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# ON THE THEORY OF INNER BREMSSTRAHLUNG IN $\beta$-PROCESS

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PHYSICS DIVISION

# ON THE THEORY OF INNER BREMSSTRAHLUNG IN $\beta$-PROCESSES

J. M. Jauch

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# ON THE THEORY OF INNER BREMSSTRAHLUNG IN $\beta$ -PROCESSES

J. M. Jauch

## I. INTRODUCTION

The low energy continuous $\gamma$ -radiation accompanying various types of $\beta$ -processes have been studied theoretically by a number of workers. Knipp and Uhlenbeck[1] and Bloch[2] developed the theory for the ordinary $\beta$ -processes. Morrison and Schiff[3] applied the theory to the case of K-capture and Chang and Falkoff[4] reviewed the earlier work and extended it to the case of other coupling types and forbidden $\beta$ -transitions.

The fundamental physical process involved is the following: In the case of $\beta$ -transitions an electronic charge is suddenly released from the nucleus and transferred to the outgoing electron. At the same time a magnetic moment of the order of a Bohr magneton is created. This sudden change of the state of motion of the charges causes a rearrangement of the electromagnetic field oscillators which results in an emission of radiation. It is clear that this $\gamma$ -emission is a general feature of all types of charge transfer processes and could be observed under many different experimental conditions. Thus for instance it was pointed out by Hayakawa and Tomonaga[5] that such radiation should occur in the high energy nuclear collisions which result in meson production. This effect was studied by L. I. Schiff[6]. A good example is also furnished by the $\mu$ -meson decay. This case has not yet been studied.

The soft $\gamma$-radiation has been observed by various experimental workers. Stahel and Guillissen[7] have compared the total radiation emitted by RaE with the theory and Wu[8] has done the same for $P^{32}$. Both found excellent agreement with the theory of Knipp and Uhlenbeck. In the case of K-capture the radiation was observed for the first time by H. Bradt and coworkers[9] with $Fe^{55}$. For K-capture processes the total number of photons emitted per capture process depends on the total energy difference for the nuclear transition. Bradt and coworkers made a rough estimate for this energy in $Fe^{55}$ and found a value $w_o = 0.15$ MeV. A more sensitive method would be the measurement of the energy spectrum and determination of the endpoint with the method of a Kurie plot.

The sensitive solid $\gamma$-ray counters now available make possible for the first time a detailed study of this radiation[10]. The quantities which can be measured are the energy spectrum and the total number of photons per nuclear process. From this can be obtained the maximum energy of the nuclear transition. This is of special interest in the case of K-capture where this process seems the only possiblility of measuring this quantity directly. One can also measure the angular correlation of the $\gamma$-rays with the electrons. Theory predicts a strong angular correlation with $\gamma$-rays coming out preferentially in the direction of the electrons.

For allowed transitions this correlation turns out to be independent of the interaction type so that in these cases no information can be gained

regarding these interactions. In the case of the $\mu$ -meson decay a small dependence on the interaction type remains which results from the restriction imposed on the four outcoming particles by conservation of energy and momentum. The effect would be very difficult to observe experimentally.

In this report we shall discuss the theory of this radiation for the cases of $\beta$ -decay and K-capture. For the case of ordinary $\beta$ -decay this is partly a repetition of previous work, the case of K-capture is a slight improvement of older work.

Preliminary measurements on the continuous $\gamma$-radiation in the K-capture of $Fe^{55}$ are analyzed and the feasibility of the method is demonstrated.

## II. INNER BREMSSTRAHLUNG FOR ALLOWED -DECAY

The inner Bremsstrahlung results from the transitions produced by two parts of the Hamiltonian in a second order process. The part $H_\beta$ may be written

$$H_\beta = G\,(\bar{\Psi}\,\Gamma^\alpha\,\Phi)(\bar{\psi}\,\gamma_\alpha\,\varphi) \tag{2.1}$$

Here the quantities $\Psi$, $\Phi$, $\bar{\psi}$, $\varphi$ are the spinor field operators referring to the proton, neutron, electron and neutrino respectively. $\Gamma^\alpha$, $\gamma_\alpha$ represent any one of the five combinations of the Dirac $\Gamma^\mu$ for nucleous and $\gamma_\mu$ for leptons which represent the five coupling types in $\beta$ -theory. The Dirac $\gamma_\mu$ are chosen so as to satisfy

$$\{\gamma_\mu \gamma_\nu\} = 2 g_{\mu\nu} \tag{2.2}$$

The spinors $\bar{\Psi}$ and $\bar{\psi}$ are the conjugate spinors defined in the following way:

