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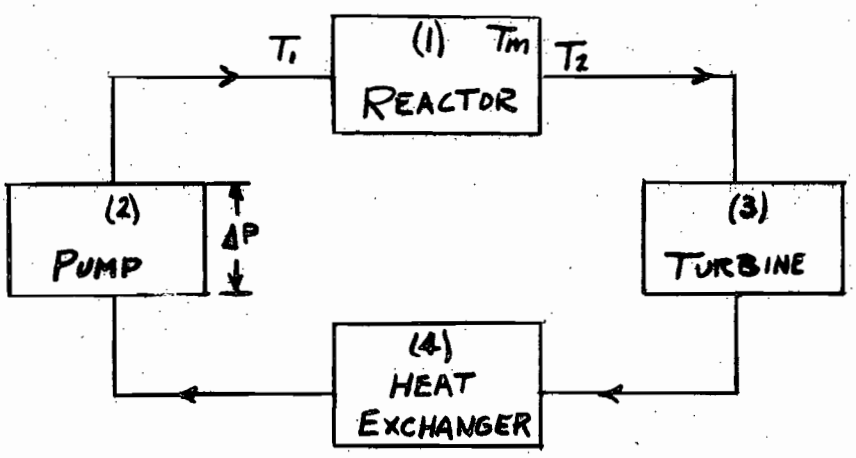
OAK RIDGE SCHOOL OF REACTOR TECHNOLOGY

Lecture Notes - Engineering A

By S. Thompson

I. General Discussion

Our problem is concerned with engineering in its application to the design and use of reactors. In order to limit the scope of the problem to material which can be covered fairly adequately in the time available, I wish to consider reactors only in one context, as part of a very simple four component system, which we can visualize schematically like this:



(1) Here we have a reactor operating at some limiting temperature T_m, taking in a quantity W per unit time of some cooling fluid at a temperature T_1 and discharging it at temperature T_2.

(2) A pump or compressor providing a pressure rise Δp sufficient to overcome the pressure losses in all other components.

(3) A turbine for developing external power and for running the pump may or may not be provided, depending on the purpose of the system.

(4) And finally a heat exchanger which reduces the temperature to a value appropriate to the requirements of the system pump.

It should be pointed out that many aspects of reactor engineering are completely neglected in this simple system and will be pretty much completely ignored in this series of lectures. For instance, our system includes no chemical processing equipment for fissionable materials or fission products.

In this context, as part of a system, we are going to discuss reactors from two aspects: (1) How do the operating characteristics of the reactor, or at least some of them, depend on the internal design of the reactor and (2) what do the design specifications for the system demand for operating characteristics of the reactor. So far as internal design of the reactor is concerned, we will need to know first about the distribution of neutrons and the requirements for criticality, and here we will borrow from the reactor theory course enough information to enable us to determine the distribution of power in the reactor as a function of space and time. In discussing the thermal output of the reactor we will combine enough heat transfer, fluid flow, thermodynamics, and theory of elasticity to establish the dependence of power output on temperatures, pressures, velocities and material characteristics in the reactor. We will try to establish limitations on power output due to (1) top temperatures allowed by materials, (2) temperature difference allowed by thermal stress, and (3) pressure drop allowed by power consumption.

The handling of thermal energy without neutron considerations is a fairly commonplace engineering problem, and a considerable amount of engineering development has been expended in determining its application to engineering problems. The handling of neutrons, however, is not yet a commonplace engineering problem, and the distribution of thermal energy in a reactor cannot

be considered as a problem completely independent from neutron considerations. The distribution of thermal energy, or even its existence, in the reactor is entirely dependent on neutron distribution. Materials requirements for neutron purposes seem to be set in antithesis to requirements for the utilization of thermal energy.

The first large reactors which were built, for instance Hanford, had as their only purpose the utilization of neutrons for the production of plutonium from uranium fissions. The economy of neutrons determined largely the reactor configuration, while the problem in handling thermal energy was to minimize its nuisance value. A considerable plant, as well as a large amount of Grand Coulee electrical power, are needed to dissipate the energy of fission into Columbia River water. In mobile power plants using nuclear energy, like the naval and aircraft applications, the conversion of thermal energy to mechanical energy is the central problem. The neutron problem, due to the use of enriched fissionable material, is important only in its bearing on achieving criticality with a reasonable amount of fissionable material and a reasonable size of reactor. Reactors in the future may have to combine the emphases of these two types. If nuclear energy is to compete economically as an extension of world sources of power, it will probably be necessary to use fissionable materials economically including both considerations of power plant efficiency in the usual sense and including the maximum use of neutrons in making new fissionable material. It will be our job here to try to understand the interrelationships between nuclear considerations and the more usual engineering problems of power plant design. In this connection, I plan to discuss in a very few lectures the overall performance of power plants as it depends on reactor characteristics. This will involve mostly an analysis of a sample power cycle from the point of view of efficiency and power output.

I would like to give a few general reference for those who wish to pursue these topics further.

1. Heat Transmission, W. H. McAdams, McGraw Hill Book Co., Inc., New York, 1942.

2. Theory of Elasticity - S. Timoshenko, McGraw-Hill Book Co., Inc., New York,

3. Thermodynamics - J. H. Keenan, John Wiley & Sons, Inc., New York, 1941.

4. Handbook of Engineering Fundamentals - Eshbach - John Wiley & Sons, New York.

5. The Science and Engineering of Nuclear Power - C. Goodman - Addison-Wesley Press, Cambridge, Mass., 1947.

II. Power Generation in Reactors

1. Heat Distribution

In a given position in a reactor the total heat generated is proportional to both the neutron flux and to the concentration of fissionable material. The energy released per fission is about 200 Mev. Since 1 ev = 1.6×10^{-19} Joules or watt sec then to obtain 1 watt of power it requires $\frac{1 \times 10^{-6}}{200 \times 1.6 \times 10^{-19}} = 3.1 \times 10^{10}$ fissions per sec.* The number of fissions per second taking place in a volume element dV of a reactor is given by (for a thermal reactor)

$$N_f \, \sigma f \, (nv)_{th} \, dV$$

where N_f is the number of fissionable atoms per unit volume of the material, σ_f is the microscopic cross section for fission for these atoms, $(nv)_{th}$ is the flux of thermal neutrons, dV is the element of volume being considered. The number of fissionable atoms per unit volume can be expressed

$$N_f = \frac{a d_f}{A}$$

*The Science and Engineering of Nuclear Power, C. Goodman, Addison-Wesley Press, Cambridge, Mass., 1947, p. 196.

where d_f is the density of fissionable material in space. If the mass M_f of uranium is uniformly distributed, in a volume V_R then

$$N_f = \frac{a\ M_f}{A\ V_R}$$

Here a is Avagadro's number ($0.6\ 02 \times 10^{24}$), and A is the mass number of fissionable uranium. The power available from the increment of volume dV, using uranium 235, is

$$\frac{.602 \times 10^{24} \times 545 \times 10^{-24}}{235 \times 3.1 \times 10^{10}} \quad (d_f\ \phi_{th}\ dV)$$

where $\phi = nv$, and the cross section for fission for U_{235} is about 545 barns. The power from volume dV is

$$0.45 \times 10^{-10} \quad d_f\ \phi_{th}\ dV \qquad \text{watts.}$$

Let us take for example a reactor in which there are 100 Kg of U^{235} more or less uniformly distributed and an average thermal flux of about 10^{13} neutrons per sq cm per sec. For this reactor then we would expect the total power output to be

$$\frac{0.45 \times 10^{-10} \times 1.0 \times 10^{5} \times 10^{13}}{10^{6}} = 45 \text{ megawatts}$$

Let us see how this energy is apt to be distributed in space and time. It has been estimated that the energy of fission is dissipated about as follows.*

	MEV
KE of fission fragments	165 ± 15
β rays	7
KE of neutrons	5
γ rays	11
Energy from neutron capture	12
	200 ± 15
Neutrinos	10

*The Elements of Nuclear Reactor Theory, Part I, Glasstone and Edlund, 1951, p.

The fission fragments are large particles and do not travel far. Their energy is absorbed very close to the point of fission, as is that of β rays. Neutrons have a much larger range and conceivably large numbers could escape from the reactor. However, for most reactors neutron economy will dictate that most of them will stay in the reactor. γ rays having very high ranges can be absorbed only by large masses of material. Most of these will escape the reactor, but for health reasons must be caught in shielding. Neutrinos have such long ranges that, so far as we are concerned, they never enter the picture at all. Of the 200 Mev per fission to be accounted for we might assume, depending on the configuration, that 5 to 10 percent will be absorbed in the shielding outside the reactor. For specific cases a detailed accounting must be made.

Of the 200 Mev of energy some 12 Mev or 6% is delayed due to decay of fission products, 7 being due to β's and 5 due to γ's. This means that immediately after a shutdown following a long period of operation, the reactor is operating at 6 percent of its total power, this power then decaying, according to the formula

$$P = 0.065\, P_o t^{-1/5} \left[1 - \frac{t}{t + t_1}^{+1/5} \right]$$

where t = time in seconds after shutdown

t_1 = time of pile operation at P_o in seconds

for periods from 10 sec $< t <$ 100 days. This must be taken into account in the design of the heat transfer system, so that sufficient heat transfer is available after shutdown to prevent damage to the reactor.

In summary we might say that 200 Mev of energy are formed per fission. Of this, 86 percent is absorbed at the point of fission, 14 percent somewhere

within the reactor, its reflector and its shield. Some 94 percent, for engineering purposes, is created immediately at the time of fission, while 6 percent is delayed, and gradually decays after shutdown, the delayed power decreasing by a factor of two for each factor of 32 in time after shutdown.

2. Dimensional Analysis of Reactors

We have seen how the generation of power locally in a reactor depends on the local densities of neutron flux and fissionable material. It would seem appropriate now to investigate relationships which determine the neutron flux distribution and density of fissionable material. This requires a consideration of criticality conditions. For this purpose we will write down three equations, which are taken from the reactor theory text. These are

$$\nabla^2 q = \frac{\partial q}{\partial \tau} \qquad (1)$$

$$\frac{1}{3\Sigma_t} \nabla^2 nv - \Sigma_a nv + p\, q_{th} = 0 \qquad (2)$$

$$q_f = f \eta \epsilon \Sigma_a nv \qquad (3)$$

These equations, subject to appropriate boundary conditions, represent criticality requirements for a nuclear reactor. The first of these equations represents the slowing down of neutrons from the fission to the thermal energy limits. The second shows the diffusion and loss by capture of thermal neutrons and the third shows the formation of new fission neutrons from the capture of thermal neutrons in fissionable material. Equations (2) and (3) provide boundary conditions for the slowing down range. Some explanation of terms is probably in order, although an attempt has been made to use terms from the Edlund notes.*

* The Elements of Nuclear Reactor Theory, Part III, Glasstone and Edlund, CF-51-9-127.

$q = nv\xi\Sigma_s/p$ = slowing down density, neuts/cm^3-sec passing energy E

n = density of neutrons

v = velocity of neutrons

ξ = average logarithmic energy loss per collision

$$\tau = \exp\left(\int_0^u \frac{1}{3\Sigma_t(\xi\Sigma_s + \gamma\Sigma_a)}\,du\right)$$ = age of neutrons in slowing down

$$p = \exp\left(-\int_0^u \frac{\Sigma_a}{\xi\Sigma_s + \gamma\Sigma_a}\,du\right)$$ = resonance escape probability

Σ_t = macroscopic transport cross section

Σ_s = macroscopic scattering cross section

Σ_a = macroscopic absorption cross section

u = lethargy = $\log E_0/E$

E = neutron energy

γ = (See Reactor Theory text, equation (6.110.2)

$f = \dfrac{\Sigma_u}{\Sigma_a}$ = thermal utilization

$\eta = \nu\dfrac{\Sigma_f}{\Sigma_u}$ = neutrons per capture in fissionable material

ϵ = fast effect, fissions from fast neutrons

ν = neutrons per fission

The limitations on the use of these equations do not particularly concern us here. What we want to do with them is to demonstrate a general method of handling such equations, and to find what we can from them about the characteristics of reactors. However, some limitations on the equations should probably be mentioned. Diffusion approximations to the Boltzmann

equation are in general not good where anything very rapid is happening to neutron flux (Marshak, Ellis).* Extrapolation length depends on transport mean free path and if this is a function of lethargy this amounts to saying that the effective size of the reactor is a function of energy. If transport cross section is a function of space variables we must write

$$\nabla \cdot \left(\frac{1}{\Sigma_t} \nabla nv \right) \text{ instead of } \frac{1}{\Sigma_t} \nabla^2 nv$$

If the reactor is not thermal, E must be defined in terms of an integral, over the slowing down range, of fast fissions. All of these effects are given serious consideration in connection with the ANP reactor, and some of them are of significance. We are going to ignore all of this with the claim (not yet entirely established) that the necessary changes can finally be taken care of by methods similar to those we are now using.

Dimensional Analysis**

Physical quantities are associated both with a number and a dimension. Dimensional analysis tries to study separately the dimensions, leaving the numbers for experiment or calculation. There are three different outlooks on dimensional analysis, depending on the kind of information available: (1) hunch or experiment; (2) differential equation; (3) algebraic equation. (1) Here we have cause to think a certain physical quantity of interest depends on certain other physical quantities. What are the possible combinations? We can never get a number to go with the dimensions. (2) Here a differential equation gives a relationship between physical quantities, but

* The Principal of Reactor Criticality, C. B. Ellis, ANP-69, March 2, 1949. Theory of the Slowing Down of Neutrons to Elastic Collisions with Atomic Nuclei, by Robert E. Marshak, Rev. Mod. Phys., V19, No. 3, July, 1947.

** Handbook of Engineering Fundamentals, O. Eshbach, John Wiley and Sons, New York.

numbers can be found only by integrating the equation. (3) A physical equation gives both physical relationships between dimensions, and numbers.

Not all equations have unitary homogeneity. Engineers are notorious for equations which do not, one well known example being the equation for the velocity of efflux from a nozzle

$$v = 223.7\sqrt{h_1 - h_2}$$

where

v = velocity in feet per second

h = enthalpy in Btu per lb

This equation has been derived from

$$v(\text{ft/sec}) = \sqrt{2g(h_1-h_2)} = \sqrt{2 \times 32.2\ (\text{ft/sec}^2)(h_1-h_2)\text{ft}}$$

In this both sides of the equation have units of ft/sec. If we are to express (h_1-h_2) in the units of Btu per lb then we have 1 Btu $\equiv$ 778 ft lbs or 1 Btu/lb $\equiv$ 778 ft. Thus (h_1-h_2)Btu/lb $\equiv$ 778 (h_1-h_2)ft and the above equation becomes

$$v = \sqrt{2 \times 32.2 \times 778\ (h_1-h_2)} = 223.7\sqrt{h_1-h_2}$$

where v is in ft/sec and (h_1-h_2) is measured not in feet but in the units of enthalpy or Btu per lb. This is not a depreciation of the non-dimensional formula since in engineering routine it is very handy, but a warning of caution in its use. In the particular case mentioned here, elaborate charts are in existence from which h_1-h_2 can be read directly in Btu/lb, while velocity is desired for engineering reasons in feet per second.

Let's see what we can find out about these equations from dimensional analysis, which is just the statement that physical laws must be given in equations all of whose terms have the same dimensions or units - the rule of

unitary homogeneity. A check which you can easily do will show that our equations are in this form, the terms in equation (1), for instance, having the dimensions of $L^{-5}\ T^{-1}$. Notice the dimensions of $\nabla^2(d^2/dx^2)$ are L^{-2}, of q are $L^{-3}T^{-1}$, of τ are L^2, where L represents the dimension of length and T of time. If we multiply through equation (1) by any quantity having the dimensions L^5T we will find that the equation has been separated into dimensionless parameters. Some of these parameters will be useful. Such a quantity, as for example, q/τ_{th} has the dimensions of $L^{-5}\ T^{-1}$ where τ_{th} is the thermal age. Since $\nabla^2 q/q$ has the dimensions $1/L^2$ is can be written

$$\frac{\nabla^2 q}{q} = -\frac{\alpha^2}{R^2}$$

where R is a critical length dimension of the reactor and α is a parameter for a particular reactor. Multiplying both sides of equation (1) by τ_{th}/q we have $\tau_{th}\ \frac{\nabla^2 q}{q} = \frac{\tau_{th}}{q}\frac{\partial q}{\partial \tau}$ which is a dimensionless equation. this may be rearranged to give

$$-\frac{\tau_{th}\alpha^2}{R^2} = \frac{\partial q}{q}\ \frac{1}{\partial\tau/\tau_{th}} = \frac{\partial(\log q)}{\partial(\tau/\tau_{th})}$$

This can be integrated with respect to energy from fission energy to thermal to give

$$-\frac{\tau_{th}}{R^2}\int_0^1 \alpha^2\, d(\tau/\tau_{th}) = \int_{q_f}^{q_{th}} d(\log q) = \log(q_{th}/q_f) \qquad (4)$$

or $$q_f/q_{th} = \exp\left(\frac{\tau_{th}}{R^2}\int_0^1 \alpha^2\, d(\tau/\tau_{th})\right)$$

Likewise equations (2) and (3) can be made dimensionless by dividing by $(\Sigma_a nv)$, giving

$$\frac{\nabla^2 nv}{3\Sigma_t \Sigma_a nv} - 1 + \frac{p\, q_{th}}{\Sigma_a nv} = 0$$

and

$$\frac{p\, q_f}{q_{th}} \frac{q_{th}}{\Sigma_a nv} = k$$

Eliminating $\frac{p\, q_{th}}{\Sigma_a nv}$ between the two equations

$$\frac{\nabla^2 nv}{3\Sigma_t \Sigma_a nv} - 1 + pf\eta\epsilon \frac{q_{th}}{q_f} = 0$$

Again, as in equation (1) ($\nabla^2 nv/nv$) has the dimensions of $1/L^2$ so let

$$\frac{\nabla^2 nv}{nv} = -\frac{\beta^2}{R^2}$$

where β is a constant for a particular system, and R is some critical length. By substitution we have

$$-\frac{\beta^2}{3\Sigma_t \Sigma_a R^2} - 1 + pf\eta\epsilon \frac{q_{th}}{q_f} = 0 \qquad (5)$$

Eliminating q_{th}/q_f between two equations (4) and (5) gives an expression for the criticality of the reactor.

$$\left(\frac{\beta^2}{3\Sigma_t \Sigma_a R^2} + 1\right) \exp\left[\frac{\tau_{th}}{R^2} \int_0^1 \alpha^2\, d(\tau/\tau_{th})\right] = pf\eta\epsilon \qquad (6)$$

It should be noticed that the form of this equation is essentially that of the Fermi age equation in which

$$\frac{\beta^2}{3\Sigma_t \Sigma_a R^2} = B^2 L^2$$

$$\frac{\tau_{th}}{R^2} \int_0^1 \alpha^2 \, d(\tau/\tau_{th}) = B^2 \tau_{th}$$

and

$$pf\eta\epsilon = k$$

We can use equation (6) to find out something about the mass of fissionable material in the reactor which is needed for criticality. For this purpose let us say that the cross section for absorption is composed of two parts; one Σ_u in uranium or other fissionable material; the other Σ_c in all other materials. Then

$$\Sigma_a = \Sigma_u + \Sigma_c$$

and from our definition of f

$$f = \frac{\Sigma_u}{\Sigma_u + \Sigma_c}$$

The macroscopic cross section for uranium is

$$\Sigma_u = \sigma_u N_u = \sigma_u \frac{M_u}{V_R} \frac{a}{A_u}$$

where σ_u = the microscopic cross section for absorption in fissionable material

M_u = the critical mass

a = Avogadro's number

V_R = volume of the reactor

A_u = mass number of fissionable material

If we let $V_R = \gamma R^3$, then

$$\Sigma_u = \frac{\sigma_u M_u a}{\gamma R^3 A_u}$$

For convenience let

$$\left(\frac{\tau_{th}}{R^2} \int_0^1 \alpha^2 \, d(\tau/\tau_{th}) \right) = I$$

Then dividing and substituting for f and substituting for Σ_a in equation (6) gives

$$\left(\frac{\beta^2}{3\Sigma_t(\Sigma_u + \Sigma_c)R^2} + 1 \right) \frac{\Sigma_u + \Sigma_c}{\Sigma_u} = p\eta\epsilon e^{-I}$$

or

$$\left(\frac{\beta^2}{3\Sigma_t \Sigma_c R^2} + \frac{\Sigma_u + \Sigma_c}{\Sigma_c} \right) = p\eta\epsilon \frac{\Sigma_u}{\Sigma_c} e^{-I}$$

or

$$\left(\frac{\beta^2}{3\Sigma_t \Sigma_c R^2} + 1 \right) = \frac{\Sigma_u}{\Sigma_c} \left(p\eta\epsilon e^{-I} - 1 \right)$$

or we obtain a dimensionless equation involving the critical mass

$$\frac{\Sigma_u}{\Sigma_c} = \frac{\sigma_u M_u a}{\Sigma_c \gamma R^3 A_u} = \frac{\dfrac{\beta^2}{3\Sigma_t \Sigma_c R^2} + 1}{p\eta\epsilon \exp\left[-\dfrac{\tau_{th}}{R^2} \displaystyle\int_0^1 \alpha^2 \, d(\tau/\tau_{th}) \right]^{-1}} \tag{7}$$

We can separate this equation into class parameters of two kinds, geometrical and physical:

geometrical: α, β, γ

physical: $\dfrac{\sigma_u M_u a}{\Sigma_c R^3 A_u}, \Sigma_t \Sigma_c R^2, \dfrac{\tau_{th}}{R^2}, p, \eta, \epsilon$

Certain general statements can be made about the geometrical factors; for instance, for all bare reactors of practical interest, α, β and γ have magnitudes close to π. Their accurate values can be found only from an

integration, generally numerical, of the differential equations, but for classes of reactors for which dimensionless parameters are the same, one calculation does for the whole class. Equation (7) can be put into a more general form by referring the dimensionless ratios to a definite value of the critical dimension, R. Since we are now going to demonstrate that a minimum value of the mass exists for one value of R, let us express the ratios in terms of this minimum. Then

$$\mu = \frac{M_u}{M_{min}} \qquad \rho = \frac{R}{R_{min}}$$

$$A = \frac{A_u \gamma \Sigma_c R_{min}^3}{\sigma_u a M_{min}} \qquad C = p \eta \epsilon$$

$$B = \frac{A_u \gamma R_{min} \beta 2}{3 \sigma_u a M_{min} \Sigma_t} \qquad D = \frac{\tau_{th}}{R_{min}^2} \int_0^1 \alpha^2 \, d\left(\frac{\tau}{\tau_{th}}\right)$$

Then equation (7) becomes

$$\mu = \frac{A\rho^3 + B\rho}{Ce^{-D/\rho^2} - 1} \tag{8}$$

A minimum occurs when

$$\frac{d(\log \mu)}{d\rho} = 0$$

and at this point μ and ρ are both 1. The minimum then occurs when

$$0 = \frac{3A + B}{A + B} - \frac{2D\, Ce^{-D}}{Ce^{-D} - 1}$$

or when

$$\frac{\frac{3A}{B} + 1}{\frac{A}{B} + 1} = \frac{2\,DC}{C - e^{D}} \tag{9}$$

From equation (8), for the minimum point

$$1 = \frac{A + B}{Ce^{-D} - 1} \tag{10}$$

Equation (9) can be solved by numerical approximation for the ratio of A/B, and hence for the critical radius. Then equation (10) can be solved for the critical mass. It is obvious that the critical mass approaches infinity for two values of ρ, one when $\rho > \infty$, for which equation (8) becomes appropriately

$$\mu = \frac{A\,\rho^3}{C - 1} \qquad \text{(for large } \rho\text{)}$$

if Σ_c is relatively large, and

$$\mu = \frac{B\,\rho}{C - 1} \qquad \text{(for large } \rho\text{)}$$

if Σ_c is relatively small. The second infinity occurs when

$$Ce^{-D/\rho_\infty^2} = 1$$

or when

$$\rho_\infty^2 = \frac{D}{\log c}$$

<u>Example</u>:

If there is no absorption of neutrons outside the uranium, then $\Sigma_c = A = 0$. Then Eq. (8) becomes

$$\mu = \frac{B\,\rho}{Ce^{-D/\rho^2} - 1}$$

Equation (9) and (10) are

$$\frac{2\,DC}{C - e^{D}} = 1$$

and

$$\frac{B}{Ce^{-D} - 1} = 1$$

Let us try this for a mixture of BeO and U_{235}, for which

$$\Sigma_c = 0 \quad \text{(for purpose of illustration only)}$$
$$\Sigma_t = 0.67 \text{ cm}^{-1}$$
$$\tau_{th} = 110 \text{ cm}^2$$
$$\sigma_u = 645 \times 10^{-24} \text{ cm}^2$$

$$A = 0$$

$$B = \frac{235 \times 4/3\ \pi^3 R_{min}}{3 \times 645 \times .602 \times 0.67\ M_{min}} = 12.5 \quad \frac{R_{min}}{M_{min}}$$

$$C = 2.15$$

$$D = \frac{\pi^2 \times 110}{R_{min}^2} = \frac{1080}{R_{min}^2}$$

Substituting the above value of C

$$\frac{2 \times 2.15 \times D}{2.15 - e^D} = 1$$

or

$$D = \tfrac{1}{2} - \frac{e^D}{4.3}$$

By trial

Let D = 0.3	then		$.3 \neq \frac{1}{2} - .314 = .186$
" D = 0.2	"		$.2 \neq .5 - .284 = .216$
" D = 0.22	"		$.22 \neq .5 - .29 = .21$
" D = 0.211	"		$.211 \neq .5 - .287 = .213$
" D = 0.212	"		$\boxed{.212} = .5 - .288 = .212$
	or		D = .212

is a solution of the above equation.

$$R_{min} = \left(\frac{1080}{.212}\right)^{\frac{1}{2}} = 71.2 \text{ cm.}$$

$$M_{min} = \frac{12.5 \times 71.2}{B}$$

$$B = 2.15 \times .81 - 1 = 0.74$$

$$M_{min} = \frac{12.5 \times 71.2}{0.74} = 1.2 \text{ kg}$$

So

$$\mu = \frac{0.74\,\rho}{2.15\, e^{\frac{-.212}{\rho^2}} - 1} \qquad \rho_\infty = \left(\frac{.212}{\log 2.15}\right)^{\frac{1}{2}} = (.277)^{\frac{1}{2}} = .527$$

ρ		μ
1		1
2	1.48/1.04	1.4
5	3.7/1.15	3.2
.6	.443/4.193	2.3

A rough plot of this result is shown in Figure 1.

As a problem I would suggest that you find approximate curves for μ and values for R_{min} and M_{min} using enriched uranium in three moderators - heavy water, beryllium oxide and graphite. For contrast you might try this also for natural water in which absorption in the moderator is relatively large. You will need the following values.

