UNCLASSIFIED

MARTIN MARIETTA ENERGY SYSTEMS LIBRARIES
3 4456 0360789 9

ORNL 751
Physics-General
59

A DIFFUSION SOLUTION FOR A BLACK THERMAL NEUTRON PIPE DUCT CONTAINING A PURE SCATTERING MATERIAL

OAK RIDGE NATIONAL LABORATORY
CENTRAL RESEARCH LIBRARY
CIRCULATION SECTION
4500N ROOM 175
LIBRARY LOAN COPY
DO NOT TRANSFER TO ANOTHER PERSON
If you wish someone else to see this report, send in name with report and the library will arrange a loan.
UCN-7969 (3 9-77)

OAK RIDGE NATIONAL LABORATORY
OPERATED BY
CARBIDE AND CARBON CHEMICALS DIVISION
UNION CARBIDE AND CARBON CORPORATION
POST OFFICE BOX P
OAK RIDGE, TENNESSEE

UNCLASSIFIED

UNCLASSIFIED ORNL-751

This document consists of 14 pages. Copy 8 of 125. Series A.

Contract No. W-7405, eng.26

PHYSICS DIVISION

A DIFFUSION SOLUTION FOR A BLACK THERMAL NEUTRON PIPE DUCT CONTAINING A PURE SCATTERING MATERIAL

David W. Whitcombe

DATE ISSUED: JUL 5 1950

OAK RIDGE NATIONAL LABORATORY

operated by

CARBIDE AND CARBON CHEMICALS DIVISION
Union Carbide and Carbon Corporation
Post Office Box P
Oak Ridge, Tennessee

UNCLASSIFIED

MARTIN MARIETTA ENERGY SYSTEMS LIBRARIES

3 4456 0360789 9

ORNL-751
Physics - General

Internal Distribution:

1	G. T. Felbeck (C&CCC)	16	E. J. Murphy	26	E. P. Blizard
2-3	706-A Library	17	C. E. Center	27	F. C. VonderLage
4	706-B Library	18	J. A. Swartout	28	Lewis Nelson
5	Biology Library	19	A. H. Snell	29	D. W. Cardwell
6	Health Physics Library	20	A. Hollaender	30	M. J. Skinner
7-8	Training School Library	21	J. S. Felton	31	N. M. Smith
9	Metallurgy Library	22	K. Z. Morgan	32	C. E. Clifford
10-13	Central Files	23	J. H. Gillette	33	L. C. Noderer
14	C. E. Larson	24	E. O. Wollan	34	M. C. Edlung
15	A. M. Weinberg	25	A. Householder	35	D. W. Whitcombe

External Distribution:

36-43	Argonne National Laboratory
44	Armed Forces Special Weapons Project
45-46	Atomic Energy Commission, Washington
47	Battelle Memorial Institute
48	Brush Beryllium Company
49-56	Brookhaven National Laboratory
57	Bureau of Ships
58-61	Carbide & Carbon Chemicals Division (K-25)
62-65	Carbide & Carbon Chemicals Division (Y-12)
66	Chicago Operations Office
67	Columbia University (J. R. Dunning)
68-70	General Electric, Richland
71	Idaho Operations Office
72-73	Iowa State College
74-77	Knolls Atomic Power Laboratory
78-80	Los Alamos Scientific Laboratory
81	Massachusetts Institute of Technology (A. Kaufmann)
82-84	Mound Laboratory
85-86	National Advisory Committee for Aeronautics
87-88	National Bureau of Standards
89-93	New York Operations Office
94	North American Aviation, Inc.
95	Patent Branch (Washington)
96	Sandia Laboratory
97	Santa Fe Operations Office
98-112	Technical Information Division (Oak Ridge)
113-117	University of California Radiation Laboratory
118-121	USAF, NEPA
122-125	Westinghouse Electric Company

A DIFFUSION SOLUTION FOR A BLACK THERMAL NEUTRON PIPE DUCT CONTAINING A PURE SCATTERING MATERIAL.

David W. Whitcombe*

Much interest has been directed toward ducts containing liquid metals as both a coolant and moderator. It is the purpose of this paper to consider the effectiveness of these metals as a thermal neutron pipe considering them to be pure scatterers. The solutions all contain the parameters r_o, the extrapolated radius and z_o, the extrapolated length of the duct. It is clear that these are the important parameters in the design of the duct, since r_o and z_o must be chosen to make the duct a poor neutron pipe and still function as a satisfactory means for carrying the heat away. Since the absorption cross sections for the materials under consideration are of the order of .0003 barns the assumption shall be made that the liquid metal is a pure scatterer and then Laplace's equation represents the diffusion. The case is analogous to the problem in steady state heat flow.

