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KINETIC CALCULATIONS FOR HOMOGENEOUS REACTORS

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Homogeneous Reactor Project

KINETIC CALCULATIONS FOR HOMOGENEOUS REACTORS

W. C. Sangren

Date Issued

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A. INTRODUCTION

The objective of the following pages is to place in one report a large portion of the work connected with the kinetics of homogeneous reactors of the kind envisioned in the Homogeneous Reactor Experiment. This report deals primarily with particular thermal, nuclear and pressurizer systems containing only bulk or average quantities and no positional coordinates. The leading reason for considering only bulk quantities is that the mathematical problems become much easier. Further, the motion is considered only for a short period of time. This again reduces considerably the complexity of the equations to be solved.

Most of the investigations have dealt with the results of step increases in reactivity. Since such step increases in homogeneous reactors seem unlikely, any calculations made with this assumption are on the conservative side. The effect of a linear function increase in reactivity has been investigated to a small extent.

It is worth noting that the following is largely a mathematical discussion. In particular, no justification is given here for the equations which are used to represent the equations of motion. Consequently, any physical conclusions reached here are only as good as the equations from which they are derived.

Since the systems of equations which arise contain one or more non-linear equations, it is necessary to resort to non-analytical methods. Two general methods have been used to obtain approximate solutions. The

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most detailed and accurate results can be obtained by using numerical integration either with desk computers or larger machines. Unfortunately, these numerical methods are somewhat lengthy and expensive, and do not lead to results or formulas which may be used for equations with different values of the parameters. Sections B and C deal in turn with "Hand Computations" and "SEAC Computations". The other general method consists of using semi-analytical methods in order to obtain approximate formulas. Sections D and E are respectively, "Analysis - J. M. Stein" and "Analysis".

The final section, F, Contains "Comments and Conclusions". There also exists an Appendix devoted to some mathematical notes. For convenience pages 87, 88, and 89 contain definitions of the symbols which are used. When necessary the values of the parameters or constants used in a particular computation are listed in the text.

B. HAND COMPUTATIONS

Various systems of differential equations have been numerically integrated by desk computers. The first set of such calculations was done in 1949 under the direction of I. Spiewak and C. B. Graham. Insufficient records remain of these investigations in order to draw conclusions concerning them.

The results of the next numerical integration appear in ORNL 925. The following system formed the basis for this integration:

$$\dot{\phi} = \lambda_o\left[(1 - \beta)\, k - k_c\right] \phi + \sum_i \lambda_i C_i$$

$$\dot{C}_i = - \lambda_i C_i + \beta_i \lambda_o k \phi \quad (i = 1, \ldots, 6)$$

$$\dot{T} = S\,(P - P_o)$$

$$P = \gamma \lambda_o k \phi$$

$$k_c = k_{co} + \alpha\,(T - T_o),$$

where the constants are given in ORNL 925. Figures 36, 37, and 38 of ORNL 925 exhibit these results and are reproduced here as Figures 1, 2 and 3.* Figure 3 indicates that the assumption of one group of delayed neutrons is a fair approximation.

Five systems of differential equations have recently been integrated. The following system is the first which was investigated:

$$\dot{P} = - \frac{\alpha}{\tau}(T - T_o)\, P + \frac{\delta - \beta}{\tau} P + \lambda C$$

$$\dot{C} = - \lambda C + \frac{\beta}{\tau} P$$

$$\dot{T} = S\,(P - P_o),$$

where $\delta = 0.01$, $\alpha = 0.001$, $\tau = 0.0001$, $\beta = 0.0075$, $\lambda = 0.08$, $P_o = 1000$, $S = 0.00573$. These results appear in ORNL 1121 and in Figures 4, 5 and 6.

Partly in order to test the assumption of a constant source of delayed neutrons as compared to one group of delayed neutrons the following reduced system was integrated.

$$\dot{T} = S\,(P - P_o)$$

$$\dot{P} = - \frac{\alpha}{\tau}(T - T_o)\, P + \frac{\delta - \beta}{\tau} P + \frac{\beta}{\tau} P_o,$$

---

* Many of these figures will not appear here, but will appear later in a supplement.

where the above constants are used. It is apparent that a constant delayed neutron source is a good assumption.

Systems 1 and 9, as listed under SEAC computations were calculated both by the SEAC (the National Bureau of Standards' Eastern electronic digital computer) and by desk computers. The hand computations were carried out before the SEAC computations since it was desirable to have these results both as a means of checking the machine results and as a method of estimating magnitudes. Figures 7 through 16 exhibit these results.

The most recent hand computation was carried out on System 14, also listed below. In this case it was thought desirable to shock the pressurizer system only and follow the early motion of the neutron and pressurizer systems. Figures 17 through 21 present these results.

All five of the above systems were integrated by the Runge-Kutta method. For the more difficult systems such as System 9, one computer can calculate at about the rate of 0.3 sec. per system per month.

C. SEAC COMPUTATIONS

The following set of equations dealing with the thermal, nuclear and pressurizer systems form the basis for the computations performed on the SEAC:

(1) $\dot{T} = S\,(P - P_o)$

(2) $\dot{P} = \frac{1}{\tau}\left[\left(\frac{\partial k}{\partial \rho}\right)(\rho - \rho_o) + \delta - \beta\right] P + \frac{\beta}{\tau} P_o$

(3) $\dot{\rho} = -\frac{A\rho_o}{V} U$

$$(4) \quad \dot{U} = \frac{A}{M_r}(p - p_o) - \frac{\beta}{M_r} U - \frac{E}{M_r} U|U|$$

$$(5) \quad \rho - \rho_o = \frac{1}{v_s^2}(p - p_o) - \left|\frac{d\rho}{dT}\right|(T - T_o)$$

From the computational viewpoint it is desirable to replace equations (1) through (5) by a corresponding set with fewer constants or parameters. Using equation (5) in equation (4) and letting $y = \frac{1}{SP_o}(T - T_o)$, $x = \frac{P}{P_o}$, $v = \frac{1}{\tau}\left(\frac{dk}{d\rho}\right)\frac{A\rho_o}{V} U$ and $z = \frac{1}{\tau}\left(\frac{dk}{d\rho}\right)(\rho - \rho_o)$ one can obtain

$$(6) \quad \dot{y} = x - 1$$

$$(7) \quad \dot{z} = - v$$

$$(8) \quad \dot{x} = zx + \frac{\delta - \beta}{\tau} x + \frac{\beta}{\tau}$$

$$(9) \quad \dot{v} = - \alpha_1 v - \alpha_2 v|v| + w_H^2 \left[z + w_n^2 y\right]$$

where $w_H^2 = \frac{A^2\rho_o}{V M_r} v_s^2$, $w_n^2 = \frac{1}{\tau}\left(\frac{dk}{dT}\right)SP_o$, $\alpha_1 = \frac{\beta}{M}$ and $\alpha_2 = \frac{E}{M_r}\frac{A\rho_o}{\tau V}\left(\frac{dk}{d\rho}\right)^{-1}$.

Thirteen such systems were computed on the SEAC. In all the systems equations (6) and (7) and the initial conditions $x(0) = 1$, $y(0) = 0$, $z(0) = 0$ and $v(0) = 0$ are kept fixed. The systems were completed by adding one of the following pairs of differential equations:

$(8)_1$	$\dot{x} = zx + 375x$	$(9)_1$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y$
$(8)_2$	$\dot{x} = zx + 300x + 75$	$(9)_2$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v$
$(8)_3$	$\dot{x} = zx + 300x + 75$	$(9)_3$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.01v\lvert v\rvert$
$(8)_4$	$\dot{x} = zx + 300x + 75$	$(9)_4$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$
$(8)_5$	$\dot{x} = zx + 100x$	$(9)_5$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y$
$(8)_6$	$\dot{x} = zx + 25x + 75$	$(9)_6$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.1v$
$(8)_7$	$\dot{x} = zx + 25x + 75$	$(9)_7$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.03v$
$(8)_8$	$\dot{x} = zx + 25x + 75$	$(9)_8$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.01v\ v$
$(8)_9$	$\dot{x} = zx + 25x + 75$	$(9)_9$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$
$(8)_{10}$	$\dot{x} = zx - 25x + 75$	$(9)_{10}$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$
$(8)_{11}$	$\dot{x} = zx - 5x + 75$	$(9)_{11}$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$
$(8)_{12}$	$\dot{x} = zx + 50x + 75$	$(9)_{12}$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$
$(8)_{13}$	$\dot{x} = zx + 100x + 75$	$(9)_{13}$	$\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y - 0.3v - 0.01v\lvert v\rvert$

where $\left(\frac{dk}{d\rho}\right) = 0.0074$, $V = 1.75$, $L = 5.9$, $A = 0.037$, $v_s^2 = 7.05(10^6)$ $w_n^2 = 50$, $w_H^2 = 2.5(10^4)$, $\tau = 0.0001$, $\beta = 0$ or 0.0075, $\alpha_1 = 0$ or 0.1 or 0.3 and $\delta = 0.0050$ or 0.0070 or 0.010 or 0.0125 or 0.0175 or 0.0375. For convenience in identification each system is named according to the number which is a subscript for its equation (8) and (9).

A fourteenth system with

$(8)_{14}$ $\dot{x} = zx - 75x + 75$ $\qquad$ $(9)_{14}$ $\dot{v} = 2.5\,(10^4)z + 1.25\,(10^6)y$

and $x(0) = 1$, $y(0) = 0$, $z(0) = 0$ and $v = -10^5$ was computed with a desk computer

The general system (6), (7), (8), (9), with initial conditions was programmed for solution by the modified Euler method on the SEAC. The parameter values, integration interval (of order 10^{-4} sec.), and the

interval at which the results were to be printed out were read into the machine independently of the main routine. Since it was not necessary to change the main routine from system to system, it allows the possibility of using in the future the main routine with different values of the parameters. The range of integration was either from $t = 0$ to $t = 0.3$ sec. or $t = 0$ to $t = 0.4$ sec. The total machine time, including one hour for checking the programming and coding, for all thirteen systems was about five hours. However, the actual computation time per system was less than twenty seconds. The remaining machine time was taken up by the printing out of the results and to a small extent by reading the routine and parameter values into the machine. One hundred values of x and fifty values for each of y, v and z were printed out. Table 1 gives a summary of these results while Figures 22 through 60 exhibit the results.

Some comments may be made concerning these results. For this purpose consider in turn power, temperature, density, velocity and pressure as represented through the symbols x, y, z, v and $\bar{p}$. Figure 61 and 62 exhibit respectively maximum x and time of maximum x versus δ. It is interesting to note that there exists a maximum time in Figure 62 which occurs at about prompt critical, i.e. $\delta = 0.0075$. This value of δ separates, of course, the two general types of motion of x. That is for $\delta > \beta = 0.0075$, x has an exponential like rise at first and is brought down only by the negative temperature coefficient, while for $\delta < \beta$, the rise in x is more step like at first and is brought down not only by the negative temperature coefficient but also by an exponential like fall. From the SEAC numbers it is clear that increasing the damping in the pressurizer increases the maximum value of x and increases the time at which x reaches this maximum. For

$\delta < 0.0025$ this effect is probably negligible. Two general effects may be noticed on the x vs. t curves as the δ increases. Although at $\delta = 0.01$ no noticeable motion due to the pressurizer appears, it does not take a much bigger δ, and in particular at $\delta = 0.0125$, before small oscillations appear on the x vs. t curve. Overdamping, while being only clearly defined for linear equations, may in the sense of never allowing a motion to oscillate about its equilibrium position be used to describe the other effect. Up to $\delta = 0.01$ the x is overdamped. At $\delta = 0.0125$ the x is barely overdamped, while at $\delta = 0.0175$ the system becomes underdamped. It may be emphasized that the suffixes over and under do not refer to the degree of the damping, i.e., being more or less damped.

The curves of temperature versus time indicate that the general motion for $\delta < 0.0125$ goes up smoothly to the new equilibrium temperature. The equilibrium temperature is given by $y_E = \frac{\delta}{\tau w_n^2}$, i.e., $T_E = \frac{\delta}{\alpha} + T_o$. Figure 63 shows the value of the temperature at t = 0.2 sec. versus δ. This curve indicates that beyond $\delta = 0.0135$ the temperature is no longer overdamped. The pressurizer motion does not show up in the temperature curves.

The curves of density versus time fall steadily toward their new equilibrium values until around $\delta = 0.012$. For larger δ the pressurizer oscillations can be seen. Figure 63 also shows the value of z at t = 0.2 sec. versus δ. Again overdamping disappears at about $\delta = 0.0135$. The equilibrium density is given by $z_E = \frac{\delta}{\tau}$, i.e., $\rho = \rho_o + \delta \left(\frac{\partial k}{\partial \rho}\right)^{-1}$.

The curves of velocity versus time give a good picture of the pressurizer oscillations and the effects of the damping coefficients. Figure 64 shows a curve of maximum velocity versus δ.

Figure 65 exhibits a curve of maximum pressure versus δ. At $\delta = 0.0225$ the pressure is seen to be 4000 lbs./sq.in., the assumed bursting pressure. Around $\delta = 0.01$ the largest maximum occurs at the second rather than the first maximum.

D. ANALYSIS - J. M. STEIN

This section will be devoted to summarizing the analytical results obtained by J. M. Stein in ORNL 630, ORNL 730 and ORNL 925 which are relevant to the previously discussed computations.

ORNL 630. Consider the case of no delayed neutrons and no pressurizer. Then

$$\dot{T} = S(P - P_o), \quad \dot{P} = \lambda_o\left[k - k_c\right]P \text{ and } k_c = k_{co} + \alpha(T - T_o),$$

where $T(0) = T_o$ and $P(0) = P_o$, form a complete system of equations defining the change of power and temperature with time. Upon eliminating time by dividing $\dot{P}$ by $\dot{T}$ one can integrate and obtain

$$P - P_o \log \frac{P}{P_o} - P_o = -\frac{\lambda_o \alpha}{2S}\left[T - T_o\right]\left[T - T_o - \frac{2e}{\alpha}\right]$$

The maximum temperature excursion is $T_o + \frac{2e}{\alpha}$ and the maximum and minimum power excursion which occur at $T = T_o + \frac{e}{\alpha}$ are given by

$$P_m - P_o \log \frac{P_m}{P_o} - P_o = \frac{\lambda_o e^2}{2\alpha S}$$

or approximately

$$P_{max} = \frac{\lambda_o e^2}{2\alpha S} \text{ and } -\log \frac{P_{min}}{P_o} - 1 = \frac{\lambda_o e^2}{2SP_o\alpha}$$

For very small values of e the oscillations are nearly sinusoidal with frequency $\frac{w_n}{2\pi} = \frac{1}{2\pi}\sqrt{\lambda_o \alpha S P_o}$. Figures 66, 67 and 68 exhibit the character of these oscillations, where $\lambda_o = 10^4$, $\alpha = 0.001$, $S = 0.00571$, $P_o = 1000$, $\frac{\delta k}{k} = \frac{e}{1.5}$, $\frac{w_n}{2\pi} = 1.2$.

If gas generation is taken into account the k required for criticality is given by $k_c = k_{co} + \alpha (T - T_o) + \epsilon (v_g - v_{go})$ and an equation for gas generation is given by $\dot{v}_g = r (P - P_o)$. Proceeding in the fashion of the above paragraph one can easily obtain that

$$P_m - P_o \log \frac{P_m}{P_o} - P_o = \frac{\lambda_o}{2(\alpha S + \epsilon r)} e^2$$

Figure 68 also exhibits for this case maximum P vs. e, where $\epsilon = 1$ and $r = 0.000004$.