	D_2O	BeO	C	H_2O
Σ_c cm^{-1}	.00008	.00074	.00036	0.017
Σ_t cm^{-1}	.435	.67	.303	.90
τ_{th} cm^2	120	110	350	33
$\sigma_u \times 10^{+24}$ cm^2	645	645	645	645

<u>Bare Sphere</u> - α, β, $3/4\,\gamma = \pi$

$$\therefore \int_0^1 \alpha^2 \, d \frac{\tau}{\tau_{th}} = \pi^2$$

$A_u = 235$

$a = 0.602 \times 10^{+24}$

$p = 1$

$\epsilon = 1$

$$\eta = \nu \frac{545}{645} = 2.5 \times \frac{545}{645}$$

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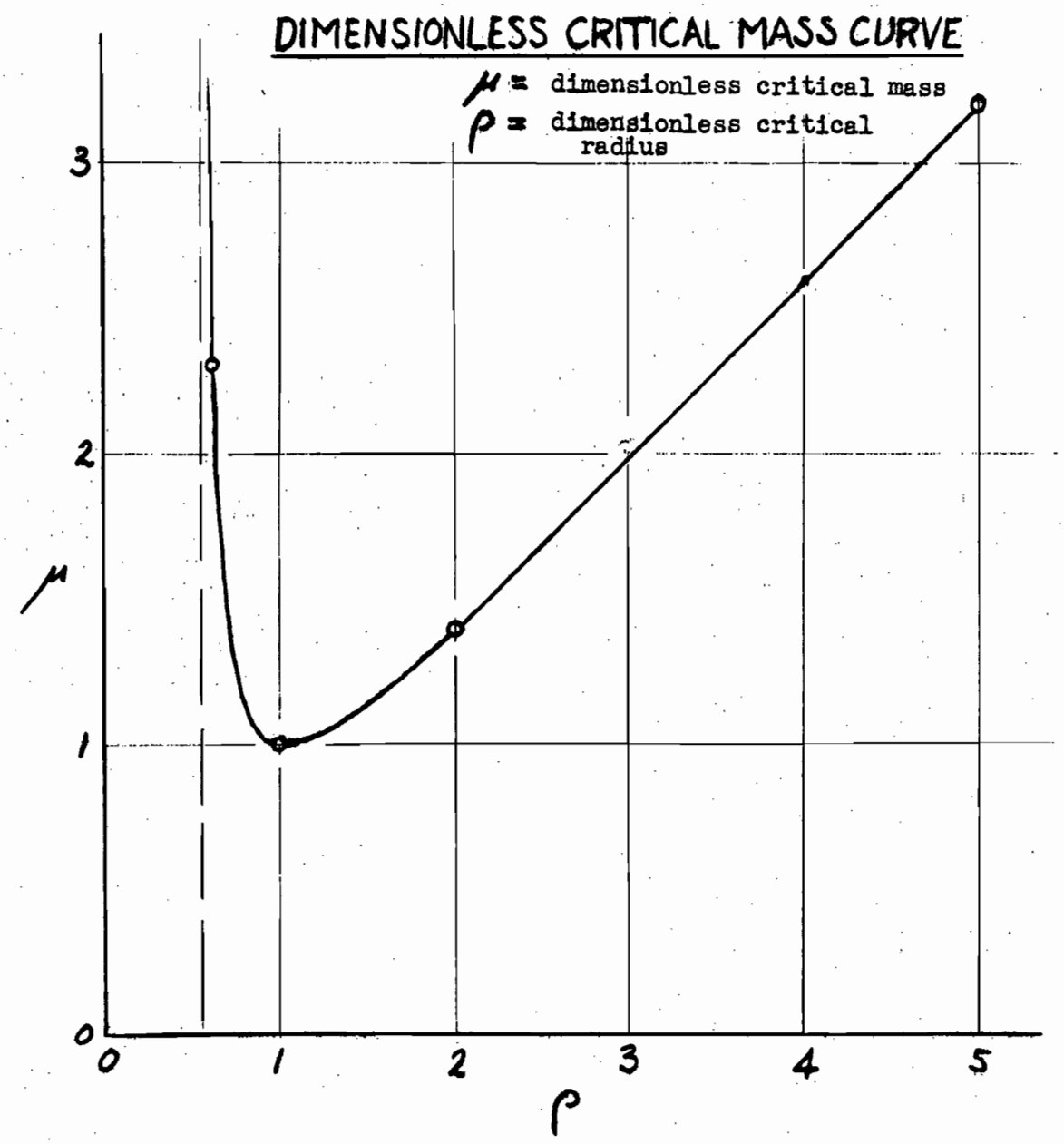

DIMENSIONLESS CRITICAL MASS CURVE
μ = dimensionless critical mass
ρ = dimensionless critical radius
3
2
1
0
μ
0
1
2
3
4
5
ρ

3. Breeding Possibilities

A subject of much interest in the field of nuclear energy now is the possibility of using nuclear energy as a means of extending our natural resources for the production of power. In this connection it is necessary to know how much fissionable material is economically available. One determining factor here involves the question whether only materials which are fissionable in their natural state, like U_{235}, are available, or whether it is feasible to convert other materials, like U_{238} which are not fissionable in the natural state, to fissionable materials in significant quantities. The importance of this question arises from the fact the U_{238} is 140 times as plentiful as U_{235}, and that large quantities of non-fissionable Th_{232} are available. It is common knowledge that the Hanford reactors and others do make Pu_{239} from U_{238} at the expense of using up U_{235}, and that Th_{232} can be converted to fissionable U_{233} by neutron bombardment. The question is whether, in using up the relatively small quantity of U_{235} we can make available a significant part of these other materials. The least we would have to ask of a system for this purpose would be that for each atom of U_{235} used up, including chemical processing losses, etc., one new atom of fissionable material would be produced. Then supposedly this new material produced could be placed in another reactor and the process continued until all the non-fissionable material could be made fissionable and burned up. If we wish to add to the currently available stockpile of fissionable material it is necessary to

produce more than one new atom for each one used up. Let us see what the limiting possibilities are.

The ratio of atoms produced to those used up can be expressed as

$$\frac{\Sigma_P}{\Sigma_u} \tag{1}$$

where Σ_P=the cross section of the production material (for instance U_{238}) and Σ_u is the absorption cross section of the fissionable material (for instance U_{235}). The fractional increase of fissionable material available is then

$$r = \frac{\Sigma_P}{\Sigma_u} - 1 - \ell \tag{2}$$

where ℓ represents the fractional loss in chemical processing, leakage, etc. For example, if 110 atoms of plutonium are produced when 100 atoms of fissionable uranium are destroyed, and if one percent of the current supply is lost in chemical processing, etc., then the gain in the system would be

$$r = \frac{110 - 100 - 1}{100} = 0.09$$

We wish to know now an upper limit on the ratio Σ_P / Σ_u. This limit is set by requirements of criticality for the reactor, and is stated for an infinite reactor (since we are taking care of leakage in the factor ℓ)

$$k = 1 \tag{3}$$

where k can be represented by

$$k = pf\eta\epsilon \tag{4}$$

If we idealize the system by the assumption that all absorptions occur in the thermal range (for a real reactor a detailed accounting is necessary), then $P = \epsilon = 1$ and Equation (3) becomes

$$f\eta = 1 \tag{4}$$

where η is the number of neutrons available per absorption in fissionable material, and where f, the thermal utilization, is by definition

$$f = \frac{\Sigma_u}{\Sigma_a} \tag{5}$$

Here Σ_a is the total cross section for absorption in all materials of which the reactor is composed. Let us assume that Σ_a is composed of three parts, absorptions in fissionable material Σ_u, absorptions involved in production material Σ_p, and non useful absorptions Σ_c (structure, moderator, poisons, etc.). Then the thermal utilization can be written

$$f = \frac{\Sigma_u}{\Sigma_u + \Sigma_p + \Sigma_c} \tag{6}$$

Since from Equation (4) $f = 1/\eta$, we can write

$$\eta = \frac{\Sigma_u + \Sigma_p + \Sigma_c}{\Sigma_u} \tag{7}$$

or

$$\frac{\Sigma_p}{\Sigma_u} = \eta - 1 - \frac{\Sigma_c}{\Sigma_u} \tag{8}$$

Substituting Equation (8) into (2)

$$r = \eta - 2 - \frac{\Sigma_c}{\Sigma_u} - \ell \tag{9}$$

Let us assume we have somehow acquired a perfect reactor, in the sense that there are no losses in non useful capture, leakage, or chemical processing. Then the upper limit on the breeding gain is given by

$$r = \eta - 2 \tag{10}$$

For a system using U_{235} we know that the number of neutrons per fission is given by $\nu = 2.5$

At thermal energy we know also that not all of the neutrons absorbed in U_{235} cause fissions, giving a value of η less than r, or

$$\eta = 2.5 \times \frac{545}{645} = 2.11$$

If it were possible to eliminate all the radiative captures in U_{235} by some means, then we might achieve

$$\eta = \nu = 2.5$$

The maximum possible gain in the system then is bounded

$$0.11 < r < 0.5$$

Actually, of course, we cannot obtain this upper limit, due to the various losses in our system.

It might be thought that it is not necessary to achieve a positive value for the breeding gain. For considering this possibility let us imagine that all the natural uranium available in the world has been separated into two neat piles, one of U_{235} and the other of U_{238} and that these are at our disposal. Let us say that our problem is to figure out the minimum breeding gain which will enable us to use up all the U_{238} at exactly the time when we run out of U_{235}. If we consider that this is to be done by a series of processes, each having a gain, r, then the requirement is that

$$1 + (1+r) + (1+r)^2 - - - (1+r)^n + - - - - = 140$$

Since there are 140 times as much U_{238} as U_{235}. If the series in infinite then it can be written

$$\frac{1}{1-(1+r)} = 140$$

or, solving for r

$$r = -\frac{1}{140}$$

This is a small number, and would be even smaller if we had been trying to use up all the thorium also. Since r is negative, this implies a dwindling nuclear economy, in which the number of operable reactors at the start, determined by our supply of U_{235}, steadily decreases until finally the last lonely reactor grows cold just as all the fissionable material is used up. Since even then r is so close to zero, it would seem that we get very little relief from our technical breeding problem in the dwindling economy, and we may as well face the necessity to make r positive.

It is generally assumed that our nuclear economy should be an expanding one in which we use our U_{235} to make as much fissionable material available as possible and as soon as possible. With a positive gain, r, a useful concept is the requirement for doubling our supply of fissionable material. Considering again a series of repetitive processes, each with a gain r. The n_{th} process is assumed to be operating with twice the original amount of fissionable material. Then

$$(1 + r)^n = 2$$

or

$$n = \frac{\log 2}{\log (1+r)}$$

For example, if r = 0.01, the number of processes required for doubling the supply of fissionable material is

$$n = \frac{0.693}{0.01} = 69$$

It is apparent from the above considerations that for the success of an expanding economy we want r as large as possible, and we certainly want it positive.

How much fissionable material is used up in breeding? It takes

3.1×10^{10} fissions per second to make one watt of power. To make one megawatt day of energy hence requires

$$3.1 \times 10^{10} \times 3.6 \times 10^{3} \times 24 \times 10^{6} = 2.68 \times 10^{21} \quad \frac{\text{fissions}}{\text{megawatt day}}$$

The mass of U_{235} used up, if η = r, is hence

$$\frac{2.68 \times 10^{21} \times 235}{.602 \times 10^{24}} = 1.24 \quad \frac{\text{gm}}{\text{megawatt day}}$$

In the process plutonium is produced at the rate

$$1.24\,(1+r) \quad \frac{\text{gm}}{\text{megawatt day}}$$

The gain in fissionable material is

$$1.24\ r \quad \frac{\text{gm}}{\text{megawatt day}}$$

If we have available a limited supply of fissionable material and wish to make as much new material as quickly as possible, it is obvious that we must burn our available material at the highest rate possible, assuming the breeding gain is positive, and incidentally assuming that we have some competitive use for the thermal energy created.

Calculating r for any specific reactor requires a detailed accounting for all neutrons and for all fissionable atoms. Until breeding has been demonstrated experimentally its possibility is a matter for conjecture, and the ultimate usefulness of atomic energy for more than military purposes probably hangs in the balance.

References:

1. Physics and Atomic Energy, H. Brooks, ORNL C. F. 50-3-180, 1950

2. Sourcebook on Atomic Energy, Glasstone, D. Van Nostrand Co., Inc. 1950 (see index)

4. Numerical Methods for Reactors

It has been shown by a dimensional analysis of the differential equations for thermal reactors that the criticality conditions can be expressed in terms of dimensionless ratios of the physical quantities which describe the reactor and numbers which describe the geometrical configuration of the reactor. For one special case, namely bare reactors the values of the geometrical numbers α and β, as given by definition

$$\frac{\nabla^2 q}{q} = -\frac{\alpha^2}{R^2}$$

and

$$\frac{\nabla^2 nv}{nv} = -\frac{\beta^2}{R^2}$$

are equal and constant. For this case it is easy to find analytical solutions. For instance it can be verified by substitution that a sphere yields

$$\alpha = \beta = \pi$$

and

$$q/A = nv/B = \frac{\sin \pi (r/R)}{(r/R)}$$

More complicated cases generally are handled more easily by numerical methods. The following discussion is intended to give some familiarity with numerical methods for this purpose, since these enter in large measure in reactor engineering calculations.

Many types of numerical methods are available for the solution of differential equations. These can be roughly separated into two classes: (1) Those which use approximate differentiations; and (2) those which use approximate integrations. It is proposed here to demonstrate the application of both types to the differential equations for the critical

reactor. This example will show the sort of approach which can be made to a reactor problem not readily susceptible to analytical methods. The equations for the reactor are written as they were before:

$$\nabla^2 q = \frac{\partial q}{\partial \tau} \tag{1}$$

$$\frac{1}{3\Sigma_t} \nabla^2 (nv)_{th} - \Sigma_a (nv)_{th} + P q_{th} = 0 \tag{2}$$

$$q_f = f \eta \epsilon (nv)_{th} \Sigma_a \tag{3}$$

To make the problem definite let us consider a sphere. In spherical coordinates, Equations (1) to (3) become

$$\frac{\partial^2 (rq)}{\partial r^2} = \frac{\partial (rq)}{\partial \tau} \tag{4}$$

$$\frac{1}{3\Sigma_t} \nabla^2 (rnv)_{th} - \Sigma_a (rnv)_{th} + p r q_{th} = 0 \tag{5}$$

$$rq_f = f \eta \epsilon (rnv)_{th} \Sigma_a \tag{6}$$

If we let $(rnv)_{th} = R A \varphi$ and $rq = A \psi$, where φ and ψ are made dimensionless by the factor A of arbitrary numerical magnitude, then if $r = \xi R$ and $\tau = \theta \tau_{th}$, Equations (4) to (6) become

$$\frac{\partial^2 \psi}{\partial \xi^2} = \frac{R^2}{\tau_{th}} \frac{\partial \psi}{\partial \theta} \tag{7}$$

$$\frac{1}{3\Sigma_t R} \frac{d^2 \varphi}{d \xi^2} - \Sigma_a R \varphi + p \psi_{th} = 0 \tag{8}$$

$$\psi_f = f \eta \epsilon \Sigma_a R \varphi \tag{9}$$

<u>Slowing Down Equation</u>

Let us now apply the finite difference procedure to Equation (7). This procedure begins with the definition of a derivative. We write

$$\frac{\partial \psi}{\partial \xi} = \mathop{\mathcal{L}}_{\Delta \xi \to 0} \frac{\psi(\xi + \Delta \xi, \theta) - \psi(\xi, \theta)}{\Delta \xi} \tag{10}$$

Likewise the second derivative is defined

$$\frac{\partial^2 \psi}{\partial \xi^2} = \mathop{\mathcal{L}}_{\Delta \xi \to 0} \frac{\mathop{\mathcal{L}}_{\Delta \xi \to 0} \frac{\psi(\xi + \Delta \xi, \theta) - \psi(\xi, \theta)}{\Delta \xi} - \mathop{\mathcal{L}}_{\Delta \xi \to 0} \frac{\psi(\xi, \theta) - \psi(\xi - \Delta \xi, \theta)}{\Delta \xi}}{\Delta \xi}$$

If we agree to consider only small intervals, $\Delta \xi$, we can write approximately (This can be verified by an expansion of $\psi(\xi + \Delta \xi, \theta)$ in a Taylor series).

$$\frac{\partial \psi}{\partial \xi} = \frac{\psi(\xi + \Delta \xi, \theta) - \psi(\xi, \theta)}{\Delta \xi} \tag{12}$$

and

$$\frac{\partial^2 \psi}{\partial \xi^2} = \frac{\psi(\xi + \Delta \xi, \theta) - 2\psi(\xi, \theta) + \psi(\xi - \Delta \xi, \theta)}{(\Delta \xi)^2} \tag{13}$$

likewise

$$\frac{\partial \psi}{\partial \theta} = \frac{\psi(\xi, \theta + \Delta \theta) - \psi(\xi, \theta)}{\Delta \theta} \tag{14}$$

Substituting (13) and (14) into Equation (7)

$$\frac{\psi(\xi + \Delta \xi, \theta) + \psi(\xi - \Delta \xi, \theta)}{(\Delta \xi)^2} = \frac{R^2}{\tau_{th}} \frac{\psi(\xi, \theta + \Delta \theta)}{\Delta \theta} + \left(\frac{2\psi(\xi, \theta)}{(\Delta \xi)^2} - \frac{R^2}{\tau_{th}} \frac{\psi(\xi, \theta)}{\Delta \theta} \right)$$

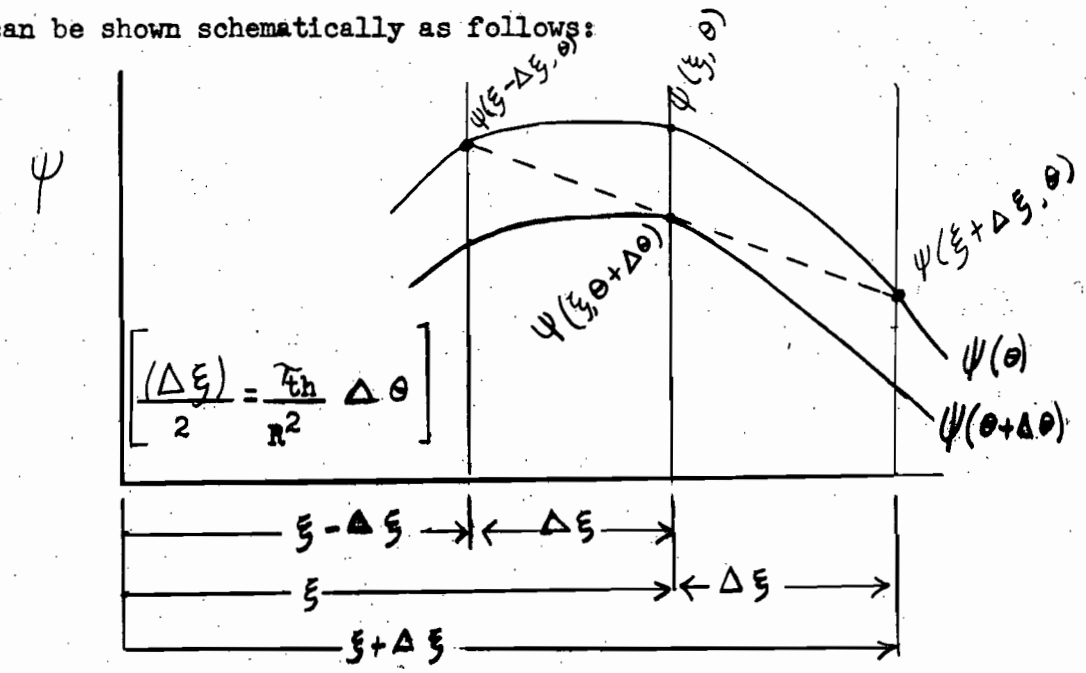
If we agree to use the relationship

$$\frac{2}{(\Delta \xi)^2} = \frac{R^2}{\tau_{th}} \frac{1}{\Delta \theta} \tag{16}$$

Equation (15) becomes

$$\frac{\psi(\xi + \Delta \xi, \theta) + \psi(\xi - \Delta \xi, \theta)}{2} = \psi(\xi, \theta + \Delta \theta) \tag{17}$$

Let us try to interpret Equation (17). It states that the value of the function ψ at a point ξ and the age $\theta + \Delta\theta$ is given by the average of its values at $\xi + \Delta\xi$ and $\xi - \Delta\xi$ at the previous age θ, where the intervals $\Delta\theta$ and $\Delta\xi$ are related by Equation (16). This can be shown schematically as follows:



Using such a graphical method of averaging is due to E. Schmidt and is called a "Schmidt plot"* It has been used principally in the solution of transient temperature distributions in solids. With it, with any given initial condition we can find any later condition by purely graphical means. The method will do more than we are asking it to do here. For instance, variable coefficients mean only variable intervals on the plot. Twc and even three dimensional problems have been solved by a numerical variation on the scheme.

For an example, let us assume a spherical reactor composed of a central core of radius $\xi = a$, with a constant density of fissionable material, and a surrounding reflector of radius, $\xi = b$, composed of material with the same moderating properties as the core.

*Heat Transfer Notes, Boelter, Cherry, Johnson and Martinelli; University of California Press, Berkeley, California; 1948, p V-36
Introduction to the Transfer of Heat and Mass, E. R. G. Eckert, McGraw-Hill Book Co., Inc., New York, 1950
Thermodynamics, E. Schmidt, Oxford Press 1949; p. 428

Let

$$\frac{R^2}{\tau_{th}} = 40$$

$$\Delta \xi = 0.1$$

$$a = 0.6, \quad b = 1.0$$

Then

$$\Delta \theta = \frac{(0.1)^2}{2} \times 40 = 0.2$$

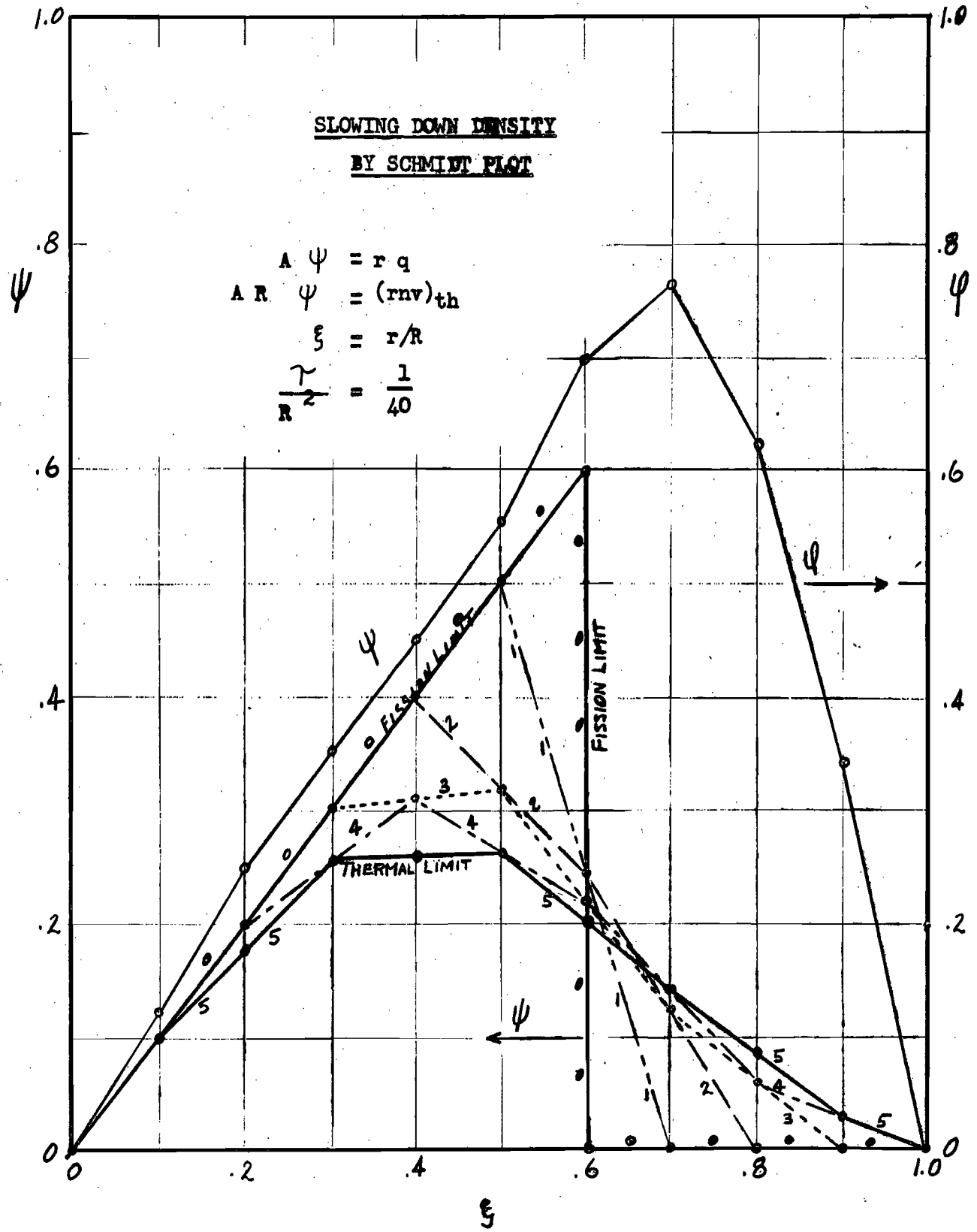
This means that there are $1/\Delta\theta = 5$ intervals in the slowing down range. The slowing down densities as found from a Schmidt plot are shown in the accompanying figure, for an arbitrary assumption of the slowing down density at the fission limit. Incidentally, since the slowing down density at the fission limit is proportional to power density, the straight line assumption for the spherical core amounts to an assumption of constant power density in the core. It should be remembered here that one of the limitations on diffusion theory makes an equation incorrect in the vicinity of the discontinuity.

Thermal Equations

To complete the problem we need to look again at the thermal equations (Equations (8) and (9)). To demonstrate the second type of numerical solution, let us use an approximate integration to find the thermal flux and to get a check on the trial values of slowing down densities found from the Schmidt plot.

Problem: Find difference form for

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = \frac{\partial q}{\partial \tau}, \qquad \Delta x = \Delta y = h, \quad \Delta \tau = k$$


SLOWING DOWN DENSITY
BY SCHMIDT PLOT
A ψ = r q
A R ψ = (rnv)th
ξ = r/R
τ/R² = 1/40
ψ
ξ
FISSION LIMIT
THERMAL LIMIT
0
.2
.4
.6
.8
1.0

Let us rearrange Equation (8) for numerical integration

$$\frac{d^2 \varphi}{d \xi^2} = 3 \Sigma_t \Sigma_a R^2 \varphi - 3 \Sigma_t R P \psi_{th} = 0 \quad (18)$$

Integrating once

$$\frac{d \varphi}{d \xi} = A + \int_0^{\xi} \left\{ 3 \Sigma_t \Sigma_a R^2 \varphi - 3 \Sigma_t R P \psi_{th} \right\} d \xi \quad (19)$$

Integrating again

$$\varphi = A \xi + B + \int_0^{\xi} d \xi_1 \int_0^{\xi_1} \left\{ 3 \Sigma_t \Sigma_a R^2 \varphi - 3 \Sigma_t R P \psi_{th} \right\} d \xi_2 \quad (20)$$

Since the flux at $\xi = 0$ must be finite

$$B = \varphi_o = 0$$

Since the flux is zero at $\xi = 1$

$$A = - \int_0^1 d \xi_1 \int_0^1 \left\{ 3 \Sigma_t \Sigma_a R^2 \varphi - 3 \Sigma_t R P \psi_{th} \right\} d \xi_2 \quad (21)$$

Incidentally if we were to integrate Equations (19) and (20) across a boundary where there was a discontinuity in Σ_t we would have to accomodate a discontinuity in $d \varphi / d \xi$ to keep current densities of neutrons the same on both sides of the boundary. This current is given by

$$\frac{v}{3 \Sigma_t} \frac{dn}{dr}$$

Equation (20) is an integral equation.* It is our problem to solve it numerically. A common method for this is the method of iterations, or successive substitutions or approximations. It is to be noted that the function φ appears on the right side of the equation under an integral and on the left of the equation. The method of iterations involves a trial

* Linear Integral Equations, W. V. Lovitt, McGraw-Hill (1924)

value function for φ which is substituted into the right side of the equation, the manipulations of integration are performed, boundary conditions are satisfied, and a new value of φ is attained at the left of the equation. This is substituted again into the right side of the equation and the process repeated. If the process converges, as it does for certain classes of problems under certain conditions, a stage will be reached where the shape of the curve for φ on the left will be as close as desired to that on the right. This then will be the solution to the problem.