THE PROBLEM

The geometry and coordinates of the problem are shown in Fig. 1.

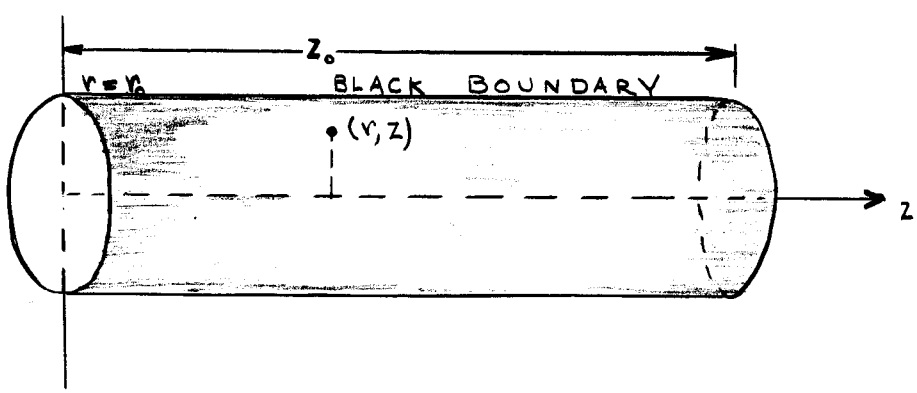

Fig. 1.

*On loan from Fairchild Engine and Airplane Corporation, NEPA Project.

The differential equations and boundary conditions are written as:

(1) $$\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{zz} = 0 \quad (0 \leq z \leq z_o,\ 0 \leq r \leq r_o)$$

(2) $$\varphi(r_o, z) = 0 \quad (0 \leq z \leq z_o)$$

(3) $$\varphi(0, z) = \text{finite} \quad (0 \leq z \leq z_o)$$

(4) $$\varphi_z(r, 0) = -n_o \quad (0 \leq r \leq r_o)$$

(5) $$\varphi(r, z_o) = 0 \quad (0 \leq r \leq r_o).$$

THE SOLUTION

The problem is solved by means of separation of variables, i.e., assume

(6) $$\varphi = R\,Z\ .$$

Then the two equations are obtained

(7) $$\begin{cases} R'' + \frac{1}{r}R' + \alpha^2 R = 0 \\ Z'' - \alpha^2 Z = 0 \end{cases}$$

where α^2 is the separation constant and α^2 may be positive or negative, i.e., α may be a pure real or a pure imaginary. The solution to (7) is

(8) $$\begin{cases} R = J_o(\alpha r) \\ Z = e^{\pm \alpha z} \end{cases} \quad \text{or} \quad \begin{cases} \text{ch}\,\alpha z \\ \text{sh}\,\alpha z \end{cases}$$

where ch, sh refer to the hyperbolic cosine and sine.

Since $\alpha = i\beta$ is still a separation constant. Other solutions are

$$(9) \qquad \begin{aligned} R &= I_o\ (\beta\ r) \\ Z &= \begin{cases} \cos\beta\ z \\ \sin\beta\ z \end{cases} \end{aligned} \quad .$$

Also $\alpha = 0$ is a separation constant but the solutions of (7) then include $R = \log r$ which is ruled out by (3), and φ = constant which is ruled out by (2) and (5). So the following is a solution to (1) for all real α and β.

$$(10) \qquad \varphi = (A_\alpha \operatorname{ch} \alpha\ z + \beta_\alpha \operatorname{sh} \alpha\ z)\ J_o\ (\alpha\ r) + $$
$$+ (C_\beta \cos\beta\ z + D_\beta \sin\beta\ z)\ I_\beta\ (\beta\ r)\ .$$

The solution corresponding to the case when the separation constant is zero has already been eliminated using boundary condition (3). Now A_α, β_α, C_β, D_β must be determined so that all the boundary conditions are satisfied. In the majority of problems one may take $A_\alpha = B_\alpha = 0$ <u>or</u> $C_\beta = D_\beta = 0$. In some special problems all four coefficients will have to be evaluated. Let us take $C_\beta = D_\beta = 0$ and see if a solution to (1) - (5) may be obtained. Now $\varphi_z = (A_\alpha\ \alpha \operatorname{sh} \alpha\ z + B_\alpha\ \alpha \operatorname{ch} \alpha\ z)\ J_o\ (\alpha\ r)$ and using boundary condition (4) this becomes