The following rough calculation is used to give the order of magnitude of the pressure rise. Taking into account the thermal expansion and compressibility of the coolant, the density may be written as

$$\rho = \rho_o \left[1 - \gamma_1 (T - T_o) + \gamma_2 (p - p_o)\right]$$

The volume of the tank may be written as

$$V = V_o \left[1 + \gamma_3 (p - p_o)\right] ,$$

where the elastic expansion due to pressure is taken into account but the thermal expansion is neglected. The mass of coolant in the tank then becomes approximately

$$M = \rho V = \rho_o V_o \left[1 - \gamma_1 (T - T_o) + (\gamma_2 + \gamma_3) (p - p_o)\right] .$$

The continuity equation when coolant is allowed to escape becomes

$$\dot{M} = - \rho_o A U .$$

Consequently

$$U = \frac{V_o}{A}\left[\gamma_1 \dot{T} - (\gamma_2 + \gamma_3) \dot{p}\right] .$$

The equation of motion for the material in the relief pipe is

$$M_r \dot{U} = p - E_1 U^2 .$$

The condition for a maximum pressure is $\dot{p} = 0$ or $U = \frac{V_o}{A} \gamma_1 \dot{T}$. Therefore, if one assumes $\dot{U} = 0$ or since $\dot{T} = S (P - P_o)$ that $\dot{P} = 0$, i.e., where P is max-

imum, then

$$p = E_1 \left(\gamma_1 \frac{V_o}{A} \dot{T}\right)^2 = E_1 \left(\beta_1 \frac{V_o}{A} S\right)^2 (P_{max} - P_o)^2.$$

Figure 69 can then be constructed by using the previous power maximum and the following constants:

$\gamma_1 = 0.00153$, $S = 0.00571$, $V_o = 3060$ in.^{3}, $A = 1.765$ in.^{2}, $P_o = 1000$, $E_1 = 0.0143 \frac{\text{psi sec}^2}{\text{ft}^2}$

<u>ORNL 730</u>. Consider the following set of equations which include gas generation and a linear function of reactivity but does not include delayed neutrons:

$$\dot{P} = \lambda_o (k - k_c)P, \quad \dot{T} = S(P - P_o), \quad \dot{v}_g = r(P - P_o),$$

$$k_c = k_{co} + \alpha(T - T_o) + \epsilon (v_g - v_{go}) \text{ and } k = k_o + c_t t.$$

Let w be defined by

$$w = k - k_c = k_o - k_{co} + c_t t - \alpha(T - T_o) - \epsilon (v_g - v_{go}).$$

Using this new variable and eliminating time one can obtain

$$\frac{dP}{dw} = \frac{\lambda_o w P}{c_t - (\alpha S + \epsilon r)(P - P_o)}$$

The solution of this equation in the phase plane is

$$P - P_i - \left[P_o + \frac{c_t}{\alpha S + \epsilon r}\right] \log \frac{P}{P_i} = \frac{\lambda_o}{2(\alpha S + \epsilon r)} (w_i^2 + w^2)$$

where P_i and w_i represent initial values. From the equation for $\dot{P}$ it is clear that the maximum and minimum power occur when $w = 0$. To obtain P_{max} therefore set $w = 0$ in the phase plane solution. Figure 70 exhibits the phase plane solution. The solution represents an undamped oscillation which for small c_t and $\delta k = k_o - k_{co}$ approaches an harmonic oscillation of frequency $\frac{w_n}{2\pi} = \frac{1}{2\pi}\sqrt{(\alpha S + \epsilon r) P_o \lambda_o}$.

Figure 72 shows a family of curves with parameter $X_i = \frac{P_i}{P_o}$ where $X_{max} = \frac{P_{max}}{P_o}$ is plotted against δk and where $c_t = 0$. For small δk the maximum power is seen to be quite sensitive to initial power level but for large δk the maximum power level is almost independent of this level. Figure 73 shows a family of curves with parameter $X_i = \frac{P_i}{P_o}$ where X_{max} is plotted against $\gamma = \frac{c_t}{(\alpha S + \epsilon r) P_o}$ and where $\delta k = 0$. The values of the constants used in Figures 71 and 72 have been given above.

<u>ORNL 925</u>. Consider the following system of equations:

$$\dot{\phi} = \lambda_o \left[(1 - \beta) k - k_c\right] \phi + \sum_i \lambda_i c_i \quad (i = 1, \ldots, 6) \quad \dot{c}_i = - \lambda_i c_i + \beta_i \lambda_o k \phi$$

$$\dot{T} = S (P - P_o), \quad P = \gamma \lambda_o k \phi, \quad k_c = k_{co} + \alpha (T - T_o).$$

The solution of the equation for $\dot{c}_i$ is given by

$$c_i(t) = c_i(0) e^{-\lambda_i t} + \beta_i \lambda_o k e^{-\lambda_i t} \int_0^t e^{+\lambda_i \tau} \phi(\tau) d\tau, \text{ where}$$

$$c_i(0) = \frac{\beta_i}{\lambda_i} \lambda_o k \phi(0).$$

Using integration by part this becomes

$$c_i(t) = \frac{\beta_i}{\lambda_i} \lambda_o \left[k_o \phi(t) - k \int_{\phi(0)}^{\phi} e^{\lambda_i \left[t(\theta) - t(\phi)\right]} d\theta\right.$$

Substituting this equation in the equation for ϕ and writing the result in terms of P, one can obtain

$$\dot{P} = \lambda_o (k - k_c) P - \lambda_o k \sum_i \beta_i \int_{P_o}^{P} e^{\left[t(s) - t(P)\right]} ds,$$

where $\lambda_o \beta (k - k_o)$ is neglected as a second order quantity. Dividing this equation by the equation for $\dot{T}$, one has

$$\frac{dP}{dT} = \frac{\lambda_o}{S} (k - k_c) \frac{P}{P - P_o} - \frac{\lambda_o k}{S} \sum_i \beta_i \frac{\int_{P_o}^{P} e^{\lambda_i \left[t(s) - t(P)\right]} ds}{P - P_o}$$

The last term on the right hand side of this equation may be evaluated for various values of time and various values of loss of delayed neutrons in the

system external to the core. By replacing the integrands by their arithmetic means the term $\eta = \sum_i \beta_i \frac{\int_{P_o}^{P} e^{\lambda_i [t(s) - t(P)]}}{P - P_o}$ becomes $\eta = \sum_i \frac{\beta_i}{2}\left[1 + e^{-\lambda_i t(P)}\right]$. This last expression may be evaluated for various values of time and various values of loss of delayed neutrons in the system external to the core. For times up to 0.1 sec., i.e., through the maximum of the power surge for the reactor of the Homogeneous Reactor Experiment, this expression for η changes on the order of five percent. Hence, one may write

$$\frac{dP}{dT} = \frac{\lambda_o}{S}(k - k_c)\frac{P}{P - P_o} - \frac{\lambda_o k \eta}{S}$$

where η is considered as a constant.

Let $Y = \frac{\lambda_o}{w_n}\left[e - \alpha(T - T_o)\right]$, $X = \frac{P}{P_o} - 1$ and $K = \frac{\lambda_o}{w_n} k\eta$ then this last equation can be written as

$$\frac{dx}{dY} + \frac{Y}{X} + Y - K = 0.$$

The corresponding equation on a time basis is

$$\ddot{Y} + (K - Y)\dot{Y} + w_n^2 Y = 0.$$

Although the above first order non-linear equation cannot be solved by the usual analytical method, it may be conveniently investigated by the method of isoclines. Any curve along which $\frac{dX}{dY}$ is constant is called an isocline. A plot of isoclines and the resulting paths is shown in Figure 74. In general the motion starts with $X = 0$ and $Y = Y_o$ and rises to a maximum value of X. If then Y_o was not too large X and Y drop to zero in infinite time never having changed signs.

Three convenient approximations may be used with the equation $\frac{dx}{dY} + \frac{Y}{X} + Y - K = 0$ which lead to analytic solutions. First from Figure 73 it may be noted that Y does not change much from Y_o during the rise of X provided Y_o is fairly large. Consequently, with large e and short time a good approximate equation is given by $\frac{dX}{dY} = -Y - \frac{Y}{X} - K\frac{Y}{Y_o}$. This has the

solution

$$\frac{Y_o}{Y_o - K} X - \left[\frac{Y_o}{Y_o - K}\right]^2 \log\left[\frac{Y_o - K}{Y_o} X + 1\right] = \frac{Y_o^2 - Y^2}{2} .$$

Figure 71 exhibits this solution when $Y_o = 20.65$ and $\eta = 9.1$, i.e., $e = 0.014$. In the region of the maximum where $X \gg 1$ the equation may be approximated by $\frac{dX}{dY} = K - Y$. The solution of this equation is $X = X_{max} - \frac{1}{2}(K - Y)^2$. A useful approximation is therefore given by $X_{max} = \frac{1}{2}(K - Y_o)^2$, where $X = 0$ when $Y = Y_o$. Figure 75 is a plot of $X_{max} = X_{max} + 1$ versus e. The third approximation applies to the case of small δk or e. In this case the isocline $\frac{dX}{dY} = 0$ is an upper bound for the maximum X and therefore one can put $X_{max} = \frac{Y_o}{K - Y_o}$.

E. ANALYSIS

E1. Linearization. Although it is not the objective here to treat linearized equations in any detail, it does seem worthwhile to note the method and to indicate the conditions for stability in the linear case for the two main complete systems considered in this report. Consider first the system formed from equations (6) through (9) subject to the usual initial conditions which were the basis for the SEAC calculations. The equilibrium conditions are found by letting the right hand sides of equations (6) through (9) be equal to zero. These conditions are $x_E = 1$, $v_E = 0$, $z_E = \frac{\delta}{\tau}$, $y_E = \frac{\delta}{\tau w_n^2}$. Assuming that $\frac{\delta}{\tau}$ and $\frac{\delta}{\tau}\frac{1}{w_n^2}$ are small, letting $\bar{x} = x - 1$, $\bar{v} = v$, $\bar{z} = z + \frac{\delta}{\tau}$ and $\bar{y} = y - \frac{\delta}{\tau}\frac{1}{w_n^2}$, and keeping only the linear terms, one obtains

$$\dot{\bar{y}} = \bar{x}$$

$$\dot{\bar{z}} = -\bar{v}$$

$$\dot{\bar{x}} = \bar{z} - \frac{\beta}{\tau}\bar{x}$$

$$\dot{\bar{v}} = -\alpha_1\bar{v} + w_H^2(\bar{z} + w_n^2\bar{y})$$

The conditions for the stability of this system of equations about the origin ($\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, $\bar{v} = 0$) may be determined in various ways. Whether one assumes a solution of the form $e^{\gamma t}$, uses the Laplace transform with parameter or uses some other technique, the problem resolves itself into finding the solutions of the equation

$$\gamma^4 + \left(\frac{\beta}{\tau} + \alpha_1\right)\gamma^3 + \left(\frac{\beta}{\tau}\alpha_1 + w_H^2\right)\gamma^2 + \frac{\beta}{\tau} w_H^2 \gamma + w_H^2 w_n^2 = 0$$

If all the solutions of this equation have negative real parts, then the system can be considered stable. The Nyquist technique of mapping by means of a polynomial gives a quick way of obtaining the conditions for no roots with positive real parts. The conditions are

$$\sqrt{(b)^2 - 4(d)} > 2\frac{(c)}{(a)} - (b) > -\sqrt{(b)^2 - 4(d)} \text{ and } (b)^2 > 4(d)$$

where $(a) = \frac{\beta}{\tau} + \alpha_1 > 0$, $(b) = \frac{\beta}{\tau}\alpha_1 + w_H^2 > 0$, $(c) = \frac{\beta}{\tau} w_H^2 > 0$ and $(d) = w_H^2 w_n^2 > 0$. For the Homogeneous Reactor if one lets $w_n^2 = 50$, $w_H^2 = 2.5\ (10^4)$, $\beta = 0.0075$ and $\tau = 0.0001$ then these conditions are not satisfied if $\alpha_1 \leqq 0.1$ but are satisfied if $\alpha_1 \geqq 0.2$.

If α_1 and β are set equal to zero, the case of no friction in the pressurizer and no delayed neutrons, then the solutions are purely oscillatory provided $w_H^2 > 4\, w_n^2$. If $\frac{1}{w_H^2}$ is set equal to zero then the fourth degree polynomial in γ becomes $\gamma^2 + \frac{\beta}{\tau}\gamma + w_n^2 = 0$ and the pressurizer system has been uncoupled.

In the case of the system $\dot{T} = S(P - P_o)$ and $\dot{P} = \frac{1}{\tau}\left[\delta - \beta - \alpha(T - T_o)\right]P + \frac{\beta}{\tau}P_o$ the equilibrium conditions are $x = \frac{P}{P_o} = 1$ and $y = \frac{1}{SP_o}(T - T_o) - \frac{\delta}{\tau w_n^2}$ Letting $\bar{x} = x - 1$ and $\bar{y} = y + \frac{\delta}{\tau}\frac{1}{w_H^2}$ and keeping only the linear terms, one obtains $\dot{\bar{y}} = \bar{x}$ and $\dot{\bar{x}} = \frac{\beta}{\tau}\bar{x} - w_n^2\bar{y}$. The conditions for stability of this system about the origin ($\bar{x} = 0$, $\bar{y} = 0$) depends as expected upon the solution of the equation

$$\gamma^2 + \frac{\beta}{\tau}\gamma + w_n^2 = 0.$$

Since $2\gamma = -\frac{\beta}{\tau} \pm \sqrt{\left(\frac{\beta}{\tau}\right)^2 - 4w_n^2}$, the solutions all have negative real parts provided $\frac{\beta}{\tau} > 0$ and $w_n^2 > 0$. This system is therefore stable for small oscillations.

E2. Iterative Formulas. It would be extremely desirable to have useful analytical solutions to the non-linear systems considered here. Unfortunately, the best one can aim for are analytical approximations to the solutions. Approximations generally apply only to certain ranges of the parameters and variables, and for this reason caution must be used both in obtaining and using approximations. The technique for obtaining approximation formulas in this portion has the dubious advantage that it could probably be shown to be a mathematically rigorous technique. On the other hand the formulas are not easy to use.

Consider again equations (6) through (9) along with their initial conditions. A first approximation for x may be obtained, since $z(0) = 0$, by setting $z(t) = 0$. Solving equation (8) gives

$$x_1 = \frac{1}{\delta - \beta}\left[\delta e^{\frac{\delta - \beta}{\tau} t} - \beta\right]$$

Using equation (6) one can then obtain

$$y_1 = \frac{\delta}{\delta - \beta}\left[\frac{\tau}{\delta - \beta}\left(e^{\frac{\delta - \beta}{\tau}} - 1\right) - t\right].$$

Assume next that α_1 and α_2 are small enough that the terms $\alpha_1 v$ and $\alpha_2 v|v|$ may be neglected in a first approximation. Actually these terms are not considered in the next approximation, but they could be. Equations (7) and (9) give then

$$\ddot{z} + w_H^2 z = - w_H^2 w_n^2 y.$$

Substituting y, in the right hand side of this equation leads to

$$z_1 = A_1 \cos w_H t + \beta_1 \sin w_H t + D_1 + E_1 t + F_1 e^{\Psi t}$$

where

$$\Psi = \frac{\delta - \beta}{\tau}, \quad A_1 = -(D_1 + F_1), \quad \beta_1 = -\frac{1}{w_H}(E_1 + \Psi F_1)$$

$$D_1 = \frac{w_n^2 \delta}{\tau \Psi^2}, \quad E_1 = \frac{\delta}{\Psi \tau} w_n^2, \quad F_1 = -\frac{\delta}{\tau \Psi^2} w_H^2 w_n^2 \left[\Psi^2 + w_H^2\right]^{-1}$$

If desired v_1 can now be obtained from equation (7) and $\bar{p}$ can be obtained from $z_1 + w_n^2 y_1$.

