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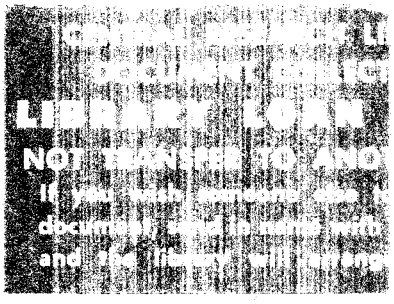
A RANGE CRITERION FOR TESTING AN OUTLYING OBSERVATION

Jack Moshman
George J. Atta



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MATHEMATICS PANEL

A RANGE CRITERION FOR TESTING AN OUTLYING OBSERVATION

by

JACK MOSHMAN
and
GEORGE J. ATTA

DATE ISSUED: FEB 18 1952

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CORRECTION TO ORNL-1020

"A Range Criterion for Testing an Outlying Observation"

Jack Moshman
George J. Atta

An error in the derivation of the critical values of g_n and g'_n has been called to the attention of the authors by C. Eisenhart, National Bureau of Standards. It was tacitly, and erroneously, assumed that, after the deletion of the suspected outlying observation in a sample of size n, the sum of squares of the deviations of the remaining $(n - 1)$ variates from their mean was equal to $(n - 1)$ times the sample variance of a random sample of size $(n - 1)$. It obviously is a smaller quantity.

Shortly after this paper was submitted for reproduction and distribution, W. J. Dixon [1] published the distribution of a quantity r_{10} which was defined as

$$r_{10} = \frac{x_n - x_{n-1}}{x_n - x_1} = 1 - \frac{1}{g_n}$$

Dixon's table of critical values of r_{10} have been converted to their equivalent in terms of g_n and appear in Table II (Corrected). The latter table should replace Table II on page 9 of the original report.

The authors acknowledge their indebtedness to Dr. Eisenhart for calling their attention to this matter.

References

[1] W. J. Dixon, "Ratios Involving Extreme Values", Annals of Mathematical Statistics 22: 68-78 (1951).

Table II (Corrected)

Critical Values of g_n and g'_n

Sample Size (n)	Level of Significance (α) 1 percent	2 percent	5 percent	10 percent
3	83.333	41.667	16.949	8.772
4	9.009	6.494	4.255	3.115
5	4.545	3.690	2.793	2.257
6	3.311	2.809	2.273	1.931
7	2.755	2.415	2.028	1.767
8	2.439	2.188	1.880	1.664
9	2.247	2.041	1.776	1.587
10	2.114	1.934	1.701	1.536
11	2.008	1.852	1.645	1.497
12	1.931	1.789	1.603	1.466
13	1.869	1.739	1.565	1.439
14	1.818	1.698	1.536	1.416
15	1.779	1.664	1.511	1.399
16	1.742	1.634	1.490	1.383
17	1.712	1.610	1.471	1.368
18	1.686	1.587	1.456	1.357
19	1.661	1.570	1.441	1.348
20	1.642	1.553	1.429	1.337
21	1.623	1.538	1.418	1.328
22	1.608	1.524	1.408	1.319
23	1.592	1.511	1.399	1.312
24	1.580	1.499	1.391	1.305
25	1.567	1.490	1.383	1.299

Table II continued

Sample Size (n)	Level of Significance (α) 1 percent (99%)	2 percent	5 percent (95%)	10 percent
26	1.555	1.479	1.376	1.294
27	1.546	1.471	1.368	1.289
28	1.536	1.462	1.362	1.282
29	1.527	1.453	1.357	1.279
30	1.517	1.447	1.351	1.274

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JACK MOSHMAN
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GEORGE J. ATTA

DATE ISSUED: JUN 12 1951

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Introduction

Of considerable importance in many fields of applied statistics is, the problem of whether or not to reject an apparently anomalous reading in a series of measurements. Most of the criteria suggested in the literature involve a statistic whose value depends on a sum of squares of deviations. In actual practice this can be an irksome task, particularly when the measurements are made to several significant figures. In this report, a criterion is proposed which is simply the ratio of two ranges, or differences.