$$-n_o = B_\alpha\ \alpha\ J_o\ (\alpha\ r)\ .$$

Now let α_i be the roots of

$$(11) \qquad J_o\ (\alpha\ r_o) = 0$$

then

$$-n_o = \sum_i B_i\ \alpha_i\ J_o\ (\alpha_i\ r)$$

and since the $J_o\ (\alpha_i\ r)$ form an orthogonal set one obtains

$$-\int_0^{r_o} n_o\ r\ J_o\ (\alpha_i\ r)\ d\ r = B_i\ \alpha_i \int_0^{r_o} r\ J_o^2\ (\alpha_i\ r)\ d\ r$$

or

$$\frac{-n_o\ r_o}{\alpha_i}\ J_1\ (\alpha_i\ r_o) = \frac{B_i\ \alpha_i\ r_o^2}{2}\ J_1^2\ (\alpha_i\ r_o)\ .$$

Hence

$$B_i = \frac{-\ 2\ n_o}{r_o\ \alpha_i^2\ J_1\ (\alpha_i\ r_o)} \tag{12}$$

$$\varphi = \sum_{i=1}^{\infty} \left(A_i\ \mathrm{ch}\ \alpha_i\ z - \frac{2\ n_o\ \mathrm{sh}\ \alpha_i\ z}{r_o\ \alpha_i^2\ J_1(\alpha_i\ r_o)}\right) J_o\ (\alpha_i\ r)\ . \tag{13}$$

It is seen that boundary condition (2) is satisfied because of (11) and it remains only to determine A_i in (13) in accord with boundary condition (5). Then write

$$0 = \sum_{i=1}^{\infty} \left[A_i\ \mathrm{ch}\ \alpha_i\ z_o - \frac{2\ n_o\ \mathrm{sh}\ \alpha_i\ z_o}{r_o\ \alpha_i^2\ J_1(\alpha_i\ r_o)}\right] J_o\ (\alpha_i\ r)$$

and from the orthogonality of the Bessel function one gets

$$A_i = \frac{2\ n_o\ \mathrm{th}\ (\alpha_i\ z_o)}{r_o\ \alpha_i^2\ J_1\ (\alpha_i\ r_o)}\ , \tag{14}$$

where th refers to the hyperbolic tangent.

The complete solution to (1) - (5) may be written in the form

$$(15) \qquad \varphi = \frac{2\, n_o}{r_o} \sum_{i=0}^{\infty} \frac{-\mathrm{sh}\,(\alpha_i\, z) + \mathrm{th}\,\alpha_i\, z_o\, \mathrm{ch}\,\alpha_i\, z}{\alpha_i^2\, J_1\,(\alpha_i\, r_o)}\, J_o\,(\alpha_i\, r)$$

but since

$$-\,\mathrm{sh}\,(\alpha_i\, z) + \mathrm{th}\,(\alpha_i\, z_o)\,\mathrm{ch}\,(\alpha_i\, z) = +\,\frac{\mathrm{sh}\,\left[\alpha_i (z_o - z)\right]}{\mathrm{ch}\,\alpha_i\, z_o}$$

the solution (15) may be written as

$$(16) \qquad \varphi\,(r,z) = \frac{+2\, n_o}{r_o} \sum_{i=0}^{\infty} \frac{\mathrm{sh}\,\left[\alpha_i\,(z_o - z)\right]}{\alpha_i^2\,\mathrm{ch}\,(\alpha_i\, z_o)}\, \frac{J_o\,(\alpha_i\, r)}{J_1\,(\alpha_i\, r_o)}$$

where α_i are roots of $J_o\,(\alpha\, r_o) = 0$, i.e., $\alpha_1\, r_o = 2.405$, $\alpha_2\, r_o = 5.52$, $\alpha_3\, r_o = 8.65$, $\alpha_4\, r_o = 11.79$, ...