The integrations for one trial are shown in the accompanying table. For this table values of ψ_{th} were read from the Schmidt plot. Integrations were done by the trapezoidal rule. A trial value for φ having the shape of a sin curve and an undetermined multiplier C for adjusting its amplitude was used. A brief description of the problem follows.

Let us write

$$\Sigma_a = \Sigma_u + \Sigma_c$$

where Σ_u is the cross section for capture in uranium and Σ_c is the cross section for capture outside uranium. For the example let $\Sigma_c = 0$. Then in the core $\Sigma_a = \Sigma_u$ and in the reflecter $\Sigma_a = 0$. Besides these conditions let us force the solution to a steady state by making ψ_f obtained from Equation (9) coincide with that of ψ_f from the Schmidt plot at some reference point, say $\xi = 0.6$. Then this requires from Equation (9)

$$\psi_f\Big)_{\xi = 0.6} = C\eta\,\Sigma_u R \left.\frac{\varphi}{C}\right)_{\xi = 0.6} = 0.6 \qquad (22)$$

Let us also force the convergence by making use of the undetermined multiplier C to make the trial and calculated values of $\varphi\Big)_{\xi = 0.6}$ agree. Then

from Equation (20)

$$\varphi\Big)_{\xi=0.6} = 0.6A + C\int_0^{0.6} d\xi_1 \int_0^{\xi_1} 3\Sigma_t \Sigma_a R^2 \left(\frac{\varphi}{C}\right) d\xi_2 - \int_0^{0.6} d\xi_1 \int_0^{\xi_1} 3\Sigma_t R p \psi_{th}\, d\xi_2 \qquad (23)$$

The following values pertain to a Be O moderated reactor

$$\tau_{th} = 110 \text{ cm}^2$$

$$R^2 = 40 \times 110 = 4400 \text{ cm}^2$$

$$\Sigma_t = 0.67 \text{ cm}^{-1}$$

$$R = (4400)^{\frac{1}{2}} = 66.3 \text{ cm.}$$

From Equation (22)

$$2.11\, \Sigma_u\, R\, C\, (.95) = 0.6$$

or

$$C\, \Sigma_u = \frac{0.6}{2.11 \times 66.3 \times .95} = 0.00452$$

Some constants needed are

$$3\, \Sigma_t\, \Sigma_u\, C\, R^2 = 3 \times 0.67 \times 0.00452 \times 4400 = 40.0$$

$$3\, \Sigma_t\, Rp = 3 \times 0.67 \times 66.3 = 133$$

From Equation (21) and the tables (7 and 13)

$$A = -40.0\ (.2612) + 133\ (.0878)$$

$$= -10.40 + 11.68$$

$$= 1.28$$

From Equation (23) and the tables (8, 14, and 15)

$$\varphi_{0.6} = -3.87 + 3.80 + 0.77$$

$$= 0.70$$

$$C = \frac{0.70}{0.95} = 0.74$$

$$\Sigma_u = \frac{0.00452}{0.74} = 0.0061 \quad \text{cm}^{-1}$$

We can calculate critical mass from the definition of Σ_u

$$\Sigma_u = \frac{M \ \sigma_u \ a}{V \ A_u}$$

or

$$M = \frac{0.0061 \times 4 \times \pi \times (4400)^{3/2} \times (0.6)^3 \times 235}{3 \times 645 \times .602 \times 10^3}$$

$$= 1.0 \text{ Kg}$$

From equation (9)

$$\psi_f = \eta \ \Sigma_u R \ \varphi$$

$$= 2.11 \times .0061 \times 66.3 \ \varphi$$

$$= 0.87 \ \varphi$$

① ξ	② $d\xi$	③ ψ_{th}	④ ③ $d\xi$	⑤ $\int$④	⑥ ⑤ $d\xi$	⑦ $\int$⑥	⑧ 133 x ⑦
0		0		0		0	0
	.1		.005		.0003		
.1		.10		.005		.003	.04
	.1		.014		.0012		
.2		.18		.019		.0015	.20
	.1		.022		.0030		
.3		.26		.041		.0045	.60
	.1		.027		.0055		
.4		.27		.068		.0100	1.33
	.1		.028		.0082		
.5		.28		.096		.0182	2.42
	.1		.025		.0109		
.6		.21		.121		.0291	3.87
	.1		.018		.0130		
.7		.15		.139		.0421	5.60
	.1		.012		.0145		
.8		.09		.151		.0566	7.53
	.1		.006		.0154		
.9		.03		.157		.0720	9.57
	.1		.002		.0158		
1.0		0		.159		.0878	11.68

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
ρ/c	(9) $d\xi$	$\int$(10)	(11) $d\xi$	$\int$(12)	40.0 x (13)	$A\xi$	φ (15) + (14) − (8)	ψ_f 0.87 x (16)
0		0		0	0	0	0	0
	.016		.0008					
.31		.016		.0008	.03	.13	.12	.10
	.045		.0039					
.59		.061		.0047	.19	.26	.25	.22
	.070		.0096					
.81		.131		.0143	0.57	.38	.35	.30
	.088		.0175					
.95		.219		.0318	1.27	.51	.45	.39
	.098		.0268					
1.00		.317		.0586	2.34	.64	.56	.49
	.098		.0366					
.95		.415		.0952	3.80	.77	.70	0.61
	0		.0415					0
.81		.415		.1367	5.46	.90	.76	0
	0		.0415					
.59		.415		.1782	7.13	1.02	.62	0
	0		.0415					
.31		.415		.2197	8.76	1.15	.34	0
	0		.0415					
0		.415		.2612	10.40	1.28	0	0

It can be seen that, by coincidence, the calculated value of ψ_f is quite close to the curve assumed in the Schmidt plot. The calculated φ is shown for comparison on the plot. The vagaries in the curve for φ are probably an accumulation of numerical errors in the integration and differentations, and if so, would be reduced by a closer interval, $\Delta\xi$.

Such factors as changing material properties, variable extrapolation lengths as a function of energy, etc., are handled nicely by a Schmidt plot by changes of spacing $\Delta\xi$, as described quite fully in the "Heat Transfer Notes" referred to previously.

In part II-2 of these lectures, on the "Dimensional Analysis of Reactors", an equation was given for the critical mass of a reactor (Equation (7)), in terms of dimensionless parameters. This equation is written here

$$\frac{\sigma_u \quad M_u \quad a}{\Sigma_c \quad \gamma \quad R^3 \quad A_u} = \frac{\dfrac{\beta^2}{3\Sigma_t \ \Sigma_c \ R^2} + 1}{p\eta\epsilon \exp\left[-\dfrac{\tau_{th}}{R^2}\int_0^1 \alpha^2 \, d(\tau/\tau_{th})\right] - 1}$$

Let us look at this equation for our present numerical example. In it we now know all quantities (from our numerical solution) except α^2 and β^2. If we knew these we could do an analysis of this from which we would know the mass curve. In other words, from our one calculated point, we could determine the characteristics of the family of "similar" reactors. For a rough evaluation let us take

$$\beta^2 = \int_0^1 \alpha^2 \, d\left(\frac{\tau}{\tau_{th}}\right) = \alpha^2$$

From our numerical example we know the following values:

$$R = 66.3 \text{ cm.}$$
$$\Sigma_t = 0.67 \text{ cm}^{-1}$$
$$\Sigma_a = \Sigma_u = 0.0061 \text{ cm}^{-1}$$
$$\tau_{th} = 110 \text{ cm}^2$$
$$P = \epsilon = 1$$
$$f = 1$$
$$\eta = 2.11$$
$$\sigma_u = 645 \times 10^{-24} \text{ cm}^2$$
$$M_u = 1.0 \times 10^3 \text{ gm}$$
$$a = 0.602 \times 10^{24}$$
$$A_u = 235$$
$$\Sigma_c = 0$$
$$\gamma = \frac{4}{3}\pi (0.6)^3 = 0.906$$

Because, in an example, we assumed $\Sigma_c = 0$, Equation (7) becomes

$$1 = \frac{\dfrac{\alpha^2 \gamma R \, Au}{3 \Sigma_t \sigma_u M_u a}}{\eta e^{-\alpha^2 \frac{\tau_{th}}{R^2}} - 1}$$

Substituting values

$$\frac{\gamma R A_u}{3 \Sigma_t \sigma_u M_u a} = \frac{0.906 \times 66.3 \times 235}{3 \times 0.67 \times 645 \times 0.602 \times 1.0 \times 10^3} = 0.018$$

$$\frac{\tau_{th}}{R^2} = \frac{110}{4450} = \frac{1}{40}$$

Then

$$1 = \frac{0.018\ \alpha^2}{2.11\ e^{-\frac{\alpha^2}{40}} - 1}$$

Approximately

$$e^{-\frac{\alpha^2}{40}} = 1 - \frac{\alpha^2}{40}$$

and

$$2.11 - \frac{2.11}{40}\alpha^2 - 1 = .018\alpha^2$$

$$\alpha^2\ (.018 + .053) = 1.11$$

$$\alpha^2 = \frac{1.11}{.071} = 15.6$$

or

$$\alpha = 3.95$$

This value of α is to be compared with that of π for the case of the bare reactor. Using this value (3.95) for both α and β would give an approximate mass curve for our reflected system.

References for Numerical Methods

Relaxation Methods in Engineering Science, R. W. Southwell; Oxford University Press, 1940.

Numerical Mathematical Analysis, J. B. Scarborough; The John Hopkins Press, Baltimore, 1930.

Numerical Methods of Analysis in Engineering, L. E. Grinter; The Macmillan Company, New York, 1949.

A Study of the Numerical Solution of Partial Differential Equations, O'Brien, Hyman, Kaplan; Journal of Mathematics and Physics, V. XXIX, No. 4, January 1951, M.I.T.

Numerical Computation of Neutron Distribution and Critical Size, A. S. Thompson; Journal of Applied Physics, V. 22, No. 10, Oct. 1951.

Problem: Vibrating String of Uniform Section

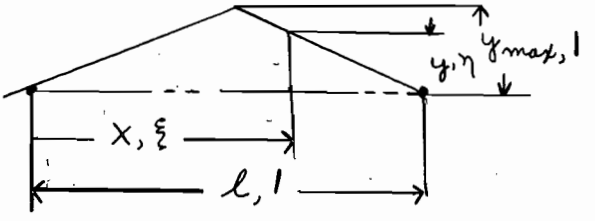
$$\frac{d^2y}{dx^2} = 2w^2 \frac{\mu}{T} y^*$$

$$\frac{d^2y}{d\xi^2} = -\frac{w^2 \mu l^2}{T} \eta \quad , \begin{cases} \eta = y/y_{max} \\ \xi = x/l \end{cases}$$

Or integrating

$$\eta = -\frac{w^2 \mu l^2}{T} \int_0^{\xi} d\xi_1 \int_0^{\xi_1} \eta \, d\xi_2 + A\xi + B$$

Iterate this equation twice, starting with the curve shown above, solving for frequency each time. Check with analytical value.



* From $\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$, by assumption of separability of functions.

III. Power Removal from Reactors

Until the present time we have been talking about the problem that is specifically new with nuclear reactors, namely the generation of power from the fission process. We might list the items we discussed:

(1) Power density is proportional to: (a) neutron flux; and (b) density of fissionable material;

(2) Some power is produced locally (at the point of fission), some 5 to 10 percent is absorbed in reflector, shielding, control rods, etc.;

(3) Most of the power is produced at the time of fission, but about 6 percent is delayed, and decays exponentially after shutdown of the nuclear reaction;

(4) The critical mass of fissionable material is a function of the critical radius of the reactor. For one critical radius the critical mass is less than for either larger or smaller critical radii. The critical mass and critical radius can be expressed in terms of dimensionaless physical and geometrical parameters;

(5) The neutron flux distribution is a function of reactor geometry. For certain special cases it can be found analytically, but must in general be found numerically;

(6) Excess neutrons over those required for criticality, can be used for converting certain non-fissionable materials into fissionable materials. It is probably on the possibility of large scale conversion of these materials that the future of atomic energy for peacetime purposes hangs.

The next task is to remove the power generated by the fission process in the reactor. If we know the distribution of this power throughout the reactor it is now our problem to remove it consistent with certain limitations, some of which are nuclear (mostly we cannot use heavy neutron absorbers for heat removal), and some of which are purely mechanical. The mechanical problems are connected primarily with heat transfer. (1) If we are dealing with large power densities then there will generally be

large temperature gradients and attendant large thermal stresses. We will try to find out to what extent we can mitigrate the thermal stress problem and to what extent we must live within its confines. (2) If we are designing a thermal power plant, its thermal efficiency will depend largely on the top temperature available at the exit from the reactor. Hence we will want to know how much we have to gain from pushing the creep resistance of high temperature materials to their limits. (3) The removal of thermal energy will generally involve the circulation of a cooling fluid. The more power we remove from the reactor, the more power we must use in pumping the fluid. We will wish to determine limitations on power removal due to pumping power requirements. For this purpose it will be necessary to discuss some aspects of fluid flow and thermodynamics.

Heat transfer is conventionally studied in three categories: (1) radiation; (2) convection; and (3) conduction. Here radiation will be ignored on the assumption that engineering applications of the immediate future will be limited by materials considerations to temperatures at which radiation effects can be considered as corrections*, of small magnitude, to the convection heat transfer. Heat transfer by convection will be studied to determine the limitations on power density due to the pressure losses which accompany it. Heat transfer by conduction will be studied primarily in its relationship to the thermal stress problem.

1. Order of Magnitude of Thermal Stress

Most materials, when their temperature is raised, experience an expansion of their dimensions, unless thay are in some manner constrained. This is expressed by an equation

$$\epsilon = \alpha \Delta T \qquad (1)$$

where ϵ is the strain in the material and α is the coefficient of linear expansion, ΔT being the change in temperature above some reference

* The Science and Engineering of Nuclear Power, Clarke Goodman; Addison-Wesley Press, Cambridge, Mass. (1947) p. 326

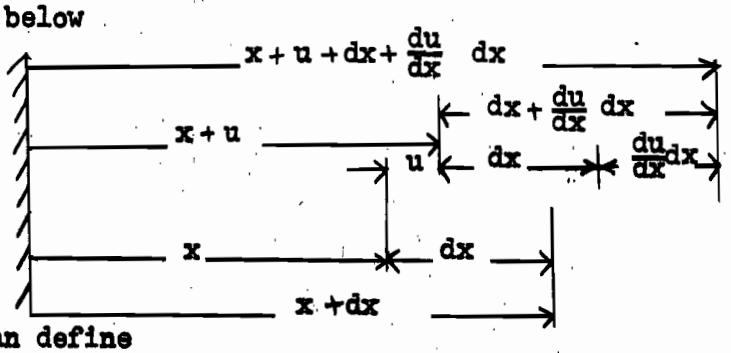
temperature. The strain, ϵ, is defined in terms of a deformation, u. Let us represent an increment of distance, dx, at a point at distance, x, from some reference point. Let us say that the point x, on deformation, moves a distance, u, from its original position. It is now a distance, x + u, from our reference point. At the same time, the point, x + dx, has moved to a new position, $x + u + dx + \frac{du}{dx}\,dx$. The change of length of the increment dx is hence

$$dx + \frac{du}{dx}\,dx - dx = \frac{du}{dx}\,dx$$

The strain ϵ, being the ratio of this change in length to the original length is

$$\epsilon_x = \frac{\frac{du}{dx}\,dx}{dx} = \frac{du}{dx}$$

This is shown below



Likewise we can define

$$\epsilon_y = \frac{dv}{dy}, \quad \epsilon_z = \frac{dw}{dz}$$

Where v and w are deformations in the yard y directions. Likewise if a stress (force per unit cross section of the material) is applied to the material, deformations will again occur and we can define a strain, which for an elastic material is proportional to the stress

$$\epsilon = \frac{\sigma}{E} \tag{2}$$

the proportionality constant E being called the "modulus of elasticity" or "Young's modulus". Equation (2) is Hooke's law relating elastic stresses and strains. Also, it has been found experimentally that when stresses

are applied a change in dimension normal to the applied stress occurs. This change is opposite in direction to that in the direction of the stress, and related to it by a factor, μ, called "Poisson's ratio". Therefore, we must write, due to stress σ_x, σ_y, and σ_z.

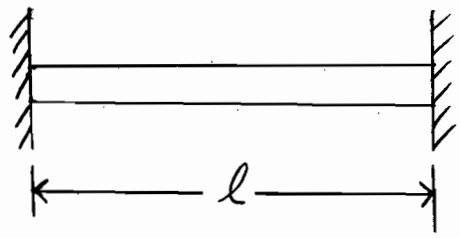
$$\epsilon_x = \frac{1}{E}\left\{\sigma_x - \mu(\sigma_y + \sigma_z)\right\}$$
$$\epsilon_y = \frac{1}{E}\left\{\sigma_y - \mu(\sigma_z + \sigma_x)\right\}$$
$$\epsilon_z = \frac{1}{E}\left\{\sigma_z - \mu(\sigma_x + \sigma_y)\right\}$$

Equations (3) are called the generalized Hooke's law for elastic strains. The factor μ has values for different materials between zero and one-half For most engineering materials it is about 0.3. For rubber it is close to one-half, the limiting value for an incompressible material. If both temperature differences and stresses exist we must add Equation (4) to each of Equations (3) to find the strains.

$$\epsilon_x = \frac{1}{E}\left\{\sigma_x - \mu(\sigma_y + \sigma_z)\right\} + \alpha\Delta T$$
$$\epsilon_y = \frac{1}{E}\left\{\sigma_y - \mu(\sigma_z + \sigma_x)\right\} + \alpha\Delta T \qquad (4)$$
$$\epsilon_z = \frac{1}{E}\left\{\sigma_z - \mu(\sigma_x + \sigma_y)\right\} + \alpha\Delta T$$

Let us use Equations (4) to find out how thermal stresses arise and to establish an order of magnitude of thermal stress. The simplest case is that of a uniform straight rod constrained between two fixes walls, so that as it is heated its length cannot change. Then the strain

$$\epsilon_x = \frac{du}{dx} = \frac{\Delta \ell}{\ell} = 0$$



There are no lateral constraints on the bar so $\sigma_y = \sigma_z = 0$.

Hence from the first of Equation (4)

$$\sigma_x = -E\alpha\Delta T \tag{5}$$

From the last two of Equations (4)

$$\epsilon_y = \epsilon_z = -\frac{\mu}{E}\sigma_x + \alpha\Delta T \tag{6}$$

Combining these two expressions we find

$$\epsilon_y = \epsilon_z = (1+\mu)\alpha\Delta T$$

For the next case, let ϵ_x and $\epsilon_y = 0$ and $\sigma_z = 0$. Then for this two dimensional stress case, from the first of Equation (4)

$$\sigma_x - \mu\sigma_y = -E\alpha\Delta T$$
$$\sigma_y - \mu\sigma_x = -E\alpha\Delta T$$

Or if $\sigma_x = \sigma_y$, then

$$\sigma_x = \sigma_y = -\frac{E\alpha\Delta T}{1-\mu} \tag{7}$$

From the third of Equations (4)

$$\epsilon_z = -\frac{\mu}{E}(\sigma_x + \sigma_y) + \alpha\Delta T$$

or

$$\epsilon_z = \frac{1+\mu}{1-\mu}\alpha\Delta T \tag{8}$$

For the three dimensional stress case, let $\epsilon_x = \epsilon_y = \epsilon_z = 0$ and $\sigma_x = \sigma_y = \sigma_z$. Then from Equation (4)

$$\sigma_x = \sigma_y = \sigma_z = -\frac{E\alpha\Delta T}{1-2\mu} \tag{9}$$

For an incompressible medium ($\mu = 0.5$), Equation (9) would give an infinite stress.

In general there will exist in a heated material some maximum stress

$$\sigma_{max} = n \frac{E \alpha \Delta T}{1 - a\mu}$$

where n has a value depending on the amount of constraint

$$0 < n < 1$$

and a depends on whether the constraints exist in one, two, or three coordinate directions

Constraints	a
1	0
2	1
3	2

The order of magnitude of thermal stress is given by

$$K = \frac{E \alpha \Delta T}{1 - a\mu}$$

Where T is considered as the maximum temperature difference existing between any two points in the material. In general, if K is small compared with the allowable stress for the material, the thermal stress problem can be ignored. If this is not the case a detailed calculation is necessary to determine the value of n, which depends on geometrical considerations. The determination of the temperature distribution will involve a study of the conduction of heat through the material. For the detailed case it is necessary to consider further the stress analysis of a specific configuration.

The section which follows will consider the conduction of heat, and will determine the temperature distribution which results.

2. Conduction of Heat

The study of the problem of conduction of heat led Fourier to the equation of heat conduction which bears his name and to the fourier series

method of its solution. The fourier equation for heat conduction is written

$$q = k A \frac{\Delta T}{\Delta x}$$

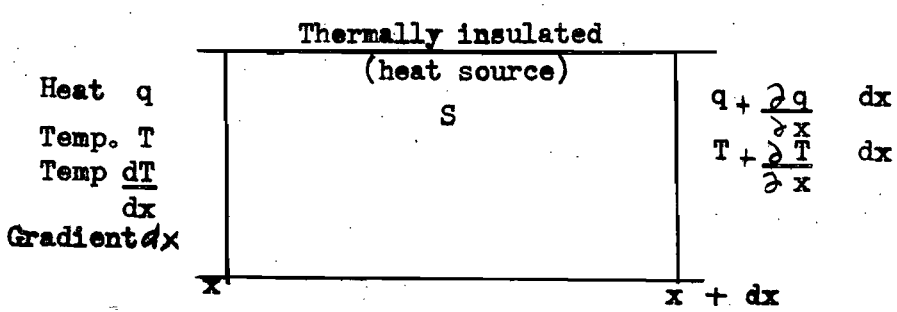
where q is the heat energy transferred across a slab of material having a cross section normal to the direction of flow of heat, A, and a temperature difference, ΔT, between two faces separated by a small distance, Δx. The coefficient k is the thermal conductivity of the material of which the slab is composed, and, for dimensional consistency, must have the dimension

$$\frac{\text{(energy)}}{\text{(time)(Temperature)(length)}}$$

If Δx is allowed to approach the limit zero, the remaining expression gives the heat transferred per unit time across a surface of area, A.

$$q = -k A \frac{\partial T}{\partial x} \qquad (2)$$

The negative sign indicates that heat flows from higher to lower temperatures. If the temperature gradient, dT/dx, and conductivity, k, are considered as functions of distance, x, a pictorial representation of the situation is



where S represents an internal energy source distributed throughout the material, and has the dimensions

$$\frac{(\text{energy})}{(\text{time})\ (\text{length})^3}$$

For an energy balance it is required that the heat which crosses the boundary at x, plus the heat created by the source, equal the heat flowing across the boundary at x + dx, plus the heat stored in the element, dx. This is expressed

$$q + S \quad A \quad dx = q + \frac{\partial q}{\partial x} \quad dx + C_p \rho \ A \quad \frac{\partial T}{\partial t} \quad dx \qquad (3)$$

where Cp is the specific heat of the material through which heat is flowing and ρ is its density

$$Cp \sim \frac{(\text{energy})}{(\text{mass})\ (\text{temperature})}$$

$$\rho \sim \frac{(\text{mass})}{(\text{length})^3}$$

Equation (3) becomes

$$AS = \frac{\partial q}{\partial x} + A\ Cp\ \rho \quad \frac{\partial T}{\partial t} \qquad (4)$$

Substituting the value of q from Equation (2)

$$\frac{\partial}{\partial x}\left(k \quad \frac{\partial T}{\partial x}\right) + S = Cp\ \rho \quad \frac{\partial T}{\partial t} \qquad (5)$$

By symmetry if we have heat flowing also in the y and z directions

$$\frac{\partial}{\partial x}\left(k \quad \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \quad \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \quad \frac{\partial T}{\partial z}\right) + S = C_p\ \rho \frac{\partial T}{\partial t} \qquad (6)$$

or in customary notation

$$\nabla \cdot (k \nabla T) + S = C_p\ \rho \quad \frac{\partial T}{\partial t} \qquad (7)$$

Equation (7) is the equation for the temperature distribution in a solid body with a distributed heat source. If the conductivity is constant, Equation (7) becomes

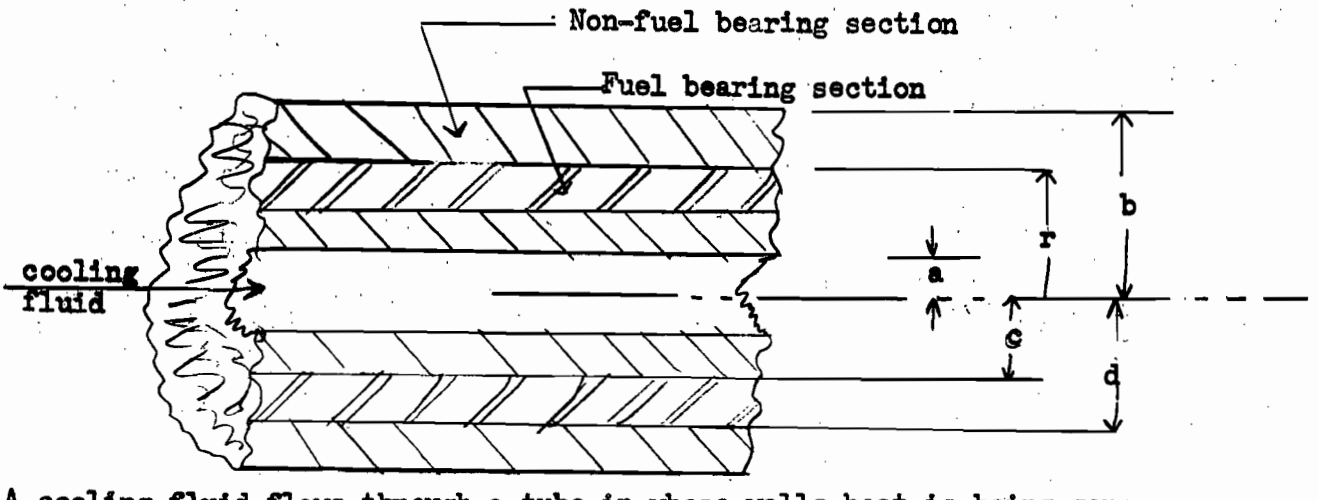
$$\nabla^2 T + \frac{S}{k} = \frac{C_p \rho}{k} \frac{\partial T}{\partial t} \qquad (8)$$

It might be noted that Equation (8), without the source term, is of the same form as the slowing down equation for neutrons. The dimensional coefficient

$$\frac{k}{C_p \rho} \sim \frac{(\text{Length})^2}{(\text{time})}$$

is called the thermal diffusivity. Being a function only of the physical properties of the heat transfer material, it can be found tabulated for many engineering materials.

For definiteness let us fix now a heat transfer model which we will use for all our discussions. This model is shown below .



A cooling fluid flows through a tube in whose walls heat is being generated. The heat which is generated within the walls must be carried to the inner surface by conduction, and carried away from the surface by convection. All variables are considered to be uniform in the tangential and axial directions, varying only with radius.

Many other models could be taken. For instance, the heat could be generated within the fluid, the tube in this case being merely a container or possibly also a moderator. This case causes some special problem of

its own which we will discuss later. Our purpose at the moment is to demonstrate the origin and magnitude of thermal stress , and the causes of high temperature in solid materials. Later this same model will be used to discuss convective heat transfer and its accompanying pressure losses. For these purposes the tubular model, having heat generation within its walls, is sufficient.

