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ORNL SUMMER SHIELDING SESSION

Gale Young, Leader

ASYMPTOTIC NEUTRON DISTRIBUTION IN A THREE COMPONENT MIXTURE WITH FORWARD HYDROGEN SCATTERING

J. W. BUTLER

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This report is one of a series on shielding and represents part of the work of a summer group working principally at Oak Ridge National Laboratory. Participants are drawn from Nuclear Development Associates (Navy contract), NEPA, Bureau of Ships and Bureau of Aeronautics (Navy Department), AEC, RAND, General Electric, Argonne and Oak Ridge National Laboratory.

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ASYMPTOTIC NEUTRON DISTRIBUTION IN A THREE COMPONENT MIXTURE WITH FORWARD HYDROGEN SCATTERING

J. W. Butler

I. Introduction

This report presents the results of an attempt to add to the generality of neutron attenuation problems which can be discussed with the aid of the assumption of straight ahead scattering by hydrogen. By the use of the Laplace transform and real Tauberian theorems it is found possible to determine the asymptotic space variation of the collision density with quite general energy dependence of the three cross sections involved. Simple bounds are also derived for the asymptotic spectral distribution function.

II. Formulation of the Problem

The problem treated is the ordinary one-dimensional one, referring to an isotropic, monoenergetic, plane source of unit strength imbedded in an infinite homogeneous medium, the medium consisting of three components:

1. "hydrogen" which scatters without angular deviation but with the correct energy spectrum after collision;
2. a substance which scatters isotropically without energy loss; and
3. a pure absorber.

For the purposes of this discussion, all cross sections are assumed to be monotonic functions of energy, but this requirement can be relaxed to some extent by closer examination of the analysis.

The appropriate form of the Boltzmann equation is

$$\frac{\mu}{\sigma(u)}\frac{\partial\Psi}{\partial x} + \Psi(x,\mu,u) = \int_0^u d\eta\,\frac{\sigma_1(\eta)}{\sigma(\eta)}\,e^{(\eta-u)}\Psi(x,\mu,\eta) + \frac{1}{2}\frac{\sigma_2(u)}{\sigma(u)}\Psi_0(x,u) + \frac{1}{2}\delta(x)\delta(u), \tag{1}$$

where

$\Psi(x,\mu,u)$ is the collision density per unit range of x, μ, and u;

$$\Psi_o(x,u) = \int_{-1}^{+1} d\mu\, \Psi(x,\mu,u);$$

$\sigma(u)$ is the total collision cross section in units of $\left(\frac{1}{\text{length}}\right)$;

$\sigma_1(u)$ - the "hydrogen" scattering cross section;

$\sigma_2(u)$ - the cross section of the isotropic scatterer'

$\delta(\cdot)$ - the Dirac delta-function;

x - the space coordinate, measured from the source plane;

μ - the direction cosine measured from the forward x direction;

$u = \log \frac{E_o}{E}$;

E is the neutron energy; and

E_o - the energy of the source neutrons.

III. Determination of the General Asymptotic Solution.

It is convenient first to separate the singular part of the solution of (1). This is effected by the representation

$$\Psi(x,\mu,u) = \Psi^{(o)}(x,\mu)\, \delta(u) + \Psi^{(ro)}(x,\mu,u) \tag{2}$$

which implies, upon integration over μ,

$$\Psi_o(x,u) = \Psi_o^{(o)}(x)\, \delta(u) + \Psi_o^{(ro)}(x,u). \tag{3}$$

Substituting (2) into (1) it is seen that $\Psi^{(o)}$ is the solution of the mono-energetic transport equation

$$\frac{\mu}{\sigma(o)} \frac{\partial \Psi^{(o)}}{\partial x} + \Psi^{(o)}(x,\mu) = \frac{1}{2} \frac{\sigma_2(o)}{\sigma(o)} \Psi_o^{(o)}(x) + \frac{1}{2}\, \delta(x), \tag{4}$$

while $\Psi^{(ro)}$ satisfies

$$\frac{\mu}{\sigma(o)} \frac{\partial \Psi^{(ro)}}{\partial x} + \Psi^{(ro)}(x,\mu,u) = \int_0^u d\eta \, \frac{\sigma_1(\eta)}{\sigma(\eta)} \, e^{(\eta-u)} \Psi^{(ro)}(x,\mu,\eta) + \frac{1}{2} \frac{\sigma_2(u)}{\sigma(u)} \Psi_o^{(ro)}(x,u)$$

