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ON ELECTRIC FOCUSSING IN THE FIXED FREQUENCY CYCLOTRON

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PHYSICS DIVISION

ON ELECTRIC FOCUSING IN THE FIXED FREQUENCY CYCLOTRON

M. E. Rose

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ON ELECTRIC FOCUSING IN THE FIXED FREQUENCY CYCLOTRON

M. E. Rose

The theory of electric focusing in the fixed frequency cyclotron has been discussed quite completely in a paper by the author published some years ago.[1] However, in view of recent interest in this problem it seems worth while to present the following alternative method in which the nature of the approximations involved will be brought out even more clearly.

We use the same notation as in reference 1. Also the same basic assumptions will be made at the start:

(A) The oscillator wave length is large compared to the dee gap. This assumption, which is very well fulfilled in practice, means that we can use quasi-stationary fields rather than the rigorous retarded solutions of Maxwell's equations.

(B) The geometry of the dees is idealized in the sense that edge effects due to the finite radius of the dees are not considered. This assumption, which is very well justified in the region where electric focusing is operative (not too large radii of curvature of the ion orbits), means that the space part of the fields are solutions of the two-dimensional static form of Maxwell's equations.

With these two assumptions we obtain an otherwise rigorous result for the orbit equation, see Eq. (17) below. In order to obtain the ion deflection in passing through a single acceleration region (lens) the further approximations

---

1. M. E. Rose, Phys. Rev. 53, 392 (1938).

that the relative energy and phase changes are small are <u>then</u> introduced.

For convenience we rewrite the equations describing the field and its symmetry properties. The potential at a point x, z and a time t is

$$V_t(x,z,t) = V(x,z) \cos(\omega t + \theta) \tag{1}$$

where t = 0 is the instant at which the ion passes the plane x = 0. Since, according to assumption (A) the space part fulfills

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

we can introduce the "stream line function" U conjugate to V and between U and V the Causby-Riemann relations exist.

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial z} \quad , \qquad \frac{\partial U}{\partial z} = - \frac{\partial V}{\partial x} \tag{2}$$

The same relations apply to U_t and V_t with

$$U_t = U \cos(\omega t + \theta)$$

The symmetry properties and boundary conditions are

$$\begin{aligned} V(x, -z) &= V(x, z) \\ V(\pm\infty, z) &= \mp V_0 \\ U(x, -z) &= -U(x, z) \\ U(\pm\infty, z) &= 0 \end{aligned} \tag{3}$$

From (2) and (3) we can also conclude that

$$\begin{aligned} U(x, 0) &= 0 \\ V(-x, z) &= -V(x, z) \\ U(-x, z) &= U(x, z) \end{aligned} \tag{3a}$$

The path of any ion with given initial conditions (at $x = -\infty$, practically) can be expressed in parametric form

$$x = x(s)$$
$$z = z(s) \tag{4}$$

where s is the distance measured along the orbit and, for definiteness, the origin of s may be taken at the point where the orbit crosses $x = 0$. Alternatively one may use the time t in the parametric form of the orbit so that x and z are certain functions of t.

What is required now is the transformation of the equation of vertical motion

$$\frac{d^2z}{dt^2} = - \frac{e}{M} \frac{\partial V_t}{\partial z}$$

into a kinematic form whereby z is expressed in terms of x. For this purpose we first transform the acceleration as follows:

$$\frac{d^2z}{dt^2} = \left(v \frac{d}{ds}\right)^2 z = v^2 \frac{d^2z}{ds^2} + v \frac{dv}{ds} \frac{dz}{ds} \tag{6}$$

Now

$$\alpha \equiv dz/dx = \frac{dz/ds}{dx/ds}$$

and

$$\frac{d^2z}{ds^2} = \left(\frac{dx}{ds} \frac{d}{dx}\right)^2 z = \left(\frac{dx}{ds}\right)^2 \frac{d^2z}{dx^2} + \alpha \frac{d^2x}{ds^2} \tag{7}$$

using $d/dx = (ds/dx)d/ds$. Since

$$(ds)^2 = (dx)^2 + (dz)^2$$

we have

$$\frac{dx}{ds} = (1+\alpha^2)^{-\frac{1}{2}} \; ; \quad \frac{dz}{ds} = \alpha(1+\alpha^2)^{-\frac{1}{2}} \tag{8}$$

and

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \tag{9}$$

Consequently,

$$\frac{d^2z}{ds^2} = \frac{1}{1+\alpha^2}\frac{d\alpha}{dx} - \alpha\frac{(dz/ds)(d^2z/ds^2)}{dx/ds}$$