For the tubular model Equation (8) becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{S}{k} = \frac{C_p \rho}{k} \frac{\partial T}{\partial t} \tag{9}$$

since in this case

$$\nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{10}$$

Steady state conditions are assumed, giving

$$\frac{1}{r} \frac{d}{d r} \left(r \frac{dT}{d r} \right) + \frac{S}{k} = 0 \tag{11}$$

Equation (11) is the integrated subject to two boundary conditions. The first states that there is no heat flow across the outer boundary, or

$$\left. \frac{d T}{d r} \right]_{r = b} = 0 \tag{12}$$

The second condition states that the temperature at some point in the material will be fixed by the capacity of the convective heat transfer system. This can be stated

$$\left. T \right]_{r = b} = T_b \tag{13}$$

Integrating Equation (11) once

$$r \frac{d T}{d r} + \frac{1}{k} \int_a^r S r_1 \, dr_1 + C_1 = 0 \tag{14}$$

From equation (12)

$$C_1 = - \frac{1}{k} \int_a^b S r_1 \, dr_1 \tag{15}$$

Substituting this value for C_1

$$r \frac{dT}{dr} + \frac{1}{k} \int_b^r Sr_1 dr_1 = 0 \qquad (16)$$

Integrating Equation (16)

$$T + \frac{1}{k} \int_a^r \frac{dr_1}{r_1} \int_b^{r_1} Sr_2 dr_2 \quad C_2 = 0 \qquad (17)$$

From Equation (13)

$$T_b + \frac{1}{k} \int_a^b \frac{dr_1}{r_1} \int_b^{r_1} Sr_2 dr_2 + C_2 = 0 \qquad (18)$$

Substracting Equation (18) from (17)

$$T - T_b + \frac{1}{k} \int_b^r \frac{dr_1}{r_1} \int_b^{r_1} Sr_2 dr_2 = 0 \qquad (19)$$

The maximum temperature difference existing in the tube wall is

$$T_a - T_b + \frac{1}{k} \int_b^a \frac{dr_1}{r_1} \int_b^{r_1} Sr_2 dr_2 = 0 \qquad (20)$$

The average power density for the space occupied by the tube is

$$\bar{S} = \frac{2\pi \int_a^b Sr_1 dr_1}{\pi b^2} = \frac{2}{b^2} \int_a^b Sr_1 dr_1 \qquad (21)$$

Let us define two dimensionless variables

$$\xi = r/b$$

$$\gamma = S/S_o$$

where So is the power density at some reference point. Then Equation (19) becomes

$$T - T_b + \frac{b^2 S_o}{k} \int_1^{\xi} \frac{d\xi_1}{\xi_1} \int_o^{\xi} \gamma \xi_2 d\xi_2 = 0 \qquad (22)$$

Equation (20) is

$$T_a - T_b + \frac{b^2 S_o}{k} \int_1^{a/b} \frac{d\xi_1}{\xi_1} \int_o^{\xi} \gamma \xi_2 d\xi_2 = 0 \qquad (23)$$

and Equation (21) is

$$\bar{S} = -2 S_o \int_1^{a/b} \gamma \xi_1 \, d\xi_1 \qquad (24)$$

Eliminating S_o between Equations (23) and (24)

$$T_a - T_b = \frac{b^2 \bar{S}}{2k} \; \frac{\int_1^{a/b} \frac{d\xi_1}{\xi_1} \int_1^{\xi_1} \gamma \xi_2 d\xi_2}{\int_1^{a/b} \gamma \xi_1 d\xi_1} \qquad (25)$$

The double integral can be integrated by parts to give

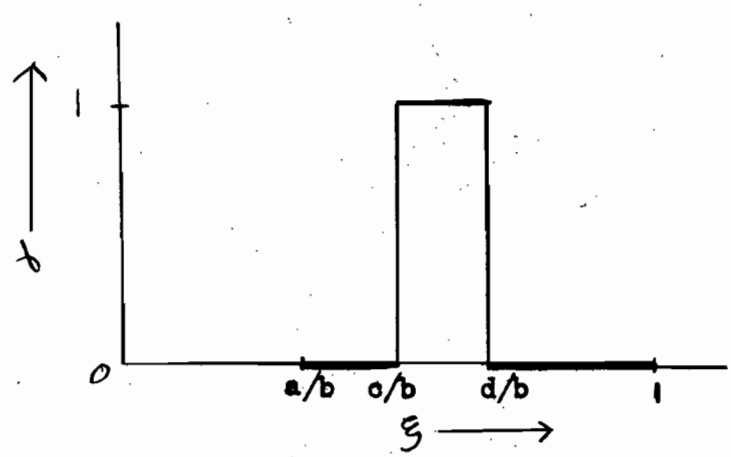
$$\int_1^{a/b} \frac{d\xi_1}{\xi_1} \int_1^{1} \gamma \xi_2 d\xi_2 = \left[\log \xi \int_1^{\xi} \gamma \xi_1 d\xi_1 \right]_1^{a/b} - \int_1^{a/b} (\log \xi_1) \gamma \xi_1 d\xi_1$$

$$= \log a/b \int_1^{a/b} \gamma \xi_1 d\xi_1 - \int_1^{a/b} (\log \xi_1) \gamma \xi_1 d\xi_1 \qquad (26)$$

Substituting into Equation (25)

$$T_a - T_b = \frac{b^2 \bar{S}}{2k} \left[\log a/b - \frac{\int_1^{a/b} (\log \xi_1) \gamma \xi_1 d\xi_1}{\int_1^{a/b} \gamma \xi_1 d\xi_1} \right] \qquad (27)$$

For our case γ has the form of a step function



For this case

$$\int_1^{a/b} \gamma \xi_1 d\xi_1 = \int_{d/b}^{c/b} \xi_1 d\xi_1 = -\left[\frac{d^2 - c^2}{2b^2} \right]$$

and

$$\int_{1}^{a/b} (\log \xi_1)\, \gamma \xi_1 d\xi_1 = \int_{d/b}^{c/b} (\log \xi_1)\, \xi_1 d\xi_1$$

Integrating by parts

$$\int^{c/b} (\log \xi_1)\, \xi_1 d\, \xi_1 = \left[(\log \xi)\, \frac{\xi^2}{2}\right]_{d/b}^{c/b} - \tfrac{1}{2}\int_{d/b}^{c/b} \xi_1 d\xi_1$$

$$= \tfrac{1}{2}\left(\frac{c}{b}\right)^2 \log\left(\frac{c}{b}\right) - \tfrac{1}{2}\left(\frac{d}{b}\right)^2 \log\left(\frac{d}{b}\right) + \left[\frac{d^2 - c^2}{4\, b^2}\right]$$

Hence Equation (27) becomes

$$T_b - T_a = \frac{b^2\, \bar{S}}{2\, k}\left[-\tfrac{1}{2} - \log\,(a/b) + \frac{d^2}{d^2-c^2}\log\,(d/b) - \frac{c^2}{d^2-c^2}\log\,(c/b)\right] \quad (28)$$

simplifying

$$T_b - T_a = \frac{b^2\, \bar{S}}{2\, k}\left[-\tfrac{1}{2} + \log\,(c/a) + \frac{\log\,(d/c)\,(d/c)^2}{(d/c)^2 - 1}\right]$$

$$= \frac{b^2\, \bar{S}}{4\, k}\left[(d/c)^2\, \frac{\log\,(d/c)^2}{(d/c)^2 - 1} - 1 + \log\,(c/a)^2\right] \quad (29)$$

Since it was found in the last section that the thermal stress depended for its order of magnitude on $\triangle T$, which here is $(T_b - T_a)$, we would like to find from Equation (29) the conditions under which, for a given average power density, $\bar{S}$, we will have the minimum temperature difference. We can see first that we wish to have a large thermal conductivity, k, and a small tube diameter, b. Looking at the quantity inside the brackets, we see that it becomes zero for $(c/a) = (d/c) = 1$. Hence we want the thinnest layer of fuel bearing material from which we can abstract the required amount of heat, and the thinnest cladding layer we can stand between the fuel and the cooling fluid. The limiting thickness for the fuel layer is of course set by the amount of fuel required.

The highest temperature difference would exist when the fuel was spread throughout the tube material. Then

$$T_b - T_a = \frac{b^2 \bar{S}}{4 k} \left[\frac{(b/a)^2 \log (b/a)^2}{(b/a)^2 - 1} - 1 \right] \tag{30}$$

For this case it is desirable to have (b/a) as close to one as possible. For this case approximately

$$T_b - T_a \sim \frac{b\, t\, \bar{S}}{2\, k} \tag{31}$$

where t is the thickness of the tube wall.

3. Thermal Stress in Power Producing Elements

In section (1) it was shown that the order of magnitude of thermal stress is given by

$$n\left(\frac{E\alpha\Delta T}{1-a\mu}\right)$$

where a depended on the number of constraints imposed on the heated material and n is a factor depending on the detailed geometry of the problem, but generally

$$0 < n < 1$$

In section (2) an expression was developed for the temperature difference ΔT for a hollow tube, in whose walls heat was generated, and in which a cooling fluid was circulated. It was shown that to some extent the temperature difference is under the control of the designer. In these two sections the purpose was mostly to find under what conditions the order of magnitude of thermal stress would be low enough to justify ignoring it. It is the purpose of the present section to demonstrate the application of the theory of elasticity to the detailed investigation of the distribution of thermal stress.

The distribution of stresses in any elastic body* must satisfy three conditions: (1) equilibrium; (2) compatibility; and (3) Hooke's law. Condition (1) states that the summation of forces on any isolated element of the body must be equal to zero. Condition (2) states that the strains in the body must add up over the whole body to fit the deformations imposed

* Theory of Elasticity, S. Timoshenko, McGraw-Hill Book Co., Inc. New York (1934) (Introduction and Chapters VI and VII)

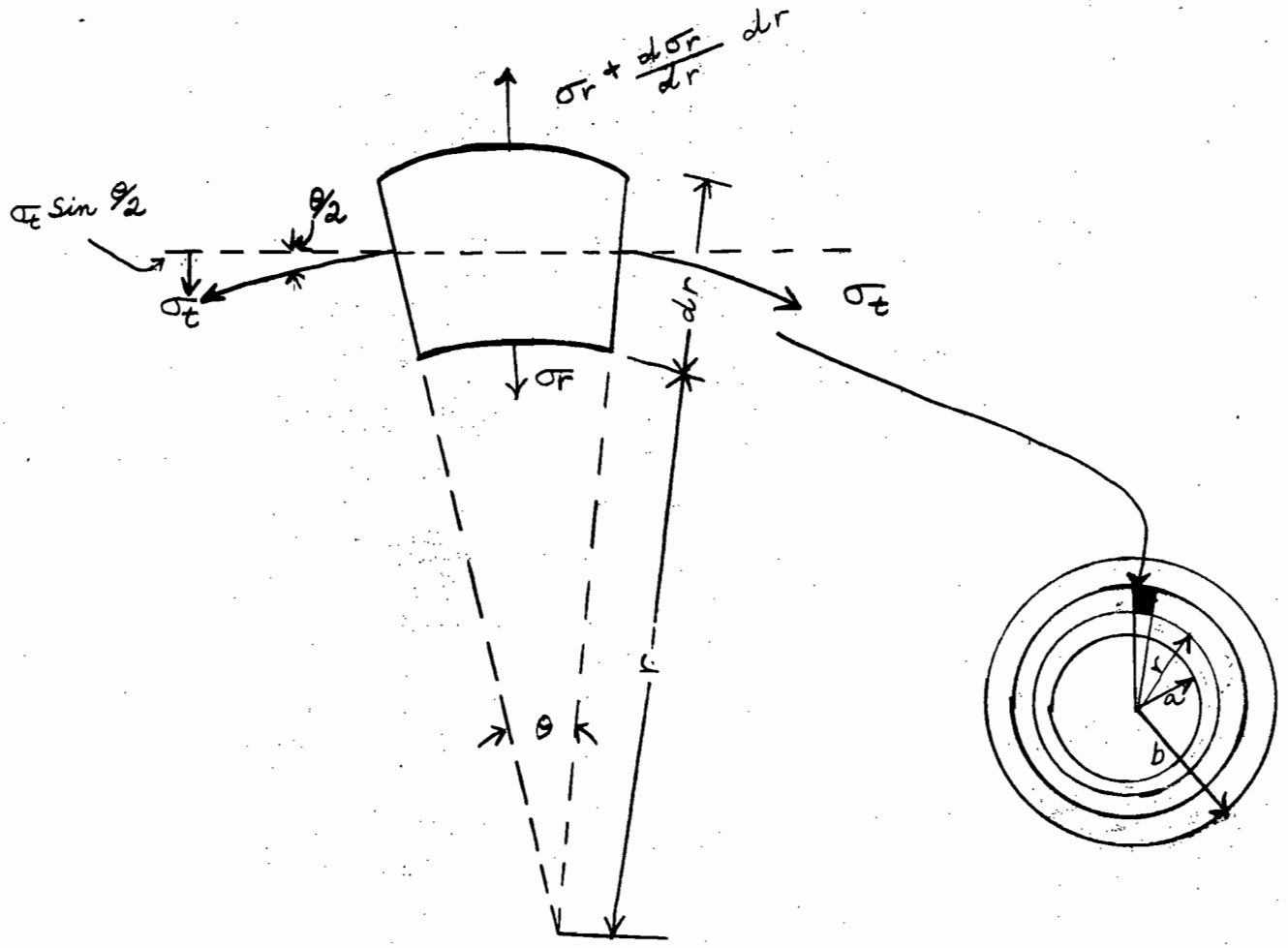
Strength of Materials, Part I, S. Timoshenko, D. VanNostrand Co., Inc. New York (1941) (Chapters I and II)

on the body. Condition (3) requires the stresses and strains in the body to be linearly interdependent. In general these three conditions will result in a linear system of equations which must be solved simultaneously. For certain classes of problems, like the bending of beams, assumptions can be made which enable one to satisfy each of these conditions separately and a great simplification results. In many of these problems one merely solves an equation for equilibrium for a statically determinate system, and is hardly aware even of the existence of other conditions.

For demonstration purposes we will write down the three conditions on elastic stresses for the heated tube which we discussed in Section (2), and will use these conditions to find the distribution of thermal stress. All variables are considered as uniform axially and tangentially, the only variation being with radius. It will be noted that a simplification results which makes the problem soluble. The directions of the principal stresses (stresses in the radial, axial and tangential directions are designated σ_r, σ_z, and σ_t respectively) coincide with the directions of the principal coordinate axes r, z, and t. This means that we do not have to calculate any shear stresses, only compressions and tensions. The theory of elasticity shows that any combination of stresses at a point can be expressed in terms of three principal stresses, with no shears, but it is not always easy to pick a coordinate system, which we can handle, which does this everywhere. Our assumption of uniformity means that we have what is called "plane strain", in which cross sections of the tube remain plane during deformation (axial strain is independent of radius). The equations we write will fit these idealizations.

Equilibrium

Let us isolate from the tube a section of solid material and find what is necessary to satisfy the equilibrium of forces on it.



We adopt here the conventions that a positive sign with a stress designates a tension, a negative sign designates compression. σ_t is tangential stress, and σ_r is axial stress. From the diagram

$$\left(\sigma_r + \frac{d\sigma_r}{dr} dr\right)(r + dr)\,\theta - \sigma_r r\,\theta - 2\,\sigma_t\, dr \sin \theta/2 = 0 \qquad (1)$$

Since we are dealing with differential increments

$$2 \sin \theta/2 = \theta$$

Simplifying Equation (1) and neglecting second order terms

$$\sigma_r\, dr + \frac{d\sigma_r}{dr} r\, dr - \sigma_t\, dr = 0 \qquad (2)$$

or

$$\frac{d(r\sigma_r)}{dr} - \sigma_t = 0 \qquad (3)$$

Compatibility

Equation (3) is the equilibrium condition on the stresses in the tube. In Section (1) we said that the strains were related to the deformations by expressions like

$$\epsilon_x = \frac{\partial u}{\partial x}$$

For our case

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{4}$$

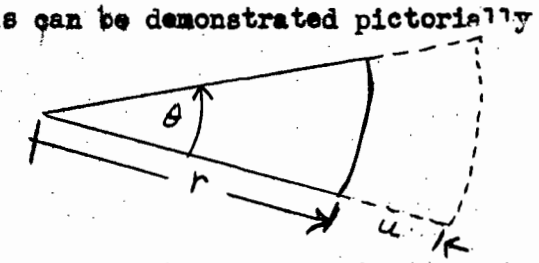
and

$$\epsilon_t = \frac{\partial v}{r \partial \theta} \tag{5}$$

The axial strain is given by

$$\epsilon_z = \frac{\partial v}{\partial s} \tag{6}$$

Since all quantities are uniform in the z directions, ϵ_z is a constant. Since tangential uniformity has been assumed we know that the tangential strain must be a function only of radius. Hence $\partial v/\partial \theta$ must be a function only of radius. This can be demonstrated pictorially



The original dimension in the tanjential direction is $(r\,\theta)$. The final dimension after deformation is $\left\{(r+u)\,\theta\right\}$. The strain is hence

$$\epsilon_t = \frac{(r+u)\,\theta - r\,\theta}{r} = \frac{u}{r} \tag{7}$$

Equations (4) and (7) can be combined by differentiating Equation (6) to eliminate u. Then

$$\frac{d}{dr}\,(r\,\epsilon_t) = \epsilon_r \tag{8}$$

Equation (8) is called the "compatibility condition" for the strains.

Hooke's Law

It and Equation (3) are two of the equations necessary to express the state of stress and strain in the deformed tube. Since Equation (3) gives a relationship for stresses while Equation (8) is for strains, another relationship is needed to relate stresses and strains. This relationship is provided by Hooke's law which we have discussed before. In terms of the principal stresses for the tube we can write for radial strain

$$\epsilon_r = \frac{1}{E}\left\{\sigma_r - \mu(\sigma_t + \sigma_z) + E\alpha\ (T - T_b)\right\} \qquad (9)$$

where, as in the last section, T_b is temperature at the outer surface of the tube. The tangential strain is

$$\epsilon_t = \frac{1}{E}\left\{\sigma_t - \mu(\sigma_z + \sigma_r) + E\alpha\ (T - T_b)\right\} \qquad (10)$$

Axial strain, which is constant, is

$$\epsilon_z = \frac{1}{E}\left\{\sigma_z - \mu(\sigma_r + \sigma_t) + E\alpha\ (T - T_b)\right\} = C \qquad (11)$$

Solving Equation (11) for σ_z gives

$$\sigma_z = \mu(\sigma_r + \sigma_t) - E\alpha\ (T - T_b) + \epsilon_z E \qquad (12)$$

Substituting Equation (12) into (9) and (10)

$$\epsilon_r = \frac{1}{E}\left[(1 - \mu^2)\sigma_r - \mu(1+\mu)\sigma_t + (1+\mu)E\alpha(T - T_b)\right] - \epsilon_z\mu \qquad (13)$$

and

$$\epsilon_t = \frac{1}{E}\left[(1 - \mu^2)\sigma_t - \mu(1+\mu)\sigma_r + (1+\mu)E\alpha(T - T_b)\right] - \epsilon_z\mu \qquad (14)$$

Substituting Equations (13) and (14) into (8) gives the compatibility condition in terms of stresses instead of strains

$$\frac{d}{dr}\left\{\frac{r}{E}\left[(1-\mu^2)\sigma_r - \mu(1+\mu)\sigma_t + (1+\mu)E\alpha\ (T - T_b)\right]\right\} \qquad (15)$$
$$= \frac{1}{E}\left[(1-\mu^2)\sigma_r - \mu(1+\mu)\sigma_t + (1+\mu)E\alpha\ (T - T_b)\right]$$

From Equations (3) and (15) we have two equations in two variables, σ_t and σ_r. Eliminating σ_t between these gives (E assumed constant and factor $(1+\mu)$ removed)

$$\frac{d}{dr}\left\{ r\left[(1-\mu)\frac{d}{dr}(r\sigma_r) - \mu\sigma_r + E\alpha(T-T_b)\right]\right\}$$

$$= (1-\mu)\sigma_r - \mu\frac{d}{dr}(r\sigma_r) + E\alpha(T-T_b) \qquad (16)$$

The second terms on both sides cancel, leaving

$$\frac{d}{dr}\left\{ r\left[(1-\mu)\frac{d}{dr}(r\sigma_r) + E\alpha(T-T_b)\right]\right\} = (1-\mu)\sigma_r + E\alpha(T-T_b) \qquad (17)$$

Differentiating the product $(r\,\sigma_r)$

$$\frac{d}{dr}\left\{ r\left[(1-\mu)\sigma_r + (1-\mu)\,r\,\frac{d\sigma_r}{dr} + E\alpha(T-T_b)\right]\right\} = (1-\mu)\sigma_r + E\alpha(T-T_b) \qquad (18)$$

Differentiating the product in the first brackets, cancelling like terms, and factoring $(1-\mu)$

$$r^2\frac{d^2\sigma_r}{d\,r^2} + 3\,r\,\frac{d\,\sigma_r}{d\,r} + \frac{E\alpha\,r}{1-\mu}\,\frac{d}{dr}(T-T_b) = 0 \qquad (19)$$

The first two terms can be combined, giving

$$\frac{d}{dr}\left(r^3\,\frac{d\sigma_r}{dr}\right) + \frac{E\alpha r^2}{1-\mu}\,\frac{d}{dr}(T-T_b) = 0 \qquad (20)$$

We substitute the value of $\frac{dT}{dr}$ from equation (16) of the last section which is

$$\frac{dT}{dr} = -\frac{1}{kr}\int_b^r S\,r_1\,dr_1 \qquad (21)$$

Equation (20) now becomes

$$\frac{d}{dr}\left(r^3\,\frac{d\sigma_r}{dr}\right) = \frac{E\alpha}{k(1-\mu)}\,r\int_b^r S r_1\,dr_1 \qquad (23)$$

Equation (23) is to be integrated twice subject to appropriate boundary conditions. Let these conditions be

$$\left.\frac{d\sigma_r}{dr}\right]_{r=b} = 0 \qquad (24)$$

and

$$\left.\sigma_r\right]_{r=0} = 0 \qquad (25)$$

Then integrating once, subject to Equation (24)

$$\frac{d\sigma_r}{dr} = \frac{E\,\alpha}{k\,(1-\mu)}\,\frac{1}{r^3}\int_b^r r_1 dr_1 \int_b^{r_1} Sr_2 dr_2 \qquad (26)$$

Integrating again subject to Equation (25)

$$\sigma_r = \frac{E\,\alpha}{k\,(1-\mu)}\int_a^r \frac{dr_1}{r_1^3}\int_b^{r_1} r_2 dr_2 \int_b^{r_2} Sr_3 dr_3 \qquad (27)$$

From equation (3) the tanjential stress is

$$\sigma_t = r\,\frac{d\sigma_r}{dr} + \sigma_r \qquad (28)$$

Substituting Equations (26) and (27) into (28)

$$\sigma_t = \frac{E\,\alpha}{k(1-\mu)}\left[\frac{1}{r^2}\int_b^r r_1 dr_1 \int_b^{r_1} Sr_2 dr_2 + \int_a^r \frac{dr_1}{r_1^3}\int_b^{r_1} r_2 dr_2 \int_b^{r_2} Sr_3 dr_3\right] \qquad (29)$$

From equation (12) if the tube is constrained against axial motion, $\epsilon_z = C = 0$, and

$$\sigma_z = \mu\,(\sigma_r + \sigma_t) - E\,\alpha\,(T - T_b) \qquad (30)$$

Substituting Equations (27) and (29)

$$\sigma_z = \frac{E\,\alpha\,\mu}{k(1-\mu)}\left[\frac{1}{r^2}\int_b^r r_1 dr_1 \int_b^{r_1} Sr_2 dr_2 + 2\int_a^r \frac{dr_1}{r_1^3}\int_b^{r_1} r_2 dr_2 \int_b^{r_2} Sr_3 dr_3\right] - E\,\alpha\,(T - T_b) \qquad (31)$$

The value of $T - T_b$ from Equation (20) of Section (2) is

$$T - T_b = -\frac{1}{k}\int_b^r \frac{dr_1}{r_1}\int_b^{r_1} Sr_2 dr_2 \qquad (32)$$

Hence the axial stress is

$$\sigma_z = \frac{E\,\alpha}{k\,(1-\mu)}\left[\frac{\mu}{r^2}\int_b^r r_1 dr_1 \int_b^{r_1} Sr_2 dr_2 + 2\mu\int_a^r \frac{dr_1}{r_1^3}\int_b^{r_1} r_2 dr_2 \int_b^{r_2} Sr_3 dr_3 + (1-\mu)\int_b^r \frac{dr_1}{r_1}\int_b^{r_1} Sr_2 dr_2\right. \qquad (33)$$

Equations (27) and (33) can be put into dimensionless form by the substitution

$$r = b\,\xi$$

$$S = S_o\,\gamma$$

and

$$S_o = -\frac{\bar{S}}{2\int_1^{a/b}\gamma\,\xi_1 d\xi_1}$$

Now equation (27) is

$$\sigma_r = -\frac{E\alpha\,\bar{S}\,b^2}{2\,k(1-\mu)\int_1^{a/b}\gamma\,\xi_1 d\xi_1}\left[\int_{a/b}^{\xi}\frac{d\xi_1}{\xi_1^3}\int_1^{\xi_1}\xi_2 d\xi_2\int_1^{\xi_2}\gamma\,\xi_3 d\xi_3\right] \quad (34)$$

Equation (29) is

$$\sigma_t = -\frac{E\alpha\,\bar{S}\,b^2}{2\,k(1-\mu)\int_1^{a/b}\gamma\,\xi_1 d\xi_1}\left[\frac{1}{\xi^2}\int_1^{\xi}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 + \int_{a/b}^{\xi}\frac{d\xi_1}{\xi_1^3}\int_1^{\xi_1}\xi_2 d\xi_2\int_1^{\xi_2}\gamma\,\xi_3 d\xi_3\right]$$

Equation (33) is

$$\sigma_z = \frac{-E\alpha\,\bar{S}\,b^2}{2k\,(1-\mu)\int_1^{a/b}\gamma\,\xi_1 d\xi_1}\left[\frac{\mu}{\xi^2}\int_1^{\xi}\xi_1 d\xi_1\int_{a/b}^{\xi_1}\gamma\,\xi_2 d\xi_2 + 2\mu\int_1^{\xi}\frac{d\xi_1}{\xi}\int_1^{\xi_1}\xi_2 d\xi_2\int_1^{\xi_2}\gamma\,\xi_3 d\xi_3 + (1-\mu)\int_1^{\xi}\frac{d\xi_1}{\xi_1}\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2\right]$$

From Equation (3), the maximum value of σ_r occurs when $\sigma_r = \sigma_t$, or from equations (34) and (35), when

$$\frac{1}{\xi^2}\int_1^{\xi}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 = 0 \quad (37)$$

This happens when $\xi = 1$, $(r = b)$. From equation (36), at $\xi = 1$, it is evident that

$$\sigma_z\Big]_{\xi=1} = 2\mu\,\sigma_r\Big]_{\xi=1} = 2\mu\,\sigma_t\Big]_{\xi=1}$$

From equation (34), then the maximum stress at $\xi = 1$ is

$$\sigma_r\Big]_{\xi=1} = -\frac{E\alpha\bar{S}\,b^2}{2k\,(1-\mu)\int_1^{a/b}\gamma\xi_1 d\xi_1}\left[\int_{a/b}^{1}\frac{d\xi_1}{\xi_1^3}\int_1^{\xi_1}\xi_2 d\xi_2\int_1^{\xi_2}\gamma\,\xi_3 d\xi_3\right] \tag{38}$$