$$+ \frac{\sigma_1(o)}{\sigma(o)} \Psi^{(o)}(x,\mu) \, e^{-u} \, . \tag{5}$$

Although the solution of (4) is well-known, its Laplace transform will now be derived for use in the subsequent discussion. Taking the bilateral Laplace transform of both sides results in

$$\left(s \frac{\mu}{\sigma(o)} + 1\right) \varphi^{(o)}(s,\mu) = \frac{1}{2} \frac{\sigma_2(o)}{\sigma(o)} \varphi_o^{(o)}(s) + \frac{1}{2} \, , \tag{6}$$

where

$$\varphi^{(o)}(s,\mu) = \int_{-\infty}^{+\infty} dx \, e^{-sx} \, \Psi^{(o)}(x,\mu)$$

and

$$\varphi_o^{(o)}(s) = \int_{-\infty}^{+\infty} dx \, e^{-sx} \Psi_o^{(o)}(x) = \int_{-1}^{+1} d\mu \, \varphi^{(o)}(s,\mu) .$$

Equation (6) yields, after a slight amount of manipulation,

$$\varphi_o^{(o)}(s) = \frac{\sigma(o)}{\sigma_2(o)} \frac{k(s,o)}{1 - k(s,o)} \tag{7}$$

and

$$\varphi^{(o)}(s,\mu) = \frac{\sigma(o)}{2} \frac{1}{\left(s\mu + \sigma(o)\right)\left(1 - k(s,o)\right)} \, , \tag{8}$$

where the definition

$$k(s,u) = \frac{\sigma_2(u)}{2s} \log \frac{s + \sigma(u)}{-s + \sigma(u)} = \frac{\sigma_2(u)}{s} \operatorname{arth} \frac{s}{\sigma(u)} \tag{9}$$

has been used.

Returning now to equation (5), the substitutions

$$\Psi^{(ro)}(x,\mu,u) = \Phi(x,\mu,u)\ \sigma(u)e^{-u}$$

and

$$\Psi_o^{(ro)}(x,u) = \Phi_o(x,u)\ \sigma(u)e^{-u} = \int_{-1}^{+1} d\mu\ \Phi(x,\mu,u)\ \sigma(u)e^{-u} \tag{11}$$

cause this equation to assume the simpler appearing form:

$$\mu \frac{\partial\Phi}{\partial x} + \sigma(u)\ \Phi(x,\mu,u) = \int_0^u d\eta\ \sigma_1(\eta)\ \Phi(x,\mu,\eta) + \frac{1}{2}\ \sigma_2(u)\ \Phi_o(x,u) + \frac{\sigma_1(o)}{\sigma(o)}\ \Psi^{(o)}(x,\mu). \tag{12}$$

Upon taking the bilateral Laplace transform of equation (12) and inserting the expression (8), the equation

$$\big(s\mu + \sigma(u)\big)\varphi(s,\mu,u) = \int_0^u d\eta\ \sigma_1(\eta)\ \varphi(s,\mu,\eta) + \frac{1}{2}\ \sigma_2(u)\varphi_o(s,u) + \frac{1}{2}\ \sigma_1(o)\ \frac{1}{\big(1-k(s,o)\big)\big(s\mu+\sigma(o)\big)} \tag{13}$$

is obtained, where

$$\varphi(s,\mu,u) = \int_{-\infty}^{+\infty} dx\ e^{-sx}\ \Phi(x,\mu,u)$$

and

$$\varphi_o(s,u) = \int_{-\infty}^{+\infty} dx\ e^{-sx}\ \Phi_o(x,u) = \int_{-1}^{+1} d\mu\ \varphi(s,\mu,u).$$

It will now be assumed that the transform $\varphi_o(s,u)$ has a finite and non-zero convergence strip and, until stated otherwise, that s has a real value interior to this strip. Under these conditions, it is permissible to differentiate equation (13) with respect to u, leading to the equations

$$\frac{\partial\varphi}{\partial u} + \frac{\sigma'(u)-\sigma_1(u)}{s\mu + \sigma(u)}\varphi(s,\mu,u) = \frac{1}{2}\frac{\frac{\partial}{\partial u}\left(\sigma_2(u)\varphi_o(s,u)\right)}{s\mu + \sigma(u)} \tag{14}$$

and

$$\varphi(s,\mu,o) = \frac{\sigma_2(o)}{2}\frac{\varphi_o(s,o)}{s\mu + \sigma(o)} \quad \frac{\sigma_2(o)}{2}\frac{1}{\left(1-k(s,o)\right)\left(s\mu+\sigma(o)\right)^2} \quad . \tag{15}$$