from which we obtain

$$\frac{d^2z}{ds^2} = \frac{1}{(1+\alpha^2)^2}\frac{d\alpha}{dx} \tag{10}$$

Then for the vertical acceleration we obtain from (6)

$$\frac{d^2z}{dt^2} = \frac{v^2}{(1+\alpha^2)^2}\frac{d\alpha}{dx} + v\frac{dv}{ds}\frac{\alpha}{\sqrt{1+\alpha^2}} \tag{11}$$

$$\begin{aligned}\frac{dv}{ds} &= \frac{1}{v}\frac{dv}{dt} = \frac{1}{v}\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial z}\frac{dz}{dt}\right) \\ &= \frac{1}{v}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\frac{dx}{ds} + v\frac{\partial v}{\partial z}\frac{dz}{ds}\right) \\ &= \frac{1}{v}\frac{\partial v}{\partial t} + \frac{1}{\sqrt{1+\alpha^2}}\left(\frac{\partial v}{\partial x} + \alpha\frac{\partial v}{\partial z}\right)\end{aligned} \tag{12}$$

Defining E as the total energy at x = 0 we have

$$\frac{Mv^2}{2} = E - eV(x, z)\cos(\omega t + \theta) \tag{13}$$

and

$$
\begin{aligned}
Mv \frac{\partial v}{\partial t} &= \omega e V \sin \varphi \\
Mv \frac{\partial v}{\partial x} &= -e \frac{\partial V}{\partial x} \cos \varphi \\
Mv \frac{\partial v}{\partial z} &= -e \frac{\partial V}{\partial z} \cos \varphi
\end{aligned}
\tag{14}
$$

where $\varphi = \omega t + \theta$.

We also transform the right hand side of (5) as follows:

$$
\frac{\partial V}{\partial z} = \frac{\partial U}{\partial x} = \frac{dU}{dx} - \frac{\partial U}{\partial z} \frac{dz}{dx}
$$

$$
\frac{\partial V}{\partial x} = \frac{dV}{dx} - \frac{\partial V}{\partial z} \frac{dz}{dx}
$$

So

$$
\frac{\partial U}{\partial x} = \frac{dU}{dx} + \frac{dV}{dx} \frac{dz}{dx} - \frac{\partial U}{\partial x} \left(\frac{dx}{dx}\right)^2
$$

or

$$
\frac{\partial U}{\partial x} = \frac{1}{1+\alpha^2} \left(\frac{dU}{dx} + \alpha \frac{dV}{dx}\right) \tag{15}
$$

Then with the use of (11), (12), (14) and (15) we obtain from (5)

$$
\frac{Mv^2}{(1+\alpha^2)^2} \frac{d\alpha}{dx} + \frac{\alpha}{\sqrt{1+\alpha^2}} \frac{\omega e V \sin \varphi}{v} - \frac{\alpha e}{1+\alpha^2} \cos\varphi \left(\frac{\partial V}{\partial x} + \alpha \frac{\partial V}{\partial z}\right)
$$

$$
= -\frac{e \cos \varphi}{1+\alpha^2} \left(\frac{dU}{dx} + \alpha \frac{dV}{dx}\right) \tag{16}
$$

Using the above for the connection between derivatives following the path and derivatives along x, we see that the last two terms on the left side of (16) cancel with the last term on the right side and for the final orbit equation we obtain the result:

$$\frac{d\alpha}{dx} = - \frac{\alpha(1+\alpha^2)^{3/2}\,\omega e V \sin\varphi}{Mv^3} - \frac{e\cos\varphi}{Mv^2}(1+\alpha^2)\frac{dU}{dx} \qquad (17)$$

It will be noted that no approximations of the kind $\alpha << 1$ have been made. If this approximation is made we obtain

$$\frac{d\alpha}{dx} = - \frac{e\cos\varphi}{Mv^2}\frac{dU}{dx} \qquad (18)$$

which is equivalent to the form in which the equation of motion was used in reference 1. If we expand $\cos\varphi$ and Mv^2 in terms of the phase change and relative energy change we get

$$\cos\varphi = \cos(\theta + \omega t) \cong \cos\theta - \omega t \sin\theta$$

and

$$1/Mv^2 = \frac{1}{2}(E - eV\cos\varphi)^{-1} \cong \frac{1}{2E}\left(1 + \frac{eV\cos\varphi}{E}\right)$$

$$\cong \frac{1}{2E}\left(1 + \frac{eV\cos\theta}{E}\right)$$

so that from (18)

$$\frac{d\alpha}{dx} \cong - \frac{e}{2E}\frac{dU}{dx}\left\{\cos\theta - \omega t\sin\theta + \frac{eV}{E}\cos^2\theta\right\} \qquad (19)$$