Theory

Grubbs [1] proposed a statistic

$$T_n = \frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where the x_i are the observations ordered in magnitude, i.e. $x_1 \leq x_2 \leq \ldots \leq x_n$, and $\bar{x}_n$ and $\bar{x}$ are the means of all but the largest and all the readings respectively, i.e.

$$\bar{x}_n = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i \qquad \text{and}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad .$$

T_n is used to test whether the largest observation may be considered too large, relative to the remaining (n-1) observations, and should consequently be rejected if the test is positive.

A similar measure

$$T_n' = \frac{S_1^2}{S^2} = \frac{\sum_{i=2}^{n} (x_i - \bar{x}_1)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where

$$\bar{x}_1 = \frac{1}{n-1} \sum_{i=2}^{n} x_i$$

may be used to test whether the smallest reading is too small. Grubbs has tabulated values of T_n and T_n' for samples of size $n = 3(1)\ 25$ and $\alpha = 1$ per cent, 2.5 per cent, 5 per cent and 10 per cent, where α is the "level of significance", or the probability of incorrectly rejecting an observation which should be included.

It was shown by Patnaik [2] that a good approximation to s, the sample standard deviation based on ν degrees of freedom is w_n/c_n where w_n is the range $(x_n - x_1)$ of the sample and c_n is an appropriate scale factor based on sample size. Coupled with w_n/c_n is ν_n, the "associated" number of degrees of freedom; in general $\nu_n < \nu$.

To evaluate ν_n and c_n for various sample sizes, we equate the first two moments of w_n/σ with those of $c_n \chi / \sqrt{\nu_n}$. Let

$$E\ (w_n/\sigma) = M_n = d_n$$

and

$$\text{Var}\ (w_n/\sigma) = V_n .$$

Now d_n is the mean value of the range of a sample of size n from a normal population and is tabulated in various places, e.g. Pearson [3]. Values of $\sqrt{V_n}$ may also be found tabulated in Pearson [3].

After expansion by Stirling's formula, the first two moments of $c_n \chi / \sqrt{\nu_n}$ are

$$M_n = c_n \left\{ 1 - \frac{1}{4\nu_n} + \frac{1}{32\nu_n^2} + \frac{5}{128_n^3} + \ldots \right\} \tag{1}$$

and

$$V_n = c_n^2 \left\{ 1 - \left(1 - \frac{1}{4\nu_n} + \frac{1}{32\nu_n^2} + \frac{5}{128\nu_n^3} + \ldots \right) \right\}$$

whence

$$\frac{V_n}{M_n^2} = \frac{1}{2\nu_n} + \frac{1}{8\nu_n^2} - \frac{1}{16\nu^3} + \ldots = k, \text{ say.} \tag{2}$$

Furthermore, we may solve (1) for c_n getting

$$c_n = M_n \left(1 + \frac{1}{4\nu_n} + \frac{1}{32\nu_n^2} - \frac{5}{128\nu_n^3} + \ldots \right). \tag{3}$$

Knowing V_n and M_n, equations (2) and (3) may be solved for ν_n and c_n to the order of approximation including third powers of $(1/\nu_n)$. Table I provides the values of $M_n = d_n$, V_n, ν_n and c_n for values of n = 2(1) 15 (4) 19 (1) 20 (4) 24 (1) 25.

TABLE I

MEANS, VARIANCES, DEGREES OF FREEDOM AND SCALE FACTORS FOR THE RANGE

SAMPLE SIZE n	MEAN d_n	VARIANCE V_n	DEGREES OF FREEDOM ν_n	SCALE FACTOR c_n
2	1.128	.727	1.006	1.487
3	1.693	.789	1.934	1.916
4	2.059	.774	2.951	2.237
5	2.326	.747	3.828	2.481
6	2.534	.719	4.692	2.672
7	2.704	.694	5.499	2.830
8	2.847	.672	6.259	2.963
9	2.970	.653	6.989	3.078
10	3.078	.635	7.689	3.179
11	3.173	.619	8.356	3.269
12	3.258	.605	9.001	3.350
13	3.336	.593	9.616	3.424
14	3.407	.581	10.227	3.491
15	3.472	.570	10.806	3.553
19	3.689	.537	12.890	3.761
20	3.735	.531	13.361	3.806
24	3.895	.507	15.204	3.960
25	3.931	.503	15.606	3.994