Let $\gamma_\mu^+$ be the Hermitian conjugate of $\gamma_\mu$ , then we define a matrix $\eta$ up to an arbitrary real factor by the conditions

$$\gamma_\mu^+ = -\eta \gamma_\mu \eta^{-1} \tag{2.3}$$

$$\eta^+ = \eta \tag{2.4}$$

Then we define

$$\bar{\psi} = \psi^* \eta \tag{2.5}$$

where $\psi^*$ is the Hermitian conjugate of $\psi$ . If $\psi$ is considered a column spinor then $\psi^*$ and $\bar{\psi}$ is a row spinor. This definition does not make use of any special representation of the $\gamma$ . For free fields (no external fields present) the $\psi$ can be decomposed in plane waves. We shall write

$$\psi(x) = \frac{1}{\sqrt{\Omega}} \sum_p \left( a_p u e^{ipx} + b_p^* v e^{-ipx} \right) \tag{2.6}$$

where $\Omega$ is the fundamental volume. p is the four vector of energy and momentum. $a_p$ is the operator which annihilates an electron in state p, $b_p^*$ creates a positron in state p. The c-number spinors u, v satisfy the equations

$$\begin{aligned} K_+ u &= 0 \\ K_- v &= 0 \end{aligned} \tag{2.7}$$

where $K_{\pm}(p) = i\gamma p \pm m$ . Since $p^2+m^2 = 0$ these operators are of rank two and allow two linearly independent solutions $u_r(p), v_s(p)$; $(r,s = 1,2)$ each. They are normalized according to

$$(\bar{u}_r u_s) = \frac{m}{E}\delta_{rs}$$
$$(\bar{v}_r v_s) = -\frac{m}{E}\delta_{rs} \tag{2.8}$$

where $E = p^0$.

From (2.8) follow the important relations

$$\sum_r u_{r\rho}\bar{u}_{r\sigma} = -\frac{1}{2E}K_{-\rho\sigma}$$
$$\sum v_{r\rho}\bar{v}_{r\sigma} = -\frac{1}{2E}K_{+\rho\sigma} \tag{2.9}$$

which we shall use often in the following calculations.

The second Hamiltonian is that describing the interaction with the electromagnetic radiation and is given by

$$H_e = ie\, A^{\mu}\,\bar{\psi}\gamma_{\mu}\psi \tag{2.10}$$

Here $A_{\mu}$ is the operator for the transverse electromagnetic field which we shall write in Fourier decomposition

$$A_{\mu}(x) = \frac{1}{\sqrt{\Omega}}\sum_{r,k}\frac{1}{\sqrt{2\omega}}\left(a_{rk}e_{\mu}^{(r)}e^{ikx} + a_{rk}^{*}e_{\mu}^{(r)}e^{-ikx}\right) \tag{2.11}$$

The operator $a_{rk}$ annihilates a photon with momentum and energy k ($k^0 = \omega$) and polarization vector $e_{\mu}$ . For reasons of formal symmetry we treat $e_{\mu}$

as a four vector with the special condition $e_0 = 0$. We also have

$$(e^{(r)} \cdot k) = 0 \qquad (r = 1, 2) \tag{2.12}$$

The constant e in (2.10) is the dimensionless interaction constant and related to the fine structure constant $\alpha = \frac{1}{137}$ by $e = \sqrt{4\pi\alpha}$. The interaction constant G in (2.1) is of dimension $cm^2$ and of order $10^{-32}$ $cm^2$. The expressions (2.1) and (2.10) are the Hamiltonian densities. The total Hamiltonians are obtained by integrating over the three dimensional volume $\Omega$.

The process we wish to calculate is obtained by a second order perturbation calculation according to the standard formula for the transition probability per unit time dP for the emission of an electron with momentum p within dp and solid angle $d\alpha$, a photon of momentum $\underline{k}$ within dk and solid angle $d\beta$ and a neutrino within the solid angle $d\gamma$.

$$dP = 2\pi \sum |H|^2 \, d\rho \tag{2.13}$$

The summation sign indicates the summation over all the final spin states and the polarization and will later also be interpreted to include integration over all the directions of the neutrino. H stands for the second order matrix element

$$H = \sum \frac{H_{fn} \, H_{ni}}{E_i - E_n} \tag{2.14}$$

and $d\rho$ denotes the density per unit energy of the final states

$$d\rho = \frac{\Omega}{(2\pi)^3} p^2 dp\, d\alpha \frac{\Omega}{(2\pi)^3} k^2 dk\, d\beta \frac{\Omega}{(2\pi)^3} \nu^2 d\gamma \tag{2.15}$$

The initial (i), final (f) and intermediate states (n) are characterized as follows

<u>Initial state</u>

Nucleus in state given by a nuclear wave function V

<u>Intermediate states</u>

a) Nucleus U

Electron $\underline{P}'$, E'

Neutrino $\underline{\nu}$, $\varepsilon$

b) Nucleus V

Positron $\underline{q}$, F

Electron $\underline{p}$, E

Photon $\underline{k}$, $\omega$

<u>Final state</u>

Electron $\underline{p}$, E

Photon $\underline{k}$, $\omega$

Neutrino $\underline{\nu}$, $\varepsilon$

Nucleus U

There are thus just two intermediate states possible in this case and H becomes

$$H = \frac{H_{fa}\, H_{ai}}{E_i - E_a} + \frac{H_{fb}\, H_{bi}}{E_i - E_b} \tag{2.15}$$