The total deflection is

$$\Delta\alpha = \int_{-\infty}^{\infty} \frac{d\alpha}{dx}\, dx = -\frac{e}{2E} \int_{-\infty}^{\infty} (\cos\theta - \frac{\omega x}{v} \sin\theta + \frac{eV}{E} \cos^2\theta)\, dx \tag{20}$$

The first term vanishes as explained in reference 1. The second term gives the deflection due to field variation, the third gives the deflection due to energy change.

There are higher terms which we have neglected: First, there are cross terms which come from simultaneous consideration of energy change and field variation, (for example, $-\frac{\omega x}{v} \sin\theta \cdot \frac{eV}{2E^2} \cos\theta$ which vanishes because x and V are odd and dU/dx is an essentially odd function[2]). Another cross term neglected arises from a more exact treatment of $\cos\varphi$ in the expansion of $1/Mv^2$. This gives $-(1/2E^2)\, e\, V \frac{\omega x}{v} \sin\theta$ which is also odd when multiplied by dU/dx. In fact, because the expansion parameters eV/E and $\omega x/v$ are odd in x we see that the first terms neglected always have opposite "parity" to the terms retained so that the first correction terms which give a non-vanishing result are of second order in the parameters of smallness as compared to the terms retained. To introduce these higher order terms in the expansions made would be inconsistent (in view of assumption A) and would be unimportant in view of their small effect. Thus, the field variation and energy-change contributions to $\Delta\alpha$ are additive to a high order of approximation.

We may now return to the fundamental Eq. (17) and examine the additional term which, to first order, is

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2. Actually $\partial U/\partial x$ is odd but dU/dx differs from it by a term of order α and it is not consistent to consider this difference in the present approximation.

$$- \alpha \omega e V \sin \varphi / Mv^3$$

First, the physical interpretation of this term is very simple. It will be observed that it is not, strictly speaking, a focusing effect in that it doesn't reverse sign with z but rather with α. Moreover, it exists even in a uniform field along x. In fact this term is simply the deflection gradient along x which arises from acceleration along x. Suppose there were no forces in the z direction but that the ion accelerates (or decelerates) along x. Then there would be a deflection gradient

$$\frac{d\alpha}{dx} = \frac{d}{dx}\frac{v_z}{v_x} = -\frac{v_z}{v_x^2}\frac{dv_x}{dx} = -\frac{\alpha}{v_x}\frac{dv_x}{dx}$$

$$\cong -\frac{\alpha}{v}\frac{dv}{dt}\frac{dt}{dx} \cong -\frac{\alpha}{v^2}\frac{dv}{dt} \cong -\frac{\alpha \omega e V \sin \varphi}{Mv^3}$$

from (14). Thus, this extra term in the deflection gradient $d\alpha/dx$ tends to collimate the orbits parallel to x in an accelerating field (and parallel to z in a decelerating field). In fact, it is this simple effect which makes the approximation $\alpha << 1$ a very good one. However, this collimation term, as we may call it, is a small correction on the deflection gradient. Its ratio to the focusing term may be simply estimated. To do this we note that in obtaining $\Delta \alpha$ we need to expand the collimation term as well as the focusing term. When this is done the main contribution from the collimation term is

$$\frac{\alpha \omega^2 x}{Mv^4} e V \cos \theta \tag{20a}$$

arising from time variation of the phase φ.

We compare this with the smaller of the two principal contributions to the focusing, the energy-change focusing, which contributes

$$- \frac{e^2 \cos \theta}{2 E^2} V \frac{dU}{dx} \tag{20b}$$

to the deflection gradient. The ratio of (20a) to (20b) is

$$R \sim \frac{\alpha \omega^2 x M}{e \, dU/dx} \approx \frac{\alpha M \omega^2 k^2}{e V_0}$$

where k is the dee gap. Since $E \sim M \omega^2 r^2$ where r is the radius of curvature of the orbit, we have

$$R \sim \alpha \frac{E k^2}{e V_0 r^2}$$

But $E \approx$ const. r^2 and when $r \sim k$, $E \sim e V_0$. Therefore, $E/eV_0 \sim r^2/k^2$. Thus $R \sim \alpha << 1$. It follows then that the collimation effect makes a contribution to the deflection which is small compared to the contribution of the energy change.