Formal solution of this system, regarding $\varphi(s,\mu,u)$ as the unknown function, yields, after some reduction,

$$\varphi(s,\mu,u) = \frac{1}{2}\int_0^u d\xi\left(\frac{1}{s\mu+\sigma(u)}\exp\int_0^u d\eta\,\frac{\sigma_1(\eta)}{s\mu+\sigma(\eta)}\right)\frac{\partial}{\partial\xi}\left(\sigma_2(\xi)\varphi_o(s,\xi)\right)$$

$$+\frac{\sigma_2(o)}{2}\frac{\varphi_o(s,o)}{s\mu+\sigma(u)}\exp\int_0^u d\eta\,\frac{\sigma_1(\eta)}{s\mu+\sigma(\eta)} + \frac{\sigma_1(o)}{2}\frac{1}{\left(1-k(s,o)\right)\left(s\mu+\sigma(u)\right)\left(s\mu+\sigma(o)\right)}\exp\int_0^u d\eta\,\frac{\sigma_1(\eta)}{s\mu+\sigma(\eta)}\,, \tag{16}$$

where the integral formula

$$\exp\int_\xi^u d\eta\,\frac{\sigma'(\eta)}{s\mu+\sigma(\eta)} = \frac{s\mu+\sigma(u)}{s\mu+\sigma(\xi)} \tag{17}$$

has been used. Partial integration of the integral term in equation (16), followed by integration of the resulting equation over $\mu(-1,+1)$, now results in the form

$$(1 - k(s,u))\,\varphi_o(s,u) = \int_0^u d\eta\,\Delta(s,u,\eta)\,\sigma_1(\eta)\,\sigma_2(\eta)\,\varphi_o(s,\eta) + \sigma_1(o)\,\frac{\Delta(s,u,o)}{1 - k(s,o)} \tag{18}$$

in which the definition (9) has been employed and the new definition

$$\Delta(s,u,\eta) = \frac{1}{2}\int_{-1}^{+1} d\mu\,\frac{1}{(s\mu + \sigma(u))(s\mu + \sigma(\eta))}\,\exp\int_\eta^u d\xi\,\frac{\sigma_1(\xi)}{s\mu + \sigma(\xi)} \tag{19}$$

has been made. A slight additional simplification may be effected by the substitution

$$\varphi_o(s,u) = \sigma_1(o)\,\frac{\Delta(s,u,o)}{(1 - k(s,o))(1 - k(s,u))}\,\chi(s,u) \tag{20}$$

and the new definition

$$H(s,u,\eta) = \frac{\Delta(s,u,\eta)}{\Delta(s,u,o)}\,\sigma_1(\eta)\,\sigma_2(\eta), \tag{21}$$

yielding the final form

$$\chi(s,u) = \int_0^u d\eta\,\frac{\Delta(s,\eta,o)}{1 - k(s,\eta)}\,H(s,u,\eta)\,\chi(s,\eta) + 1 \tag{22}$$

which will serve as a basis for most of the remaining discussion.

Two additional restrictions will now be placed on the cross section functions $\sigma(u)$ and $\sigma_2(u)$. These are

$$\sigma'(u) \geqslant 0 \qquad (u \geqslant 0) \tag{23}$$

$$\frac{\sigma'(u)}{\sigma(u)} - \frac{\sigma_2'(u)}{\sigma_2(u)} > 0 \qquad (u \geqslant 0) \tag{24}$$

and may be shown to be sufficient for the truth of the following statements about the functions $\Delta(s,u,\eta)$ and $k(s,u)$:

(a) $\Delta(s,u,\eta)$ and $\frac{\partial}{\partial\eta}\Delta(s,u,\eta)$ are continuous functions of s, u, and η on the set $(|s| < \sigma(o),\ u \geqslant 0,\ \eta \geqslant 0)$ and are therefore bounded on any interval $(|s| \leqslant \sigma(o) - \epsilon,\ \epsilon > 0)$ for finite values of u and η.