For the energies we have

$E_i = \Delta =$ maximum energy available for the $\beta$ -transition

$$E_a = E' + \varepsilon$$
$$E_b = \Delta + E + F + \omega \tag{2.16}$$

Energy conservation requires

$$\Delta = E + \varepsilon + \omega \tag{2.17}$$

and we obtain

$$E_i - E_a = W - F$$
$$E_i - E_b = -W - F \tag{2.18}$$

with

$$W = E + \omega$$

The matrix elements involved in this transition are*

$$H_{ai} = \frac{G}{\Omega} \bar{u}(p') M^{\alpha} \gamma_{\alpha} v(\nu)$$
$$H_{fa} = \frac{ie}{\sqrt{\Omega 2\omega}} \bar{u}(p)(\not{\epsilon}) u(p')$$
$$H_{bi} = \frac{ie}{\sqrt{\Omega 2\omega}} \bar{u}(p)(\not{\epsilon}) v(q) \tag{2.19}$$
$$H_{fb} = -\frac{G}{\Omega} \bar{v}(q) M^{\alpha} \gamma_{\alpha} v(\nu)$$

---

*The difference in sign between the $H_{ai}$ and $H_{fb}$ arises from the different order in which the emission and absorption processes occur in the two intermediate states a and b. Indeed the relevant operators which multiply these matrix elements are for the two cases $a_p^* a_{p'} a_{p'}^*$ for (a) and $b_q a_p^* b_q^*$ for (b). We see that the emission and absorption operators are in a different order. By bringing them in the same order we see that the relative sign of the two terms is opposite since the operators anticommute. The absolute sign is arbitrary and irrelevant. We have chosen +1 for case (a) and - 1 for case (b). One sees then immediately from expression (2.21) that this charge conjugate treatment adopted here is equivalent with the "naive" negative energy picture where virtual electrons are emitted into negative energy states and no account is taken of the exclusion

(continue on next page)

with $$M^{\alpha} = (v \Gamma^{\alpha} u) \tag{2.20}$$

Substituting these expressions in H (2.15) we obtain

$$H = \frac{ieG}{\sqrt{\Omega^3 2\omega}} \left\{ \frac{S_u}{W-F} + \frac{S_v}{W+F} \right\} \tag{2.21}$$

with

$$\begin{aligned} S_u &= \sum M^{\alpha} \bar{u}(p)(\gamma e) u(p') \bar{u}(p') \gamma_{\alpha} v(\nu) \\ S_v &= \sum M^{\alpha} \bar{u}(p)(\gamma e) v(q) \bar{v}(q) \gamma_{\alpha} v(\nu) \end{aligned} \tag{2.22}$$

and the summation is extended over the spin states of the electron (p') and positron (q). With the help of the relations (2.9) we find for these sums

$$\begin{aligned} S_u &= -\frac{1}{2F} M^{\alpha} \bar{u}(p)(\gamma e) K_{-}(p') \gamma_{\alpha} v(\nu) \\ S_v &= -\frac{1}{2F} M^{\alpha} \bar{u}(p)(\gamma e) K_{+}(q) \gamma_{\alpha} v(\nu) \end{aligned} \tag{2.23}$$

Here use was made of the momentum conservation in the intermediate states

$$\begin{aligned} \underline{p}' &= \underline{p} + \underline{k} \\ \underline{q} &= -\underline{p} - \underline{k} \end{aligned} \tag{2.24}$$

from which follows $E' = F$ (2.25)

---

* (continued)
principle for the supposedly filled negative energy states. This situation is familiar from the theory of the Compton effect. As a curious fact we mention here that the final result for $d\underline{P}$ comes out the same whether the sign in (2.19) for $H_{fb}$ is correct or not.

which was used in (2.23).

We can now substitute (2.23) into the curly bracket in (2.21) and obtain

$$\frac{S_u}{W-F} + \frac{S_v}{W+F} = \frac{1}{W^2-F^2}\Big(W(S_u+S_v) + F(S_u-S_v)\Big)$$

$$= \frac{1}{W^2-F^2}\,\bar{u}(p)(\not{e})K_-(g)M^\alpha\gamma_\alpha\, v(\nu) \tag{2.26}$$

where g has the four components

$$\underline{g} = \underline{p}' = \underline{p} + \underline{k}$$

$$g^0 = W = E + \omega \tag{2.27}$$

We now find from (2.21) and (2.26) for $|H|^2$

$$\bar{\Sigma}|H|^2 = \frac{e^2G^2}{\Omega^3\, 2\omega}\,\frac{1}{(W^2-F^2)^2}\,X \tag{2.28}$$

where

$$X = \bar{\Sigma}\,|\bar{u}(p)(\not{e})K_-(g)M^\alpha\gamma_\alpha\, v(\nu)|^2 \tag{2.29}$$

A closer examination of this expression shows that as long as we consider only the unrelativistic approximation to the nuclear wave function (this means we disregard the small components in the nuclear matrix elements). We can factor out the nuclear matrix element for all five covariants and obtain

$$X = -\frac{|M|^2}{\varepsilon E} \sum Y \tag{2.30}$$

$$Y = \frac{1}{4} \operatorname{tr} (\gamma e)(i\gamma g - m)(i\gamma \nu)(i\gamma g - m)(\gamma e)(i\gamma p - m) \tag{2.31}$$

where we have made use again of (2.9).