It is apparent that the above first approximation can be useful only for the early rise in x before the z has any importance. The next step is to use this first approximation in order to obtain a second and better approximation. The simplest way to do this is to put x_1 and z_1 into the right hand side of equation (8) and then integrate. This gives

$$x_2 = f_1 + f_2 t + f_3 t^2 + f_5 e^{\Psi t} + f_6 e^{2\Psi t} + f_7 t e^{\Psi t}$$

where $f_1 = -\left[f_5 + f_6 - 1\right]$, $f_2 = -\frac{\beta}{\delta} f_7$, $f_3 = -\frac{\Psi \beta}{2} f_7$,

$$f_5 = \left[\frac{\delta}{\tau} + \frac{\beta \delta}{\tau^2 \Psi^3} w_H^2 w_n^2 \frac{1}{\Psi^2 + w_H^2}\right] \frac{1}{\Psi}, \quad f_6 = -w_n^2 \frac{w_H^2}{2} \frac{\delta^2}{\tau^2 \Psi^4}$$

$$f_7 = w_n^2 \frac{\delta^2}{\tau^2 \Psi^3}$$

Letting $y_2 = a_1 + a_2 t + a_3 t^2 + a_4 t^3 + a_5 e^{\Psi t} + a_6 e^{2\Psi t} + a_7 t e^{\Psi t}$ and

$$z_2 = w_n^2 (c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{\Psi t} + c_6 e^{2\Psi t} + c_7 t e^{\Psi t} + A_2 \cos w_H t + B_2 \sin w_H t)$$

one can then show using the equations $\dot{y} = x - 1$ and $\ddot{z} + w_H^2 z = -w_H^2 w_n^2 y$ that

$$a_6 = \frac{1}{2\Psi} f_6, \quad a_7 = \frac{1}{\Psi} f_7, \quad c_1 = \frac{2}{w_H^2} a_3 - a_1, \quad c_2 = \frac{6}{w_H^2} a_4 - a_2, \quad c_3 = - a_3,$$

$$c_4 = - a_4, \quad c_5 = - \frac{w_H^2 \; a_5}{\left[w_H^2 + \Psi^2\right]} + \frac{2\Psi w_H^2 \; a_7}{\left[w_H^2 + \Psi^2\right]^2}, \quad c_6 = \frac{w_H^2 \; a_6}{4\Psi^2 + w_H^2},$$

$$c_7 = - \frac{w_H^2 \; a_7}{w_H^2 + \Psi^2}$$

$$A_2 = -\left[c_1 + c_5 + c_6\right], \quad B = - \frac{1}{w_H}\left[c_2 + \Psi c_5 + 2\Psi c_6 + c_7\right].$$

Note again that no friction in the pressurizer has been taken into account by these formulas, although formulas with friction terms could easily be derived. Again v_2 can be obtained from equation (7) and $\bar{p}_2$ can be obtained from $\bar{p}_2 = z_2 + w_n^2 y_2$.

The above formulas for x, y, z, v, and $\bar{p}$ were first evaluated using the constants that were used with the SEAC computations. The formulas with subscript 1 lead to numbers which are close to the SEAC numbers only during the time from the beginning of the x or power rise to about half the total x rise. The formulas with subscript 2 lead to numbers which are close to the SEAC numbers to times slightly beyond the maximum x. This time proves to be sufficient to also obtain maximum $\bar{p}$ values. The SEAC results and the formula results are compared in an overall fashion in Figures 61, 62 and 65.

Using the above formulas Figures 76 and 77 exhibit the result of changing β, the percentage of delayed neutrons, on the maximum of x versus δ and the time of maximum x versus δ.

These formulas have also been used for calculating such things as power and pressure maximums for intermediate and large homogeneous reactors. Table 6 indicates the constants used and the values of maximum power and pressure for each reactor with various δ's. Table 7 indicates the corresponding time of power maximums. Figures 82 through 88 exhibit the results for the six foot diameter in further detail. Figures 89, 90 and 91 exhibit the results indicated in Table 6 and 7.

It is interesting to compare the curves of pressure versus time as given in Figures 87 and 88. For $\delta < \beta$, as represented in Figure 87, the pressure reaches its maximum on its first upswing. For $\delta > \beta$, as represented in Figure 88, the pressure reaches only a relative maximum on its first upswing, the actual maximum being reached on the third upswing. The time of maximum pressure is determined largely by the time of maximum $\dot{x}$. For $\delta < \beta$, the time of maximum $\dot{x}$ is very small, but for $\delta > \beta$ the time of maximum $\dot{x}$ is appreciable.

Table 2 contains the iterative formulas.

It is possible, of course, to use the formulas with subscript 2 in order to obtain still better formulas. This seems to be unnecessary for sufficient accuracy and undesirable for evaluational purposes. One may note, however, that another iteration would extend the time of validity of such formulas. For $\delta > \beta$ the formula for x_1 indicates through the term with $e^{\Psi t}$ the exponential rise in power while the term $e^{2\Psi t}$ in the formula for x_2 indicates the decline in power due to the temperature coefficient.

The solution of the system of equations $\dot{T} = S(P - P_o)$ and $\dot{P} = \frac{1}{\tau}\left[\delta - \beta - \alpha(T - T_o)\right] + \frac{\beta}{\tau} P_o$ may be approximated in the same fashion. In the case one has again $x_1 = \frac{1}{\delta - \beta}\left[\delta e^{\Psi t} - \beta\right]$ and $y_1 = \frac{\delta}{\delta - \beta}\left[\frac{\tau}{\delta - \beta}(e^{\Psi t} - 1) - t\right]$. The second approximation formulas become:

$$x_2 = g_1 + g_2 t + g_3 t^2 + g_5 e^{\Psi t} + g_6 e^{2\Psi t} + g_7 t e^{\Psi t}$$

and

$$y_2 = b_1 + b_2 t + b_3 t^2 + b_4 t^3 + b_5 e^{\Psi t} + b_6 e^{2\Psi t} + b_7 t e^{\Psi t},$$

where

$$g_1 = -\left[g_5 + g_6 - 1\right], \quad g_2 = -\frac{\beta}{\delta} g_7, \quad g_3 = -\frac{\Psi\beta}{2\delta} f_7$$

$$g_5 = \left[\frac{\delta}{\tau} + \frac{\delta\beta w_n^2}{\tau^2\Psi^3}\right]\frac{1}{\Psi}, \quad g_6 = -\frac{\delta^2 w_n^2}{2\Psi^4\tau^2}, \quad g_7 = \frac{\delta^2 w_n^2}{\tau^2\Psi^3},$$

$$b_1 = -\left[b_5 + b_6\right], \quad b_2 = f_1 - 1, \quad b_3 = \frac{1}{2} f_2, \quad b_4 = \frac{1}{3} f_3$$

$$b_5 = \frac{1}{\Psi}\left[g_5 - \frac{1}{\Psi} g_7\right], \quad b_6 = \frac{1}{2\Psi} g_6, \quad b_7 = \frac{1}{\Psi} g_7.$$

These formulas are very much like the previous formulas for x_2 and y_2. The only difference occurs through the factor $w_H^2\left[w_H^2 + \Psi^2\right]^{-1} = \left[1 + \frac{\Psi^2}{w_H^2}\right]^{-1}$.

In cases then where $|\Psi| \ll w_H$, i.e., $\frac{\delta - \beta}{\tau}$ is much smaller than the pressurizer frequency, the pressurizer may be neglected in calculating power.

<u>E3. Short Formulas</u>. The results of this portion are primarily due to T. A. Welton. Consider again equations (1) through (5). Upon differentiating on both sides of equation (3) and making use of equations (4) and (5), equations (1) through (5) are seen to be equivalent to

$$\dot{P} = \frac{1}{\tau}\left[(\quad - \beta) + \frac{\partial k}{\partial \rho}(\rho - \rho_o)\right] P + \frac{\beta}{\tau} P_o$$

$$\dot{T} = S(P - P_o)$$

$$\ddot{\rho} + \alpha\dot{\rho} + w_H^2(\rho - \rho_o) = -w_H^2\left|\left(\frac{\rho}{T}\right)\right|(T - T_o).$$

The quantities $\rho - \rho_o$ and $T - T_o$ are more convenient to work with than ρ and T. Keeping in mind that ρ and T are actually $\rho - \rho_o$ and $T - T_o$, the following equation may be used:

$$\dot{P} = \frac{1}{\tau}\left[(\delta - \beta) + \left(\frac{\partial k}{\partial \rho}\right)\rho\right] P + \frac{\beta}{\tau} P_o$$

$$\dot{T} = S(P - P_o)$$

$$\ddot{\rho} + \alpha_1 \dot{\rho} + w_H^2 \rho = - w_H^2 \left(\frac{\partial \rho}{\partial T}\right)\Big| T.$$

The objective of this part is to obtain approximate solutions or formulas which are easily evaluated for the above system subject to certain initial or external conditions. The numbers resulting from these formulas may then be compared with the SEAC calculations.

The following three types of initial or external conditions are considered:

(A) $P = P_o$ up to time $t = 0$, at which time a reactivity step δ is introduced.

(B) P = constant up to time $t = 0$, at which time a step in P is introduced, i.e., $P = P_o$ for $t < 0$, $P(0) = P_o - \pi$.

(C) δ is a linear function of time passing through zero at $t = 0$.

Besides these conditions, the following two simplifications are used:

(1) $\beta = 0$, i.e., no delayed neutrons

(2) $\frac{1}{w_H^2} = 0$, i.e., uncoupled pressurizer.

Consider first the case where conditions <u>(1) and (2) apply,</u> i.e., $\beta = 0$ and $\frac{1}{w_H^2} = 0$. The equations (1) become

$$\dot{P} = \frac{1}{\tau}\left[\delta + \left(\frac{\partial k}{\partial \rho}\right)\rho\right] P, \quad \dot{T} = S(P - P_o), \quad \rho = -\left|\left(\frac{\partial \rho}{\partial T}\right)\right| T.$$

Letting $Q = \log \frac{P}{P_o}$ and substituting for ρ, one can obtain

$$\dot{Q} = \frac{1}{\tau}\left[\delta - \alpha T\right. , \quad \dot{T} = SP_o(e^Q - 1)$$

(11) or

$$\ddot{Q} = - w_n^2 \ (e^Q - 1) + \frac{1}{\tau} \ \dot{\delta}$$

The (A) condition may be written as: $Q(0) = 0$, $\dot{Q}(0) = \frac{\delta}{\tau}$, $\dot{\delta} = 0$.
The (B) condition becomes: $Q(0) = \log\ (1 - \frac{\pi}{P_o})$, $\dot{Q}(0) = 0$, $\dot{\delta} = 0$.
Condition (C) is determined by setting $\dot{\delta}$ equal to an appropriate value as well as choosing $Q(0)$ and $\dot{Q}(0)$ in some manner yet to be decided.

For all three cases the energy argument may be used effectively. Equation (11) has the form

$$\frac{d}{dt}\left[\frac{\dot{Q}^2}{2} + w_n^2 e^Q - w_n^2 \ (1 + \frac{\dot{\delta}}{\tau w_n^2})\ Q\right] = 0$$

Consequently, if the quantity $E(Q, \dot{Q}) = \dot{Q}^2/2 + w_n^2 e^Q - w_n^2 \ (1 + \frac{\dot{\delta}}{\tau w_n^2})Q$ is evaluated from the assumed initial conditions, its value remains the same thereafter. In particular the maximum power P can be obtained by putting $\dot{P}$ or $\dot{Q}$ equal to zero. For case (A) one has

$$e^Q - Q = \frac{\delta^2}{2w_n^2\tau^2} + 1$$

or

$$\frac{P}{P_o} - \log\frac{P}{P_o} - 1 = \frac{\delta^2}{2w_n^2\tau^2}$$

This formula for power extremum was obtained previously by J. M. Stein in ORNL 630 (see page 7 in this report) by another method. For large power surges e is much larger than Q and 1, and therefore

$$P_{max} = P_o e^Q = \frac{\delta^2 P_o}{2w_n^2\tau^2}$$

Since the quantity w_n^2 is proportional to P_o, P_{max} is actually independent of P_o.

For case (B) the power extremums may be obtained from

$$e^Q - Q = 1 - \frac{\pi}{P_o} - \log\left(1 - \frac{\pi}{P_o}\right)$$

or

$$\frac{P}{P_o} - \log\frac{P}{P_o} = 1 - \frac{\pi}{P_o} - \log\left(1 - \frac{\pi}{P_o}\right)$$

For large power surges the power maximum is given by

$$P_{max} = P_o e^Q = P_o - \pi - P_o \log\left(1 - \frac{\pi}{P_o}\right).$$

For case (C) one puts $\dot{\delta} = d$, a constant. Since the initial conditions have not yet been specified many solutions of this type are possible. However, there exists one solution which is steady. By putting

$$w_n^2 (e^Q - 1) = \frac{1}{\tau} d \text{ and } \dot{Q} = 0$$

at $t = 0$ one has, since $\ddot{Q}$ is then zero at $t = 0$, that $w_n^2 (e^Q - 1) = \frac{1}{\tau} d$ for all $t > 0$. Therefore

$$P = P_o \left(1 + \frac{d}{w_n^2 \tau}\right)$$

where P_o was, and remains, the power being removed. Therefore, such a steady rise of reactivity will give a power which is steadily above the power being removed.

In general, one can obtain power maximums from

$$w_n^2 e^Q - w_n^2 \left(1 + \frac{d}{\tau w_n^2}\right)Q = \frac{1}{2}\dot{Q}_i^2 + w_n^2 e^{Q_i} - w_n^2 \left(1 + \frac{d}{\tau w_n^2}\right)Q_i,$$

or

$$\frac{P}{P_o} - \left(1 + \frac{d}{\tau w_n^2}\right) \log\frac{P}{P_o} = \frac{1}{2w_n^2}\left(\frac{P_i}{P_i}\right)^2 + \frac{P_i}{P_o} - \left(1 + \frac{d}{\tau w_n^2}\right) \log\frac{P_i}{P_o}$$

This, of course, includes all of the above special cases. In general, the plot of power versus time can be described as a conservative non-linear oscillation about a steadily changing equilibrium point.

Although the above method of obtaining these formulas is quite satisfactory when $\beta = 0$, it is not applicable to the case when $\beta \neq 0$. The

above cases will therefore be investigated with another method. Since in case (A) $\dot{T}(0) = 0$ and $T(0) = 0$, T must start off at least as t^2. As Q or P increases $\dot{T}$ becomes rapidly larger and the net excess reactivity $\delta - \alpha T$ must appear about like that in Figure 78. The excess reactivity is approximately equal to the constant δ for a time and then rather suddenly drops to zero at a time $t = \Delta$. At this time the power is a maximum since $\dot{P}$ is then zero. The reactivity will therefore be approximated by a constant up to time Δ and then a perpendicular drop to zero. Up to time Δ, one has then $Q = \frac{\delta}{\tau} t$ and consequently $\dot{T} = SP_o\,(e^{\frac{\delta}{\tau}t} - 1)$. Since for large $\frac{\delta}{\tau}$, 1 may be neglected in comparison with $e^{\frac{\delta}{\tau}t}$, it follows that $T = \frac{SP_o\tau}{\delta} e^{\frac{\delta}{\tau}t}$. Consequently, the excess reactivity becomes $\delta - \frac{\tau^2}{\delta} w_n^2\, e^{\frac{\delta}{\tau}t}$. Δ may now be determined by setting the excess reactivity equal to zero, i.e., $\delta - \frac{\tau^2}{\delta} w_n^2\, e^{\frac{\delta}{\tau}\Delta} = 0$. The maximum power in turn becomes

$$P_{max} = P_o e^{\frac{\delta}{\tau}\Delta} = \frac{\delta^2 P_o}{\tau^2 w_n^2}$$

This approximation is seen to overestimate the previous approximate formula by a factor of about two.