(b) $\Delta(s,u,\eta)$ and $\frac{\partial}{\partial\eta}\Delta(s,u,\eta)$ satisfy the inequalities

$$\Delta(s,u,\eta) > 0 \tag{25}$$

and

$$\frac{\partial}{\partial\eta}\Delta(s,u,\eta) < 0 \tag{26}$$

for $|s| < \sigma(o)$, $u \geqslant 0$, and $\eta \geqslant 0$.

(c) $k(s,u)$ and $\frac{\partial}{\partial u}k(s,u)$ satisfy the inequalities

$$0 < k(s,u) < 1 \tag{27}$$

and

$$\frac{\partial}{\partial u}k(s,u) < 0 \tag{28}$$

on the half strip $(|s| < s_o,\ u \geqslant 0)$, if $\sigma_2(u) \neq 0$.

(d) The equation

$$1 - k(s,o) = 0 \tag{29}$$

has simple roots at $s = \pm s_o$, where s_o (> 0) satisfies the inequality

$$s_o < \sigma(o). \tag{30}$$

(e) The equation

$$1 - k(-s_o,u) = 0 \tag{31}$$

has a simple root at $u = 0$.

If the symbol $J_H(\cdot)$ is used to denote the linear integral operator

$$J_H(f(u)) = \int_0^u d\eta\, \frac{\Delta(s,\eta,o)}{1 - k(s,\eta)}\, H(s,u,\eta)\, f(\eta), \tag{32}$$

then the Neumann solution of equation (22) is

$$\chi(s,u) = \sum_{n=0}^{\infty} J_H^n(1) \tag{33}$$

where $J_H^n(\cdot)$ is the nth iterate of the operator (32). By making use of (a) above and inequalities (27) and (30), it is now possible to show from the series (33) that $\chi(s,u)$ is a continuous function of s and u in the region ($|s| \leqslant s_o$, $u \geqslant 0$) and is therefore bounded on any interval ($|s| \leqslant s_o - \epsilon, \epsilon > 0$) for finite u. It is also seen from the series (33), by employing (e) and inequalities (25) and (27), that

$$\chi(s,u) \longrightarrow +\infty \qquad (s \longrightarrow -s_o +) \tag{34}$$

for $u > 0$.

According to the definition (21),

$$H(s,u,o) = \sigma_1(o)\ \sigma_2(o), \tag{35}$$

and by using (25) and (26) the inequality

$$0 < H(s,u,\ ) < \sigma_1(\eta)\ \sigma_2(\eta) \qquad (|s| < \sigma(o),\ u \geqslant 0,\ \eta > 0) \tag{36}$$

is established. It is therefore possible to find functions $h_1(s,\eta)$ and $h_2(s,\eta)$ such that

$$h_1(s,o) = h_2(s,o) = \sigma_1(o)\ \sigma_2(o) \tag{37}$$

and, for a preassigned neighborhood of $u = o$,

$$o < h_1(s,\eta) \leqslant H(s,u,\eta) \leqslant h_2(s,\eta) \leqslant \sigma_1(\eta)\ \sigma_2(\eta) \qquad (|s| < \sigma(o),\ u \geqslant \eta \geqslant 0); \tag{38}$$

and if $\chi_1(s,u)$ and $\chi_2(s,u)$ are the solutions of the equations

$$\chi_i(s,u) = \int_0^u d\eta\ \frac{\Delta(s,\eta,o)}{1 - k(s,\eta)}\ h_i(s,\eta)\ \chi_i(s,\eta) + 1 \qquad (i = 1,\ 2), \tag{39}$$

it may be seen by comparing the Neumann series solutions or otherwise that

$$\chi_1(s,u) \leqslant \chi(s,u) \leqslant \chi_2(s,u) \qquad (|s| < s_o) \tag{40}$$

over the range of u values for which (38) holds. Equations (39) are easily solved by differentiation, resulting in

$$\chi_i(s,u) = \exp \int_0^u d\eta \, \frac{\Delta(s,\eta,o)}{1 - k(s,\eta)} \, h_i(s,\eta) \qquad (i = 1, 2). \tag{41}$$