When $\xi = a/b$, it can be seen that $\sigma_r = 0$. At this point

$$\sigma_t = -\frac{E\alpha\bar{S}\,b^2}{2k\,(1-\mu)\int_1^{a/b}\gamma\xi_1 d\xi_1}\left[\frac{b^2}{a^2}\int_1^{a/b}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2\right] \tag{39}$$

and

$$\sigma_z = -\frac{E\alpha\bar{S}\,b^2}{2k\,(1-\mu)\int_1^{a/b}\gamma\xi_1 d\xi_1}\left\{\mu\frac{b^2}{a^2}\int_1^{a/b}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 + (1-\mu)\int_1^{a/b}\frac{d\xi_1}{\xi_1}\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2\right\} \tag{40}$$

Usually the largest stress at $\xi = a/b$ is the tangential stress σ_t. These equations can be simplified by partial integrations. First

$$\int_1^{\xi}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 \overset{*}{=} \frac{1}{2}\,\xi^2\int_1^{\xi}\gamma\,\xi_1 d\xi_1 - \frac{1}{2}\int_1^{\xi}\gamma\,\xi_1^3 d\xi_1 \tag{41}$$

Next

$$\int_1^{\xi}\frac{d\xi_1}{\xi_1^3}\int_1^{\xi_1}\xi_2 d\xi_2\int_1^{\xi_2}\gamma\,\xi_3 d\xi_3 = \frac{1}{2}\left[\int_1^{\xi}\frac{d\xi_1}{\xi_1}\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 - \int_1^{\xi}\frac{d\xi_1}{\xi_1^3}\int_1^{\xi_1}\gamma\,\xi_1^3 d\xi_1\right]$$

$$= \frac{1}{2}\log\xi\int_1^{\xi}\gamma\,\xi_1 d\xi_1 - \frac{1}{2}\int_1^{\xi}\gamma\,\xi_1\log\xi_1 d\xi_1 + \frac{1}{4\xi^2}\int_1^{\xi}\gamma\,\xi_1^3 d\xi_1 - \frac{1}{4}\int_1^{\xi}\gamma\,\xi_1 d\xi_1 \tag{42}$$

From Equation (41)

$$\int_1^{a/b}\xi_1 d\xi_1\int_1^{\xi_1}\gamma\,\xi_2 d\xi_2 = \frac{1}{2}\left(\frac{a}{b}\right)^2\int_1^{a/b}\gamma\,\xi_1 d\xi_1 - \frac{1}{2}\int_1^{a/b}\gamma\,\xi_1^3 d\xi_1 \tag{43}$$

From Equation (42)

$$\int_{a/b}^{1}\frac{d\xi_1}{\xi_1^3}\int_1^{1}\xi_2 d\xi_2\int_1^{2}\gamma\,\xi_3 d\xi_3 = \left(\frac{1}{4}-\frac{1}{2}\log a/b\right)\int_1^{a/b}\gamma\,\xi_1 d\xi_1 + \frac{1}{2}\int_1^{a/b}\gamma\,\xi_1\log\xi_1 d\xi_1 - \frac{1}{4}\left(\frac{b}{a}\right)^2\int_1^{a/b}\gamma\,\xi_1^3 d\xi_1 \tag{44}$$

---

* Let $v = \int_1^{\xi}\xi_1 d\xi_1$, $u = \int_1^{\xi}\gamma\,\xi_1 d\xi_1$, then $\int_1^{\xi} u\,dv = \left[uv\right]_1^{\xi} - \int_1^{\xi} v\,du$

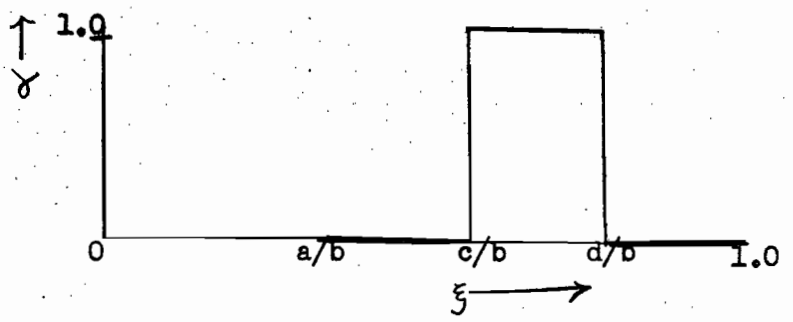
Substituting Equation (43) into (39), the maximum stress at $\xi = a/b$ is

$$\sigma_{max}\Big]_{\xi = a/b} = -\frac{E\alpha\bar{S}b^2}{4k(1-\mu)}\left[1 - \frac{b^2}{a^2}\,\frac{\int_1^{a/b}\gamma\xi_1^3\,d\xi_1}{\int_1^{a/b}\gamma\xi_1 d\xi_1}\right] \qquad (45)$$

Substituting Equation (44) into (38), the maximum stress at $\xi = 1$ is

$$\sigma_{max}\Big]_{=1} = -\frac{E\alpha\bar{S}b^2}{4k(1-\mu)}\left[\frac{1}{2} - \log a/b + \frac{\int_1^{a/b}\gamma\xi_1\log\xi_1 d\xi_1 - \frac{1}{2}\frac{b^2}{a^2}\int_1^{a/b}\gamma\xi_1^3 d\xi_1}{\int_1^{a/b}\gamma\xi_1 d\xi_1}\right] \qquad (46)$$

The maximum stress in the tube is hence given by Equation (45). Let us evaluate this for the case of the step distribution of power



For this case

$$\int_1^{a/b}\gamma\xi_1 d\xi_1 = \int_{d/b}^{c/b}\xi_1 d\xi_1 = -\frac{d^2 - c^2}{2b^2}$$

and

$$\int_1^{a/b}\gamma\xi_1^3 d\xi_1 = \int_{d/b}^{c/b}\xi_1^3 d\xi_1 = -\frac{d^4 - c^4}{4b^4}$$

From equation (45) the maximum stress is hence

$$\sigma_{max}\Big]_{=a/b} = -\frac{E\alpha\bar{S}b^2}{4k(1-\mu)}\left[1 - \frac{b^2}{a^2}\,\frac{d^4 - c^4}{4b^4}\,\frac{2b^2}{d^2 - c^2}\right]$$

$$= \frac{E\alpha \bar{S}}{4k(1-\mu)} \quad \frac{b^2}{a^2} \quad \frac{(d^2 + c^2 - 2a^2)}{2}$$

The positive sign designates a tensile stress. We can write

$$\frac{d^2 + C^2 - 2a^2}{2} = \frac{(d+c)(d-c)}{2} + (c+a)(c-a)$$

Let $t_1 = d - c$ be the thickness of the fuel bearing layer, and $t_2 = c - a$ be the thickness of the cladding. Let $c_1 = \frac{d+c}{2}$ be the radius to the center of the fuel bearing layer and $c_2 = \frac{c+a}{2}$ be the radius to the center of the cladding layer. Also let $f = \frac{a^2}{b^2}$ be the ratio of free flow area for coolant to tube cross section. Then the maximum stress is

$$\sigma_{max} = \frac{E\alpha}{4k(1-\mu)} \quad \frac{\bar{S}}{f} \quad (t_1 c_1 \quad 2 t_2 c_2)$$

The ratio $\frac{E\alpha}{4k(1-\mu)}$ depends only on the physical properties of the tube material, the factor S is a measure of the power output of the reactor, while the other terms are geometrical factors.

For a low stress with a given power density it is important to have

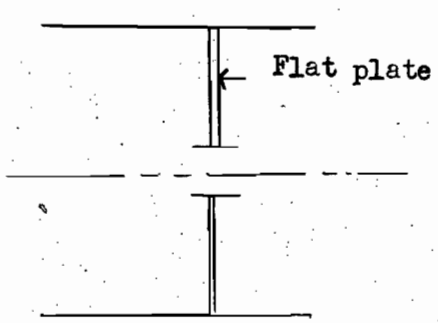
1. Low modulus of elasticity, E
2. Low coefficient of thermal expansion, α
3. Thin tube (b), (a), (f)
4. Thin fuel layer and cladding (t_1 and t_2)
5. High thermal conductivity, k
6. Low value of Poisson's ratio, μ

It is evident that the required strength of the material is somehow dependent on σ_{max}. For a really brittle material σ_{max} could not be allowed to exceed the tensile strength of the material without rupture. For a very ductile material, and only a few cycles of heating and cooling, plastic flow of the material will prevent σ_{max} from exceeding the failure stress,

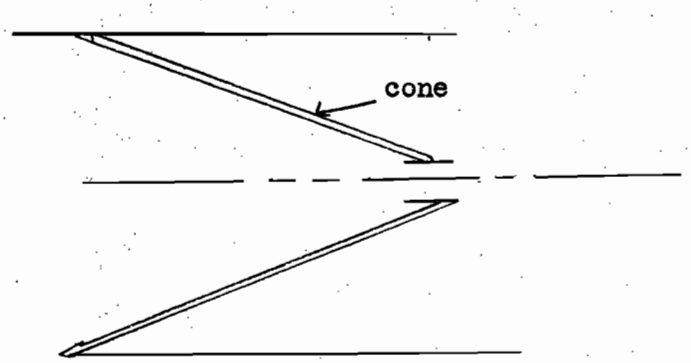
even for very large calculated values of elastic stress. This is explained by saying that thermal stresses are "displacement stresses", and any plastic flow which occurs will "relieve" the stress. For large numbers of repeated cycles over the yield stress, the alternating direction of the plastic flow will eventually result in a "fatigue" failure, very much like the failure of a turbine blade due to mechanical vibrations. The criterion here must be that the alternating heating and cooling will not result in cyclic plastic flow. A good rule which can be shown to hold for this case limits the calculated elastic stress for cyclic operation to less that the sum of the yield strengths in tension and compression. For ductile materials, where the tensile and compressive strength are about equal, this means that the calculated elastic stress must not exceed twice the yield strength.

We have seen here that certain factors which determine thermal stress are to some extent under the control of the designer, both properties of materials and design dimensions. Another feature of importance in limiting thermal stress in certain cases is under the control of the designer. It can be shown (from considering Hooke's law combined with the compatibility condition) that the criterion for all thermal stresses to be zero is that the distribution of temperature in a rectangular coordinate system should be a linear function of the coordinates. Consider the following two methods of mounting a cold bearing in a hot shell:

(1)



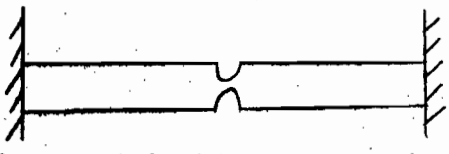
(2)



In method (1) thermal stresses will be determined entirely by the difference in temperature between inside and outside of the flat plate. In method (2) it is possible to design the cone so that the axial distribution of temperatures is linear, so that there are no thermal stresses.

Some mention of stress concentrations should be made. It is obvious that if a bar with a notch is placed between two fixed walls, and heated, the maximum stress will be much

$$\sigma_{max} >>> E\alpha\Delta T$$



greater than that anticipated for a uniform bar. In general, for designs involving the flow of large quantities of heat it does not do to have large changes of cross sections or thickness. Two parts of radically different thickness are not generally suited to be welded together for this reason. Some comparative numbers for properties of materials are given below. It should be remembered that these numbers vary considerably even for materials supposed to be similar, and the ones used here are only indicative.

	Thermal Conductivity	Modulus of Elasticity	Thermal Expansion Coeff
	Btu/hr °F ft	psi	1° F
Steel	15	30×10^6	7×10^{-6}
Aluminum	125	10×10^6	15×10^{-6}
Graphite	100	1×10^6	1×10^{-6}
Alumina	2	50×10^6	5×10^{-6}

Poisson's ratio is taken as 0.3 for all materials. The ratio of $\frac{E\alpha}{4k(1-\mu)}$ is given below for these values of physical properties.

	$\left(\frac{E\alpha}{4k(1-\mu)}\right)$			Representative Tensile Strengths psi
Steel	$\frac{30 \times 7}{4 \times 15 \times .7}$	$\sim$	5.	30,000
Aluminum	$\frac{10 \times 15}{4 \times 125 \times .7}$	$\sim$	0.4	10,000
Graphite	$\frac{1 \times 1}{4 \times 100 \times .7}$	$\sim$	0.004	3,000
Alumina	$\frac{5 \times 50}{4 \times 2 \times .7}$	$\sim$	45	10,000

From the values given above it can be seen that the choice of material has a very important bearing on thermal stress.

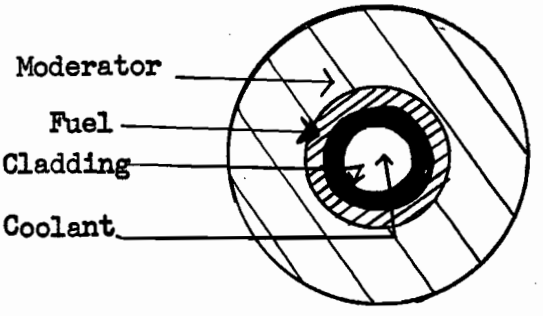
Problem: Stainless Steel Sandwich

$F = 0.5$

$\bar{s} = 1$ kw/cc

$C_1 = C_2 = 0.5$ in.

$t_1 = t_2 = 0.010$ in.



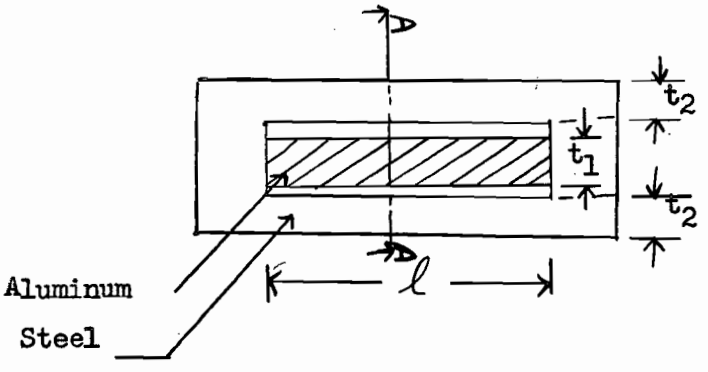
Find maximum stress. IS this allowable if yield point of materials is 50,000 psi?

Problem:

$\ell = 24$

$t_1 = 0.3$

$t_2 = 0.2$



Both aluminum and steel members are heated uniformly to 100° F above initial temperature at which members fit with no stresses. What are the stresses and strains in the aluminum and the steel at Section "A".

4. Convective Heat Transfer

In discussing convective heat transfer we will be interested primarily in two quantities, heat transfer efficiency and power density, and in their dependence on the geometry of apparatus and physical properties of cooling fluids. Equations will be developed for heat transfer into viscous fluids and for the pressure losses which necessarily accompany the heat transfer. The relationship between heat transfer and pressure losses determines to some extent a limitation on the power density available in a reactor, and the efficiency of operation of the system of which the reactor is a part. It will be found that, to some extent, this limitation on power density, like the one due to thermal stress is under the control of the designer.

The problem of heat transfer to viscous fluids will be divided into two main parts, the first involving the pressure losses in the fluid, the second involving the transfer of heat to the flowing fluid.

Pressure Losses in Viscous Flow

Newton is reported to have proposed the first law for resistance to flow of a fluid past a body, based on momentum considerations. The resistance to motion was supposed to depend on the number of particles coming into contact with the body ($\rho A w$) and the change in velocity imparted to the particles (w'). The change in velocity (w') was assumed to depend linearly on the original velocity (w), or $w' = f w$, giving for the resistance to flow

$$f \rho A w^2$$

It has been found since that the resistance to flow depends on the whole shape of the body, not just on the frontal area. Nevertheless the form of Newton's expression is preserved, at the expense of some complication in defining the factors of which it is composed.

We are interested in the flow of viscous fluids through tubes (with some modification through other configurations) and will write the expression for the pressure loss per unit length of tube

$$\frac{dP}{dx} = \frac{4 f \gamma}{D} \frac{w^2}{2 g} \tag{1}$$

where

p = pressure loss in distance, dx

D = diameter of tube

w = average velocity of flow in tube

g = accelearation of gravity

f = friction factor

γ = weight density of fluid

We will be primarily concerned with establishing values for the factor, f, which are adequate for engineering purposes.

For non-circular tubes, D becomes the hydraulic diameter and is defined as four times the ratio of the cross section, A, of the fluid flow passage to the wetted perimeter , S, or

$$D = \frac{4 A}{S} = 4 m \tag{2}$$

where m is the hydraulic radius. It is to be noted that the hydraulic radius, m, for a circular tube is one-half the actual radius. Authors define f sometimes based on diameter, sometimes on radius, and sometimes on hydraulic radius. Definitions of f by different authors hence differ by factors of two or four, and the reader must often be exceedingly wary to infer from the context which definition is being used.

The pressure loss can be alternatively written in terms of the shearing stress, τ_a, at the wall

$$\frac{dP}{dx} = \frac{S \tau_a}{A} = \frac{4 \tau_a}{D} \tag{3}$$

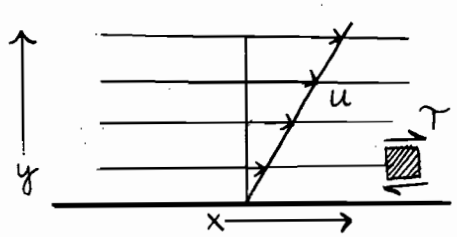
Eliminating dp/dx between Equations (1) and (3)

$$\tau_a = f \gamma \frac{w^2}{2 g}$$

To find the appropriate values for f, it is necessary to find the shear stress, τ_a, which exist at the walls of the heat transfer passage. It will be found that the shear stress depends on the type of flow-laminar or turbulent. In laminar flow the fluid can be considered as moving in layers with relatively little mixing of adjacent layers. Turbulent flow is accompanied by an agitation of the fluid by eddies which thoroughly mix the fluid. For the laminar case an analytical solution for τ_a can be found, while the turbulent case must be handled experimentally, with some help from dimensional analysis.

Laminar Flow of a Viscous Fluid

When a viscous fluid flows past a solid surface, shearing stresses are set up in the fluid between moving layers of fluid and the stationary wall. This situation is shown schematically below



The viscosity of the fluid is defined by the expression

$$\tau = \mu \frac{\partial u}{\partial y} \tag{5}$$

where τ = shearing stress in the fluid

u = velocity of the fluid

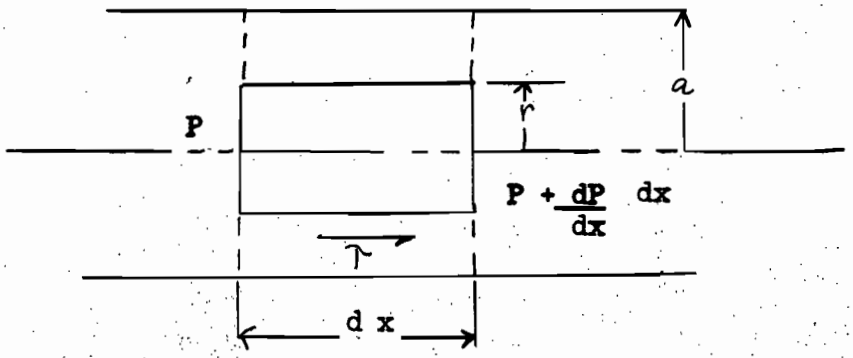
y = distance from wall

μ = viscosity

For dimensional consistency μ must have the dimensions

$$\mu \sim \frac{(\text{force})(\text{time})}{(\text{length})^2}$$

Let us isolate, in a fluid in laminar flow through a tube, a tubular segment of the fluid. The flow is assumed to be incompressible (no inertia forces in the straight tube).



The force on the segment due to the difference in pressure across it is (pressure assumed independent of r)

$$\pi r^2 \frac{dp}{dx} dx \tag{6}$$

The frictional force on the segment is

$$2 \pi r \, dx \, \tau \tag{7}$$

since these forces must be equal

$$\pi r^2 \frac{dp}{dx} dx = 2 \pi r \, dx \, \tau \tag{8}$$

or

$$\frac{r}{2} \frac{dp}{dx} = \tau \tag{9}$$

From equation (5)

$$\tau = \mu \frac{\partial u}{\partial r} \tag{10}$$

Substituting in Equation (5)

$$\frac{r}{2} \frac{dp}{dx} = \mu \frac{\partial u}{\partial r} \tag{11}$$

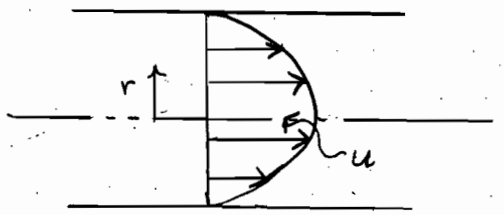
Equation (11) is to be integrated subject to the condition that the velocity at the wall is zero, or

$$u \Big]_{r = a} = 0 \tag{12}$$

Performing this integration

$$u = \frac{1}{4\mu} \frac{dp}{dx} (a^2 - r^2) \tag{13}$$

Equation (13) states that the velocity profile across a laminar stream in a tube is a parabola.



The maximum velocity in the stream is

$$u_{max} = \frac{1}{r\,\mu} \quad \frac{d\,p}{d\,x} \quad a^2 \tag{14}$$

The average velocity in the stream is

$$w = \frac{2\,\pi \int_0^a u\,r\,dr}{\pi\,a^2} = \frac{1}{8\,\mu} \quad \frac{d\,p}{d\,x} \quad a^2 \tag{15}$$

From equation (9) the shear stress at the wall is

$$\tau_a = \frac{a}{2} \quad \frac{d\,p}{d\,x} \tag{16}$$

Substituting this expression into (15) to eliminate $\frac{dp}{dx}$

$$\tau_a = 4\,\mu\,\frac{w}{a} \tag{17}$$

Equation (17) gives the shear stress at a tube wall due to the flow of a viscous fluid through the tube with an average velocity, w. From equation (4) then the friction factor for a laminar flow is

$$f = 4\,\mu\,\frac{w}{a} \Big/ \gamma\left(\frac{w^2}{2g}\right) \tag{18}$$

Let a = D/2, and $\rho = \gamma/g$. Then equation (18) becomes

$$f = 16\left(\frac{\mu}{w\,D\,\rho}\right) \tag{19}$$

In equation (19) the dimensionless ratio

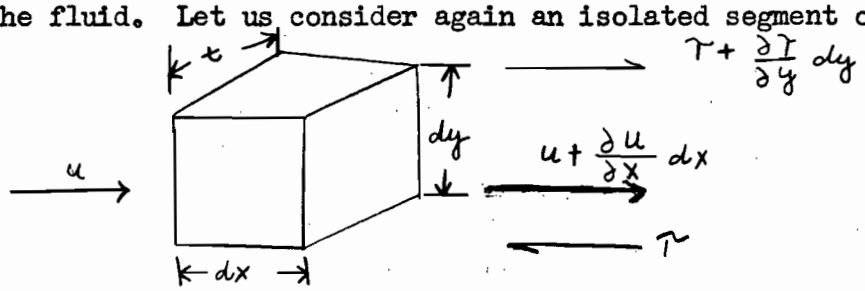
$$\frac{w\,D\,\rho}{\mu} = (Re) \tag{20}$$

is called the Reynolds number. This number is found to be an important criterion

for the characteristics of the flow of viscous fluids. It will appear many times in our subsequent studies. With this definition Equation (14) becomes

$$f = \frac{16}{(Re)} \tag{21}$$

The Reynolds number arises as a criterion in connection with the flow of fluids where the viscous forces become of significance compared to the inertia forces in the fluid. For many cases of fluid flow, the ratio of inertia forces to viscous forces in the fluid provides a criterion for the behavior of the fluid. Let us consider again an isolated segment of the fluid.



The inertia force in the fluid is given by the mass of the fluid times its acceleration as it traverses the element ds.

$$F_1 = \rho\, t\, d\, x\, d\, y\;\; u\;\; \frac{\partial u}{\partial x} \tag{22}$$

where ρ is the mass density of the fluid. The friction (viscous) force on the element is given by the difference between the shear forces on the upper and lower surfaces of the element, or

$$F_2 = t\, d\, x\, d\, y\;\; \frac{\partial \tau}{\partial y} \tag{23}$$

From Equation (5) we get for Equation (23)

$$F_2 = t\, d\, x\, d\, y\, \mu\;\; \frac{\partial^2 u}{\partial y^2} \tag{24}$$

The ratio of the inertia to the friction force is hence

$$\frac{F_1}{F^2} = \frac{\rho u \frac{\partial u}{\partial x}}{\mu \frac{\partial^2 u}{\partial y^2}} \tag{25}$$

Define dimensionless quantities

$$u = V\,\theta$$
$$x = L\,\xi \qquad (26)$$
$$y = L\,\eta$$

where V is a characteristic velocity and L a characteristic linear dimension of the flow passage. Then Equation (25) can be written

$$\frac{F_1}{F_2} = \alpha \frac{\rho V L}{\mu} = \alpha\,(Re) \qquad (27)$$

where

$$\alpha = \frac{\theta \dfrac{\partial \theta}{\partial \xi}}{\dfrac{\partial^2 \theta}{\partial \eta^2}} \qquad (28)$$

In Equation (27), aside from the dimensionless Reynolds number, there is only the dimensionless geometry factor, α. It is necessary to exercise a considerable degree of caution in the choice of units to make Reynolds number dimensionless, using data from experimental compilations. In Equation (5), to fit the definition of viscosity, μ was assigned the units.

$$\frac{(\text{force})(\text{time})}{(\text{length})^2}$$

The most commonly quoted unit for viscosity is the poise, which has the dimension

$$\frac{\text{gm}}{\text{cm sec}} = \frac{(\text{Mass})}{(\text{time})(\text{length})}$$

or the centipoise which is one-one hundredth of a poise. This would require using the weight density, γ instead of the mass density, ρ. It should be remembered that the "kinematic viscosity", μ/ρ must have the dimensions of diffusivity or

$$\frac{\mu}{\rho} \sim \frac{(\text{length})^2}{(\text{time})}$$

The dimensions used should be consisted with this requirement.