Since the α_i increase so rapidly it may be found that one or two terms of the series is sufficient. It is noted that $\alpha_2 > 2\,\alpha_1$ and since the α's occur in an exponent one term will be sufficient in the case when $z \approx z_o$ and $z_o >> 1$. Then in this case

$$(17) \qquad \varphi\,(r,\, z) = \frac{2\, n_o}{r_o\, \alpha_1^2}\, \frac{\mathrm{sh}\,\alpha_1\,(z_o - z)}{\mathrm{ch}\,\alpha_1\, z_o}\, \frac{J_o\,(\alpha_1\, r)}{J_1\,(\alpha_1\, r_o)}$$

Since $J_1(\alpha_2 r_o)$ will be negative the above expression represents an upper bound. Now $\alpha_1 \cdot r_o = 2.4$ so (17) may be written as

$$(18) \qquad \varphi(r,z) = \frac{2\, n_o\, r_o\, J_o\left(\frac{2.4r}{r_o}\right)}{(2.4)^2\, J_1(2.4)} \quad \frac{\mathrm{sh}\left[\frac{2.4}{r_o}(z_o - z)\right]}{\mathrm{ch}\,\frac{2.4\, z_o}{r_o}} .$$

Only the extrapolated boundaries have been used in the above analysis, and not the actual dimensions which will be designated by z_1 and r_1. This was a convenient manipulation because it allowed the use of boundary condition (5) rather than the following more complicated expression for the return current, J_-, which is

$$(19) \qquad J_- = \frac{\varphi(r, z_1)}{4} + \frac{\lambda}{6}\, \varphi_z(r, z_1) .$$

Let us now determine z_1 assuming (5) is equivalent to (19). Then

$$(20) \qquad \varphi(r, z_1) + \frac{2}{3}\lambda\, \varphi_z(r, z_1) = 0 .$$

Now, writing (18) as

$$(21) \qquad \varphi(r,z) = Q(r)\ \mathrm{sh}\ \frac{2.4\,(z_o - z)}{r_o}$$

where

$$Q(r) = \frac{2\, n_o\, r_o\, J_o\left(\frac{2.4\, r}{r_o}\right)}{(2.4)^2\, J_1(2.4)\ \mathrm{ch}\,\frac{2.4\, z_o}{r_o}}$$

and cancelling the Q(r) (20) becomes

$$\text{(22)} \qquad \operatorname{sh} \frac{2.4\ h}{r_o} - \frac{2}{3} \lambda \frac{2.4}{r_o} \operatorname{ch} \frac{2.4\ h}{r_o} = 0$$

where the extrapolation length $h = z_o - z_1$. Now (22) may be written as,

$$\frac{1.6\lambda}{r_o} = \operatorname{th} \frac{2.4\ h}{r_o} .$$

Since $\operatorname{th} \frac{2.4\ h}{r_o}$ must be less than one, a condition on the above analysis is that r_o be greater than 1.6λ. The condition, $r_o > 1.6\lambda$, is not a troublesome one since the diffusion theory doesn't apply in cases where a boundary separation is of the order of a mean free path.

Then h may always be solved for, to obtain

$$\text{(23)} \qquad h = \frac{r_o}{2.4} \operatorname{th}^{-1} \frac{1.6\lambda}{r_o}, \quad r_o > 1.6\lambda .$$

Now for the design of a neutron pipe one needs to know the relation between r_o and z_o that gives the exit current at $z = z_1$ and $r = 0$. The expression for this exit current, j_+, is

$$\text{(24)} \qquad j_+ = \frac{\varphi(0, z_1)}{4} - \frac{\lambda}{6} \varphi_z (0, z_1)$$

$$= \frac{\varphi(0, z_o - h)}{4} - \frac{\lambda}{6} \varphi_z (0, z_o - h)$$

and using (21) this may be written

$$(25) \qquad j_+ = \frac{Q(0)}{4}\left[\frac{\text{sh } 2.4\ h}{r_o} + \frac{2}{3}\lambda\ \frac{2.4}{r_o}\ \text{ch}\ \frac{2.4\ h}{r_o}\right]$$

but by (22) this becomes

$$(26) \qquad j_+ = \frac{Q(0)}{4}\ 2\ \text{sh}\ \frac{2.4\ h}{r_o}$$

$$= \frac{n_o\ r_o}{(2.4)^2} \cdot \frac{\text{sh}\left(\frac{2.4\ h}{r_o}\right)}{J_1\ (2.4)\ \text{ch}\ \frac{2.4\ z_o}{r_o}} .$$

The parameter h may easily be eliminated from (26) to obtain an expression for j_+ involving r_o, z_o, λ.

Since

$$\text{sh}\ \frac{2.4\ h}{r_o} = \text{sh th}^{-1}\ \frac{1.6\lambda}{r_o} = \frac{1.6\lambda}{r_o}\ \frac{1}{\sqrt{1 - (\frac{1.6\lambda}{r_o})^2}}$$

$$= \frac{1.6\ \lambda}{\sqrt{r_o^2 - (1.6\lambda)^2}}$$

one obtains

$$(27) \qquad j_+\ (0,\ z) = \frac{1.6\lambda\ \ n_o\ r_o}{(2.4)^2\ J_1\ (2.4)\ \sqrt{r_o^2 - (1.6\lambda)^2}\ \text{ch}\ (\frac{2.4\ z_o}{r_o})}$$

an expression for the exit current at the points $r = 0$, $z = z_1$ ($= z_o - h$).