These functions may be written in the forms

$$\chi_i(s,u) = \exp \left\{ a \int_0^u d\eta \, \frac{-\frac{\partial}{\partial\eta} k(s,\eta)}{1 - k(s, \ )} + \int_0^u d\eta \, \frac{1}{1 - k(s,\eta)} \left(\Delta(s,\eta,o) h_i(s,\eta) + a \frac{\partial}{\partial\eta} k(s,\eta) \right) \right\}$$

or

$$\chi_i(s,u) = \left(\frac{1 - k(s,u)}{1 - k(s,o)} \right)^a g_i(s,u), \tag{42}$$

where

$$g_i(s,u) = \exp \int_0^u d\eta \, \frac{1}{1 - k(s,\eta)} \left(\Delta(s,\eta,o) h_i(s,\eta) + a \frac{\partial}{\partial\eta} k(s,\eta) \right), \tag{43}$$

and if the constant a is chosen by the formula

$$a = \left. \frac{\Delta(s,u,o) h_i(s,u)}{-\frac{\partial}{\partial u} k(s,u)} \right|_{s = -s_o,\ u = 0} \tag{44}$$

(by formula (37) this is independent of the index i), then application of (31) shows that the functions $g_i(s,u)$ remain bounded and >0 as $s \longrightarrow -s_o+$. Thus $\chi(s,u)$ can be presented in the form

$$\chi(s,u) = \left(\frac{1 - k(s,u)}{1 - k(s,o)} \right)^a G(s,u) \tag{45}$$

where $G(s,u)$ satisfies the inequality

$$g_1(s,u) \leqslant G(s,u) \leqslant g_2(s,u) \tag{46}$$

and hence is also bounded and $\not\longrightarrow 0$ as $s \longrightarrow -s_o+$. From (25), (28), and (38), a is seen to be necessarily positive, thus verifying (34).

Introduction of (45) into (20) yields

$$\varphi_o(s,u) = \sigma_1(o) \frac{\Delta(s,u,o)\ G(s,u)}{\left(1 - k(s,o)\right)^{1+a}\left(1 - k(s,u)\right)^{1-a}}, \tag{47}$$

and if this is written as

$$\varphi_o(s,u) = \sigma_1(o) \frac{1}{(s + s_o)^{1+a}} \left\{ \left(\frac{s + s_o}{1 - k(s,o)}\right)^{1+a} \frac{\Delta(s,u,o)\ G(s,u)}{\left(1 - k(s,u)\right)^{1-a}} \right\}, \tag{48}$$

then the quantity in $\{\}$ is seen to remain bounded and > 0 as $s \longrightarrow -s_o+$ $(u > 0)$ and the asymptotic relation

$$\varphi_o(s,u) \sim \sigma_1(o) K^{1+a} \frac{1}{(s + s_o)^{1+a}} \left(\frac{\Delta_o(u)\ G_o(u)}{(1 - k(s,o)^{1-a}}\right) \quad (s \longrightarrow -s_o+,\ u > 0) \tag{49}$$

is obtained, where

$$\frac{1}{K} = - \frac{d}{ds} k(s,o) \Big|_{-s_o} \tag{50}$$

$$\Delta_o(u) = \Delta(-s_o,u,o), \tag{51}$$

$$k_o(u) = k(-s_o,u), \tag{52}$$

and $$G_o(u) = G(-s_o,u). \tag{53}$$

Application of a suitable Tauberian theorem to (49) and reversal of the substitution (11) now yields the final result

$$\Psi_o^{(ro)}(x,u) \sim \sigma_1(o) \frac{K^{1+a}}{\Gamma(1+a)} x^a e^{-s_o x} \sigma(u) e^{-u} \frac{\Delta_o(u)\, G_o(u)}{(1 - k_o(u))^{1+a}} \qquad (x \longrightarrow +\infty,\ u > 0). \tag{54}$$