For case (B) the initial conditions are $Q(0) = \log\,(1 - \frac{\pi}{P_o})$, $\dot{Q}(0) = 0$, $T(0) = 0$, $\dot{T}(0) = -\pi S$ and $\delta - \alpha T = \delta = 0$. Since $\dot{T} = S(P - P_o) = -\pi S$, the temperature at first drops linearly going as $T = -\pi St$ and the system therefore has a steadily increasing excess reactivity. Consequently, $\dot{Q} = + w_n^2 \frac{\pi}{P_o} t$ and therefore $Q = \log\,(1 - \frac{\pi}{P_o}) + \frac{\pi}{P_o} \frac{w_n^2 t^2}{2}$. An improved expression for T can be derived by substituting this result in $\dot{T}$ and obtaining $\dot{T} = SP_o\left[(1 - \frac{\pi}{P_o})\, e^{w_n^2 \frac{\pi}{P_o} \frac{t^2}{2}} - 1\right]$. To derive a simple expression, let $\frac{\pi}{P_o}$ be nearly unity so that the exponential term will not outweigh the constant term until large values of the exponent are reached. For large exponents a fairly accurate integral is given by

$$T = SP_o t\left[\frac{P_o}{\pi}\frac{1}{w_n^2 t^2}\left(1 - \frac{\pi}{P_o}\right)e^{\frac{\pi}{P_o} w_n^2 \frac{t^2}{2}} - 1\right]$$

The excess reactivity is therefore zero at a time Δ defined by

$$e^{\frac{\pi}{P_o} w_n^2 \frac{\Delta^2}{2}} = \frac{2}{1 - \frac{\pi}{P_o}}\frac{\pi}{P_o}\frac{w_n^2\Delta^2}{2}$$

or $e^{\sigma} = \gamma\sigma$ where $\gamma = \dfrac{2}{1 - \frac{\pi}{P_o}}$ and $\sigma = \dfrac{\pi}{P_o}\dfrac{w_n^2\Delta^2}{2}$.

With γ much greater than one the root may be found by iteration, i.e., since $z = \log + \log z$ one lets $z_1 = \log \alpha$, $z_2 = \log \alpha + \log\log \alpha$, $z_3 = \log \alpha + \log(\log \alpha + \log\log \alpha)$, etc. Since the first iteration gives the uninteresting result $P = 2P_o$ and the third iteration is probably more accurate than is justified by the crudeness of the other approximations, the second iteration formula will be used. The peak power then becomes

$$P = P_o\left(1 - \frac{\pi}{P_o}\right)e^{\sigma} = P_o\left(1 - \frac{\pi}{P_o}\right)\frac{2}{1 - \frac{\pi}{P_o}}\log\frac{2}{1 - \frac{\pi}{P_o}} = 2P_o \log\frac{2}{1 - \frac{\pi}{P_o}}$$

Comparing this with the previous formula for case (B), this formula differs by being about a factor of two too large. Since this was also the situation in case A one can adopt the procedure of solving the e act equation by these approximate methods and finally dividing all peak powers by the "fudge" or "experimental" factor of two.

In case (C) a simple argument may be used to obtain the previous result. Corresponding to P a constant there must be a linear increasing temperature just canceling the applied reactivity change, i.e. $\dot{\delta} = \alpha\dot{T}$. In

order to maintain this steady increase of temperature, the difference between the reactor power and the power removed must be such that $S(P - P_o) = \dot{T} = \frac{\delta}{\alpha}$. Therefore $P = P_o\left(1 + \frac{\delta}{w_n^2\tau}\right)$, the same formula previously obtained.

Consider now the case where condition (1) applies but condition (2) no longer applies, i.e., $\beta = 0$ and $\frac{1}{w_H^2} \neq 0$. For case (A) the initial conditions are $\dot{Q}(0) = \frac{\delta}{\tau}$, $Q(0) = 0$, $T(0) = 0$, $\rho(0) = 0$ and $\dot{\rho}(0) = 0$. Just as before the excess reactivity may be taken as equal to δ up to time Δ. Therefore as before $T = \frac{SP_o\tau}{\delta} e^{\frac{\delta}{\tau}t}$. The density ρ can now be calculated from $\left[\frac{1}{w_H^2}\frac{d^2}{dt^2} + \frac{\alpha}{w_H^2}\frac{d}{dt} + 1\right]\rho = \left(\frac{\partial\rho}{\partial T}\right)T = \left(\frac{\partial\rho}{\partial T}\right)\frac{SP_o\tau}{\delta} e^{\frac{\delta}{\tau}t}$.

For large $\frac{\delta}{\tau}t$ one has $\rho = \dfrac{\frac{SP_o\tau}{\delta}\left(\frac{\partial\rho}{\partial T}\right)e^{\frac{\delta}{\tau}t}}{1 + \frac{\alpha\delta}{w_H^2\tau} + \frac{\delta^2}{w_H^2\tau^2}}$,

the damped oscillatory terms being considered negligible. The time Δ for which the excess reactivity becomes zero can be found from

$$\frac{\frac{w_n^2\tau^2}{\delta} P_o e^{\frac{\delta}{\tau}\Delta}}{1 + \frac{\alpha\delta}{w_H^2\tau} + \frac{\delta^2}{w_H^2\tau^2}} - \delta = 0$$

Peak power in turn is given then by

$$P = P_o e^{\frac{\delta}{\tau}\Delta} = \frac{\delta^2 P_o}{w_n^2\tau^2}\left(1 + \frac{\alpha\delta}{w_H^2\tau} + \frac{\delta^2}{w_H^2\tau^2}\right)$$ and upon dividing by the

" experimental " factor two by

$$P = \frac{\delta^2 P_o}{2w_n^2\tau^2}\left(1 + \frac{\alpha\delta}{w_H^2\tau} + \frac{\delta^2}{w_H^2\tau^2}\right).$$

Comparing this with the case where $\frac{1}{w_H^2} = 0$ one can see that this result differs only in the factor $(1 + \frac{\alpha\delta}{w_H^2\tau} + \frac{\delta^2}{w_H^2\tau^2})$. This factor tends, as it should, to one as $\frac{1}{w_H^2}$ tends to zero. The physical origin of this increase in peak power lies in the delay introduced in the expansion process by the inertia and friction of the fluid pipe leading to the pressurizer. The relative effects of inertia and friction may be compared by comparing $\frac{\delta}{\tau}$ and α.

For case (B) the initial conditions are $T(0) = 0$, $\rho(0) = 0$, $\dot{\rho}(0) = 0$, $Q(0) = \log(1 - \frac{\pi}{P_o})$, $\dot{Q}(0) = 0$ and $\dot{T} = -\pi S$. Consequently,

$$\left[\frac{1}{w_H^2}\frac{d^2}{dt^2} + \frac{\alpha}{w_H^2}\frac{d}{dt} + 1\right]\rho = -S\pi\left(\frac{\partial\rho}{\partial T}\right)t.$$

Since the change in temperature is slow, density changes are slow and the friction and inertia terms will be neglected, i.e., let $\rho = -S\pi\left(\frac{\partial\rho}{\partial T}\right)t$. This leads to the same formula for power and temperature as a function of time as those obtained when $\frac{1}{w_H^2} = 0$ and in particular

$$T = SP_o\left[\frac{1}{\frac{\pi}{P_o}w_n^2 t}\left(1 - \frac{\pi}{P_o}\right)e^{\frac{\pi}{P_o}\frac{w_n^2t^2}{2}} - t\right]$$

Substituting this value of T into the equation $\frac{1}{w_H^2}\ddot{\rho} + \frac{\alpha}{w_H^2}\dot{\rho} + \rho = \left(\frac{\partial\rho}{\partial T}\right)T$

one can obtain as an approximation

$$\rho = -SP_o\left(\frac{\partial\rho}{\partial T}\right)\left[t - \frac{1 - \frac{\pi}{P_o}}{\frac{\pi}{P_o}w_n^2 t}\,\frac{e^{\frac{\pi}{P_o}\frac{w_n^2t^2}{2}}}{1 + \frac{\alpha}{w_H^2}\frac{\pi}{P_o}w_n^2 t + \frac{1}{w_H^2}\left(\frac{\pi}{P_o}w_n^2 t\right)^2}\right.$$

where as before the oscillatory terms are neglected. When $\rho = 0$, $t = \Delta$. Letting $\sigma = \frac{\pi}{P_o}\frac{w_n^2\Delta^2}{2}$, Δ may then be found from

$$e^z = \frac{2z}{1 - \frac{\pi}{P_o}} \left[1 + \frac{\alpha}{w_H^2} \sqrt{\frac{\pi}{P_o} w_n^2 z} + \frac{1}{w_H^2} \frac{\pi}{P_o} w_n^2 z \right].$$

With the previously used iteration method one has $z_1 = \log \frac{2}{1 - \frac{\pi}{P_o}}$ and

$$z_2 = \log \frac{2}{1 - \frac{\pi}{P_o}} + \log \log \frac{2}{1 - \frac{\pi}{P_o}} + \log \left[1 + \frac{\alpha}{w_H^2} \sqrt{\frac{2\pi}{P_o} w_n^2 \log \frac{2}{1 - \frac{\pi}{P_o}}} \right.$$

$$\left. + \frac{w_n^2}{w_H^2} \frac{\pi}{P_o} \log \frac{2}{1 - \frac{\pi}{P_o}} \right]$$

Dividing by two the "experimental" number the peak power becomes

$$P = \frac{1}{2} P_o \left(1 - \frac{\pi}{P_o}\right) e^z = P_o \log \frac{2}{1 - \frac{\pi}{P_o}} \left(1 + \frac{\alpha}{w_H^2} \sqrt{\frac{2\pi}{P_o} w_n^2 \log \frac{2}{1 - \frac{\pi}{P_o}}} \right.$$

$$\left. + \frac{w_n^2}{w_H^2} \frac{\pi}{P_o} \log \frac{2}{1 - \frac{\pi}{P_o}} \right)$$

As expected this formula differs from the previous formula for case (B) only by a factor which tends to one as $\frac{1}{w_H^2}$ tends to zero.

For case (C) the condition of steady power is satisfied if the excess reactivity $\delta + \left(\frac{\partial k}{\partial \rho}\right)\rho$ is equal to zero. It follows that $-\left(\frac{\partial k}{\partial \rho}\right) \mathring{\rho} = \mathring{\delta} = d$. Since

$$\left[\frac{1}{w_H} \frac{d^2}{dt^2} + \frac{\alpha}{w_H^2} \frac{d}{dt} + 1 \right] \mathring{\rho} = \left(\frac{\partial \rho}{\partial T}\right) \mathring{T}$$

$\mathring{T}$ must also be constant and be given by $\mathring{T} = \frac{d}{\alpha}$. As before one has then $P = P_o \left(1 + \frac{\delta}{w_n^2 \tau}\right)$. That this result should be the same as before follows from the fact that the steady pressurizer motion has no finite frequency.

When condition (1) does not apply, i.e. $\beta \neq 0$, the obtaining of simple formulas for maximum powers becomes much more difficult. Formulas only for case (A) will be derived here, although undoubtedly formulas for case (B) and (C) could also be derived. Starting with the equations $\dot{T} = S(P - P_o)$ and $\dot{P} = \frac{1}{\tau}\left[\delta - \beta - \alpha T\right] P + \frac{\beta}{\tau} P_o$ one obtains by approximating the reactivity by $\delta - \beta$ up to time Δ that

$$P = \left(\frac{\delta}{\delta - \beta} e^{\frac{\delta - \beta}{\tau} t} - \frac{\beta}{\delta - \beta}\right) P_o, \quad T = SP_o \frac{\delta}{\delta - \beta}\left[\frac{\tau}{\delta - \beta}\left(e^{\frac{\delta - \beta}{\tau} t} - 1\right) - t\right].$$

Setting $\dot{P}$ equal to zero and substituting for T

$$\left[\delta - \beta - \frac{w_n^2 \delta \tau}{\delta - \beta}\left(\frac{\tau}{\delta - \beta} e^{\frac{\delta - \beta}{\tau}\Delta} - \frac{\tau}{\delta - \beta} - \Delta\right)\right] + \beta \frac{P_o}{P} = \left[\delta - \beta - \frac{w_n^2 \tau^2}{\delta - \beta}\left(\frac{P}{P_o} - 1 - \frac{\delta}{\tau}\Delta\right)\right]$$

$$+ \beta \frac{P_o}{P} = 0 \text{ . Therefore}$$

$$\frac{P}{P_o} = 1 + \frac{\delta}{\tau}\Delta + \frac{(\delta - \beta)^2}{w_n^2 \tau^2} + \frac{\beta(\delta - \beta)}{w_n^2 \tau^2} \frac{P_o}{P} .$$

The above equation for P leads to $\frac{\delta}{\tau} t = \frac{\delta}{\delta - \beta} \log\left(\frac{\delta - \beta}{\delta} \frac{P}{P_o} + \frac{\beta}{\delta}\right)$

Using the experimental factor of two the formula for obtaining maximum power becomes

$$\frac{P}{P_o} = 1 + \frac{(\delta - \beta)^2}{2w_n^2 \tau^2} + \frac{\delta}{\delta - \beta} \log\left(\frac{\delta - \beta}{\delta} \frac{P}{P_o} + \frac{\beta}{\delta} + \frac{\beta(\delta - \beta)}{2w_n^2 \tau^2} \frac{P_o}{P}\right) .$$

This formula with $\beta = 0$ reduces to the previously obtained formula for the case $\beta = 0$. As δ increases first the term in $\frac{P_o}{P}$ and then the logarithm term may be neglected.

Proceeding in the same fashion for the case $\frac{1}{w_H^2} \neq 0$, the first equations for P and T remain the same. The density ρ can now be determined

from the equation

$$\ddot{\rho} + w_H^2\rho = \left(\frac{\partial\rho}{\partial T}\right)T = \left(\frac{\partial\rho}{\partial T}\right)SP_o \frac{\delta}{\delta-\beta}\left[\frac{\tau}{\delta-\beta}\left(e^{\frac{\delta-\beta}{\tau}t} - 1\right) - t\right].$$

One obtains

$$\frac{1}{\tau}\left(\frac{\partial k}{\partial\rho}\right)\rho = z = A_1 \cos w_H t + B_1 \sin w_H t + D_1 + E_1 t + F_1 e^{\frac{\delta-\beta}{\tau}t},$$

where $A_1 = -(D_1 + F_1)$, $B_1 = \frac{1}{w_H}\left(E_1 + \frac{\delta-\beta}{\tau}F_1\right)$, $D_1 = \frac{w_n^2\tau^2}{(\delta-\beta)^2}$

$$E_1 = \frac{\delta}{\delta-\beta}w_n^2, \quad F_1 = -\frac{\delta\tau}{(\delta-\beta)^2}w_n^2\left[1 + \frac{(\delta-\beta)^2}{\tau}\frac{1}{w_H^2}\right]^{-1}.$$

Setting $\dot{P}$ equal to zero now gives

$$\left[\delta - \beta - w_n^2\tau\left(A_1 \cos w_H t + B_1 \sin w_H t + D_1 + E_1 t + F_1 e^{\frac{\delta-\beta}{\tau}t}\right)\right] + \beta\frac{P_o}{P} = 0.$$

Using the approximations $\cos w_H t = 1$ and $\sin w_H t = w_H t$ this equation reduces to

$$\left[\delta - \beta + w_n^2\tau F_1\left(e^{\frac{\delta-\beta}{\tau}\Delta} - 1 - \frac{\delta-\beta}{\tau}\Delta\right)\right] + \beta\frac{P_o}{P} =$$

$$\left[\delta - \beta + w_n^2 F_1\frac{\tau}{\delta}(\delta-\beta)\left(\frac{P}{P_o} - 1 - \frac{\delta}{\tau}\Delta\right)\right] + \beta\frac{P_o}{P} = 0$$

Therefore

$$\frac{P}{P_o} = 1 + \frac{\delta}{\tau}\Delta - \frac{\delta}{w_n^2\tau F_1(\delta-\beta)}\left[\delta - \beta + \beta\frac{P_o}{P}\right].$$

Again using $\frac{\delta}{\tau}\Delta = \frac{\delta}{\delta-\beta}\log\left(\frac{\delta-\beta}{\delta}\frac{P}{P_o} + \frac{\beta}{\delta}\right)$ and the factor two the maximum power formula becomes

$$\frac{P}{P_o} = 1 + \frac{(\delta-\beta)^2}{2w_n^2\tau^2}\left[1 + \frac{(\delta-\beta)^2}{w_H^2\tau^2}\right] + \frac{\delta}{\delta-\beta}\log\left[\frac{\delta-\beta}{\delta}\frac{P}{P_o} + \frac{\beta}{\delta}\right.$$

$$\left. + \frac{\beta(\delta-\beta)}{2w_n^2\tau^2}\left[1 + \frac{(\delta-\beta)^2}{\tau^2 w_H^2}\right]\frac{P_o}{P}\right].$$

When $\frac{1}{w_H} = 0$ this reduces to the previous formula. Letting $\beta = 0$ one obtains

$$\frac{P}{P_o} = 1 + \frac{\delta^2}{2w_n^2\tau^2}\left[1 + \frac{\delta^2}{w_H^2\tau^2}\right] + \log\frac{P}{P_o}.$$

It is interesting to note that here as with the iterative formulas the pressurizer introduces the factor $\left[1 + \frac{(\delta - \beta)^2}{w_H^2\tau^2}\right]$.