The expression for Y can be evaluated with standard methods of trace calculations and gives

$$\begin{aligned} Y = {} & e^2(\nu p)(g^2+m^2) - 2e^2(\nu g)(gp+m^2) \\ & - 2(pe)(\nu e)(g^2+m^2) + 4(pe)(ge)(\nu g) \end{aligned} \tag{2.32}$$

We now carry out the summation over the photon polarization which gives

$$\overline{e^2} = 2 \qquad \overline{(pe)(ge)} = \underline{p}\,\underline{g} - (\underline{p}\,\underline{\kappa})(\underline{g}\,\underline{\kappa})$$

$$\underline{\kappa} = \frac{1}{\omega}\underline{k}$$

and the integration over all directions of the neutrino

$$\bar{\nu} = 4\pi\,(\varepsilon, 0, 0, 0) \tag{2.33}$$

The third term in (2.32) is then zero since $e^0 = 0$ and for the others we obtain

$$\begin{aligned} \sum Y = 4\pi\varepsilon \Big\{ & -2E(g^2+m^2) + 4W(gp+m^2) \\ & - 4W(\underline{p}\,\underline{g} - \underline{p}\underline{\kappa}\,\underline{g}\underline{\kappa}) \Big\} \end{aligned} \tag{2.34}$$

We finally express the result in terms of the variables E, $\omega$, $W = E + \omega$ , $\Theta = E - p\cos\vartheta$.

The following formulae are useful in this calculation

$$
\begin{aligned}
g^2 + m^2 &= -2\omega\,\Theta \\
gp + m^2 &= -\omega\,\Theta \\
(\underline{p}\,\underline{g}) - (\underline{p}\,\underline{k})(\underline{g}\cdot\underline{k}) &= -\Theta^2 + 2E\Theta - m^2 \\
W^2 - F^2 &= 2\omega\,\Theta
\end{aligned}
\tag{2.35}
$$

This gives

$$\Sigma\, y = 16\pi\,\varepsilon\, W\left\{-\frac{E^2+W^2}{W}\,\Theta + \Theta^2 + m^2\right\} \tag{2.36}$$

and

$$X = 16\pi\,|M|^2\,\frac{W}{E}\left\{\frac{E^2+W^2}{W}\,\Theta - \Theta^2 - m^2\right\} \tag{2.37}$$

$$\Sigma\,|H|^2 = 2\pi\,\frac{e^2G^2}{\Omega^3\omega^3}\,|M|^2\,\frac{W}{E}\left\{\frac{E^2+W^2}{W}\,\frac{1}{\Theta} - \frac{m^2}{\Theta^2} - 1\right\} \tag{2.38}$$

For the final expression we may then write

$$d\underline{P} = \underline{P}_0\, F(\vartheta)\, d\beta\, dE\, d\omega \tag{2.39}$$

where

$$P_0\, dE = (4\pi)^2 \frac{G^2}{(2\pi)^5} |M|^2 W (W^2 - m^2)^{1/2} (\Delta - W)^2 dE \tag{2.40}$$

is the probability for emission of an electron with energy $W = E + \omega$ within dE per unit time and

$$F(\vartheta)\, d\beta\, d\omega = \frac{\alpha}{4\pi^2} \frac{p}{(W^2 - m^2)^{1/2}} \left\{ \frac{E^2 + W^2}{W} \frac{1}{\Theta} - \frac{m^2}{\Theta^2} - 1 \right\} \tag{2.41}$$

is the probability that such an electron is accompanied by a photon at angle $\vartheta$ within solid angle $d\beta$ and energy $\omega$ within $d\omega$. This is the result obtained by Knipp and Uhlenbeck for the Fermi type interaction (Vector).

For the total probability of emitting a photon with energy $\omega$, within $d\omega$ in any direction together with an electron of energy E within dE. We obtain by integrating (2.41) over the solid angle

$$F\, d\omega = \int_\beta F(\vartheta)\, d\beta\, d\omega$$

$$F\, d\omega = \frac{\alpha}{\pi} \frac{p}{(W^2 - m^2)^{1/2}} \left\{ \frac{E^2 + W^2}{pW} \ln \frac{W + p}{m} - 2 \right\} \frac{d\omega}{\omega} \tag{2.42}$$

The total number of photons per unit time of energy $\omega$ within d $\omega$ emitted regardless of the electron energy is given by

$$\Phi(\omega)d\omega = \int_{m+\omega}^{\Delta} dW \, F \, P_0 \tag{2.43}$$

From this we can obtain the behaviour of the photon spectrum near the maximum energy in the form

$$\Phi(\omega)d\omega \simeq \frac{C}{3}\eta^3 d\omega \tag{2.44}$$

$$\eta = \omega_0 - \omega \qquad \omega_0 = \Delta - m$$

$$C \simeq \frac{\alpha G^2}{2\pi^4}|M|^2 \frac{m^2+\Delta^2}{\Delta} \ln\frac{\Delta}{m} \tag{2.45}$$

The following two limiting forms of formula (2.42) are also useful.