Turbulent Flow of a Viscous Fluid

It was found experimentally by Hagen that, if the velocity of a fluid stream is increased indefinitely a point is reached above which the flow is no longer laminar, but breaks up into vortices which cause violent agitation of the stream. A criterion for the onset of this turbulent motion was found by Reynolds to be the value of the parameter which has since been named for him

$$(Re) = \frac{V\,D\,\rho}{\mu}$$

When the Reynolds number is greater than some limiting value about 1000, turbulent flow may exist depending on initial disturbances in the stream. Under most engineering conditions turbulent flow will exist if

$$(Re) > 2000$$

In the turbulent region the pressure drop due to friction is expressed by the same equation (Equation (1)) as for the laminar case

$$\frac{d\,\rho}{d\,x} = \frac{4\,f\,\gamma}{D}\;\frac{w^2}{2\,g} \qquad (29)$$

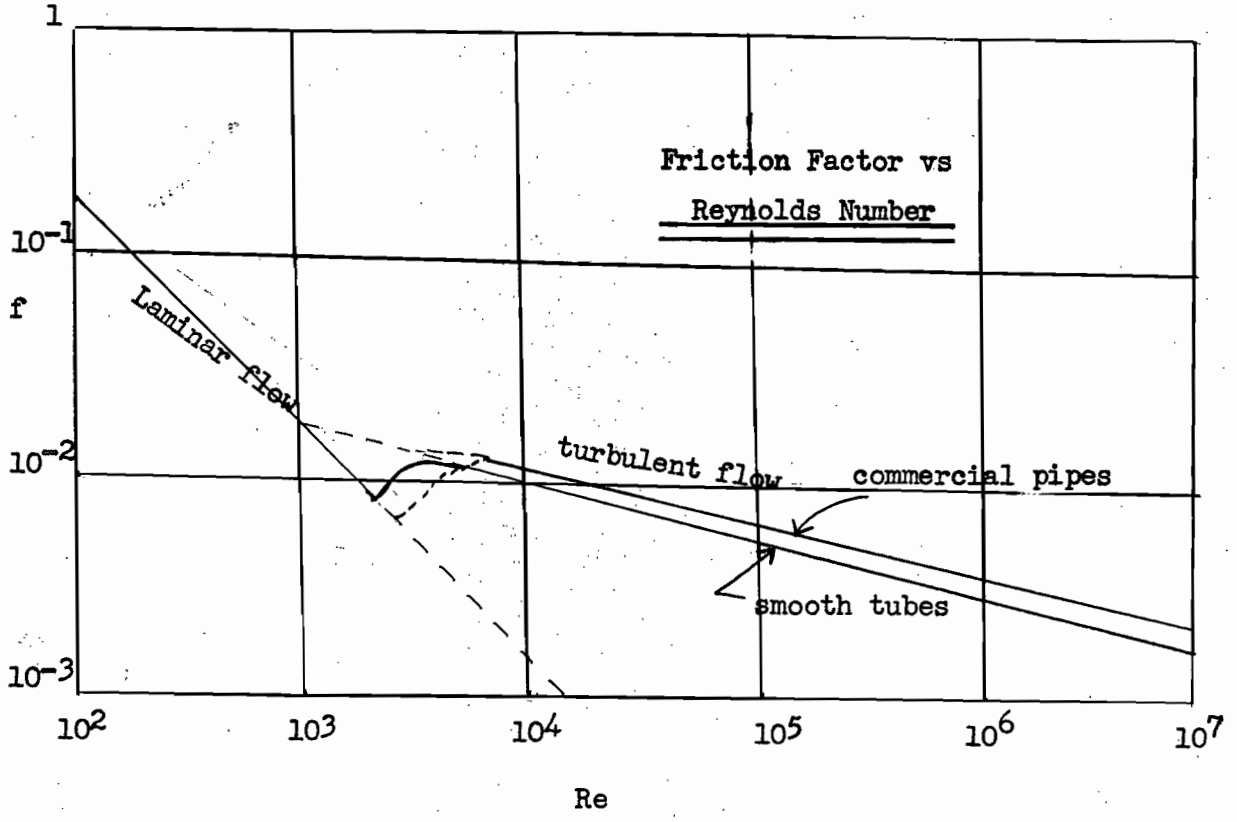
Now however, it is found that f is inversely proportional no longer to the first power of the Reynolds number, but to the one-fifth power approximately

$$f = \frac{0.046}{(Re)^{0.2}}\;^{*} \qquad (5000 < Re < 200{,}000) \qquad (30)$$

Roughness of surfaces acts to increase the value of f. The curve for f (from McAdams) is shown below as a function of Reynolds number.

* Blasius formula is

$$f = \frac{0.067}{(Re)^{0.25}}$$



The Navier-Stokes equations express the velocity and presure relationship in a viscous fluid. They will not be used here however since no solution for them has been found for the case of turbulent flow. By experimental methods it has been found that the turbulent flow of a fluid can be approximated by considering it to be composed of a boundary layer in which viscious and inertia effects are important and a central portion in which the inertia effects predominate. By a dimensional analysis of the Navier-Stokes equations for two dimensional flow Prandtl has shown that the boundary layer must have a thickness of the order of

$$\delta \sim \frac{L}{(Re)^{1/2}} \tag{31}$$

When it is remembered that for turbulent flow the Reynolds number is generally greater than 2000 it can be seen that the boundary layer is generally small

compared to the characteristic dimension, L.

The boundary layer is treated as consisting of two parts, a laminar sub-layer and a buffer layer. It has been found that the velocity profile, though it cannot be found analytically, can be correlated experimentally for turbulent flow by plotting dimensionless fluid velocity against dimensionless distance from the wall. The dimensionless velocity is defined

$$u^+ = u/\sqrt{\tau_a/\rho}$$

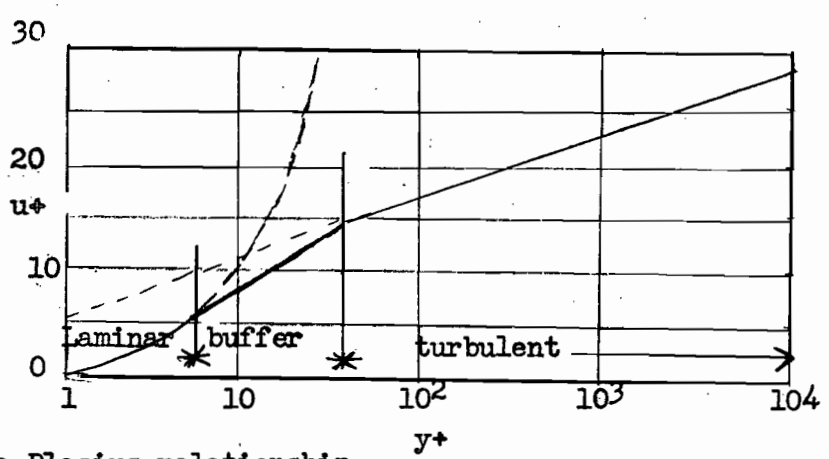
where u is the local fluid velocity, τ_a is the shear stress at the wall, and ρ is the mass density of the fluid. The dimensionless distance from the wall is

$$y^+ = \frac{y\,\rho}{\mu}\sqrt{\tau_a/\rho}$$

where y is the distance from the wall and μ is the fluid viscosity. From Equation (4) it is seen that

$$\sqrt{\tau_a/\rho} = w\sqrt{f/2}$$

where w is the average stream velocity and f is the friction factor. Such a plot, (here taken from McAdams), is shown below.



Using the Blasius relationship

$$f = \frac{0.067}{(Re)^{0.25}}$$

Prandtl has shown that the experimental distribution of velocity in the turbulent core is nearly satisfied by the relationship

$$u = u_{max} \left(1 - \frac{r}{a}\right)^{1/7}$$

where u_{max} is the velocity at the center of the tube, a is the radius of the tube and r is the distance from the center of the tube for Reynolds numbers from 5,000 to 50,000. For higher Reynolds numbers the 1/7 power must be changed, for instance at (Re) = 200,000 to a 1/8 power.

From Equation (1) the losses due to friction can be expressed

$$\frac{dp}{dx} = \frac{4 f \gamma}{D} \left(\frac{w^2}{2g}\right)$$

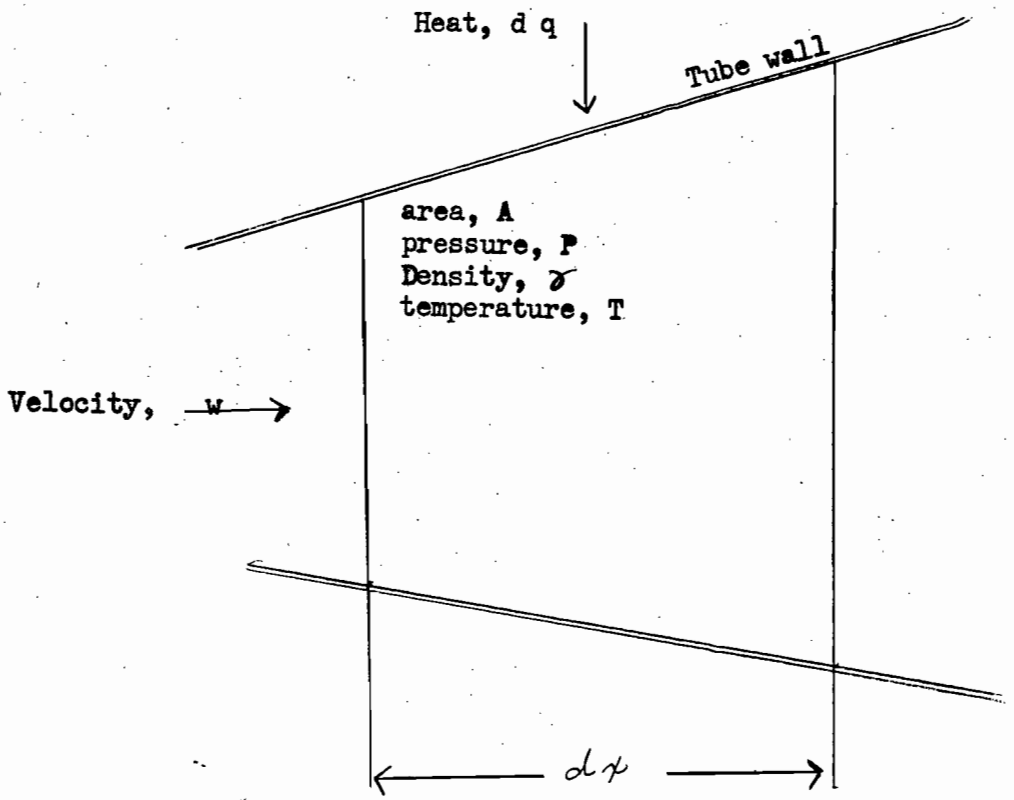
where we have found that f is one function of Reynolds number for laminar flow, another function of Reynolds number for turbulent flow. There is another source of pressure loss besides friction, which is due to entrance and exit losses from tubes, or to sudden contractions or expansions in the heat transfer passage. These losses also are expressed in terms of the velocity head

$$\Delta_p = \alpha \left(w^2/2g\right)$$

where α depends on the area ratio of the expansion or contraction. All pressure losses due to these sources are additive to the frictional losses due to viscosity.

Momentum Equation for Viscous Fluids

The equation for the conservation of momentum requires that the summation of forces over a fluid element br zero. These forces are due to (1) acceleration of the element, (2) pressure differences across the element, and (3) viscous friction forces along the wall of the element. The fluid element is shown below



The momentum equation for the tubular element is

$$A\,dx\,\frac{\gamma}{g}\,\frac{dw}{dt} + \left(p + \frac{\partial p}{\partial x}\,dx\right)A - pA + \frac{\partial F}{\partial x}\,dx = 0 \qquad (31)$$

where dF is the force due to friction on the walls. If the fluid is accelerating then

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\,\frac{dx}{dt} \qquad (32)$$

Or if coordinate distances are expressed in terms of distance along the stream tube, $\frac{dx}{dt} = w$, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + g \frac{\partial}{\partial x} \left(\frac{w^2}{2g} \right) \qquad (33)$$

Substituting in Equation (31)

$$\frac{1}{g} \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} \left(\frac{w^2}{2g} \right) + \frac{1}{\gamma} \frac{\partial p}{\partial x} + \frac{1}{A\gamma} \frac{\partial F}{\partial x} = 0 \qquad (34)$$

For steady flow, $\frac{\partial w}{\partial t} = 0$, and

$$\frac{d}{dx} \left(\frac{w^2}{2g} \right) + \frac{1}{\gamma} \frac{dp}{dx} + \frac{1}{A\gamma} \frac{dF}{dx} = 0 \qquad (35)$$

Equation (35) is the expression for the conservation of momentum. Using the expression for pressure loss due to friction from Equation (1), Equation (35) becomes

$$\frac{d}{dx} \left(\frac{w^2}{2g} \right) + \frac{1}{\gamma} \frac{dp}{dx} + \frac{4f}{D} \left(\frac{w^2}{2g} \right) = 0 \qquad (36)$$

This is the momentum equation in the form in which we will find it useful. Equation (36) is essentially Bernonilli equation. The factor, f has been discussed previously.

Conservation of Mass

The equation of continuity of mass states that fluid is neither created nor destroyed inside the confines of the element. The flow per unit time through the tube is

$$\gamma w A$$

Its conservation requires that

$$\frac{\partial}{\partial x} (\gamma w A) \, dx = A \frac{\partial \gamma}{\partial t} \, dx \qquad (37)$$

Or for steady state conditions

$$\gamma w A = \text{constant} \qquad (38)$$

For a constant tube area

$$\gamma w = \text{constant} = G g \tag{39}$$

Equation (39) states that, for a constant area tube, the mass flow per unit cross section of tube per unit time, G, is constant. Rewriting Equation (36)

$$dp + \frac{\gamma w}{g} \left[d w + \frac{2 f}{D} w \, dx \right] = 0 \tag{40}$$

From equation (39)

$$dp + G \left[d w + \frac{2 f}{D} w \, d x \right] = 0 \tag{41}$$

Integrating Equation (41) to a point, x, of the heat transfer passage

$$P_1 - P = G \left[(w - w_1) + \int_0^x \frac{2 f}{D} w \, d x \right] \tag{42}$$

where the subscript (1) designates the beginning of the tube. If the subscript (2) designates the end of the tube, and ℓ the length of the tube,

$$P_1 - P_2 = G \left[(w_2 - w_1) + \int_0^{\ell} \frac{2 f}{D} w \, d x \right]$$

Since both f and w are functions of the temperature along the heat transfer passage, which we have not yet brought into our equations, this equation for pressure loss will be left at this point for now.

Heat Transfer to Flowing Fluids

The quantity of heat transferred per unit time across an interface between a solid and a flowing fluid is expressed by Newton's law of cooling. This equation is written

$$Q = h A' (Tw - Tt) \tag{43}$$

Where Q = heat flowing per unit time

A' = heat transfer area

Tw = temperature of wall at interface

Tt = average total temperature of fluid

h = heat transfer coefficient

This equation looks like the Fourier conduction equation, except that due to lack of detailed information concerning gradients in the fluid boundary, the coefficient, h, has replaced $k/\Delta x$ and $(Tw - Tt)$ has replaced ΔT. The linear form of the equation has been saved at the expense of inventing h, which in general is far from constant.

For dimensional consistency, h must have the dimension

$$h \sim \frac{(\text{energy})}{(\text{length})^2\ (\text{time})(\text{temperature})}$$

Newton apparently anticipated that the heat transfer coefficient would depend only on physical properties of the cooling fluid. Actually h has been found to have a very complicated dependence on a combination of geometrical and physical parameters.

The quantity, T_t, was defined as the total temperature of the fluid. This is a fictitious temperature which would exist in a fluid if the kinetic energy were changed adiabatically to internal energy (if the fluid were completely stopped adiabatically). Analytically this is expressed

$$T_t = T + \frac{1}{C_p}\left(\frac{w^2}{2\,g}\right) \qquad (44)$$

where T is the temperature in the moving stream which would be sensed by a thermometer moving with the stream, C_p is the specific heat at constant pressure, w is the velocity of the stream, and g is the acceleration of gravity. For large velocities in the fluid (close to the velocity of sound), the second term in Equation (2) becomes of importance. T_t and T can be used interchangeably except for high velocity gas streams.

A considerable discussion* can be found in heat transfer publications concerning what temperature should be taken as representative of the heat tranfer fluid. It is found that, due to conductivity of the fluid, the temperature which actually exists in a gas stream which has been stopped

adiabatically at a wall is not T_t, that it rises generally only some 85 percent of the anticipated amount. Considering the state of knowledge of heat transfer, this difference is, for most purposes, insignificant.

Many empirical and semi-empirical formulae can be found for the heat transfer apparatus, the physical characteristics of the fluid, and the velocity of flow. For h as for f it is necessary first to prescribe whether the fluid flow is laminar or turbulent. The criterion for laminar or turbulent flow is the Reynold's number

$$Re = \left(\frac{w\ D\ \gamma}{\mu} \right)$$

No attempt will be made here to provide an exhaustive list of heat transfer coefficients pertinent to specific heat transfer and coolants. A designer must use considerable caution to make sure that the experimental data which he uses is pertinent to his needs. However certain fairly general statements can be made about heat transfer coefficients. The most general of these statements is that the heat transfer is very sensitive to the Prandtl number

$$(P_r) = \frac{C_p \mu g}{k}$$

where C_p = specific heat at constant pressure of coolant

μ = viscosity of coolant

k = conductivity of coolant

The Prandtl number is a measure of the ratio of the eddy diffusivity μ/ρ to the thermal diffusivity $k/C_p \rho g$. It is a measure of the ratio of the heat carried across the boundary layer by eddy convection to that carried by thermal conduction. For liquid metals, then, which have a very high thermal conductivity, Prandtl number is low, (of order 1/100) while oils

which have a low thermal conductivity, Prandtl number is high. The Prandtl number for gases is about unity, while water is somewhat above this, depending on the temperature. For oils the Prandtl number is of the order of 100. For materials having a Pradntl number corresponding to gases or higher the Colburn equation has been found experimentally to give a good fit for turbulent flow. This is expressed dimensionlessly by

$$(N_u) = \frac{h\,D}{k} = 0.023\ (R_e)^{0.8}\ (P_r)^{0.4} \tag{45}$$

where the Nusselt number is

$$(N_u) = \frac{h\,D}{k} \tag{46}$$

It is often advantageous to divide both sides of this equation by the Peclet number

$$(P_e) = (R_e)\ (P_r) \tag{47}$$

giving

$$(St) = \frac{h}{\rho\, w\, C_p\, g} = \frac{0.023}{(R_e)^{0.2}\ (P_r)^{0.6}} \tag{48}$$

where the Stanton number is

$$(St) = \frac{h}{\rho\, w\, C_p\, g} = \frac{h}{G\, C_p\, g} \tag{49}$$

The reasons for using Equation (48) instead of (46) for some cases are: (1) Since the Reynolds number occurs only to the one-fifth power, the results in this form are relatively independent of the Reynolds number; (2) For a straight tube G is a constant (mass flow per unit area) and is easy to measure; and (3) the conductivity which is hard to measure for many materials, or unknown, appears only in the Prandtl number for which fairly good values are known for many materials.

For liquid metals, Lyon gives the expression for tubes

$$(N_u) = 7 + 0.025\ (R_e)^{0.8}\ (P_r)^{0.8}$$

$$= 7 + 0.025\ (P_e)^{0.8} \qquad (50)$$

The constant factor, 7, accounts for effect of the high conductivity on the heat transfer coefficient at low Reynolds numbers, and Prandtl numbers.

For laminar flow in tubes McAdams recommends the equation

$$(N_u) = 1.62\ \left(P_e\ \frac{D}{L}\right)^{1/3} \left(\frac{\mu}{\mu_s}\right)^{0.14} \qquad (51)$$

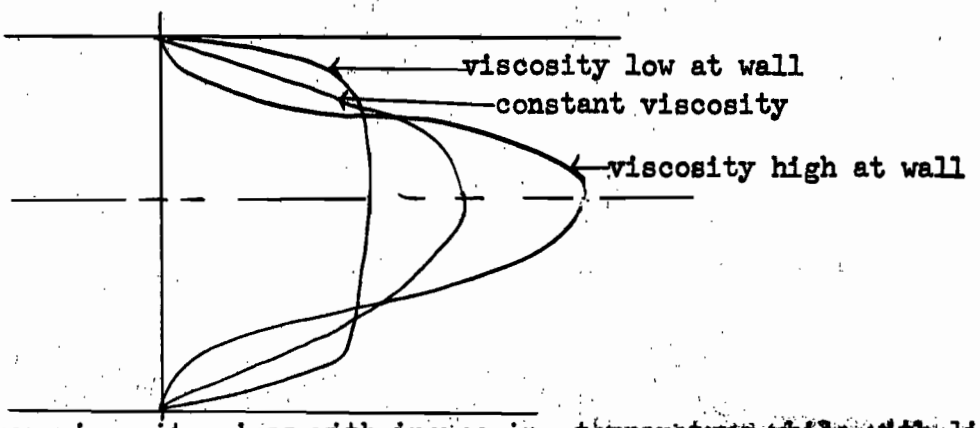
where D = tube diameter

L = tube length (heated portion)

μ = viscosity of fluid in core

μ_s = viscosity of fluid at surface of tube

The viscosity correction is due to the warpage of the profile from parabolic due to the variable viscosity with temperature



with gases viscosity rises with increasing temperature while with liquids it decreases with rising temperature.

Correlations have been established between heat transfer and friction coefficients by means of "heat-transfer analogies". As a demonstration of this for one case, let us look at Equation (30) and (48). Taking the ratio of these two equations gives

$$S_t \Big/ \frac{f}{2} = \frac{1}{(P_r)^{0.6}} \qquad (52)$$

For cases to which these equations are appropriate, Equation (52) establishes a very convenient analogy between h and f, which is independent of the Reynolds number. For gases and low-conductivity liquids Equation (52) represents experimental data quite well. For a Prandtl number close to unity

$$(S_t) \Big/ \frac{f}{2} = 1 \tag{53}$$

This is the Reynolds analogy which is good for gases. The worst departures from Equation (52) occur for heat transfer to liquid metals at low Reynolds numbers, and for heat transfer in laminar flow (see Equation (50) and (51)).

Several investigators have carried the heat transfer analogy further based on the approximate differential equations for the conservation of heat and momentum for a two-dimensional fluid stream*. It is found that the differential equations for velocity and temperature distribution across the fluid stream can be solved to give a relationship for the Nusselt or Stanton numbers in terms of the Prandtl and Reynolds numbers. Martinelli has given one such relationship. In the accompanying figure, from Martinelli, is shown a plot of the ratio of Stanton number to one half the friction factor for a Reynolds number of 50,000 according to Martinelli's formula, and for comparison a plot of Equation (52). (Note that Martinelli uses f/8 instead of our f/2 because his definition of f differs from ours by a factor of four). It is seen that the two curves coincide for Prandtl number of unity. Also for comparison two points are shown for other Reynolds numbers and a Prandtl number of 0.01. Martinelli's equation is

$$(S_t) = \frac{a\sqrt{f/2}\left(\frac{tw - tc}{tw - tm}\right)}{5\left[a\,(P_r) + \ln(1 + 5a(P_r)) + 0.5\,F\,\ln\left(\frac{(Re)}{60}\sqrt{f/2}\right)\right]} \tag{54}$$

* Heat Transfer to Molten Metals, by R. C. Martinelli, Transactions of the American Society of Mechanical Engineers, November 1947.

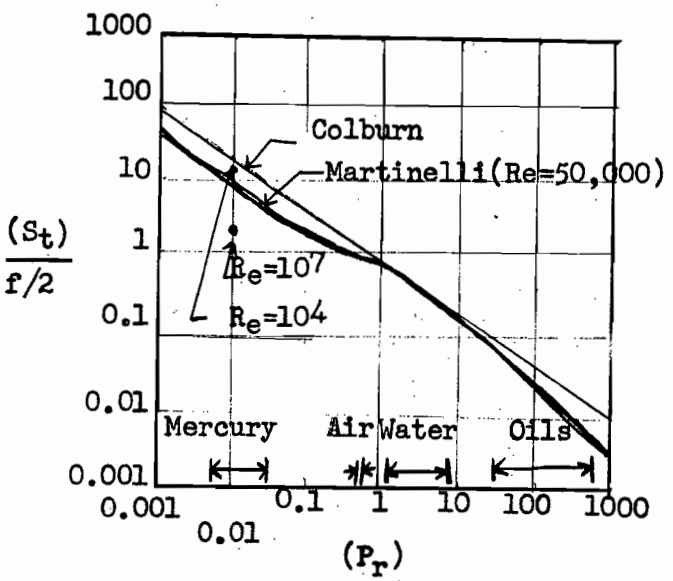
where α = ratio of eddy diffusivities for heat and momentum (taken equal to unity in his plot)

t_w = temperature of tube wall

t_c = temperature of fluid at center of tube

t_m = mixed mean temperature of fluid

F = numerical parameter defined by Martinelli ($0 < F < 1$)



It can be seen that for liquid metals the ratio of (St) to f/2 is quite dependent on (R_e) as well as (P_r). Equation (52) is pertinent primarily to gases and to water. For a more general representation we must write

$$(S_t)\Big/\frac{f}{2} = \lambda \qquad (55)$$

where λ is a function of Reynolds number and Prandtl number, to be determined experimentally. Equation (55) will be used later in our calculations of convective heat transfer efficiency.

Equation (43) gave an expression for the heat transferred per unit time across an area A'. This area is given by the product of the wetted perimeter of the heat transfer passage, S, and the length of element, dx. Since we have previously defined a hydraulic diameter, D, as four times the ratio

of the cross section, A, of the fluid flow passage to the wetted perimeter, Equation (43) can be written

$$Q = \frac{4\,h\,A}{D} \quad (T_w - T_t)\,dx \tag{56}$$

The heat absorbed per unit weight of cooling fluid is

$$dq = \frac{Q}{\gamma\,w\,A} = \frac{4\,h}{\gamma w\,D} \quad (T_w - T_t)\,dx \tag{57}$$

Equation (57) expresses the amount of heat flowing per unit weight of cooling fluid if the wall temperature is T_w and the fluid temperature is T_t.

Conservation of Energy in Heated Fluid

The thermodynamic equation for the conservation of energy for the moving fluid to which external heat is added and from which external work is extracted is given by

$$dq - dw = dh + dk + de \tag{58}$$

where dq = external heat added per unit weight of fluid

dw = external work done by fluid

dh = change in enthalpy per unit weight of fluid

dk = change in kinetic energy

de = change in potential energy

If for our tubular system no external work is done by the fluid, and changes in potential energy are negligible, then Equation (58) becomes

$$dq = dh + dk \tag{59}$$

Enthalpy is, by definition

$$h = u + p/\gamma \tag{60}$$

where u = internal energy per unit weight of fluid

p = pressure in fluid

γ = weight density of fluid

The kinetic energy is

$$k = w^2/2g \tag{61}$$

where w = velocity of fluid

g = acceleration of gravity

Equation (59) is hence

$$dq = du + d\,(p/\gamma) + d\,(w^2/2g) \tag{62}$$

The term $d\,(p/\gamma)$ is often called by engineers "flow work".

<u>Note</u>: It is interesting to note in passing that combining Equation (36) and (62) to eliminate $d(w^2/2g)$ gives the expression

$$dq = du + p\,d\,(1/\gamma) - \frac{4f}{D}\,(w^2/2g)\,dx \tag{63}$$

which is the usual statement of the first law of thermodynamics except for the last term. This last term is made necessary by the irreversibility involved in the viscous friction. Since entropy is defined

$$Tds = du + p\,d\,(1/\gamma) \tag{64}$$

Equation (63) can be written

$$Tds = dq + \frac{4f}{D}\,(w^2/2g)\,dx \tag{65}$$

where again the last term accounts for irreversibilities. The usual statement of the first law applies only to reversible processes and hence is not applicable to a great many problems in engineering such as this one. For our problem Equation (65) replaces the usual statement

$$Tds \geqq dq \tag{66}$$

For perfect gases the enthalpy is a function only of temperature, being independent of pressure. For ideal incompressible liquids, also, since pressure obviously can have no effect, enthalpy is also a function only of temperature. Actually many engineering fluids can be treated approximately on the basis that enthalpy is a function only of temperature, (within certain

pressure ranges). This can be expressed

$$dh = \left.\frac{\partial h}{\partial T}\right)_p dT + \left.\frac{\partial h}{\partial P}\right)_T dp$$

if $\left.\frac{\partial h}{\partial p}\right)_T = 0$, then

$$dh = C_p dT \tag{67}$$

where $C_p = \left.\frac{\partial h}{\partial T}\right)_p$

The obvious engineering case which cannot easily be handled by Equation (67) is that of evaporation in which, although the temperature remains constant there is a large change in the enthalpy (called "latent heat"). Here the specific heat in Equation (67) would be required to be infinite to give a finite change in enthalpy, since dT is zero. The boiling phenomenon involves a change of state and requires a special treatment. Let us consider here only cases for which Equation (67) is applicable. Then equation (62) becomes

$$dq = C_p\, dT + d\,(w^2/2g) \tag{68}$$

From the definition of total temperature in Equation (44) we see that Equation (68) is just

$$dq = C_p\, dT_t \tag{69}$$

Equation (69) is a much simplified form of the expression for the conservation of energy, applicable only to fluids for which the enthalpy is essentially independent of pressure (not necessarily to constant pressure processes - like boiling).