For $\delta < \beta$ a rather simple approximation for maximum power can be made. Again approximating the reactivity by $\delta - \beta$, or rather $\beta - \delta$, up to time Δ one has

$$P = \left(\frac{\delta}{-\beta} e^{\frac{\delta - \beta}{\tau} t} - \frac{\beta}{\delta - \beta}\right)P_o = \left(\frac{\beta}{\beta - \delta} - \frac{\delta}{\beta - \delta} e^{-\frac{\beta - \delta}{\tau} t}\right) P_o.$$

$\frac{\beta}{\beta - \delta}$ is clearly an upper bound for P and in case δ is much smaller than β it is not much larger than maximum P. Therefore, one has for a formula

$$P_{max} = \frac{\beta}{\beta - \delta}, \quad \delta \ll \beta.$$

Next a formula can be obtained for the time of maximum power. In obtaining the above equation for maximum power the equation

$$\left[\delta - \beta + w_n^2\tau F_1 \left(e^{\frac{\delta - \beta}{\tau}\Delta} - 1 - \frac{\delta - \beta}{\tau}\Delta\right)\right] + \beta\frac{P_o}{P} = 0$$ was used. It only remains to solve this equation for Δ. On rearranging one has

$$e^{\frac{\delta - \beta}{\tau}\Delta} = 1 + \frac{(\delta - \beta)^2}{\tau^2 w_H^2}\left[1 + \frac{(\delta - \beta)^2}{\tau^2 w_H^2}\right]\left[\delta - \beta + \beta\frac{P_o}{P}\right] + \frac{\delta - \beta}{\tau}\Delta$$

Since for large δ, small $\delta - \beta$ and small δ the quantity $\frac{1}{\delta}\left[\delta - \beta + \beta\frac{P_o}{P}\right]$ is close to one, it may be approximated by one. Therefore

$$\Delta = \frac{\tau}{|\delta - \beta|} \log \left\{ 1 + \frac{(\delta - \beta)^2}{\tau^2 w_n^2} \left[1 + \frac{(\delta - \beta)^2}{\tau^2 w_H^2} \right] + \frac{\delta - \beta}{\tau} \Delta \right\}$$

$$\cong \frac{\tau}{|\delta - \beta|} \log \left\{ 1 + \frac{\Psi^2}{w_n^2} \left(1 + \frac{\Psi^2}{w_H^2} \right) + \log \left[1 + \frac{\Psi^2}{w_n^2} \left(1 + \frac{\Psi^2}{w_H^2} \right) \right] \right\}$$

is a fairly good formula for the time of maximum power.

Finally a short formula can be obtained for obtaining pressure maximums. Consider again equations (6) through (9). Differentiating equation (7) and using equation (9) one obtains the pressurizer equation

$$\ddot{z} + \alpha_1 \dot{z} + \alpha_2 \dot{z}|\dot{z}| = - w_H^2 (z + w_n^2 y) = - w_H^2 \bar{p}.$$

The pressure is therefore proportional to $\ddot{z} + \alpha_1 \dot{z} + \alpha_2 \dot{z}|\dot{z}|$. The $\ddot{z}$ represents an inertia force and the $\alpha_1 \dot{z} + \alpha_2 \dot{z}|\dot{z}|$ represents a frictional force. Under the constant pressure assumption or upon differentiating $\bar{p} = z + w_n^2 y$ one obtains

$$\dot{z} = - w_n^2 \dot{y} \text{ and } \ddot{z} = - w_n^2 \ddot{y}.$$

Now equation (6) states that $\dot{y} = X - 1$ and consequently

$$\dot{z} = - w_n^2 (X - 1) \text{ and } \ddot{z} = - w_n^2 \dot{X}.$$

Although the maximum X and $\dot{X}$ occur at different times, an upper bound for $\bar{p}$ should be given by $\bar{p} = \bar{p}_i + \bar{p}_f$, where $\bar{p}_i = \frac{w_n^2}{w_H^2} \dot{X}_{max}$ and

$$\bar{p}_f = \alpha_1 \frac{w_n^2}{w_H^2} (X - 1)_{max} + \alpha_2 \frac{w_n^4}{w_H^2} (X - 1)^2_{max}$$

The formula for $\bar{p}_i$ requires a formula for $\dot{X}_{max}$. This is most easily obtained by starting with equation (8). With the usual trick of

setting z = 0 up to time Δ, one has $\dot{X} = \frac{\delta - \beta}{\tau} X + \frac{\beta}{\tau}$ or for large δ that $\dot{X} \cong \frac{\delta - \beta}{\tau} X$. Since $\dot{X}_{max}$ does not occur at X_{max} a proportionality factor may be used. Therefore let $\dot{X}_{max} = 0.42 \frac{\delta - \beta}{\tau} X_{max}$. The constant 0.42 was chosen first because it agrees pretty well with the SEAC computations and second because superficially one would expect $\dot{X}_{max}$ to occur when X has reached about half its maximum (see Table 8). For δ much smaller than β the formula $\dot{X} = \frac{\delta - \beta}{t} X + \frac{\beta}{\tau}$ indicates that $\dot{X}_{max} = \frac{\delta}{\tau}$ is a very good approximation since $\dot{X}$ has its maximum value almost at t = 0.

The question now arises as to how well the numbers given by these formulas agree with SEAC computations and the iterative formula computations. Tables 4, 5, 6, and 7 indicate the comparisons, while Figures 61, 62, and 65 exhibit the comparisons for the HRE case. The short formulas are seen to give results which agree pretty well with the other results. Of course, the agreement is best for large δ, since most of the formulas are developed for large δ. The short formula for pressure leads to values about one-half to three-fourths of the SEAC and iterative formula values. This can be explained by either choosing another proportionality constant between $\dot{X}_{max}$ and X_{max} or better by noting that the pressurizer oscillation might be expected to increase the pressure above that predicted by the formulas.

Table 3 gives the list of short formulas.

F. COMMENTS AND CONCLUSIONS

Again, let it be emphasized that this is largely a mathematical discussion of the equations of motion which are thought to represent the important features of the kinetics of homogeneous reactors of the HRE type. This report deals primarily with the non-linear features of these equations. This aspect of reactor kinetics is probably third in order of development and importance. First, of course, is the problem of obtaining equations of motion which approximate the reactor motion sufficiently well and can be mathematically investigated. The next problem is to investigate the "stability" of the motion determined by the selected equations. In particular, this means a careful investigation of linear stability. Only after stability questions are answered does it appear worthwhile to investigate, as this report has partially done, the actual motion under various initial conditions. Since a large step change in reactivity seems highly unlikely most of this report deals with a greatly idealized motion.

Having indicated the relative unimportance of the type of calculations made here, it seems only proper to give some justification for spending time and effort on calculations of the type in this report. The following four reasons may be given as partial justification:

1. Certain safety calculations can be based on such calculations.
2. The design or selection of a reactor might require such calculations.
3. Other investigations of reactor kinetics might profit from such results.
4. In contrast to the linear techniques, the non-linear techniques are less straightforward, less explored, more difficult, and more interesting.

A number of physical assumptions have been made before equations (1) through (5) were written down and they may be noted as follows: (1) constant heat removal, (2) the effects of gas may be neglected, (3) the essential features of the motion are determined in a short time, (4) constant delayed neutron source, (5) around the loop effects may be neglected, (6) bulk coordinates are used and no account need be taken of positional variation of the symbols, (7) the only non-linear effects of importance occur through the reactivity and the velocity of the fluid in the relief pipe, (8) step changes in reactivity will give results on the safe side, and (9) a linear equation of state may be used. There are undoubtedly other assumptions made but it is believed that these are the main ones.

The latest HRE design has constants which differ slightly from those used in the SEAC calculations, e.g., $w_H{}^2$, $w_n{}^2$, τ and α_1 are larger in the present HRE design. It may be noted in this regard that the SEAC calculations were carried out with practically dimensionless quantities. Each calculation therefore could be applied to many reactor models.

It is probably better to deal with entropy rather than temperature. However, for convenience of understanding the term temperature is used here.

From the "practical" viewpoint of homogeneous reactors of the HRE design the short formulas are probably very good rules of thumb. If more detailed results are desired and a greater assurance of the proper magnitudes of the results is asked for, then the iterative formulas appear to be useful. Finally, if complete assurance for details and magnitudes are desired, one must resort to numerical integration. The regular usage of high speed computing machines will undoubtedly make it economically

feasible to choose numerical integration. At present, however, it seems economical to develop short formulas.

This hierarchy of approximate solutions to systems of non-linear differential equations could probably be developed for usage with other reactor types, e.g., boiling reactors.

APPENDIX I

Short Derivation of Power Equation

The following short and rather novel derivation of the power equation, $\dot{P} = \frac{1}{\tau}\left[\delta - \beta - \alpha T\right]P + \frac{\beta}{\tau}\bar{P}$, is due to T. A. Welton. Consider the equation

$$P(t) = \int_0^\infty K(s)\, P(t - s)\, ds,$$

where s is a delay time and K(s) is a kernel indicating in what manner the past effects the present power. The kernel K(s) may be considered to be the sum of two kernels, K_p and $K_d(s)$, one due to prompt neutrons and the other due to delayed neutrons. Therefore:

$$P(t) = \int_0^\infty K(s)\, P(t - s)ds = \int_0^\infty P(t - s)\, K_p(s)ds + \int_0^\infty P(t - s)K_d(s)ds.$$

Since for the prompt kernel $P(t - s)$ can be approximated closely by the first two terms of a Taylor series, one has

$$P(t) = \int_0^\infty \left[P(t) - s\dot{P}(t)\right] K_p(s)ds + \int_0^\infty P(t - s)\, K_d(s)ds.$$

Upon solving this equation for $\dot{P}(t)$, one obtains

$$\dot{P}(t) = \frac{\int_0^\infty K_p(s)ds - 1}{\int_0^\infty sK_p(s)ds} P(t) + \frac{\int_0^\infty P(t - s)\, K_d(s)ds}{\int_0^\infty sK_p(s)ds}$$

Defining $\bar{s}$ by the quotient of $\int_0^\infty sKp(s)ds$ and $\int_0^\infty Kp(s)ds$ and defining $k(1 - b)$ as equal to $\int_0^\infty Kp(s)ds$, this last equation becomes

$$\dot{P}(t) = \frac{1 - \frac{1}{k(1 - b)}}{\bar{s}} P + \frac{\frac{1}{k(1 - b)}}{\bar{s}} \int_0^\infty P(t - s)\, K_d(s)ds$$

$$\text{or} \quad \dot{P}(t) = \frac{k - 1 - kb}{\tau} P + \frac{1}{\tau} \int_0^\infty P(t - s)\, K_d(s)ds$$

where $\tau = k(1 - b)\, \bar{s}$. Since $\int_0^\infty K_d(s)ds = kb$ if $\int_0^\infty K_p(s)ds = k(1 - b)$ and $\int_0^\infty K(s)ds) = k$,

one may write

$$\int_0^\infty P(t - s)\ K_d(s)ds = kb\overline{P}$$

where $\overline{P}$ is an average value for P(t) over its past history. Letting $k - 1 = \delta - \alpha T$ and $kb = \beta$ one therefore has

$$\dot{P} = \frac{1}{\tau}\left[\delta - \alpha T - \beta\right]P + \frac{\beta}{\tau}\overline{P}\ .$$

In the case of a δ step function the power is assumed to have been at P_o for a long part of its past history and consequently $\overline{P}$ can be replaced by P_o.

APPENDIX II

Factorization of Certain Second Order Non-Linear Differential Equations

Many second order non-linear differential equations, including some of interest in the treatment of reactor kinetics, fall into one of the following three classes:

(1) $\left[\frac{d}{dx_1} + f(x_1)\right]\left[h(x_1, x_2, x_3)\right] = 0,$

(2) $\left[\frac{d}{dx_1} + g(x_1, x_2, x_3)\right]\left[G\left\{g(x_1, x_2, x_3)\right\}\right] = 0$

and (3) $\left[\frac{d}{dt} + k(t, y)\right]\left[K\left\{k, \frac{dk}{dt}, t\right\}\right] = 0$

where x_1, x_2, x_3 represent t, y, $\frac{dy}{dt}$ in any order.

For class (1) the equation may be written as $\frac{dv}{dx_1} + f(x_1)v = 0$, where $v = h(x_1, x_2, x_3)$ is considered to be a function of x_1. The solution of $\frac{dv}{dx_1} + f(x_1)v = 0$ is

$$v = ce^{-\int^{x_1} f(z)dz}$$

where c is an integration constant. Consequently,

$$h(x_1, x_2, x_3)e^{\int^{x_1} f(z)dz} = c.$$

The equations of class (1) can therefore be solved if one can solve this last first order equation.

For class (2) the equation may be written as $\frac{dw}{dx_1} + s(x_1)w = 0$, where $s = g(x_1, x_2, x_3)$, and consequently, $w = G\ g(x_1, x_2, x_3)$ are considered to be functions of x_1. Using the integrating factor

$$r(x_1) = e^{\int^{x_1} s(z)dz},$$

one obtains $r(x_1)w(x_1) = c$, where c is an integration constant. Since $\frac{dr}{dx_1} = sr$ it follows that

$$rG\left(\frac{\frac{dr}{dx_1}}{r}\right) = c$$

for $r(x_1)$ and then solves the first order equation

$$s(x_1) = g(x_1, x_2, x_3) = \frac{\frac{dr}{dx_1}}{r} \quad \text{for } y.$$

For class (3) the equation has the form $\frac{dw}{dt} + s(t)w = 0$, where $s(t) = k(t, y)$ and $w(t) = K(k, \frac{dk}{dt}, t)$. Using the integrating factor

$$r(t) = e^{\int^t s(z)dz}$$

one obtains $r(t)w(t) = c$ or since $\dot{r}(t) = s(t)r(t)$ and $s = k(t, y)$ one has

$$r(t)K\left\{\frac{\dot{r}}{r}, \frac{\ddot{r}}{r} - \frac{r^2}{r^2}, t\right\} = c.$$

The equations of class (3) can therefore be solved if one first solves this last second order equation for $r(t)$ and then substitutes for r and $\dot{r}$ into the equation $k(t, y) = \frac{\dot{r}}{r}$ in order to obtain y either implicitly or explicitly.