For E $\sim$ m $\quad \omega$ arbitrary, we obtain

$$F(\omega) \simeq \frac{\alpha}{\pi} \frac{v\omega}{(2m\omega+\omega^2)^{1/2}(m+\omega)} \tag{2.46}$$

( $v$ = volicity in units c = $\frac{p}{m}$ )

For E arbitrary $\quad \omega \ll m$

$$F \simeq \frac{\alpha}{\pi}\frac{1}{\omega}\left\{\left(\frac{2E}{p}\ln\frac{E+p}{m} - 2\right) + \omega\left(\frac{E}{p^2}\left[-\frac{2E}{p}\ln\frac{E+p}{m} + 2\right] + \frac{2E}{p(E+p)}\right)\right\} \tag{2.47}$$

As an illustration and a check we shall give a derivation of the result (2.39) with the Feynman-Dyson graph method[11]. A slight generalization is needed which is easily obtained with Dyson's reduction of the multiple integral over P-brackets.

We calculate directly the S-matrix in the form

$$S = S_0 + S_1 + S_2 + \cdots \tag{2.48}$$

$$S_0 = I \qquad \text{(unit matrix)}$$

We are only interested in the terms in $S_2$ which give rise to the transition in question. These terms are contained in

$$S_2 = \frac{(-i)^2}{2!}\iint dx_2\, dx_1\, P\left(H_\beta(x_2)\, H_e(x_1) + H_e(x_1)\, H_\beta(x_2)\right) \tag{2.49}$$

Here P is the time-ordering operator defined by Dyson[11]. A closer examination reveals that the two integrals are the same and thus just cancel the factor 2 in front. Thus we also have

$$S_2 = -\iint dx_2\, dx_1\, P\left(H_\beta(x_2)\, H_e(x_1)\right) \tag{2.50}$$

When we select out of this double integral one matrix element corresponding to a given transition, it appears in the form

$$S_2' = R\,(2\pi)^4\,\delta(P_0 - P) \tag{2.51}$$

where R is the matrix element depending on the initial and final momenta of all the particles involved and $P_0$, P are the initial and final momentum

four vectors. In our problem we have an initial nucleus of momentum $K_0$ and internal state function $U_0$. The final state is characterized by a nuclear momentum K, wave function U, electron p, photon k, neutrino v. Thus

$$R = (K, p, k, \nu \,|R|\, K_0) \tag{2.52}$$

$$P_0 = K_0$$

$$P = K + p + k + \nu \tag{2.53}$$

The transition probability per unit time is then obtained according to the formula

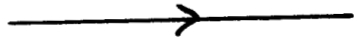
$$Q = \int |R|^2 (2\pi)^4 \, \delta(P^\circ - P) (2\pi)^{-3n} d^3K \, d^3p \, d^3k \, d^3\nu \tag{2.54}$$

where n is the number of outgoing particles, in this case $n = 4$, and the integration is extended over the respective volume elements characterizing the final state. It may also include the summation over spins and polarization.

The problem is thus reduced to calculate R. This can be done by drawing the graph corresponding to the process in question and then obtaining the corresponding analytical expression by a generalization of the rules derived in Dyson[12].

The graph in question contains four types of lines and two vertices. The types of lines are

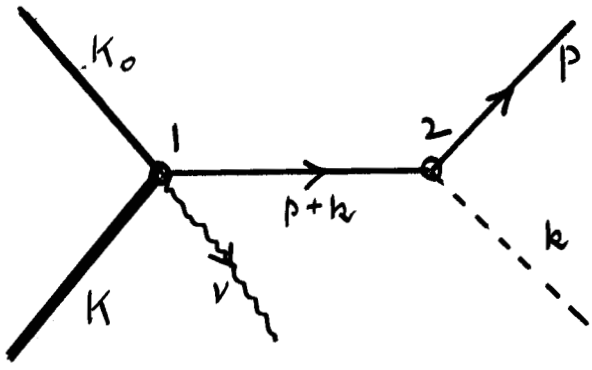
1) Heavy directed for the nucleus

2) Light directed for the electron 

3) Wavy directed for the neutrino

4) Dotted undirected for the photon

The graph which represents the inner Bremsstrahlung in $\beta$ -decay is given by




(2.55)

The incoming and outgoing nucleus is represented by $GM^{\alpha} = G(\bar{u}\,\Gamma^{\alpha} u_0)$ the nuclear matrix element. The outgoing neutrino by $v(\nu)$ , the vertex 1 by $\gamma_\alpha$ the link 1 → 2 by the propagation function $\frac{i\not{(p+k)} - m}{(p+k)^2 + m^2}$ the outgoing electron by u(p) and the outgoing photon by $\frac{1}{\sqrt{2\omega}} e^{\mu}$ and finally the vertex 2 by $\gamma_\mu$ . Thus the analytical form of (2.55) is

$$R = \frac{-ieG}{\sqrt{2\omega}} M^{\alpha} \left( \bar{u}(p)(e\not{\epsilon}) \frac{i\not{(p+k)} - m}{(p+k)^2 + m^2} \gamma_\alpha v(\nu) \right) \qquad (2.55)'$$

In order to see that this reduces to the same result (2.39) we need only to remark that according to (2.27)

$$p + k = q \qquad (p+k)^2 + m^2 = E^2 - W^2$$

so that

$$\frac{i(p+k)\gamma^2 - m}{(p+k)^2 + m^2} = \frac{1}{F^2 - W^2} K_-(g) \tag{2.56}$$

in the previous notation. Comparison of (2.55), (2.56) with (2.21) and (2.26) shows that the agreement is complete. (The fundamental volume $\Omega = 1$ in this calculation.)