A designer of a duct is interested in knowing for any z_o whether or not there is an optimum value for the radius, v_m, say that makes j_+ a minimum. Intuitively it is known that such a value does not exist but (27) appears to exhibit a v_m. It is found however that this value exists for imaginary r_o and the above analysis does not apply; there is no $v_m > 1.6\lambda$.

Before this theory may be applied to a calculation of an actual pipe of radius r_1, say, it will be necessary to know r_o in terms of r_1. This may be found using a procedure similar to the one above. Write

$$(28) \qquad j_-(r_1, z) = \varphi(r_1, z) + \frac{2}{3}\lambda\,\varphi_r(r_1, z) = \varphi(r_o, z) = 0$$

then

$$J_o\left(\frac{2.4\,r_1}{r_o}\right) - \frac{2}{3}\lambda\,\frac{2.4}{r_o}\,J_1\left(\frac{2.4\,r_1}{r_o}\right) = 0$$

or

$$(29) \qquad J_o\left(\frac{2.4\,r_1}{r_o}\right) = \frac{1.6\lambda}{r_o}\,J_1\left(\frac{2.4\,r_1}{r_o}\right)$$

where r_1 is chosen less than r_o.

The solution for r_1 in (29) may be obtained from either the graphs or the tables in Jahnke and Emde.

An idea of the sizes of the pipes that can be treated using the above theory can be found. The minimum radius r_{min} may be found by assuming $r_o = 1.6\lambda$. Then since

$$J_o(x) = J_1(x) \quad \text{when} \quad x \approx 1.43$$

it is clear that

$$r_{min} = \frac{r_o}{2.4} \cdot 1.43 = \frac{1.6\lambda}{2.4} 1.43 \approx \lambda \, . \tag{30}$$

If the radius of the pipe is of the order of a mean free path, then the flux is given by the first collision density, i. e.,

$$\varphi(r_1, z) = \varphi_o e^{-\sigma z} \tag{31}$$

where σ is the scattering cross section for the pure scattering medium. The assumption in this formulation is that the radius is sufficiently small that any collision will deflect a straight-through neutron into the wall, where it is absorbed.

SAMPLE PROBLEM

Let us find the neutron current out of the end of a neutron pipe similar to one that might occur in an actual design. Assume the parameters

$$r_1 = 24 \text{ cm}$$

$$z_1 = 240 \text{ cm} \approx 5 \text{ ft.}$$

$$\lambda = 4 \text{ cm}$$

$$n_o = 1 \quad .$$

Part (a)

Find r_o, the extrapolated radius. Using (29) the following relation

for r_o is obtained

$$J_o\left(\frac{57.6}{r_o}\right) = \frac{6.4}{r_o} J_1\left(\frac{57.6}{r_o}\right) . \tag{32}$$

To simplify the solution for r_o, write

$$\frac{57.6}{r_o} = x$$

then (32) may be written as (33)

$$J_o(x) = .111\, x\, J_1(x) . \tag{33}$$

Now (33) must be solved for a value of $x < 2.4$. This value is found by trial and error to be

$$x = 2.155$$

and

$$r_o = \frac{57.6}{2.155} = 26.73 .$$

The extrapolated radius is then 2.73 cm longer than the actual radius.

<u>Part (b)</u>

Find h, the z-extrapolation length, and z_o the z-extrapolated boundary.

From (23) one may write

$$h = \frac{r_o}{2.4} \operatorname{th}^{-1} \frac{1.6\lambda}{r_o} = \frac{26.73}{2.4} \operatorname{th}^{-1} \frac{6.4}{26.73}$$

$$= 11.14 \text{ th}^{-1} .2393 = 2.62 \text{ cm.}$$

And

$$z_o = z_1 + h = 242.62 \text{ cm.}$$

Part (c)

Find the exit current j_+ on the axis, at the end of the pipe, i.e., find $j_+ (0, z_1)$.

Using (26) one obtains

$$j_+ (0, z_1) = \frac{26.73}{(2.4)^2} \quad \frac{\text{sh} \dfrac{2.4(2.62)}{26.73}}{J_1 (2.4) \text{ ch} \dfrac{2.4(242.62)}{26.73}}$$

$$= 1.4 \times 10^{-8} .$$