Particular solutions of class (1), (2) and (3) can be found by merely setting the inner factors equal to zero, i.e., solve for y after putting respectively either $h(x_1, x_2, x_3)$ or $G\left\{g(x_1, x_2, x_3)\right\}$ or $K\left\{k, \frac{dk}{dt}, t\right\}$ equal to zero.

The following three elementary examples illustrate respectively the above methods for classes (1), (2) and (3).

(a) $y'' - \frac{1}{y}y'^2 + ay^2 + b = 0$

$$\left[\frac{d}{dy} - \frac{2}{y}\right]\left[\frac{1}{2}y'^2 + ay^3 - by\right] = 0$$

$$\frac{1}{2}y'^2 + ay^3 - by = cy^2$$

The integration may now be completed by the use of elliptic functions.

(b) $2y'y'' = -y'^4$ or $2y'' = -y'^3$

$$\left[\frac{d}{dt} - \frac{2}{y}\right]\left[y'^2\right] = 0$$

$$r(t)w(t) = c_1, \text{ where } r(t) = e^{\int^t \dot{y}^2(z)dz} \text{ and } w(t) = \dot{y}^2(t).$$

$$\therefore \dot{r}(t) = c_1$$

$$r(t) = c_1 t + c_2$$

$$\dot{y}^2 = \frac{\dot{r}(t)}{r(t)} = \frac{c_1}{c_1 t + c_2}$$

$$y = \pm 2\sqrt{t + c_3} + c_4$$

(c) $2y\ddot{y} + 2\dot{y}^2 + 6y^3\dot{y} + y^6 = 0$

$$\left[\frac{d}{dt} + y^2\right]\left[2y\dot{y} + y^4\right] = 0$$

$$r(t)w(t) = c, \text{ where } r = e^{\int^t y^2(z)dz} \text{ and } w = 2y\dot{y} + y^4$$

$$r(t)\left[\frac{\ddot{r}}{r} - \frac{\dot{r}^2}{r^2} + \frac{\dot{r}^2}{r^2}\right] = \ddot{r} = c_1$$

$$r = c_1 t^2 + c_2 t + c_3$$

$$\frac{\dot{r}}{r} = y^2 = \frac{2t + c_4}{t^2 + c_4 t + c_5}$$

If T is eliminated between the equations $\dot{T} = S(P - P_o)$ and $\dot{P} = -\left[\alpha T + \beta\right]P$ the equation $\frac{\ddot{P}}{P} - \frac{\dot{P}^2}{P^2} = -\alpha\, S(P - P_o)$ is obtained. This equation is a special case of class (1).

APPENDIX III

Quadrature in the Phase Plane For Certain Second Order Non-Linear Differential Equations

Certain non-linear second order differential equations can be solved or at least their first integral can be found by replacing the second order equation by two first order equations which may be integrated directly by quadrature in the resulting phase plane. The following three classes of differential equations fall into this category:

$$(1)\quad \ddot{y} = \frac{\dot{r}}{r(t)}\, y + \frac{dh(y)}{dy}\, \frac{y^2}{h(y)} + F\left(\frac{\dot{y}}{rh}\right) G(y)\, r^2,$$

$$(2)\quad \ddot{y} = \frac{\dot{r}}{r(t)}\, \dot{y} + F\left(\frac{y}{rh}\right) r^2 h(y)\, \frac{dk}{dy},$$

$$(3)\quad \ddot{y} = \frac{\dot{r}}{r(t)}\, \dot{y} + G(y)\, \frac{1}{h^2(y)}\, \dot{y}^2 .$$

The corresponding sets of first order equations are:

$$(1)'\quad \dot{y} = xh(y)\, r(t),\ \dot{x} = \frac{F(x)\, G(y)}{h(y)}\, r(t)$$

$$(2)'\quad \dot{y} = xh(y)\, r(t),\ \dot{x} = \left[F(x) - x^2\right] r(t)\, \frac{dh}{dy}$$

$$(3)'\quad \dot{y} = xh(y)\, r(t),\ \dot{x} = x^2 r(t)\, \frac{\left[G(y) - \frac{dh}{dy}\, h(y)\right]}{h(y)}$$

The phase plane equations therefore become:

$$(1)''\quad \frac{dy}{dx} = \frac{h^2(y)}{G(y)}\, \frac{x}{F(x)}$$

$$(2)''\quad \frac{dy}{dx} = \frac{x}{F(x) - x^2}\, \frac{h(y)}{\left(\frac{dh}{dy}\right)}$$

$$(3)''\quad \frac{dy}{dx} = \frac{1}{x}\, \frac{h^2(y)}{\left[G(y) - \frac{dh}{dy}\, h(y)\right]}$$

The corresponding solutions are:

$$(1)''' \quad \int^{y} \frac{G(z)}{h^2(x)}\, dz - \int^{x} \frac{z}{F(z)}\, dz = c$$

$$(2)''' \quad \log h(y) - \int^{x} \frac{z}{F(z) - z^2}\, dz = c$$

$$(3)''' \quad \log x + \log h(y) - \int^{y} \frac{G(z)}{h^2(z)}\, dz = c$$

APPENDIX IV

Flow Chart for SEAC Calculations

The following chart is included merely to indicate a portion of the thinking that goes into the preparation of a problem for high speed digital machines. This flow chart indicates in block diagram form the sequence of operations which the machine is to perform. With this chart as a guide the programmer translates the sequence of operations into machine language. Members of the National Bureau of Standards and B. M. Drucker, ORINS Fellow attached to the Mathematics Panel, did the programming and coding for the SEAC calculations.

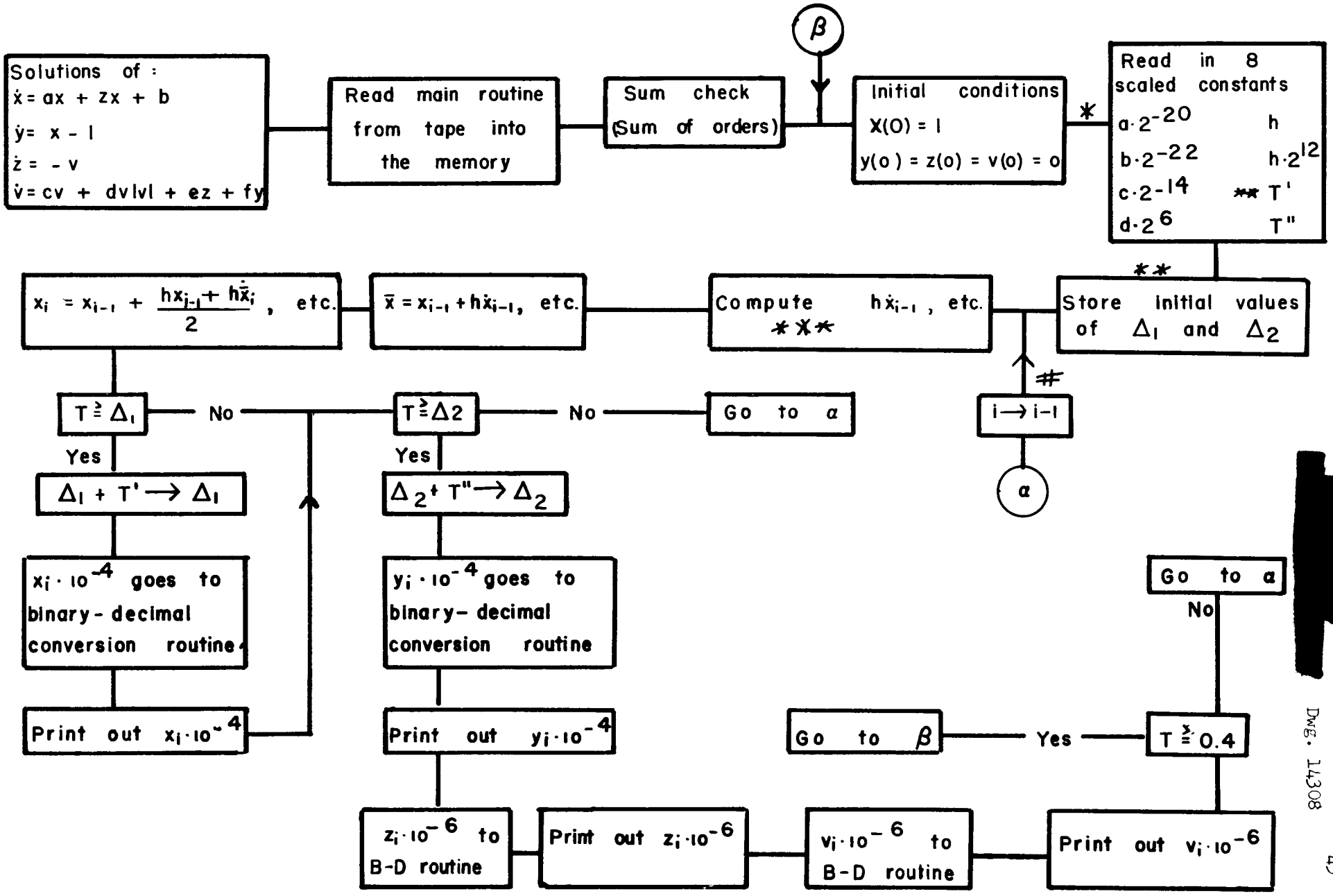
Note: These notes apply to the flow chart.

* Machine automatically stops here to allow auxiliary tape containing 8 constants (of Box IV) to be read in.

** T' denotes time interval increment for which the x values are to be printed out.
T'' denotes time interval increment for which the y, z, v values are to be printed out.
Δ_1 = time at which the next computed value of x_i is to be printed out.
Δ_2 = time at which the next computed values of y_i, z_i, v_i are to be printed out.

*** $\dot{x}_{i-1}$ denotes the derivative computed at $t = t_{i-1}$, etc.

\# denotes a purely notational box. No operations are performed.



[redacted]

Dwg. 14308

TABLE OF FIGURES

x - Power
y - Temperature
z - Density
v - Velocity
$\bar{p}$ - Pressure

Figure		Hand Integrations		
* 1	x vs t	$\delta = .0140$	$\frac{1}{w_H^2} = 0$	
* 2	y vs t	$\delta = .0140$	$\frac{1}{w_H^2} = 0$	
* 3	C_i vs t	$\delta = .0140$	$\frac{1}{w_H^2} = 0$	
* 4	x vs t	$\delta = .0100$	$\frac{1}{w_H^2} = 0$	
* 5	y vs t	$\delta = .0100$	$\frac{1}{w_H^2} = 0$	
* 6	C vs t	$\delta = .0100$	$\frac{1}{w_H^2} = 0$	
* 7	System 9	$\delta = .0100$	x vs t	
* 8			y vs t	
* 9			z vs t	
*10			v vs t	
*11			$\bar{p}$ vs t	
*12	System 1	$\delta = .0375$	x vs t	$\beta = 0$
*13			y vs t	
*14			z vs t	
*15			v vs t	
*16			$\bar{p}$ vs t	
*17	System 14	$v(0) = -10^5$	x vs t	
*18			y vs t	
*19			z vs t	
*20			v vs t	
*21			$\bar{p}$ vs t	

Figure	SEAC Calculations			
*22	System 1	δ = .0375	x vs t	β = 0
*23			y vs t	
*24			z vs t	
*25			v vs t	
*26			$\bar{p}$ vs t	
*27	Systems 2, 3, 4	δ = .0375	x vs t	
*28			y vs t	
*29			z vs t	
*30	System 2		v vs t	
*31	Systems 3, 4		v vs t	
*32	System 2		$\bar{p}$ vs t	
*33	Systems 3, 4		$\bar{p}$ vs t	
*34	System 5	δ = .0100	x vs t	β = 0
*35			y vs t	
*36	System 5		z vs t	
*37			v vs t	
*38			$\bar{p}$ vs t	
39	Systems 6,7,8,9	δ = .0100	x vs t	
40			y vs t	
41			z vs t	
42			v vs t	
43			$\bar{p}$ vs t	
44	System 10 System 11	δ = .0050 δ = .0070	x vs t	
*45			y vs t	
*46			z vs t	

Figure			
*47	System 10		v vs t
*48	System 11		v vs t
*49	System 10		$\bar{p}$ vs t
50	System 11		$\bar{p}$ vs t
*51	System 12	$\delta = .0125$	x vs t
*52			y vs t
*53			z vs t
*54			v vs t
55			$\bar{p}$ vs t
*56	System 13	$\delta = .0175$	x vs t
*57			y vs t
*58			z vs t
*59			v vs t
*60			$\bar{p}$ vs t
63	y vs δ, Z vs δ	t = 0.2	
64	Maximum v vs δ		

Formula Calculations

66	Schematic P vs t
67	Schematic P vs T
*68	x vs e
*69	Δp vs x_{max}
*70	Schematic P vs w
*71	X vs Y
*72	x vs δk parameter x_i
*73	x vs rise rate parameter x_i
*74	Isocline solution of $\frac{dx}{dy} + \frac{Y}{x} + Y - K = 0$
*75	x_{max} vs δ

Figure			
76	Max x vs δ	$\beta = 0$, $\beta = .0040$, $\beta = .0075$	
77	t of max vs δ	$\beta = 0$, $\beta = .0040$, $\beta = .0075$	
*78	System 12	$\delta = .0125$	x vs t
*79			y vs t
*80			z vs t
*81			$\bar{p}$ vs t
82	6 Ft. Core	Max x vs δ	
*83	6 Ft. Core	t of max x vs δ	
84	6 Ft. Core	Max v vs δ	
85	6 Ft. Core	t of max v vs δ	
86	6 Ft. Core	Max $\bar{p}$ vs δ	
87	6 Ft. Core	$\delta = .0025$	$\bar{p}$ vs t
88	6 Ft. Core	$\delta = .0100$	$\bar{p}$ vs t
89	Max x vs δ for 6, 10, 15, 20, 30 ft. core		
90	t of max x vs δ for 6, 10, 15, 20, 30 ft. core		
91	Max Δp vs δ for 6, 10, 15, 20, 30 ft. core		
92	Reactivity vs t		

Comparisons

61	Max x vs δ from SEAC and formulas
62	t of max x vs δ from SEAC and formulas
65	Max $\bar{p}$ vs δ from SEAC and formulas

(Figures 66 through 75 are from ORNL 630, 730, 925.)

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* These figures will appear in a supplement.