## III. INNER BREMSSTRAHLUNG IN K-CAPTURE

In the K-capture process we have also a sudden change of motion of a charge and the process should be accompanied by an inner Bremsstrahlung. Since in this case charge is only transferred from the distribution in the K-shell to the nucleus it is expected to be a less intense radiation than for $\beta$-emission. This is borne out by the calculation.

Besides this difference in energy release the process differs from the radiation in $\beta$-emission also qualitatively. First the succession of intermediate states is a different one since we start out with one external electron and end with one photon and one neutrino. The nucleus can first emit a virtual positron and a real neutrino. The positron is then subsequently annihilated by the electron in the K-shell. Alternatively, the K-electron can first get scattered into a different state under emission of a photon

and is subsequently absorbed by the nucleus under emission of the neutrino. A second difference arises from the fact that the electron in the bound state has a finite extension in $\chi$ -space. This allows transitions to intermediate states with a continuum range of the momentum. Thus we have not just a summation over the spin states but also an integration over various momentum values in the intermediate states.

For the Hamiltonian which is responsible for the transitions, we can again take

$$H = H_\beta + H_e \tag{3.1}$$

$$H_\beta = G\,(\bar{\Phi}\,\Gamma^\alpha\,\Psi)\,(\bar{\varphi}\,\gamma_\alpha\,\psi) \tag{3.2}$$

$$H_e = ie\,A^\nu\,(\bar{\psi}\,\gamma_\nu\,\psi) \tag{3.3}$$

with the same notation as in Section II.

We use Equ. (2.13), (2.14) of the previous section. For the density of final states we have instead of (2.15)

$$d\rho = \frac{\Omega}{(2\pi)^3}\,k^2\,dk\,d\beta\,\frac{\Omega}{(2\pi)^3}\,\nu^2\,d\gamma \tag{3.4}$$

since there are only two outgoing particles.

We characterize the initial and final nuclear wave function by V and U respectively and denote the nuclear matrix elements with

$$M^{\alpha} = (\bar{u}\,\Gamma^{\alpha} v) \tag{3.5}$$

For the initial electronic wave function we write $u_k(\underline{x})$ and for its Fourier transform

$$u_K(q) = \int u_K(x)\, e^{-i\underline{q}\underline{x}}\, d^3x \tag{3.6}$$

The $u_k(\underline{x})$ and $u_k(\underline{q})$ are spinor functions representing the state vector of the electron in the K-shell. For the actual application the small components can be neglected. The initial, final and intermediate states are then characterized as follows:

<u>Initial state:</u>

Nucleus V

Electron $u_k$

<u>Intermediate states:</u>

a) Electron $u_k$

Positron $\underline{q}$, F

Nucleus U

Neutrino $\underline{v}$, $\mathcal{E}$

b) Electron $\underline{p}$, E

Nucleus V

Photon $\underline{k}$, $\omega$

<u>Final state:</u>

Nucleus U

Neutrino $\underline{\nu}$, $\mathcal{E}$

Photon $\underline{k}$,

For the energies in these states we have

$$E_i = \Delta + K \tag{3.7}$$

Here $\Delta$ represents the energy of the initial nucleus (not the atom) and K the energy of the K-electron including its rest energy. We take again the zero point for the nuclear energy, the energy of the final <u>nucleus</u>.

$$\begin{aligned} E_a &= K + F + \mathcal{E} \\ E_b &= E + \omega + \Delta \end{aligned} \tag{3.8}$$

Energy conservation gives

$$\Delta + K = \mathcal{E} + \omega \tag{3.9}$$

and for the energy denominators we obtain

$$\begin{aligned} E_i - E_a &= -W - F \\ E_i - E_b &= W - E \end{aligned} \tag{3.10}$$

$$W = K - \omega \tag{3.11}$$

For the matrix elements of these processes we obtain

$$H_{ai} = \frac{G}{\Omega}\,\bar{u}(\nu)(M\gamma)\,v(q)$$

$$H_{fa} = \frac{ie}{\sqrt{\Omega 2\omega}}\,\frac{1}{\sqrt{\Omega}}\,\bar{v}(q)(e\gamma)\,u_K(-q+k) \tag{3.12}$$

$$H_{bi} = \frac{ie}{\sqrt{\Omega 2\omega}}\,\frac{1}{\sqrt{\Omega}}\,\bar{u}(p)(e\gamma)\,u_K(p+k)$$