We now have two equations for dq: Equations (57) and (69). Eliminating dq between them gives

$$\frac{d\,T_t}{Tw - T_t} = \frac{4\,h}{\gamma\, w\, D\, C_p}\, dx \tag{70}$$

From Equation (70) we have an expression for the temperature distribution in a heated passage. It can be rearranged by use of Equation (55) for the ratio of $(ST)/f/2 = \lambda$, giving

$$\frac{d\,T_t}{T_w - T_t} = 2\,\lambda\,f\,\frac{dx}{D} \tag{71}$$

Since λ and f can be assigned values for given conditions, Equation (71) can be integrated to give the temperature distribution in the fluid, assuming the wall temperature is given (or the heat added as a function of length along the tube). Several cases of temperature distribution have been considered in connection with reactor work, among them (1) constant power output per unit length of passage, (2) constant wall temperature, and (3) a sine distribution of power along the heat transfer passage. The first of these, constant power distribution, requires (assuming h constant in Equation (57))

$$T_w - T_t = \Delta T = \text{constant} \tag{72}$$

This represents a desirable situation from the point of view of heat transfer and is approximately achieved in certain reflected reactors where the fuel density and thermal flux are nearly constant over the core. For this case Equation (71) can be integrated to give (assuming λ and f constant)

$$\frac{T_t - T_{to}}{T_w - T_t} = 2\,\lambda\,f\,\frac{x}{D} \tag{73}$$

where X = distance along heated passage

T_{to} = total temperature at inlet to passage

For the total length of passage

$$\frac{T_{tl} - T_{to}}{T_{wl} - T_{tl}} = 2\,\lambda\,f\,\frac{\ell}{D} \tag{74}$$

where T_{tl} = total temperature at outlet from passage

T_{wl} = wall temperature at outlet

Equation (74) can be rearranged to give

$$\frac{T_{t1}}{T_{to}} - 1 = \left(\frac{T_{w1}}{T_{to}} - 1\right) \frac{2\lambda f \ \ell/D}{1 + 2\lambda f \ \ell/D} \tag{75}$$

The second condition, constant wall temperature, requires that $T_w = T_{w1}$ = constant. This represents a condition which would be advantageous, if attainable, in that every part of the reactor would be operating at the maximum temperature set by materials considerations. It is unattainable in practice and is of interest only in representing a limiting situation. For this case Equation (71) integrates to

$$\log \left(\frac{T_{w1} - T_t}{T_{w1} - T_{to}}\right) = 2\lambda f \ \frac{x}{D} \tag{76}$$

Or, with some rearranging, for the whole length of passage

$$\frac{T_{t1}}{T_{to}} - 1 = \left(\frac{T_{w1}}{T_{to}} - 1\right)\left(1 - e^{-2\lambda f \frac{\ell}{D}}\right) \tag{77}$$

The third, or sine distribution, condition requires

$$T_w - T_t = A \sin \ \pi x/\ell \tag{78}$$

where A is a constant depending on the maximum power density in the reactor. This does not represent a favorable situation from the point of view of heat transfer or nuclear considerations since the ends of the reactor passage do not contribute any power. However this situation does exist in bare reactors if nothing is done to alleviate it. For this case Equation (71) becomes

$$T_t - T_{to} = \frac{2\lambda f A}{\pi} \ \frac{\ell}{D} \ (1 - \cos \frac{\pi x}{\ell}) \tag{79}$$

The temperature difference over the tube is

$$\frac{T_{t1}}{T_{to}} - 1 = \frac{4\lambda f A}{\pi T_{to}} \ \frac{\ell}{D} \tag{80}$$

Substituting Equation (79) into (78)

$$T_w - T_{to} = A \left[\sin \frac{\pi x}{\ell} + \frac{2 \lambda f}{\pi} \frac{\ell}{D} \left(1 - \cos \frac{\pi x}{\ell} \right) \right] \quad (81)$$

The maximum wall temperature occurs when $\frac{dTw}{dx} = 0$, or when

$$\cos \frac{\pi x}{\ell} + \frac{2 \lambda f}{\pi} \frac{}{D} \sin \frac{\pi x}{\ell} = 0 \quad (82)$$

simplifying

$$\cot \frac{\pi x}{\ell} = - \frac{2 \lambda f}{\pi} \frac{\ell}{D} \quad (83)$$

Apparently the highest wall temperature occurs not at the end of the passage length, but somewhere in the last 1/2 of passage length. From Equation (81)

$$A = (T_w \max - T_{to}) K \quad (84)$$

Where k is to be determined from Equation (83). Substituting this into Equation (80)

$$\frac{T_{tl}}{T_{to}} - 1 = \left(\frac{T_{w\,max}}{T_{to}} - 1 \right) \frac{4 \lambda f K}{\pi} \frac{\ell}{D} \quad (85)$$

Equations (75), (77), and (85) give the relationship among reactor inlet temperature, T_{to}, outlet temperature, T_{tl}, and maximum reactor wall temperature, T_{wl} or $T_{w\,max}$, in terms of λ, f, and ℓ/D for the three cases of temperature distribution considered here. Cases where λ and f could not be considered constant would require a more detailed treatment. Let us generalize these equations and write

$$\frac{T_{tl}}{T_{to}} - 1 = 2 \left(\frac{T_{w\,max}}{T_{to}} - 1 \right) \alpha \lambda f \; \ell/D \quad (86)$$

where α is a parameter depending on the details of temperature distribution.

Power Density in Reactors

The average power density (power per unit volume) generated in a reactor can be written

$$\delta = \frac{F\ G\ g\ C_p\ T_{to}}{\ell} \left(\frac{T_{tl}}{T_{to}} - 1 \right) \tag{87}$$

where F is the free flow ratio (ratio of flow area for coolant through reactor to cross section of reactor normal to direction of flow). Substituting the value for the parenthesis from Equation (86)

$$\delta = \frac{2\ F\ G\ g\ C_p\ T_{to}\ \alpha\ \lambda\ f}{D} \left(\frac{T_{w\ max}}{T_{to}} - 1 \right) \tag{88}$$

Equation (88) gives the average power density attainable in a reactor in terms of the maximum wall temperature allowable and various other parameters. In the section which follows some of these other parameters will be interpreted in terms of pressure losses (heat tranfer efficiency) using Equation (42). It can be seen from Equation (88) that large power density requires large mass flow, G, (incompatible with high efficiency, or low pressure loss), large specific heat, C_p, large value of λ (or small Prandtl number, as with liquid metals), and large temperature difference, $(T_{w\ max} - T_{to})$, and small hydraulic diameter.

Heat Transfer Efficiency

The power used in pumping coolant through a reactor can be found from Equation (41). This power is used partly in over coming viscous friction along passage walls (the second term on the right) and partly in end losses which can be expressed in terms of velocity head. The first term on the

right of Equation (41) represents a change in kinetic energy and is not changeable to pumping power. Let us write for the part of Equation (41) which is chargeable to pumping power.

$$dp = -(1+\beta)\ \frac{2 f \ell}{D}\ G w\, d\xi \tag{89}$$

where the factor β represents the increase in losses due to end effects, and where $\xi = x/\ell$. The pumping power required per unit volume of reactor is

$$d\eta = -\frac{dp\, F w}{\ell} \tag{90}$$

where F is again the free flow ratio. Combining (89) and (90)

$$d\eta = +2(1+\beta)\ \frac{F f G}{D}\ w^2\, d\xi \tag{91}$$

Integrating Equation (91), again assuming f constant

$$\eta = 2(1+\beta)\ \frac{F f G}{D} \int_0^1 w^2\, d\xi \tag{92}$$

The efficiency of pumping can be defined

$$\epsilon = 1 - \eta/\delta \tag{93}$$

Substituting Equation (88) and (92)

$$\epsilon = 1 - \frac{(1+\beta) \int_0^1 w^2\, d\xi}{g\, C_p\, T_{to}\, a\, \lambda \left(\frac{T_{w\,max}}{T_{to}} - 1\right)} \tag{94}$$

For a high heat transfer efficiency, the last term on the right of equation (94) should be as small as possible. This means that the "mean square" velocity should be as small as possible (this is incompatible with obtaining maximum power density), the specific heat should be as large as possible, the difference ($T_{w\,max} - T_{to}$) should be as large as possible, and λ should be as large as possible. Since λ is an inverse function of the Prandtl number, the Prandtl number should be as small as possible. This means that

liquid metals are better heat transfer media than water, and water is better than oils.

For gases the velocity of soundis given by

$$a^2 = g\,k\,R\,T$$

where k = the ratio of specific heats C_p/C_v

R = gas constant

and the Mach number is defined as

$$M = \frac{W}{a}$$

If we assume that the Mach number at inlet is low enough that $T_{to} = T_o$, then Equation (94) becomes for gases

$$\epsilon = 1 - \frac{(1+\beta)\int_0^1 \left(\frac{W_1^2}{W_o}\right) d\xi \quad (k-1)\,M_o^2}{\alpha\,\lambda\left(\frac{T_{w\,max}}{T_{to}} - 1\right)} \tag{95}$$

For gases then high efficiency means low Mach number.

The actual solution for the average velocity for Equation (42) (or average square velocity for Equation (94) for gases involves the use of the equation of state for gases

$$P/\gamma = R\,T \tag{96}$$

and an approximate or iterative solution to the equation. Examples can be found in the published heat transfer literature, and since the results are not needed here this work will not be duplicated.* For a liquid, since the velocity in a cylindrical passage is constant, Equation (94) becomes

$$\epsilon = 1 - \frac{(1+\beta)\,w^2}{g\,C_p\,T_{to}\,\alpha\,\lambda\left(\frac{T_{w\,max}}{T_{to}} - 1\right)} \tag{97}$$

Equation (42) becomes for liquids

$$P_1 - P_2 = 2\,f\,G\,w\,\ell/D \tag{98}$$

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* Flow of Heated Gases, A. S. Thompson, Journal of Applied Mechanics, March 1950, pp 91-98
The Mechanics and Thermodynamics of Steady one Dimensional gas flow, Shapiro and Hawthorne, Journal of Applied Mechanics, Dec. 1947, pp A-317, A-351
Compressibility Effects on Heat Transfer and Pressure Drop in Smooth Cylindrical Tubes, J. N. Nielsen, NACA-ARR No. L4C16, Oct. 1944
On the One-Dimensional Theory of Steady Compressible Fluid Flow in Ducts with Friction and Heat Addition, Hicks, Montgomery and Wasserman, Oct. 1947 p 891 Journal of Applied Physics.

Example : Find the heat transfer efficiency, and the power density, δ , for the following cases: ($\beta = 0.5$)

1. Liquid sodium cooled reactor for which

$$F = 0.2$$

$$w = 30 \text{ fps}$$

$$T_{w\,max} - T_{to} = 100^\circ \text{ F}$$

$$D = 0.2 \text{ in.}, \ \ell = 30 \text{ in.}$$

2. Water cooled reactor for which

$$F = 0.8$$

$$w = 30 \text{ fps}$$

$$T_{w\,max} - T_{to} = 100^\circ \text{ F}$$

$$D = 0.2 \text{ in.}, \ \ell = 30 \text{ in.}$$

3. Air cooled reactor for which

$$F = 0.5$$

$$\int_0^1 \left(\frac{w}{w_o}\right)^2 = d\xi \, M_o^2$$

$$T_{w\,max} - T_{to} = 700^\circ \text{ F}$$

$$T_{to} = 500^\circ \text{ F}$$

$$D = 1 \text{ in.}, \ \ell = 30 \text{ in.}$$

4. Find heat transfer coefficient for above cases. What is outlet temperature of fluid from reactor.

5. Plot temperature distribution of wall and fluid along heat transfer passage for (1) constant power density,(2) constant wall temperature, and (3) sinusoidal distribution of power, for the last examples.

Thermal Properties for use in Example

	H_2O	Na(700° C)	Air	
C_p	1	.302	.24	cal/gm° C
μ	.010	.0018	.00018	gm/cm sec
k	.0014	.15	.000056	cal/ sec cm° C
ρ	1.0	0.78	.00129	gm/cm^3

Summary of Convective Heat Transfer

The section on convective heat transfer has been concerned with pressure losses and heat transfer when cooling fluids flow over heated surfaces. An equation for pressure losses was given

$$\frac{dp}{dx} = \frac{4 f \gamma}{D} \frac{w^2}{2g} \qquad (1)$$

Formulae were given for the friction factor, f, for laminar flow ((Re)< 2000)

$$f = \frac{16}{(Re)} \qquad (19)$$

and for turbulent flow (5,000 < (Re) < 200,000)

$$f = \frac{0.049}{(Re)^{0.2}} \qquad (30)$$

An equation was given for the heat transferred to a flowing fluid

$$\frac{dq}{dx} = \frac{4h}{\gamma w D} (T_w - T_t) \qquad (57)$$

Equation (55) was given to relate the friction factor, f, and the heat transfer coefficient, h.

$$(St)/f/2 = \lambda \qquad (55)$$

Heat transfer analogies for the determination of λ were discussed. It was found that λ is an inverse function of the Prandtl number, being large when Prandtl number is small, and vice-versa.

The heat transfer analogy, in combination with conservation laws for the cooling fluid, was used to give a relationship between fluid and passage wall temperatures.

$$\frac{T_{t1}}{T_{to}} - 1 = 2\left(\frac{T_{w\ max}}{T_{to}} - 1\right) \alpha \lambda f \quad \ell/D \tag{86}$$

Also using the heat transfer analogy expressions were derived for the power density in a reactor

$$\delta = \frac{2\ F\ G\ g\ C_p\ T_{to}\ \alpha\ \lambda\ f}{D} \left(\frac{T_{w\ max}}{T_{to}} - 1\right) \tag{88}$$

The heat transfer efficiency was defined as one minus the ratio of pumping power to heat removed, or for a gas

$$\epsilon = 1 - \frac{(1 + \beta)\left(\frac{\overline{w}^2}{w_o}\right)(k - 1)}{\alpha\ \lambda \left(\frac{T_{w\ max}}{T_{to}} - 1\right)} \quad M_o^2 \tag{95}$$

For a liquid

$$\epsilon = 1 - \frac{(1 + \beta)\ w^2}{g\ C_p\ \alpha\ \lambda\ (T_{w\ max} - T_{to})} \tag{97}$$

It was found that for large power density and high efficiency it was necessary to have high specific heat, C_p, high λ (or low Prandtl number), large temperature difference $(T_{w\ max} - T_{to})$. For large power density the hydraulic diameter should be as small as possible (large ratio of surface to volume), and the density of fluid as large as possible. However, one contradictory requirement was found that high velocity give high power density, but low efficiency. There are other limitations on high velocity, such as erosion and corrosion problems, which are often more important than the limitation due to efficiency. The resolution of these contradictory requirements can be achieved only as a result of engineering experience.

Boiling Heat Transfer

A topic in convective heat transfer which requires somewhat special treatment is that of heat transfer to boiling liquids. As was indicated earlier, no useful information is added by the statement, for a boiling process, of Equation (69)

$$dq = C_p \, d \, T_t$$

For the boiling liquid, since the heat added causes negligibly small changes in temperature for engineering processes, the specific heat would be required to be infinite, and the heat carried away by the fluid is from this point of view indeterminate.

Experimental determinations have been made of heat transfer coefficients which are appropriate to various situations in boiling heat transfer. A typical plot* of the results of experiments is shown below for water boiling at one atmosphere and at 100 psi gage under natural convection circulation past a submerged chromel wire. Here h is defined by the equation

$$q = h \, A \, \Delta T$$

where q = heat transferred per unit time

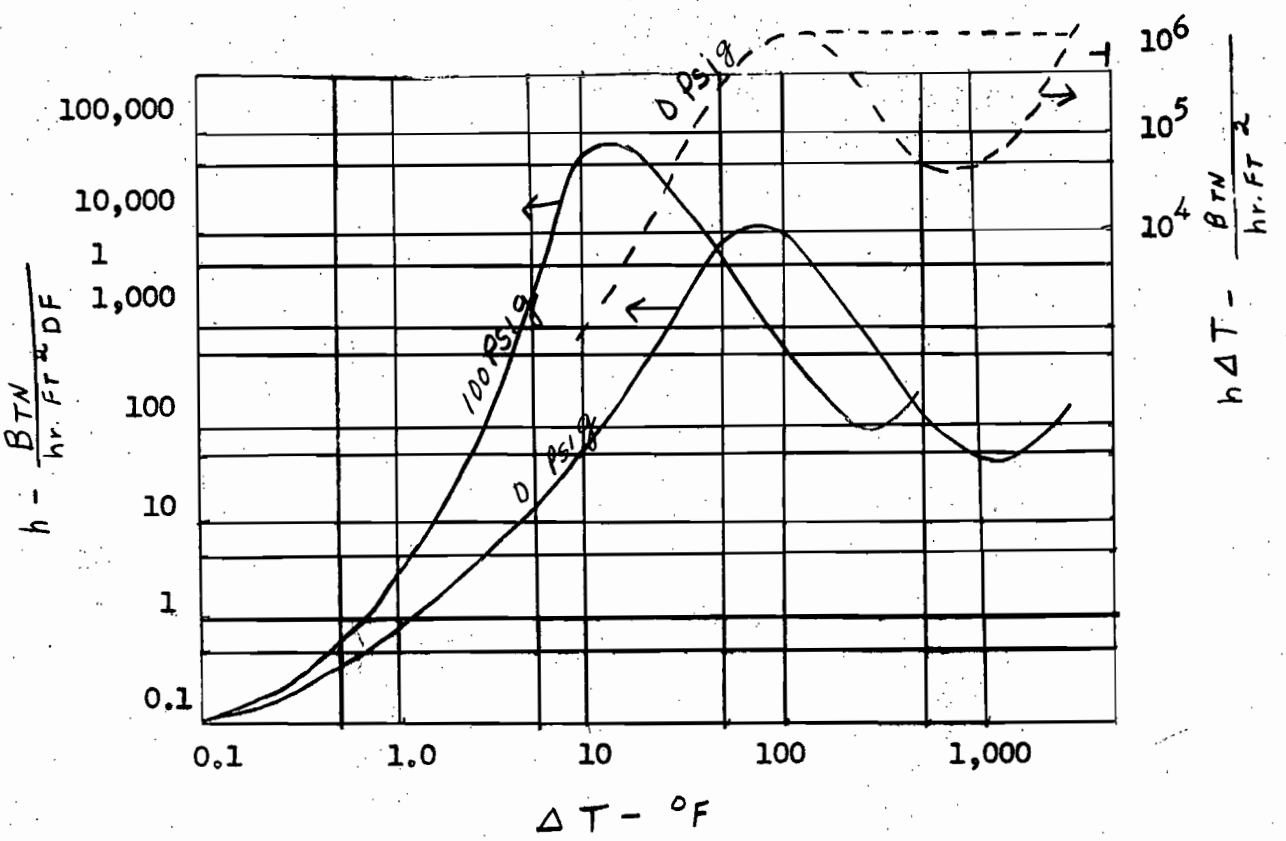
h = heat transfer coefficient

A = heated surface area

ΔT = difference between temperature of wall and boiling temperature of liquid

For appraisal of the potentialities of boiling heat transfer, part of the curve for $h \Delta T$ is sketched in, in dashed lines

* Heat Transfer to Water Boiling Under Pressure, Farber and Scorah, Trans. ASME, May 1948, p 369



One counterpart, in common experience, of the boiling phenomenon represented by the plot is the bubble on the top of a hot stove. If the temperature of the stove surface is near the boiling point of the water, the water drop will flatten out on the stove and evaporate quietly. If the temperature of the surface is considerably over the boiling temperature a film of vapor is formed between the bubble and the surface. Heat transfer into the bubble in the later case occurs only through this film and the bubble may evaporate more slowly than on the cooler surface.

As the temperature of a surface, submerged in a liquid, is increased above the boiling point, the heat transfer coefficient increases considerably as shown in the plot. In this region heat transfer occurs by a process called nucleate boiling, in which small bubbles are formed immediately at the surface, diffusing from there into the surrounding fluid where they are recondensed. The shape of bubble formed and the rate of formation (hence the

heat transfer coefficient) are dependent on the properties of the fluid and the condition of the surface-particularly as these affect surface tension and surface wetting. An oil on the surface will considerably retard the heat transfer.

As the temperature of the surface is raised a limit is reached where the heat passing through the surface can no longer be accomodated by the nucleate boiling process. Here the liquid breaks away from the surface, forming a film of vapor between liquid and surface. Since the amount of heat passing through the surface requires a larger temperature difference between liquid and surface after the break away, the surface temperature for equilibrium suddenly rises to a new value as indicated by the horizontal dashed line in the plot. This new temperature is in general so high as to result in failure of the heat transfer apparatus by melting. Since reactors are to some extent constant heat output devices, this may be a problem for boiling reactors.

Before this phenomenon had been recognized experimentally there had been some boiler failures, due mostly to local hot spots in boiler tubes where film boiling occurred. One of the deterrents to pushing boiling heat trans- to its limit is fear of this unstability, particularly with hot spots. Also there is always fear that one tube, which for some reason gets more heat, will generate more steam than its neighbor; thus its flow will be impeded, it will become hotter still, and eventually will pass over into film boiling and hence failure. Added to these difficulties, so far as reactors are concerned is the possibility of nuclear instabilities to change in density of moderators or absorbers.

The interest in nucleate boiling is sufficiently great, due to high potential heat transfer coefficients, to cause considerable interest, particularly under conditions of forced circulation. It has been found experimentally that

the heat transfer coefficients can be greatly increased by pressurizing the boiling system.

5. Thermal Power Cycles

There is much discussion, in connection with proposed applications of nuclear energy, of the effect of temperature on the performance of nuclear power plants, and of the desirability, or otherwise, of attaining high temperature. It is known that the electrical power industry has pushed continually for higher temperatures, because higher temperatures have meant higher efficiency and higher output from a given size of plant, and hence lower cost of delivered electricity. The builders of mobile power plants also have been interested in higher temperature, primarily for higher output from a given size of plant (particularly for performance of military applications) and for a saving in the fuel load which must be carried. In the case of nuclear energy the fuel load provided for burnup of fissionable material is negligible, and the cost of fuel burned up may be small (or subsidized by military programs). The arguments for high efficiency for nuclear applications may well degenerate to the following, where these are pertinent:

(1) More power from a given size and weight of plant;

(2) Conservation of nuclear fuel; and

(3) More time before reprocessing, and hence less processing cost.

For many applications (stationary power plants and some others) size and weight are a disadvantage only in so far as they are reflected in cost, and it is often cheaper to use increased weight and size to reduce the necessity for engineering development. Conservation of fuel is generally not a prime consideration in military applications, and processing cost may well be secondary. The time before reprocessing may determine the usefulness of

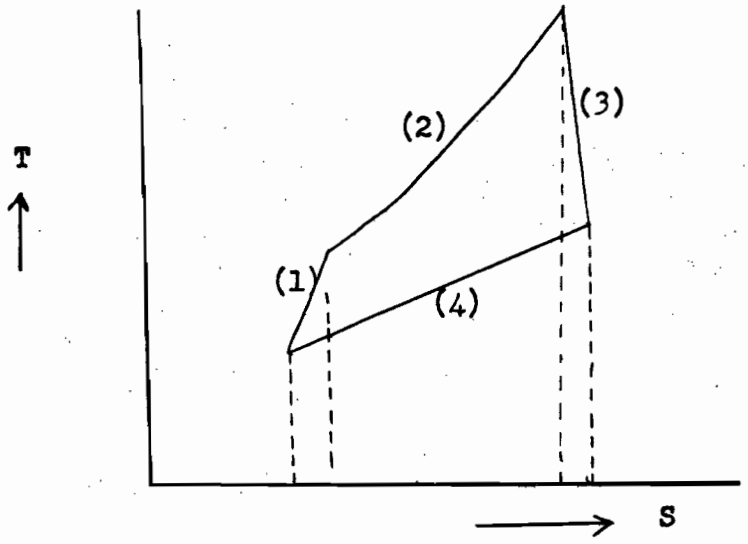
certain mobile applications of nuclear energy which require range. For some applications space and weight are important considerations - particularly aircraft.

In most cases there will be some desire for high efficiency which will be in conflict with other requirements, the conflict to be decided for the closest approximation to the objectives of some overall design. The present section will be devoted to the development of thermodynamic expressions for power plant performance, particularly as it depends on the performance of a reactor in the system. The criteria of importance will be considered here to be cycle efficiency and power output. These criteria will be related to temperature distributions and frictional pressure losses in the system.

Four Component System

For simplicity the cycle considered will be limited to four components: (1) a compressor in which the pressure level of the working fluid is raised above ambient; (2) a heater (the reactor) in which the fluid is heated to some top temperature ;(3) a turbine by which shaft work is done; and (4) a heat exchanger which cools the fluid again to ambient temperature for entrance to the compressor. This system is considered because (1) it is the simplest thermodynamic system capable of producing an appreciable excess of external work, and is fundamental to more complicated systems; (2) some general conclusions can be drawn concerning the dependence of various quantities of interest in preliminary analysis; and (3) some actual power systems in common use correspond quite closely to this system.

It is assumed that the physical states of the working fluid are completely determined by the assignment of values to two variables, here temperature and entropy. A plot of a typical cycle is then as shown below, where T is temperature and S is entropy.



(1) Compression
(2) Heating
(3) Expansion
(4) Heat Rejection

This is the same simple four-component system ϵ_o which the reactor was related in the first lecture.

If the cycle is to operate continuously, the plot of $S = S(T)$, as shown above must be a closed figure. This can be expressed analytically by stating that the requirements for the existence of a thermodynamic cycle are

$$\oint d\,T = 0 \qquad (1)$$

and

$$\oint \frac{d\,S(T)}{d\,T}\, d\,T = 0 \qquad (2)$$

To use Equations (1) and (2) it is necessary to establish the relationship $S = S(T)$. It will be found that this involves friction losses in components which will be used to define a component efficiency. The cycle efficiency and power output will then be expressed in terms of component efficiency and temperature ratios across the components.

Component Performance

For any thermodynamics process we can write an incremental expression for the conservation of energy

$$d\,q - d\,w = d\,h + d\,k + d\,e \qquad (3)$$

where

$d\,q$ = heat added to unit weight of fluid

$d\,w$ = external work done by fluid

$d\,h$ = change in enthalpy of fluid

$d\,k$ = change in kinetic energy of fluid

$d\,e$ = change in potential energy

The conservation of momentum requires

$$-\,dw = dk + de + \frac{1}{\gamma}\,dp + df \tag{4}$$

where γ = weight density of fluid

p = pressure in fluid

df = frictional energy dissipated per unit weight of fluid

Since enthalpy is by definition

$$dh = du + \frac{1}{\gamma}\,dp + pd\left(\frac{1}{\gamma}\right) \tag{5}$$

where du = change in internal energy of the fluid, Equation (3) can be written

$$dq - dw = du + pd\left(\frac{1}{\gamma}\right) + \frac{1}{\gamma}\,dp + dk + de \tag{6}$$

Subtracting Equation (4) from Equation (6)

$$dq + df = du + pd\left(\frac{1}{\gamma}\right) \tag{7}$$

By definition the change in entropy, ds, is given as

$$ds = \frac{du + pd\left(\frac{1}{\gamma}\right)}{T} \tag{8}$$

Combining Equations (7) and (8)

$$ds = \frac{dq + df}{T} \tag{9}$$

In Equation (9) it is necessary to express the frictional loss df in terms of the known energy changes in the cycle and then to express these known energy changes in terms of temperatures in the cycle.