[redacted] Dwg. 14186

SYSTEMS 6, 7, 8, 9 $\delta = .0100$ X (t)

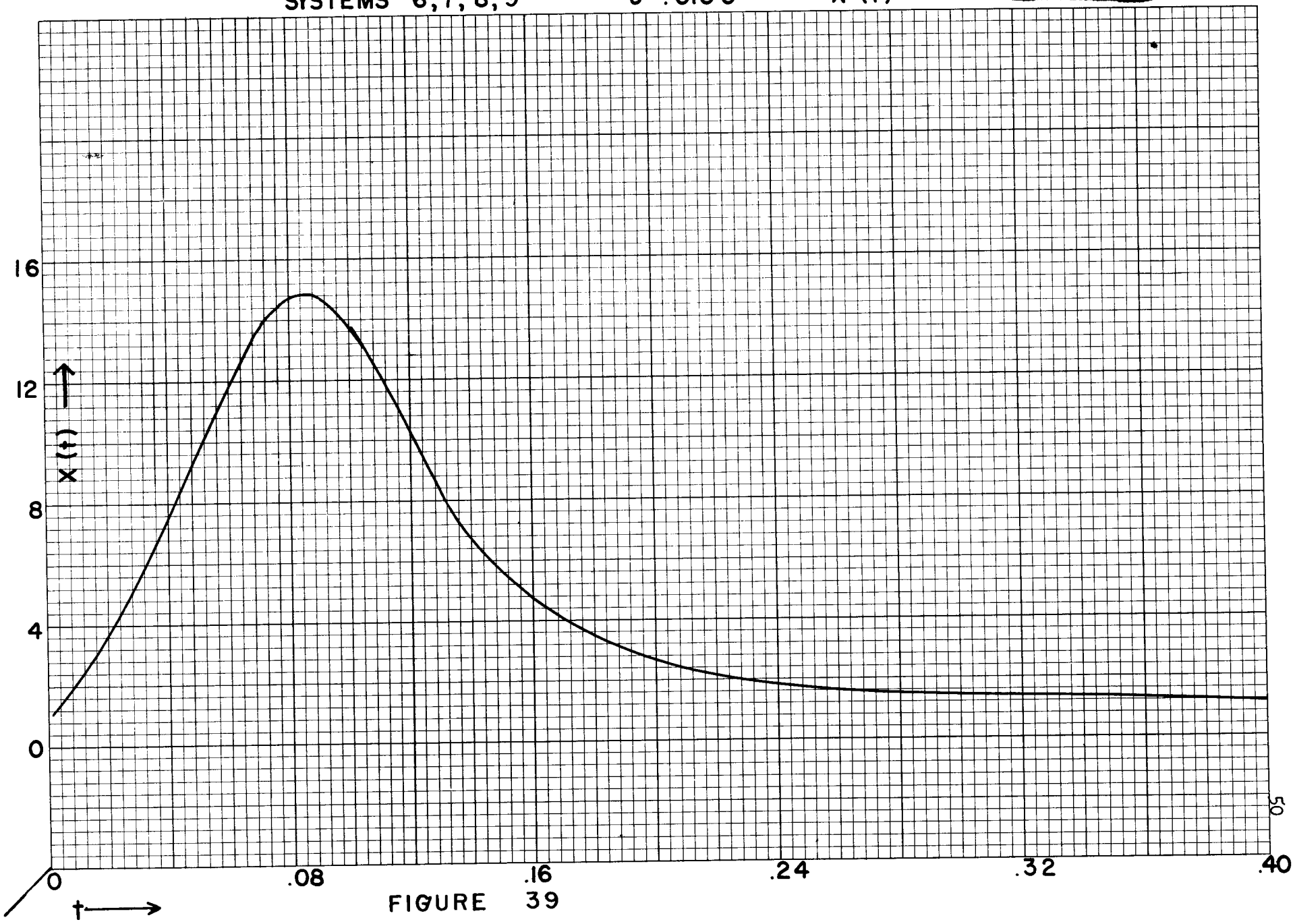

FIGURE 39

SYSTEMS 6, 7, 8, 9
δ = .0100
Y(t)
[redacted]
Dwg. 14187
°C
T_E = 250°
2.0
1.6
1.2
0.8
0.4
0
Y(t)
250°
245°
240°
0
.08
.16
.24
.32
.40
t

FIGURE 40

[redacted]

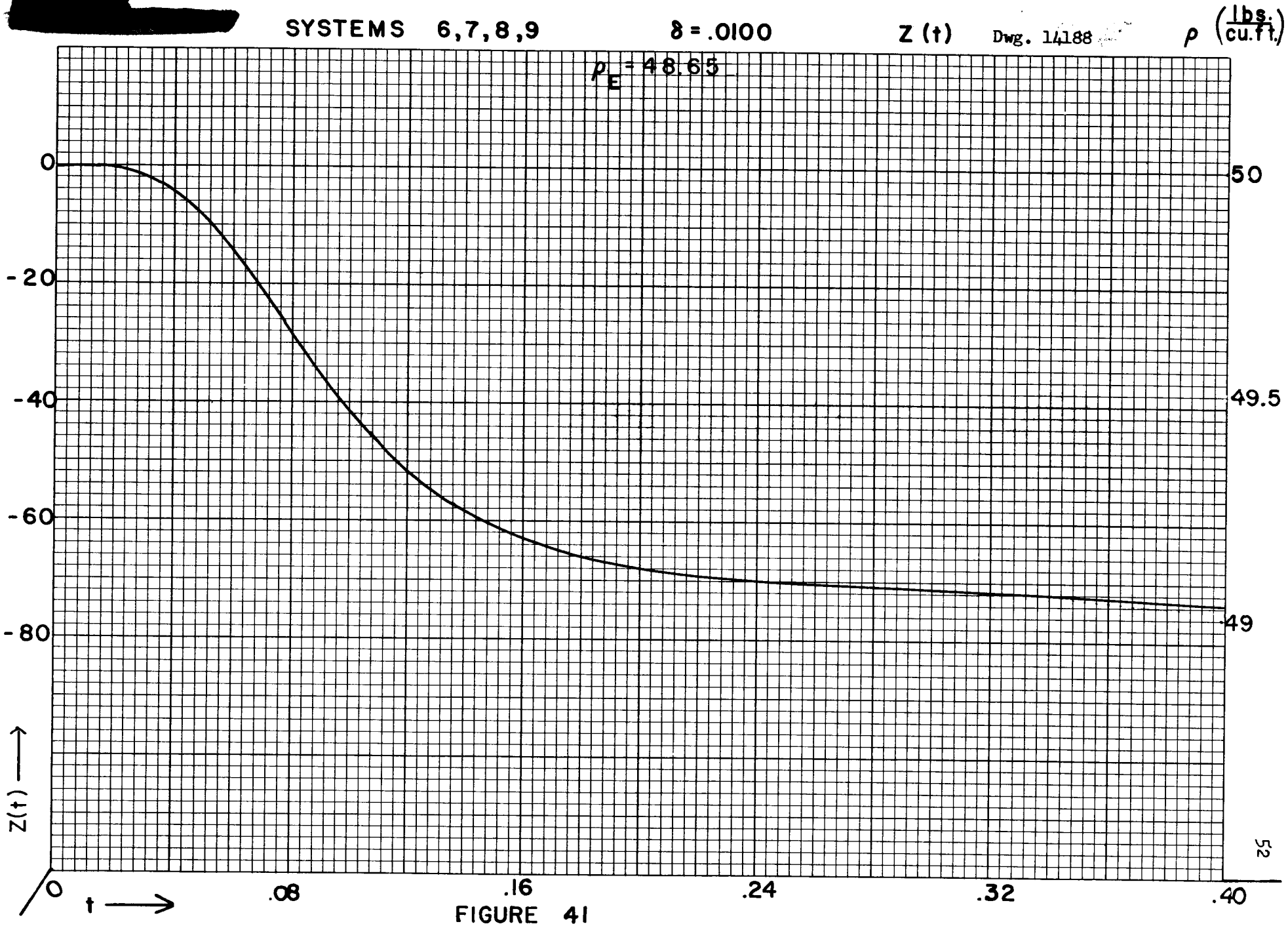

FIGURE 41

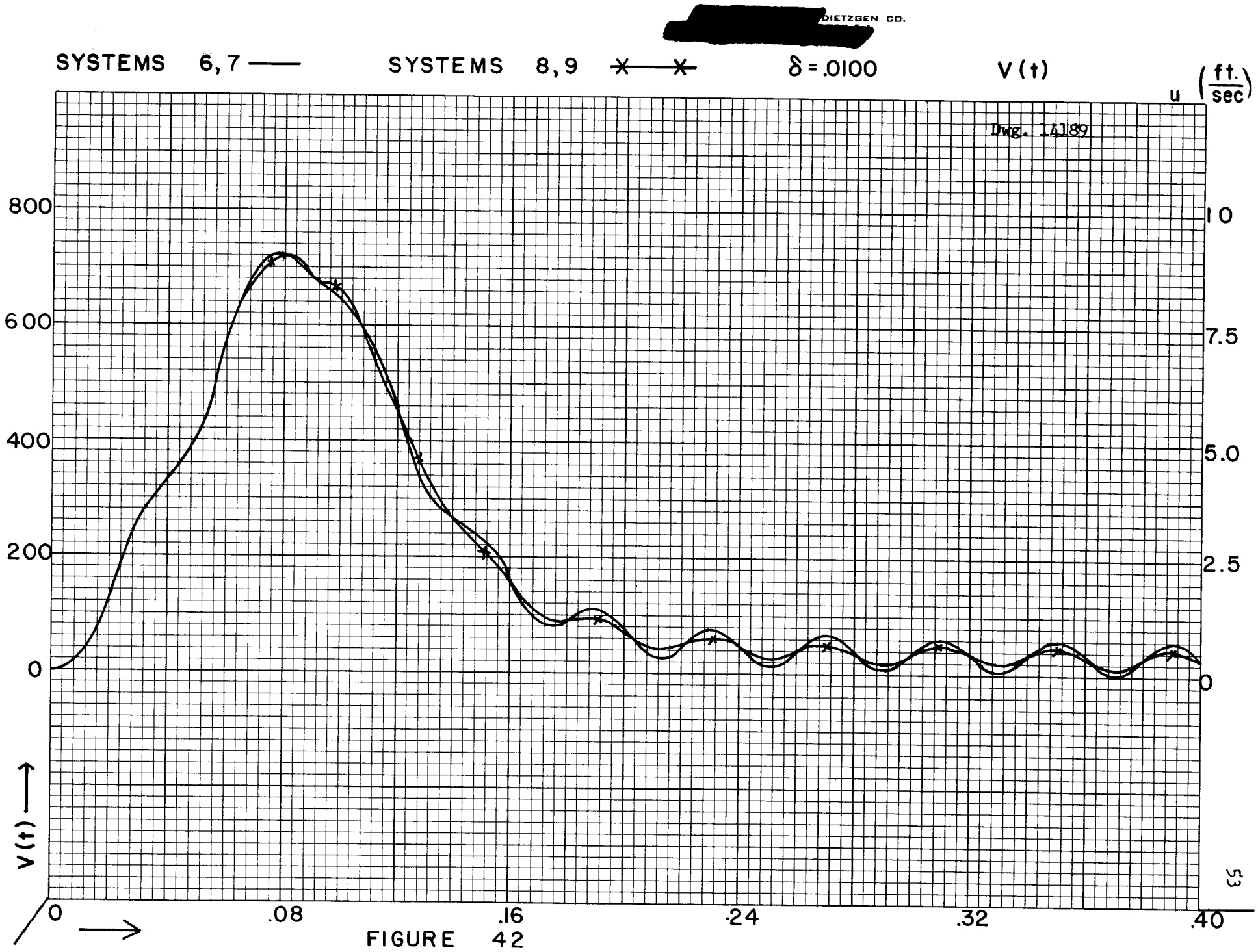
[redacted] DIETZGEN CO.
SYSTEMS 6, 7 ——
SYSTEMS 8, 9 ✕——✕
δ = .0100
V(t)
u (ft./sec)
Dwg. 14189
800
600
400
200
0
10
7.5
5.0
2.5
0
V(t)
0
.08
.16
.24
.32
.40

FIGURE 42

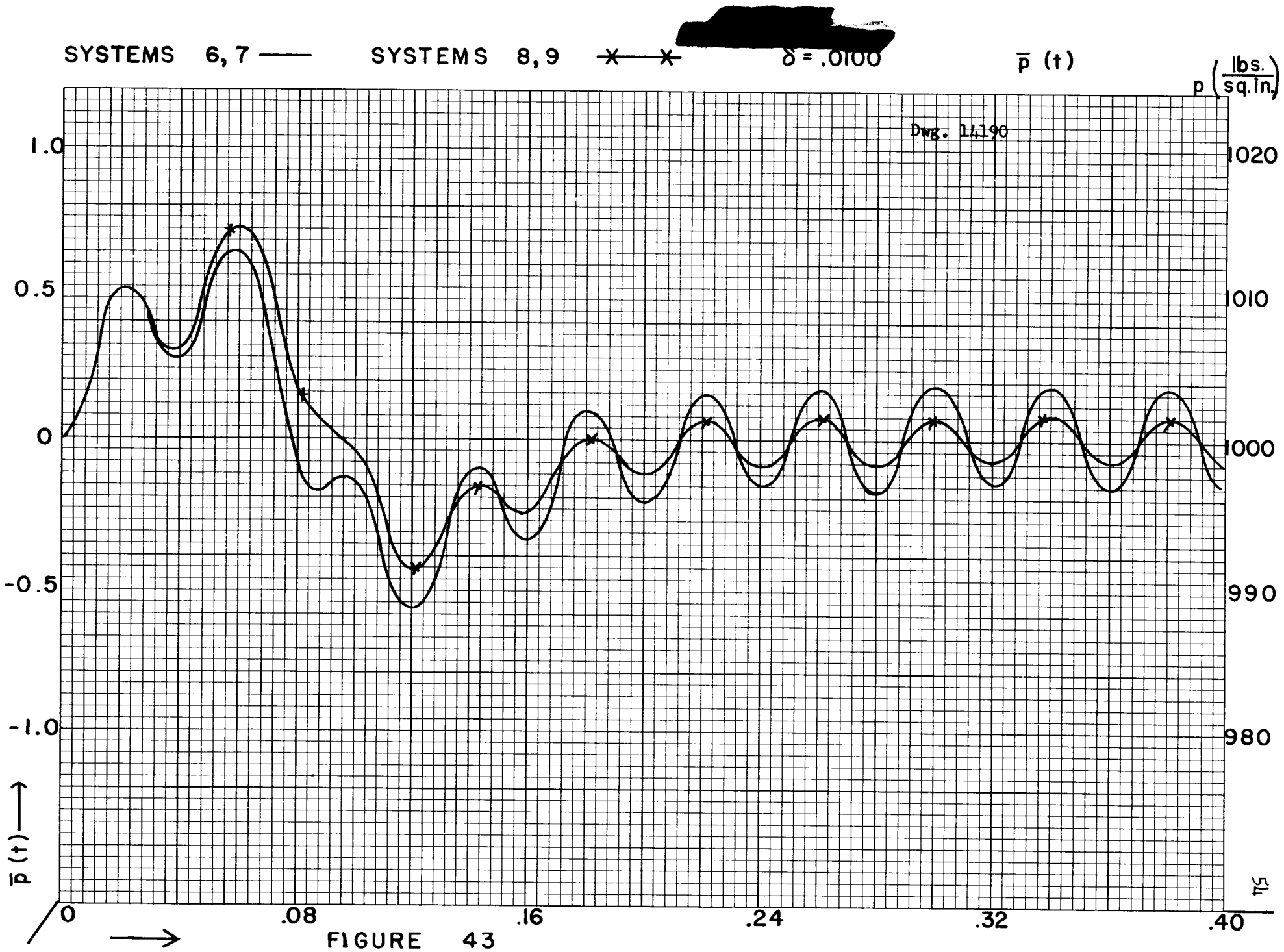
SYSTEMS 6, 7 ——
SYSTEMS 8, 9 —x—x—
[redacted]
δ = .0100
p̄ (t)
p (lbs./sq.in.)
Dwg. 14190
1.0
0.5
0
-0.5
-1.0
1020
1010
1000
990
980
p̄ (t)
0
.08
.16
.24
.32
.40