$$H_{fb} = -\frac{G}{\Omega}\,\bar{u}(\nu)(M\gamma)\,u(p)$$

This gives for

$$H = \sum_a \frac{H_{fa}\,H_{ai}}{E_i - E_a} + \sum_b \frac{H_{fb}\,H_{bi}}{E_i - E_b}$$

$$H = \frac{ieG}{\Omega^2\sqrt{2\omega}}\sum_p \bar{u}(\nu)(M\gamma)\,\frac{K_-(g)}{W^2-E^2}\,(e\gamma)\,u_K(p+k) \tag{3.13}$$

where

$$\underline{g} = \underline{p}$$

$$g^0 = W$$

Introducing the notation $\quad u_K(x) = \omega(x)\,\chi$

$$\rho_i = \frac{1}{(2\pi)^3}\int \frac{p_i}{W^2-E^2}\,\omega(\underline{p}+\underline{k})\,d^3p$$

$$\rho^0 = \frac{1}{(2\pi)^3}\int \frac{W}{W^2-E^2}\,\omega(\underline{p}+\underline{k})\,d^3p \tag{3.14}$$

$$\mu = \frac{1}{(2\pi)^3}\int \frac{m}{W^2-E^2}\,\omega(\underline{p}+\underline{k})\,d^3p.$$

We obtain finally for the probability of the emission of a photon of energy $\omega$ within $d\omega$

$$dP = \frac{2\alpha}{\pi^2} G^2 |M|^2 \left[ \underline{\rho}^2 + (\rho^0 - \mu)^2 \right] \omega (W_0 - \omega)^2 d\omega \qquad (3.15)$$

where $W_0 = \Delta + K$ is the maximum energy for the photons.

The relative probability per emission process is obtained by dividing the expression for dP with $P_0$ the K-capture probability per unit time

$$P_0 = \frac{1}{\pi} G^2 |M|^2 |\omega(0)|^2 W_0^2 \qquad (3.16)$$

This gives

$$\frac{dP}{P^0} = \frac{2\alpha}{\pi} \frac{\underline{\rho}^2 + (\rho^0 - \mu)^2}{|\omega(0)|^2} \omega \left(1 - \frac{\omega}{W_0}\right)^2 d\omega \qquad (3.17)$$

If the photon energy is large compared to the binding energy of the K-electron $\omega \gg K - m$ we may approximately evaluate the integrals in (3.14) by putting the K-state wave function = const. = 1.

From (3.6) we get

$$u_K(q) = (2\pi)^3 \delta(q) \qquad (3.18)$$

Thus

$$\rho_i \simeq \frac{k_i}{2m\omega}$$

$$\rho^0 \simeq \frac{\omega - K}{2m\omega} \qquad (3.19)$$

$$\mu \simeq -\frac{m}{2m\omega}$$

and

$$\rho^2 + (\rho^0 - \mu)^2 \simeq \frac{1}{2m^2}$$

Thus

$$\frac{dP}{P^0} = \frac{\alpha}{\pi} \frac{\omega\, d\omega}{m^2} \left(1 - \frac{\omega}{W_0}\right)^2 \tag{3.20}$$

approximation valid for $\omega \gg K - m$ thus especially at the upper end of spectrum. The total number of photons emitted per K-capture process is very nearly given by

$$n \sim \int \frac{dP}{P_0} = \frac{\alpha}{12\pi} \left(\frac{W_0}{m}\right)^2 \tag{3.21}$$

