s+\alpha+ik\mu} \tag{14}$$

Equation (12) becomes:

$$R(k,\alpha) = \frac{\sigma_s}{k}\left(\tan^{-1}\frac{k}{\sigma_s+\alpha}\right)R(k,\alpha) + \frac{1}{2}$$

$$R(k,\alpha) = \frac{\frac{1}{2}}{1-\frac{\sigma_s}{k}\tan^{-1}\frac{k}{\sigma_s+\alpha}} \tag{15}$$

The function Q becomes:

$$Q(x,\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{1}{1-\frac{\sigma_s}{k}\tan^{-1}\frac{k}{\sigma_s+\alpha}}\; \frac{1}{2}\int_{-1}^{1}\frac{d\mu}{\sigma_s+\alpha+ik\mu}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{1}{\frac{k}{\tan^{-1}\frac{k}{\sigma_s+\alpha}} - \sigma_s} \tag{16}$$

This function has been studied by several authors. For small x, it yields the first collision density, while as x increases the solution goes over into a diffusion type density. The diffusion density arises from the pole of the integrand of (16) in the upper half-plane, while the solution for small x arises from the integral along the cut required to make the integrand single-valued. We make the explicit assumption that x is sufficiently large so that the pole contribution dominates the cut contribution. The condition on x for this to be true will be examined later.

The pole is at $k = iK$, with:

$$\frac{K}{\tanh^{-1}\dfrac{K}{\sigma_s + \alpha}} = \sigma_s \tag{17}$$

Evaluating the residue, we obtain:

$$Q(x,\alpha) = \frac{K}{\sigma_s} \frac{e^{-Kx}}{\dfrac{\sigma_s/\sigma_s+\alpha}{1 - \dfrac{K^2}{(\sigma_s+\alpha)^2}} - 1} \tag{18}$$

Combining our results thus far, the final answer is to be written:

$$\psi(x,E) = F\left(E, E_o, \frac{\partial}{\partial\alpha}\right) \frac{K}{\sigma_s} \frac{e^{-Kx}}{\dfrac{\sigma_s/\sigma_s+\alpha}{1 - \dfrac{K^2}{(\sigma_s+\alpha)^2}} - 1} \tag{19}$$

with α set equal to $\sigma_H(E_o)$ after the performance of the indicated differentiations. It must be remembered that K is a function of α, in performing the differentiations.

It is now desirable to indicate several generalizations of the previous work. First, a constant, real, capture cross-section (which may be introduced to simulate the effect of inelastic scattering by heavy elements) is simply treated by evaluating (19) for $\alpha = \sigma_H(E_o) + \sigma_c$. A variable capture cross-section can be taken into account by modifying the function $S(E,\ell)$ appropriately.

Second, n need not be taken to be an integer. The function F is then a non-terminating power series, which converges for all values of the argument.

Its properties are well known (see Reference 3), and it possesses an asymptotic expansion which (for n not unduly large) is extremely convenient to use. If n is of order unity and $\left[\sigma_H(E) - \sigma_H(E_o)\right] x$ is somewhat larger than unity, then:

$$F(E,E_o,x) = \frac{1}{\Gamma(n)} \frac{\sigma_H(E_o)}{E_o} \left[\sigma_H(E) - \sigma_H(E_o)\right]^{n-1} x^n \left\{1+O\left(\frac{1}{x}\right)\right\} \tag{20}$$

It is clear (by consideration of the cases of integral n, for example) that for large x, the dominant term in (19) is obtained by taking all differentiations to operate on the exponential. The resulting series will have for its asymptotic form just:

$$\psi(x,E) = \frac{1}{\Gamma(n)} \frac{\sigma_H(E_o)}{E_o} \left[\sigma_H(E) - \sigma_H(E_o)\right]^{n-1} \frac{K/\sigma_s}{\dfrac{\sigma_s/\sigma_s+\alpha}{1-\dfrac{K^2}{(\sigma_s+\alpha)^2}} - 1} \left(\frac{\partial k}{\partial \alpha}x\right)^n \cdot e^{-kx} \tag{21}$$

A further useful extension is to include the effect of anisotropic elastic scattering by recalculating the function $Q(x,\alpha)$ with the appropriate scattering law. This is quite simple to carry out if only the "pole" contribution to the solution is required, and if a few spherical harmonics will suffice in the expansion of the anisotropic differential cross-section. This has not been done, but seems capable of giving, in a simple way, reliable information on the effect of "shadow" scattering, for example.

It is well to examine the conditions under which the "pole" solution

will suffice. This question amounts to comparing the first collision density from the source with the expression (18). The first collision density (and in fact the exact pole contribution) vanishes more rapidly than exponentially, in fact just as the exponential integral with argument equal to $(\sigma_s + \alpha)\, x$. $Q(x, \alpha)$ varies exponentially with $K < \sigma_s + \alpha$. The coefficient of the exponential in Q is less than unity and becomes very small as $\frac{\alpha}{\sigma_s + \alpha}$ approaches unity. If we call $\frac{\alpha}{\sigma_s + \alpha} = f$, for any f (in the range zero to unity) and sufficiently large x, the pole solution will dominate. For $f = .5$, this will occur at an x of approximately one mean free path, while for $f = .6$, x must be about eight mean free paths. The situation rapidly becomes unmanageable past this point.

For water at 10 Mev, f is so large that the "pole" contribution is completely useless at any reasonable distance, and Q must therefore be written in terms of a few collision densities. The first collision density is simple, and a few more terms are probably manageable. With this treatment, the "relaxation length" will be very nearly the total mean free path at the source energy, and the general picture will be very similar to the common approximation to the slowing kernel for water by means of a transport kernel with a Gaussian correction.

Finally, it is useful to indicate the order of magnitude of the error incurred by the use of the "straight-ahead" approximation. From some work of Wick, it is possible to show that for large ℓ, the function S should go as $\ell^{n'} e^{-\sigma_H(E_0)\ell}$, where n' would be n if there were no angular

effects. Following Wick, it can be shown that for 8 Mev neutrons, n' is about 0.83, whereas n is 1.43 .

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