Explicit formulas for a and K are easily found from the definitions; they are

$$a = \frac{\sigma_1(o)\ \sigma_2(o)}{\sigma'(o)\ \sigma_2(o) - \dfrac{\sigma_2'(o)}{\sigma_2(o)} \left(\sigma^2(o) - s_o^2\right)} \tag{55}$$

and

$$K = \frac{s_o}{\dfrac{\sigma_2(o)\ \sigma(o)}{\sigma^2(o) - s_o^2} - 1}\ , \tag{56}$$

and if the notations

$$h_{io}(u) = h_i(-s_o, u) \tag{57}$$

and

$$g_{io}(u) = g_i(-s_o, u) \tag{58}$$

are introduced, then the inequality to be satisfied by $G_o(u)$ appears, according to (46), in the form

$$g_{1o}(u) \leqslant G_o(u) \leqslant g_{2o}(u) \tag{59}$$

where

$$g_{io}(u) = \exp \int_0^u d\eta\, \frac{\Delta_o(\eta) h_{io}(\eta) + a k_o'(\eta)}{1 - k_o(\eta)} \tag{60}$$

by (43). Suitable functions $h_{io}(u)$ may be found analytically or numerically in various ways. One such set, which satisfies the inequality (38) (with $s = -s_o$)

for all values of u, is

$$h_{io}(u) = \sigma_1(u)\ \sigma_2(u)\ \frac{(-1)^i s_o + \sigma(o)}{(-1)^i s_o + \sigma(u)} \exp\left(-\int_0^u d\eta\ \frac{\sigma_1(\eta)}{(-1)^i s_o + \sigma(\eta)}\right) \quad (i = 1,2). \tag{61}$$

IV. Tabulation of Special Cases.

Most of the previous results on this problem appear to have been obtained under assumptions which cause $H(s,u,\eta)$ to be independent of u or, in other words, to satisfy the differential equation

$$\frac{\partial}{\partial u} H(s,u,\eta) = 0. \tag{62}$$

Equation (22) can, of course, be converted into an ordinary differential equation whenever $H(s,u,\eta)$ satisfies any differential equation of the form

$$\left(\sum_{i=1}^{n} f_i(s,u)\ \frac{\partial^i}{\partial u^i}\right) H(s,u,\eta) = 0, \tag{63}$$

and some solutions of this type with $n > 1$ have been obtained by Welton. These will not be considered here.

There appear to be four simple cases in which equation (62) is satisfied; these are characterized by the necessary additional assumptions presented in the following tabular form.

Case	$\sigma'(u)$	$\sigma_1(u)$	$\sigma_2(u)$
A.		$= 0$	$= 0$
B.		$= 0$	
C.	> 0		$= 0$
D.	$= \sigma_1(u)$	$= \sigma'(u)$	

These cases will now be investigated in some detail.

Case A.

This is rather trivial but is included for completeness. Substitution of $\sigma_1(u) = 0$ into (54) yields

$$\Psi_o^{(ro)}(x,u) = 0, \tag{64}$$

while use of $\sigma_2(u) = 0$ in (7) leads to

$$\varphi_o^{(o)}(s) = \frac{\sigma(o)}{s} \operatorname{arth} \frac{s}{\sigma(o)} \cdot \tag{65}$$

The exact inversion of (65) is

$$\Psi_o^{(o)}(x) = \frac{\sigma(o)}{2} E_1(|\sigma(o)x|) = \frac{\sigma(o)}{2} \int_1^\infty d\eta \frac{1}{\eta} e^{-\eta|\sigma(o)x|} \tag{66}$$

which has the asymptotic behavior

$$\Psi_o^{(o)}(x) \sim \frac{1}{2x} e^{-\sigma(o)x} \qquad (x \longrightarrow +\infty). \tag{67}$$

Therefore, from (3),

$$\Psi_o(x,u) \sim \frac{1}{2x} e^{-\sigma(o)x} \delta(u). \qquad (x \longrightarrow +\infty). \tag{68}$$

Case B.

Here (64) again holds and $\Psi_o^{(o)}(x)$ is given by the known inversion of (7), the asymptotic part of which is

$$\Psi_o^{(o)}(x) \sim \frac{\sigma(o)}{\sigma_2(o)} K e^{-s_o x} \qquad (x \longrightarrow +\infty) \tag{69}$$

where K is again given by (56). By (3), the result for this case is then

$$\Psi_o(x,u) \sim \frac{\sigma(o)}{\sigma_2(o)} K e^{-s_o x} \delta(u) \qquad (x \longrightarrow +\infty). \tag{70}$$

Case C.