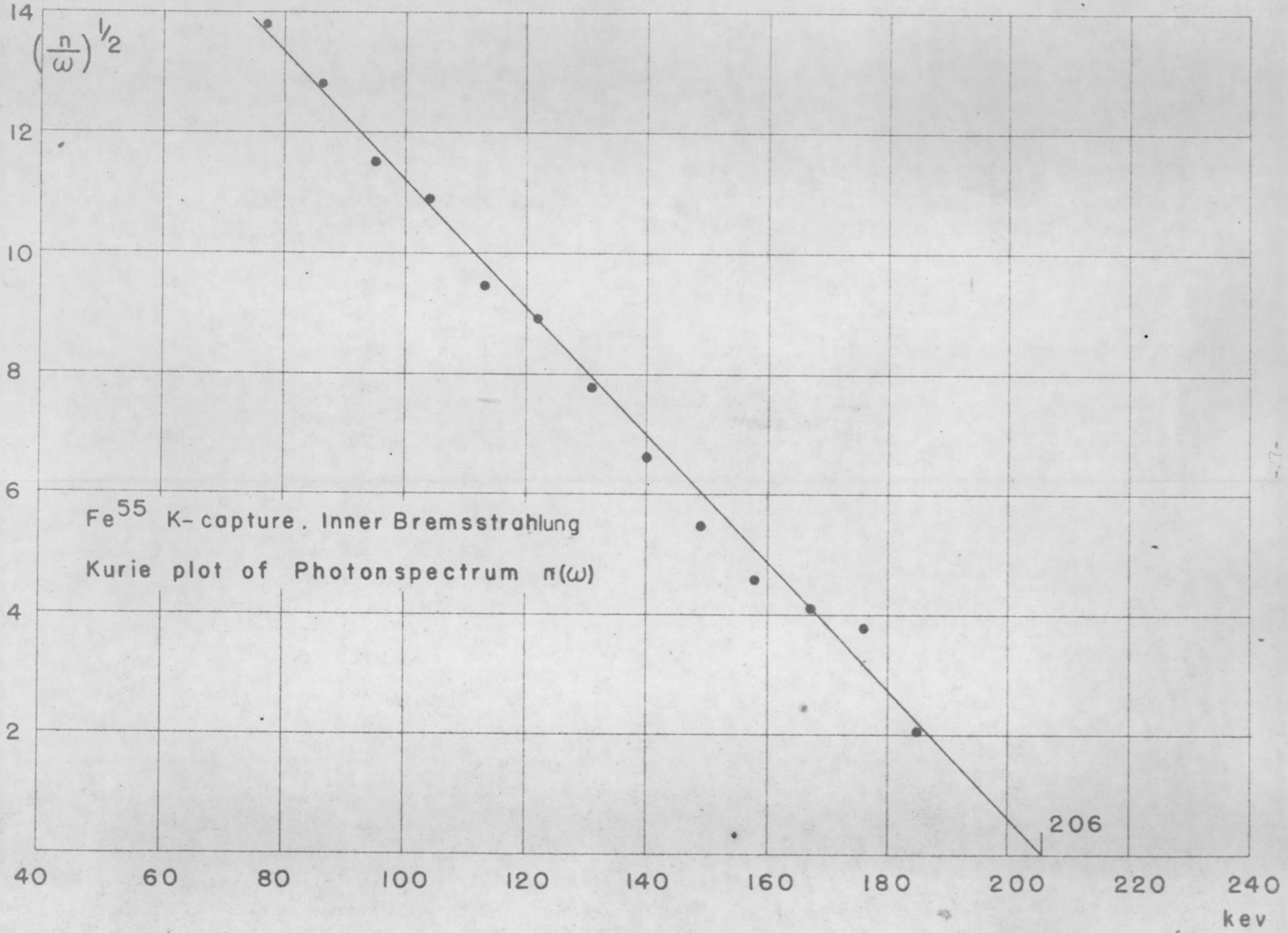
Thus, for instance, for $Fe^{55}$ $W_0 \simeq 0.210$ MeV

$$n \sim 3.2 \times 10^{-5} \tag{3.22}$$

From formula (3.20) one deduces that the photon spectrum is expected to have its maximum at

$$\omega_{max} = \frac{1}{3} W_0 \tag{3.23}$$

In order to test the feasibility of the method here proposed for measuring the energy release in K-capture processes, P. R. Bell and J. Cassidy have made a test run on $Fe^{55}$ which goes by pure K-capture to $Mn^{55}$. This was the same substance used by Bradt[9] and coworkers who made a crude determination of this energy. They give a value of 150 keV.


$\left(\frac{n}{\omega}\right)^{1/2}$
14
12
10
8
6
4
2
$Fe^{55}$ K-capture. Inner Bremsstrahlung
Kurie plot of Photonspectrum $n(\omega)$
206
40
60
80
100
120
140
160
180
200
220
240
kev

The graph shows a plot of $\left(\frac{n}{\omega}\right)^{1/2}$ where $n(\omega)d\omega$ is the number of photons of energy $\omega$ within $d\omega$. The plot according to (3.20) is supposed to be a straight line. The intersection with the abscissa gives the maximum energy and turns out to be $W_0 = 206$ keV.

This compares well with the independent determination of this energy by the measurement of the Q value in the reaction $Mn^{55}(P,N)Fe^{55}$. This was done by Stelson and Preston[13] who find $Q = 1.006 \pm 0.010$. This gives $W_0 = 219$ keV $\pm$ 10 keV.

## REFERENCES


1. J. K. Knipp, G. E. Uhlenbeck, Physica 3, 425 (1936).
2. F. Bloch, Phys. Rev. 50, 272 (1936).
3. P. Morrison, L. I. Schiff, Phys. Rev. 58, 24 (1940).
4. C. S. W. Chang, D. L. Falkoff, Phys. Rev. 76, 365 (1949).
5. S. Mayakawa, S. Tomonaga, Prog. Theor. Phys. 2, 161 (1947).
6. L. I. Schiff, Phys. Rev. 76, 89 (1949).
7. E. Stahel and J. Guillissen, Journ. de phys. et le rad. 1, 12 (1940).
8. C. S. Wu, Phys. Rev. 59, 481 (1941).
9. H. Bradt et al., Helv. Phys. Acta 19, 222 (1946).
10. L. Madansky and F. Rasetti, Phys. Rev. 83, 187 (1951).
11. F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949).
    R. P. Feynman, Phys. Rev. 76, 769 (1949).
12. C.f. C. G. Wick, Phys. Rev. 80, 268 (1950).
13. P. H. Stelson and W. M. Preston, Phys. Rev. 83, 469 (1951).