Now we let

$$S^2 = n s^2$$

$$S_n^2 = (n-1) s_n^2$$

and substitute

$$s^2 = w_n^2 / c_n^2 = \left(\frac{x_n - x_1}{c_n} \right)^2 ,$$

$$s_n^2 = w_{n-1}^2 / c_{n-1}^2 = \left(\frac{x_{n-1} - x_1}{c_{n-1}} \right)^2 ,$$

and

$$s_1^2 = w'^2_{n-1} / c_{n-1}^2 = \left(\frac{x_n - x_2}{c_{n-1}} \right)^2$$

where w_{n-1} is the range of the (n-1) remaining values after eliminating x_n, and w'_{n-1} is the range of the (n-1) values remaining after the elimination of x_1.

Therefore

$$T_n = \frac{(n-1) s_n^2}{(n) s^2} = \frac{(n-1) w_{n-1}^2 / c_{n-1}^2}{(n) w_n^2 / c_n^2} = \frac{(n-1) w_{n-1}^2 c_n^2}{(n) w_n^2 c_{n-1}^2}$$

whence

$$\left(\frac{w_n}{w_{n-1}} \right)^2 = \frac{(n-1) c_n^2}{n c_{n-1}^2} \qquad \frac{1}{T_n}$$

or

$$\frac{w_n}{w_{n-1}} = \frac{c_n}{c_{n-1} \sqrt{T_n}} \sqrt{\frac{n-1}{n}} = g_n , \text{ say.}$$

Similarly

$$\frac{w_n}{w'_{n-1}} = \frac{c_n}{c_{n-1}\sqrt{T'_n}} \sqrt{\frac{n-1}{n}} = g'_n \ .$$

Making use of the foregoing approximations, critical values of g_n and g'_n, which are equal, may be found by multiplying the square root of the reciprocal of Grubbs' T_n values by the factor

$$\frac{c_n}{c_{n-1}} \sqrt{\frac{n-1}{n}} \ .$$

This has been done and appears in Table II.

TABLE II

CRITICAL VALUES OF g_n AND g'_n

SAMPLE SIZE (n)	LEVEL OF SIGNIFICANCE (α)			
	1 per cent	2.5 per cent	5 per cent	10 per cent
3	105.237	39.782	20.256	10.076
4	10.117	6.415	4.545	3.232
5	4.719	3.494	2.783	2.222
6	3.231	2.583	2.176	1.846
7	2.577	2.158	1.890	1.654
8	2.222	1.915	1.714	1.534
9	2.000	1.759	1.602	1.455
10	1.839	1.652	1.517	1.403
11	1.727	1.571	1.457	1.365
12	1.647	1.512	1.410	1.328
13	1.580	1.456	1.374	1.303
14	1.524	1.421	1.340	1.279
15	1.480	1.388	1.316	1.255
20	1.348	1.277	1.226	1.196
25	1.272	1.221	1.191	1.151

Numerical Example

Eight readings were taken of the current through a new type resistor. These were, in milliamperes, when ranked in order of magnitude:

12.107

12.112

12.133

12.148

12.151

12.152

12.159

12.202

The highest measurement appears to be too high. As a test we compute

$$g_8 = \frac{12.202 - 12.107}{12.159 - 12.107} = \frac{.95}{.52} = 1.827.$$

From Table II we see that when $n = 8$, a value of $g = 1.827$ would occur only between 2.5 per cent and 5 per cent of the time by chance, if all readings were taken from the same underlying population. Hence one could reject the reading 12.202 with a risk of less than 5 per cent of being in error.

REFERENCES

1. Grubbs, Frank E., "Sample Criteria for Testing Outlying Observations", The Annals of Mathematical Statistics, Vol. 21, No. 1 (March 1950) p. 27.

2. Patnaik, P. B., "The Use of the Mean Range as an Estimation of Variance in Statistical Tests", Biometrika, Vol. 37 (1950) p. 79.

3. Pearson, E. S., "The Percentage Limits for the Distribution of Range in Samples from a Normal Population ($n \leq 100$)", Biometrika, Vol. 24 (1932) p. 404.