The mechanical efficiency of a process is defined as the ratio of the energy output from the process, E_o, to the energy input to the process E_{in}

$$\mu = -\frac{E_o}{E_{in}} \tag{10}$$

For instance, in a compressor, shaft work is the input and an enthalpy change in the gas is the output. In a turbine the converse is true, the enthalpy change in the gas going through the turbine is the input and shaft work is the output. For a given process - adding or subtracting heat, or doing work - the values of energy input and energy output must differ by the frictional losses, E_f, so that

$$E_o + E_{in} + E_f = 0 \tag{11}$$

Equations (10) and (11) can be combined to eliminate either E_{in} or E_o in solving for E_f. For the process where the input is from an external source (compressor or reactor), eliminating E_o

$$E_f = -(1-\mu)\, E_{in} \tag{12}$$

For the process where the output is external (turbine or heat exchanger), eliminating E_{in}

$$E_f = -(1-\mu^{-1})\, E_o \tag{13}$$

Now df can be defined for a process involving work (Compressors or turbine)

$$df_1 = -(1-\mu_1^{\pm 1})\, dw \tag{14}$$

where the positive sign indicates compression, the negative expansion. For a process involving heat (reactor or heat exchanger)

$$df_2 = (1-\mu_2^{\pm 1})\, dq \tag{15}$$

where the positive sign indicates heat added to the cycle (reactor) the negative sign heat taken from the cycle. The efficiency μ, used in Equation (14) corresponds to what is sometimes called by gas turbine designers "small stage efficiency". It is the limit of the efficiency for a stage as the number of stages to perform a given compression or expansion goes to infinity.

For a process involving both heat and work simultaneously we can write*

$$df = -(1 - \mu^{\pm 1})\, dw + (1 - \mu_2^{\pm 1})\, dq \tag{16}$$

Substituting Equation (18) into (9)

$$ds = \frac{-(1 - \mu_1^{\pm 1})\, dw + (2 - \mu_2^{\pm 1})\, dq}{T} \tag{17}$$

According to our original assumption concerning power media - that $S = S(T)$ - the changes, dw and dq, must be expressible as functions of temperature

$$dq = -\psi_1 (T)\, dT \tag{18}$$

and

$$dq = \psi_2 (T)\, dT \tag{19}$$

If the processes are considered to extend only over small enough intervals that ψ_1 and ψ_2 are essentially constant

$$dw = -C_1\, d\,T \tag{20}$$

and

$$dq = C_2\, d\,T \tag{21}$$

Substituting Equations (22) and (23) into (19)

$$d\,S = K\,\frac{dT}{T} \tag{22}$$

where

$$K = \left[(1 - \mu_1^{\pm 1})\, C_1 + (2 - \mu_2^{\pm 1})\, C_2\right] \tag{23}$$

For any one process, Equation (22) can be integrated to give

$$S_m - S_{m-1} = \log\left(\tau_m^{K}\right) \tag{24}$$

where

$$\tau_m = \frac{T_m}{T_{m-1}}$$

and T_m = the temperature at the end of the m_{th} process.

For the case considered here, where there are four discrete processes,

* For a more general treatment see " Power Plant Analysis", by A. S. Thompson NAA-SR-19, October 26, 1948

Equation (1) can be expressed

$$\sum_{m=1}^{4} (T_m - T_{m-1}) = 0 \tag{25}$$

This can be rewritten

$$\sum_{m=1}^{4} T_m = \sum_{m=0}^{3} T_m \tag{26}$$

Cancelling identical terms

$$\frac{T_4}{T_o} = 1 \tag{27}$$

Since

$$\frac{T_4}{T_o} = \prod_{m=1}^{4} \tau_m$$

Equation (27) becomes

$$\prod_{m=1}^{4} \tau_m = 1 \tag{28}$$

Equation (28) takes the place of Equation (1) for the four process cycle.

For the discrete processes Equation (2) becomes

$$\sum_{m=1}^{4} (S_m - S_{m-1}) = 0 \tag{29}$$

Substituting Equation (24) into (29)

$$\sum_{m=1}^{4} \log (\tau_m^{K}) = 0 \tag{30}$$

From the summation property of logarithium

$$\log \left\{ \prod_{m=1}^{4} (\tau_m^{K}) \right\} = 0 \tag{31}$$

or

$$\prod_{m=1}^{4} (\tau_m^{K}) = 1 \tag{32}$$

Equations (28) and (32) are the conditions for the existence of a power cycle in terms of temperature ratios, and process efficiencies which determine

the factor, k (Equation (23)). For evaporation processes, involving a change of phase where the temperature remains constant during a process, a different approach is more tractable.* Equations (28) and (32) apply particularly to gaseous media. They will now be considered for a simple gas turbine cycle. With Equations (20) and (21) a complete thermodynamic description of the cycle will be obtained.

It is necessary to evaluate the constants C_1 and C_2 for each of the four processes. This will be done using Equation (3). For a compression or expansion in which no heat is added and changes in kinetic and potential energy are negligible Equation (3) becomes

$$dw = - dh \tag{33}$$

Since for a perfect gas

$$dh = C_p \, dT \tag{34}$$

Then

$$dw = - C_p \, dT \tag{35}$$

Comparing Equations (35) and (20)

$$C_1 = C_p \tag{36}$$

Likewise for the case of heating or cooling with no external work

$$dq = dh = C_p \, dT \tag{37}$$

Hence in Equation (21)

$$C_2 = C_p \tag{38}$$

Now these values of C_1 and C_2 from Equations (36) and (38) can be used to evaluate k in equation (23) and the work done, Δw, and the heat added, Δq. This will be carried out for: (1) an adiabatic compressor; (2) a reactor; (3) an adiabatic turbine; and (4) a heat exchanger. The particular combination of processes is that of the simple gas turbine, and is used here for demonstration purposes. The conditions for the existence of a cycle will be applied to the processes and expressions for efficiency and power

* Liquid-Vapor Power System, By A. S. Thompson, NAA-SR-26, Feb. 3, 1949

output will be developed.

Compressor

Equation (23) for k hence becomes for a compression

$$K_c = (1 - \mu_c) \; C_p \tag{39}$$

The work done in driving the compressor is

$$\Delta w_c = - C_p (T_1 - T_o)$$
$$= - C_p T_o (\tau_c - 1) \tag{40}$$

Reactor

From Equation (23)

$$K_R = (2 - \mu_R) \; C_p \tag{41}$$

The heat added by the reactor is

$$\Delta q_R = C_p (T_2 - T_1)$$
$$= C_p T_o \tau_c (\tau_R - 1) \tag{42}$$

Turbine

From Equation (23)

$$K_T = (1 - \frac{1}{\mu_T}) \; C_p \tag{43}$$

The work done by the turbine is

$$\Delta w_T = - C_p (T_3 - T_2)$$
$$= - C_p \tau_c \tau_R T_o (\tau_T - 1) \tag{44}$$

Heat Exchanger

From Equation (23)

$$K_H = (2 - \frac{1}{\mu_H}) \; C_p \tag{45}$$

The heat dumped to the heat exchanger is

$$\Delta q = C_p (T_4 - T_3)$$
$$= C_p T_o \tau_c \tau_R \tau_T (\tau_H - 1) \tag{46}$$

Let us define a temperature ratio, τ, equivalent to the ratio of the top temperature at entrance to the turbine to ambient temperature, T_o. Then

$$\tau = \tau_c \, \tau_R \tag{47}$$

Then the heat added to the cycle, with which it is to be charged is, from Equation (42)

$$Q = \Delta q_R = C_p T_o (\tau - \tau_c) \tag{48}$$

The net work done by the cycle is the sum of the turbine work and compressor work, or

$$W = - C_p T_o \left\{ \tau_c - 1 + \tau (\tau_T - 1) \right\} \tag{49}$$

The efficiency is

$$E = \frac{W}{Q} = \frac{\tau_c - 1 + \tau (\tau_T - 1)}{\tau - \tau_c} \tag{50}$$

These are subject to conditions from Equations (28) and (32), or

$$\tau \, \tau_T \, \tau_H = 1 \tag{51}$$

$$\tau_c^{K_c - K_R} \, \tau^{K_R} \, \tau_T^{K_T} \, \tau_H^{K_H} = 1 \tag{52}$$

Combining Equations (51) and (52) to eliminate τ_H

$$\tau_T^{K_T - K_H} \, \tau_c^{K_c - K_R} \, \tau^{K_R - K_H} = 0 \tag{53}$$

Or solving for τ_T

$$\tau_T = \tau_c^{-\frac{K_c - K_R}{K_T - K_H}} \, \tau^{-\frac{K_R - K_H}{K_T - K_H}} \tag{54}$$

Substituting Equation (54) into (49)

$$W = - C_p T_o \left\{ \tau_c - 1 - \tau + \tau \frac{K_T - K_R}{K_T - K_H} \tau_c - \frac{K_c - K_R}{K_T - K_H} \right\} \tag{55}$$

The work done by the cycle will be maximum when

$$\frac{d W}{d \tau_c} = 0 \tag{56}$$

or when

$$\frac{K_c - K_R}{K_T - K_H} \tau^{\frac{K_T - K_R}{K_T - K_H}} \tau_c^{-\frac{K_c - K_R + K_T - K_H}{K_T - K_H}} = 1 \qquad (57)$$

Solving Equation (57) for τ_c

$$\tau_c = \left(\frac{K_c - K_R}{K_T - K_H}\right)^{\frac{K_T - K_H}{K_c - K_R + K_T - K_H}} \tau^{\frac{K_T - K_H}{K_c - K_R + K_T - K_H}} \qquad (58)$$

Let $$a = \frac{K_T - K_R}{K_c - K_R + K_T - K_H} \qquad (59)$$

$$b = \frac{K_T - K_H}{K_c - K_R + K_T - K_H} \qquad (60)$$

Then Equation (58) becomes

$$\tau_c = \left(\frac{1-b}{b}\right)^b \tau^a \qquad (61)$$

Equation (54) becomes

$$\tau_T = \tau_c^{-\left(\frac{1-b}{b}\right)} \tau^{+\left(\frac{a-b}{b}\right)} \qquad (62)$$

Substituting Equation (61) into (62)

$$\tau_T = \left(\frac{1-b}{b}\right)^{-(1-b)} \tau^{(a-1)} \qquad (63)$$

Substituting Equations (61), (59), and (60) into (55)

$$W = -C_p T_o \left\{ \left(\frac{1-b}{b}\right)^b \tau^a - 1 + \tau \left[\left(\frac{1-b}{b}\right)^{-(1-b)} \tau^{a-1} - 1\right]\right\} \qquad (64)$$

Equation (64) can be simplified by adding and subtracting the quantity ($2\ \tau^{1/2}$) to complete a square

$$W = C_p T_o \left\{ \left[\tau^{1/2} - 1\right]^2 - \left[\frac{1}{b^b\ (1-b)^{(1-b)}} \tau^a - 2\ \tau^{1/2}\right]\right\} \qquad (65)$$

From Equation (50) then the efficiency of the cycle is

$$E = \frac{\left[\tau^{1/2} - 1\right]^2 - \left[\frac{1}{b^b\ (1-b)^{(1-b)}} \tau^a - 2\tau^{1/2}\right]}{\tau - \left(\frac{1-b}{b}\right)^b \tau^a} \qquad (66)$$

The values of a and b are given by Equation (59) and (60). For the case where the efficiencies of all components are the same and the specific heat is constant throughout the cycle, some simplifications result.

<u>Constant specific heat, C_p, and component efficiency the same for all components</u>

For this case Equation (60) gives

$$b = 1/2 \tag{67}$$

Using this value for b, Equation (61), the condition for maximum power output from the cycle gives for the temperature ratio across the compressor

$$\tau_c = \tau^{a} \tag{68}$$

The temperature ratio across the turbine is

$$\tau_T = \tau^{(a-1)} \tag{69}$$

The work output from the cycle is

$$W = C_p T_o \left\{ \left[\tau^{1/2} - 1 \right]^2 - 2 \left[\tau^{a} - \tau^{1/2} \right] \right\} \tag{70}$$

The cycle efficiency is

$$E = \frac{\left[\tau^{1/2} - 1 \right]^2 - 2 \left[\tau^{a} - \tau^{1/2} \right]}{\tau - \tau^{a}} \tag{71}$$

Also for this case, from Equation (59)

$$a = \frac{1 + \frac{1}{\mu} - \mu}{2} \tag{72}$$

Or solving Equation (72) for the efficiency

$$\mu = 1/2 \left(1 - 2a + \sqrt{5 - 4a + 4a^2} \right) \tag{73}$$

Or approximately, for μ not too much less than unity

$$\mu \sim 1.5 - a \tag{74}$$

Equation (74) gives a physical interpretation to a. It is approximately 1.5 less the "average" component efficiency, for component efficiencies not too far from unity. This hypothetical case of equal efficiencies for all components is found to be a good indicator of cycle performance. The value

μ, to be used is actually very close to the numerical average of the actual component efficiencies for the cycle. The results using b = 1/2 are quite close even for cases where b is quite different from 1/2. For specific cases actual values should be taken, but for our purposes the results will be good enough.

In the accompanying figure is shown a plot of the dimensionless power output from the simple gas cycle, $W/C_pT_o = W'$, and the cycle efficiency, E, for the case where b = 1/2. For comparison the Carnot efficiency is also shown. The Carnot efficiency is just

$$E = \frac{\tau - 1}{\tau}$$

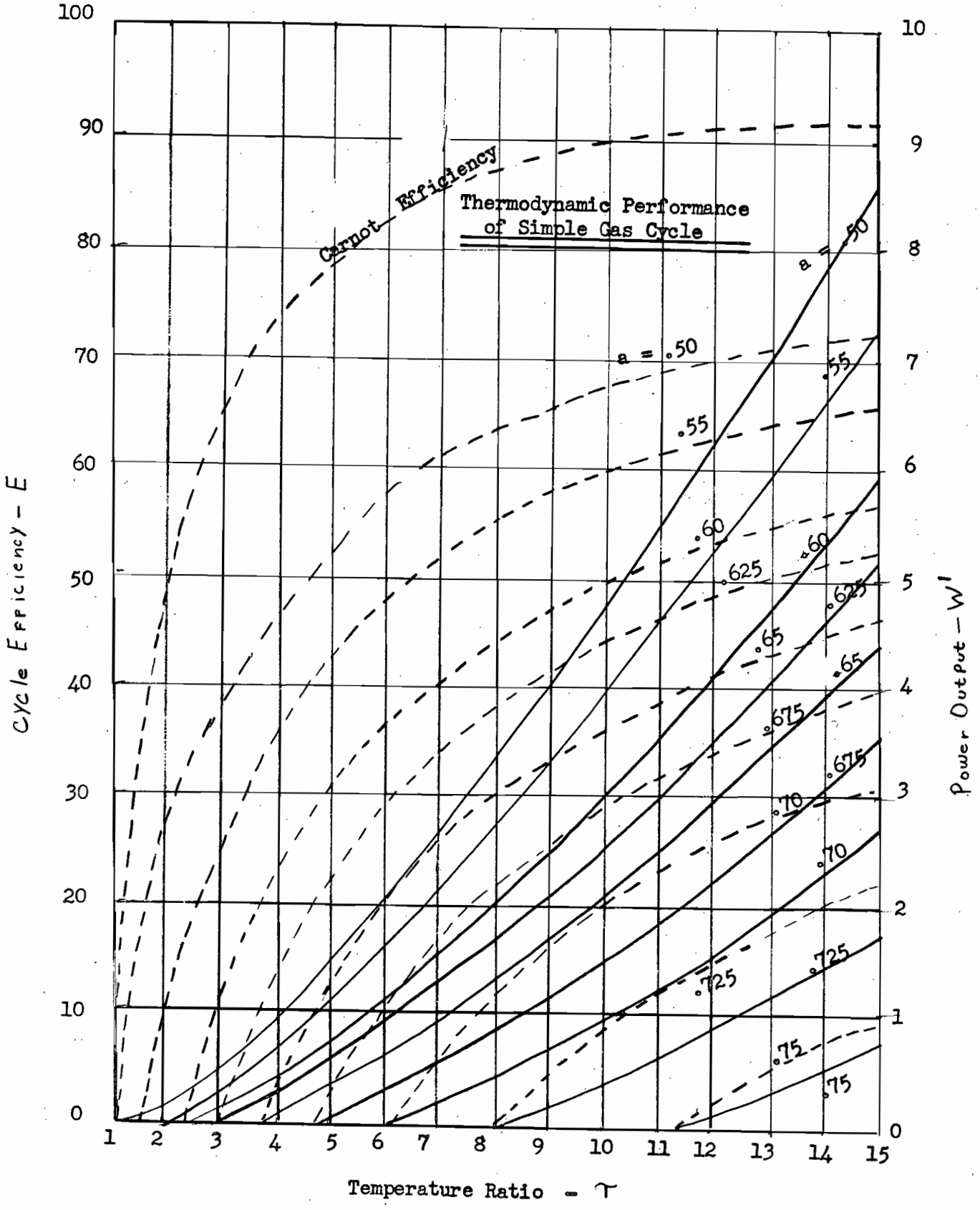
For our gas cycle, from Equation (71), for a = 1/2, the efficiency with component efficiencies of 100 percent would be

$$E = \frac{\tau^{1/2} - 1}{\tau^{1/2}}$$

By adding various complicating features such as "intercooling" and "reheating" the efficiency of the gas cycle can be made to approach more closely that of the Carnot cycle, if one works only on paper. However, in actual experience it is generally found difficult to make these complications pay their way. There is always a tendency for miscellaneous pressure losses to dissipate the possible advantage of such a scheme.

The plot shows that efficiency reaches a region of vanishing return as temperature is increased indefinitely. The power output, within the limits of our assumptions, does not seem to. However it should be remembered that it may not be possible to increase temperature along a line of constant component efficiency, but that the component efficiency will fall off as temperature is increased. This is a specific design problem involving the design of compressors, turbines, and heat exchangers, and their matching.

Inspection of the plot of cycle performance will show the justification for certain approximations, such as that of Equation (74). It can be seen


Thermodynamic Performance
of Simple Gas Cycle
Carnot Efficiency
a = .50
.55
.60
.625
.65
.675
.70
.725
.75
Cycle Efficiency - E
Power Output - W'
Temperature Ratio - τ
100
90
80
70
60
50
40
30
20
10
0
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
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10
11
12
13
14
15

that if the parameter, a, is as low as .75, the temperature ratio required for the system to have greater than zero output is about 11.5. If the ambient temperature were 500° R (Rankine - absolute temperature on engineering scale - 460° R = 0° F), then the top temperature of the cycle would be 500 x 11.5 = 5750° R. This would seem like an unwarranted struggle to achieve marginal performance. Hence component efficiencies to be of any engineering interest must have an average at least greater than 75 percent. Since present gas turbine practice uses temperature ratios around 3 or 4, as limited by present materials, the lowest average component efficiency must be somewhere around 85 percent. Representative values in current jet-engine practice are probably about

$\mu_c = .80$

$\mu_R = .95$ (Combustion Chamber)

$\mu_T = .85$

$\mu_H = 1.00$ (discharge to atmosphere- no heat exchanger)

Average $\mu = .90$

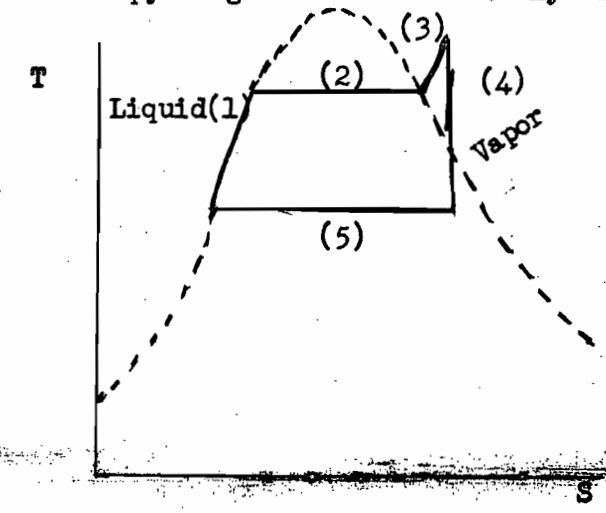
From the plot, $\mu = .90$ corresponds to a minimum temperature ratio of operation of about 2.4. With an ambient temperature of 530° R (70° F), this corresponds to a top temperature at the inlet to the turbine of 1270° R (810° F) An additional problem should be mentioned here. Power systems are designed for "design point" operation. For a system with a top temperature at the design point of 1200° F, the component efficiencies will generally be arranged to have maximum values as near as possible to this design point. At all other conditions the components efficiencies will be less than the maximum value. Hence the system having an average component of .90 at the design point of 1200° F would probably not operate at all at 810° F as shown above because of deterioration of component efficiencies. In airplanes "off-design" operation is required by changes in speed, altitude, ambient

temperature, cargo load, etc.

It should be noted in the plot that a marginal design is subject to two very serious limitations. The first is that, if the design point is missed because a component is not as good as had been supposed, the marginal design will not operate at all. The second is that the marginal design will have poor off design permormance, which, in the case of a gas turbine, means that it will be hard to start. It should be noted also that a small change in component efficiency or temperature ratio results in a large change of cycle efficiency (and power output) in the region of present operation of gas turbines.

It is not planned here to discuss in detail liquid-vapor power systems.* In general it can be remarked that liquid-vapor systems (for instance steam power plants) tend to approach much closer than gas turbine cycles to the Carnot efficiency and hence are not subject to many of the limitations of the gas turbine system. For example the overall performance is not so sensitive to changes in component efficiency or to changes in temperature as the gas turbine cycle - primarily because the efficiency is already fairly high. Pumping power in the liquid vapor system is low and therefore the starting problem is almost insignificant.

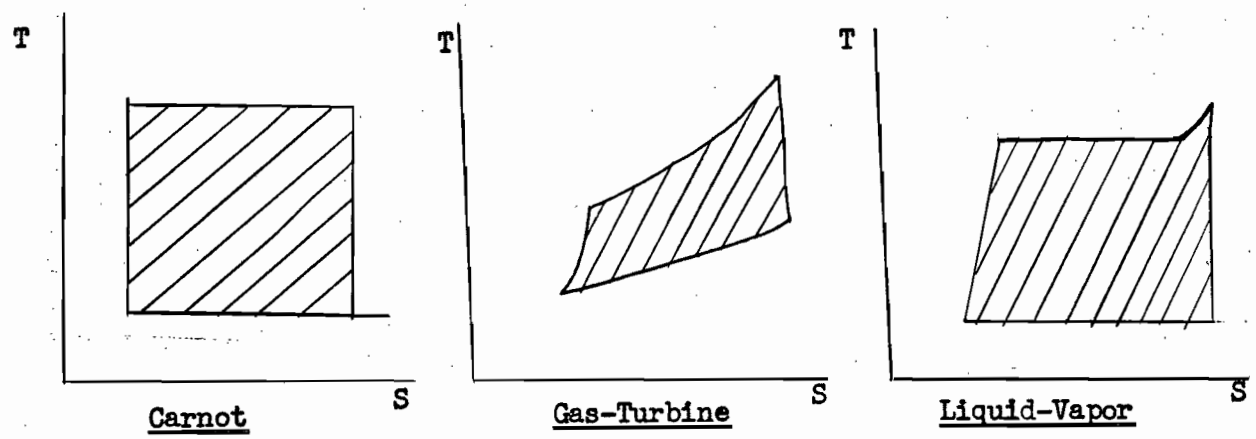
The Rankine cycle is a good approximation for a liquid-vapor system. Its temperature-entropy diagram is schematically as shown below.



(1) Liquid Heating
(2) Vaporization
(3) Superheating
(4) Expansion in turbine
(5) Heat rejection

* Liquid Vapor Power Systems, A. S. Thompson, NAA-SR-26, Feb. 3, 1949

It can be seen that the cycle is a closer approach to a Carnot cycle than the gas turbine. Changes in temperature and entropy due to pumping (between (1) and (5)) are so small as to be negligible and do not show upon the diagram.



The cycle discussed here for the gas turbine was designed for maximum work. It could have been designed (and for some cases would have been) for maximum efficiency or for maximum output per unit weight or size of plant, and the general approach would have been the same.

Examples

1. Consider a gas turbine cycle consisting of four processes having the following values of parameters:

Compressor

$C_{pc} = 0.24\ B_{tu}/lb$ $\qquad K_c = 0.24\ (1 - 0.80)$

$\mu_c = 0.80$ $\qquad = 0.048$

Heater

$C_{ph} = 0.25\ B_{tu}/lb$ $\qquad K_{hR} = (2 - 0.85)\ 0.25$

$\mu_R = 0.85$ $\qquad = 0.288$

Turbine

$C_{pt} = 0.25\ B_{tu}/lb$ $\qquad K_t = 0.25\ \left(1 - \frac{1}{0.90}\right)$

$\mu_t = 0.90$ $\qquad = -\ 0.028$

Heat Exchanger (Atmosphere)

$C_{pe} = 0.24\ B_{tu}/lb$ $\qquad K_H = \left(2 - \frac{1}{1.0} \right) 0.24$

$\mu_H = 1.00$ $\qquad = 0.240$

For this example

$$a = \frac{-0.028 - 0.288}{0.048 - 0.288 - 0.028 - 0.240} = 0.622$$

$$b = \frac{-0.028 - 0.240}{0.048 - 0.288 - 0.028 - 0.240} = 0.527$$

$$b^b\ (1-b)^{(1-b)} = (.527^{.527})\ (.473^{.473}) = .712 \times .702 = 0.50$$

For this example then, Figure 1 can be used, for = 5, the value of the maximum work output is given by

$$W' = 0.62$$

The efficiency of the cycle is

$$E' = 25 \text{ per cent}$$

For an overall temperature ratio of $\tau = 10$, $W' = 2.9$ and $E' = 46$ per cent.

2. Calculate the specific thrust (lb of thrust per lb of air per second) for a jet engine designed for maximum thrust at a flight speed of 1,000 fps ($M_o \simeq 1.0$) at a flight altitude of 36,000 ft. ($T_o = 390°$ R) and conforming to the conditions of Example 1. It is assumed here that the efficiency of compression is the same in the diffuser and compressor, and that the efficiency of expansion is the same in the nozzle and in the turbine.

Assuming that the mechanical energy expended by the engine on the atmosphere is used entirely to create a change in velocity across the engine, then this energy is

$$P = \frac{v_2^2 - v_o^2}{2g}$$

The specific thrust of the engine is given by

$$F = \frac{v_2 - v_o}{g}$$

Eliminating v_2 between these equations, the specific thrust is

$$F = \frac{2\,g\,P + v_o^2 - v_o}{g}$$

The jet efficiency is then

$$\frac{F\,v_o}{P}$$

For $\tau = 5$

$$P = (390)(0.24)(778)(0.62) = 45{,}000 \ \frac{\text{ft. lb.}}{\text{lb.}}$$

$$F = \frac{(2)(32.2)(45{,}000) + (1{,}000)^2 - 1000}{32.2} = 30.3 \ \frac{\text{lb. sec.}}{\text{lb.}}$$

$$\text{Jet efficiency} = \frac{(100)(30.3)(1{,}000)}{45{,}000} = 67 \text{ per cent}$$

For $\tau = 10$

$$P = (390)(0.24)(778)(2.9) = 211{,}000 \ \frac{\text{ft. lb.}}{\text{lb.}}$$

$$F = \frac{(2)(32.2)(211{,}000) + (1{,}000)^2 - 1000}{32.2} = 87.2 \ \frac{\text{lbs sec.}}{\text{lb.}}$$

$$\text{Jet efficiency} = \frac{(1000)(87.2)(1{,}000)}{211{,}000} = 41 \text{ per cent}$$

3. Compute the change in performance in Example 1 due to substitution of a heat exchanger having 80 per cent mechanical efficiency. For this case only K_H changes, its new value being

$$K_H = (2 - 1/8)(0.24) = 0.180$$

Then

$$a = \frac{-0.028 - 0.288}{0.048 - 0.288 - 0.028 - 0.180} = 0.705$$

$$b = \frac{-0.028 - 0.180}{0.048 - 0.288\ 0\ 0.028 - 0.180} = 0.463$$

The values of power output and cycle efficiency may again be found very closely from Figure 1.

$\tau = 5$

W' = _____(negative)

E' = _____(negative)

$\tau = 10$

W' = 1.1

E' = 21 per cent