In this case the expression (54) becomes meaningless and it is necessary to substitute $\sigma_2(u) = 0$ directly into the definitions (9) and (21), yielding

$$k(s,u) = 0, \tag{71}$$

$$H(s,u,\eta) = 0, \tag{72}$$

and, according to (22) and (20);

$$\chi(s,u) = 1 \tag{73}$$

and

$$\varphi_o(s,u) = \sigma_1(o)\,\Delta(s,u,o). \tag{74}$$

By III(a), $\Delta(s,u,o)$ is continuous function of s on the interval $(|s| < \sigma(o))$ for $u \geqslant 0$, and analysis similar to that used in discussing (41) may be used to show that

$$\Delta(s,u,o) \sim \frac{1}{2\sigma(o)a}\,\frac{1}{(s+\sigma(o))^a}\,\frac{Q_o(u)}{(\sigma(u)-\sigma(o))^{1-a}} \qquad (s \longrightarrow -\sigma(o)+,\ u > 0), \tag{75}$$

where a is given by

$$a = \frac{\sigma_1(o)}{\sigma'(o)} \tag{76}$$

and

$$Q_o(u) = \exp\int_0^u d\eta\,\frac{\sigma_1(\eta) - a\sigma'(\eta)}{\sigma(\eta) - \sigma(o)}\,. \tag{77}$$

Therefore

$$\Psi^{(ro)}_{(o)}(x,u) \sim \frac{1}{2}\,\frac{\sigma_1(o)}{\sigma(o)}\,\frac{1}{\Gamma(1+a)}\,x^{a-1}\,e^{-\sigma(o)x}\,\sigma(u)e^{-u}\,\frac{Q_o(u)}{(\sigma(u)-\sigma(o))^{1-a}}$$
$$(x \longrightarrow +\infty,\ u > 0) \tag{78}$$

and, since the singular part of the solution is evidently given by (67), the complete result is

$$\Psi_o(x,u) \sim \frac{1}{2x} e^{-\sigma(o)x} \delta(u) + \frac{1}{2} \frac{\sigma_1(o)}{\sigma(o)} \frac{1}{\Gamma(1+a)} x^{a-1} e^{-\sigma(o)x} \sigma(u) e^{-u} \frac{Q_o(u)}{\left(\sigma(u) - \sigma(o)\right)^{1-a}}$$

$$(x \longrightarrow +\infty). \tag{79}$$

This is equivalent to a result obtained by Hurwitz for a different angular distribution of the source.

Case D.

The general solution (54) holds in this case and an exact formula may be written for $G_o(u)$. It is easily verified by direct differentiation of (19) that, under the assumptions for this case,

$$\frac{\partial}{\partial u} \Delta(s,u,\eta) = 0. \tag{80}$$

Hence (62) holds,

$$H(s,u,\eta) = H(s,o,\eta), \tag{81}$$

and the functions $h_{io}(u)$ may be selected as

$$h_{1o}(u) = h_{2o}(u) = H(-s_o,o,u). \tag{82}$$

Moreover, use of the integral

$$\Delta(s,0,o) = \frac{1}{\sigma^2(o) - s^2} \tag{83}$$

leads to

$$H(-s_o,o,u) = (\sigma^2(o) - s_o^2) \Delta(-s_o,o,u) \sigma_1(u) \sigma_2(u) \tag{84}$$

and also

$$\Delta_o(u) = \Delta_o(0) = \frac{1}{\sigma^2(o) - s_o^2}; \tag{85}$$

and substitution of (84) and (85) into (60) yields, by the use of (59)

$$G_o(u) = \exp\int_0^u d\eta \frac{1}{1 - k_o(\eta)} \left(\Delta(-s_o,o,\eta)\ \sigma_1(\eta)\ \sigma_2(\eta) + ak_o'(\eta)\right) . \tag{86}$$

Here the singular component is given by (69) and the complete asymptotic formula is therefore

$$\Psi_o(x,u) \sim \frac{\sigma(o)}{\sigma_2(o)} Ke^{-s_o x}\ \delta(u) + \sigma_1(o) \frac{1}{\sigma^2(o) - s_o^2} \frac{K^{1+a}}{\Gamma(1+a)} x^a e^{-s_o x} \sigma(u) e^{-u} \frac{G_o(u)}{(1 - k_o(u))^{1-a}}$$

$$(x \longrightarrow +\infty) \tag{87}$$

with $G_o(u)$ given by (86). This result includes a case discussed by Murray in which the assumptions were

$$\sigma_1(u) = \sigma_1(o)e^u ,$$

$$\sigma_2(u) = \sigma_2 = \text{constant},$$

and $$\sigma(u) = \sigma_1(u) + \sigma_2 .$$