FIGURE 43

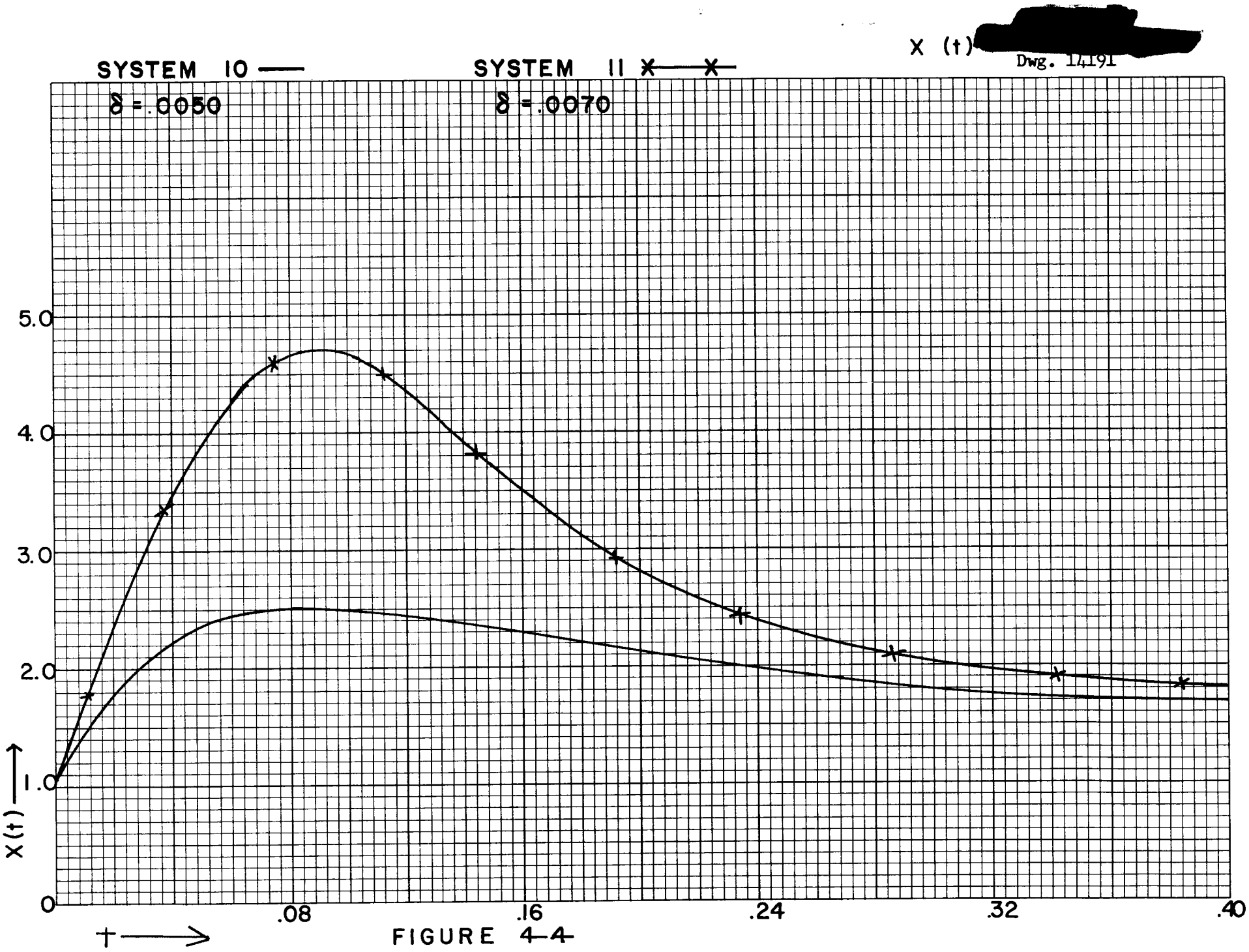
X (t) [redacted]
Dwg. 14191
SYSTEM 10
δ = .0050
SYSTEM 11
δ = .0070
5.0
4.0
3.0
2.0
1.0
0
X(t)
.08
.16
.24
.32
.40
t

FIGURE 4-4

SYSTEM II $\delta = .0070$ $\bar{p}(t)$ [redacted] Dwg. 14192 $p \left(\frac{lbs.}{sq\ in.}\right)$

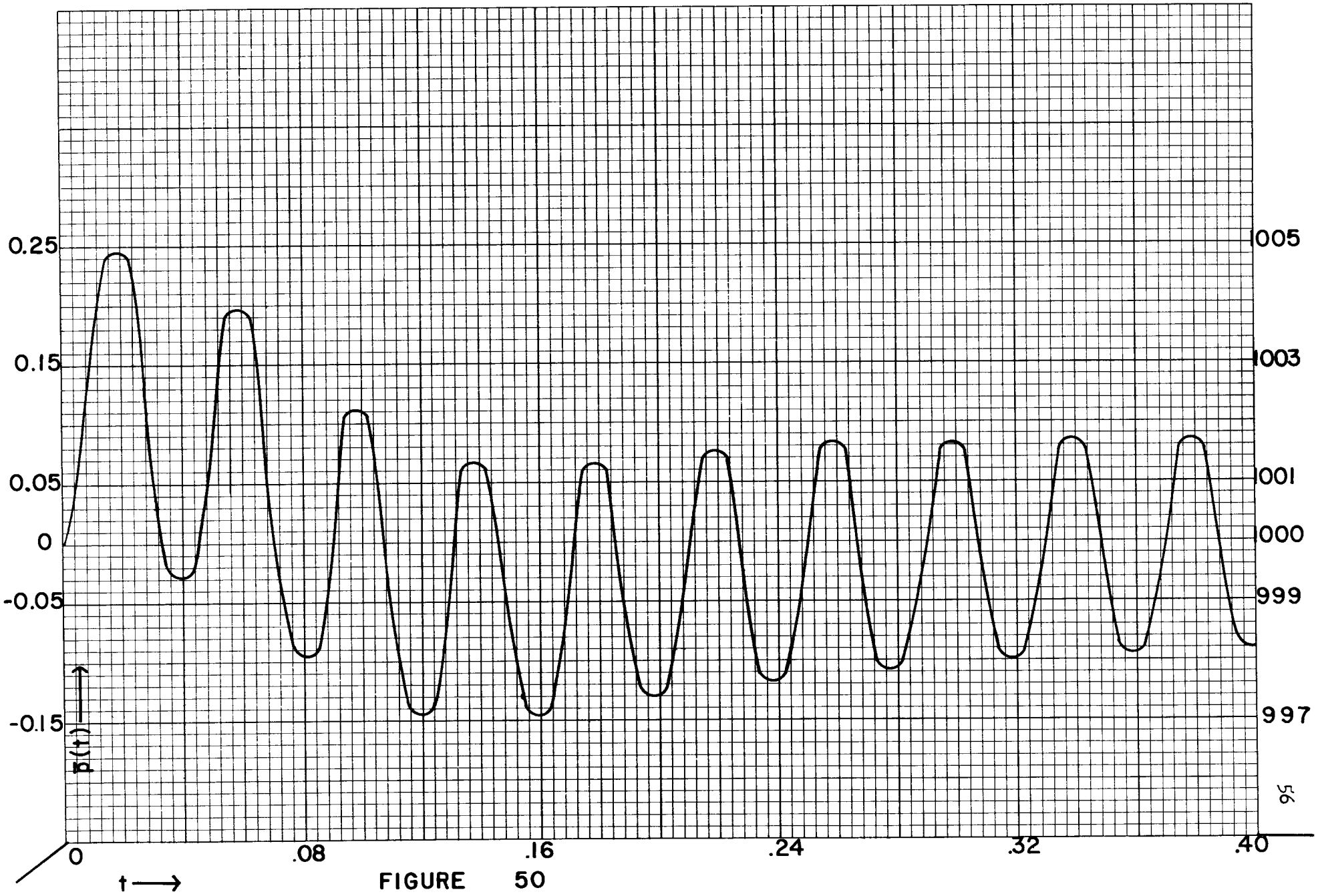

FIGURE 50

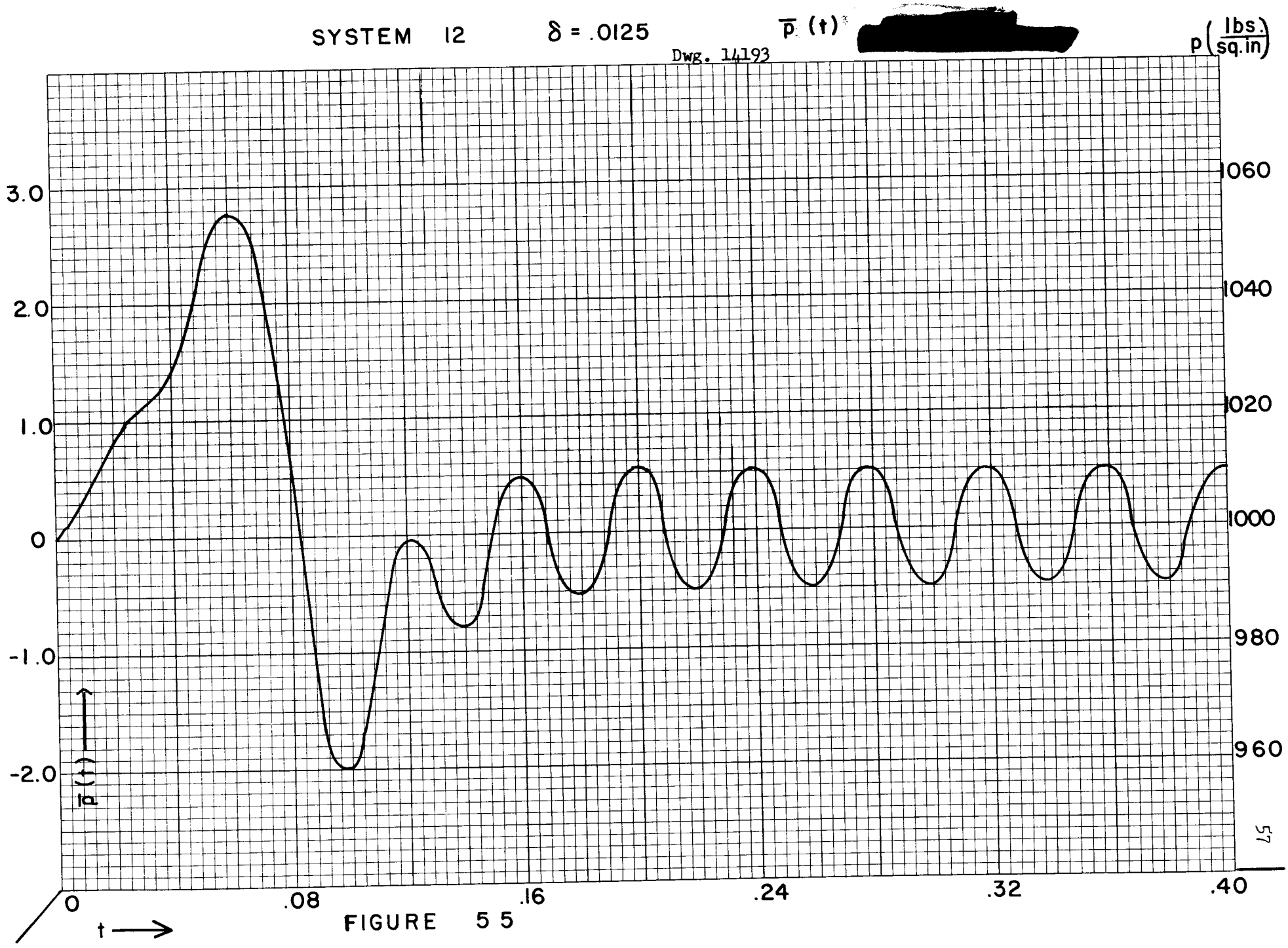
SYSTEM 12
δ = .0125
p̄ (t)
[redacted]
Dwg. 14193
p (lbs./sq.in)
3.0
2.0
1.0
0
-1.0
-2.0
1060
1040
1020
1000
980
960
0
.08
.16
.24
.32
.40
t
p̄ (t)

FIGURE 55

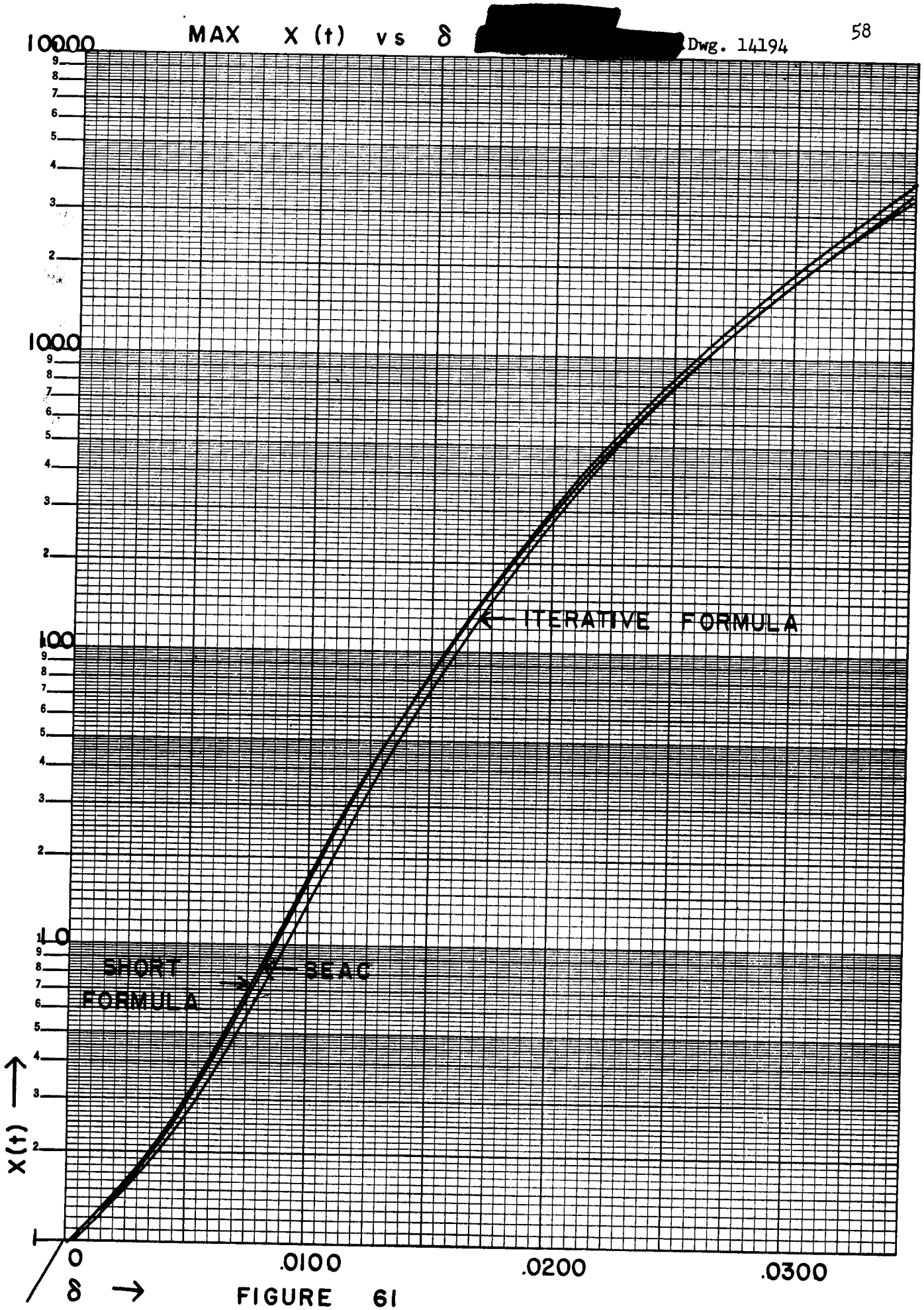
MAX X (t) vs δ [redacted] Dwg. 14194
10000
1000
100
10
1
ITERATIVE FORMULA
SHORT FORMULA
SEAC
X(t)
0
.0100
.0200
.0300
δ

FIGURE 61

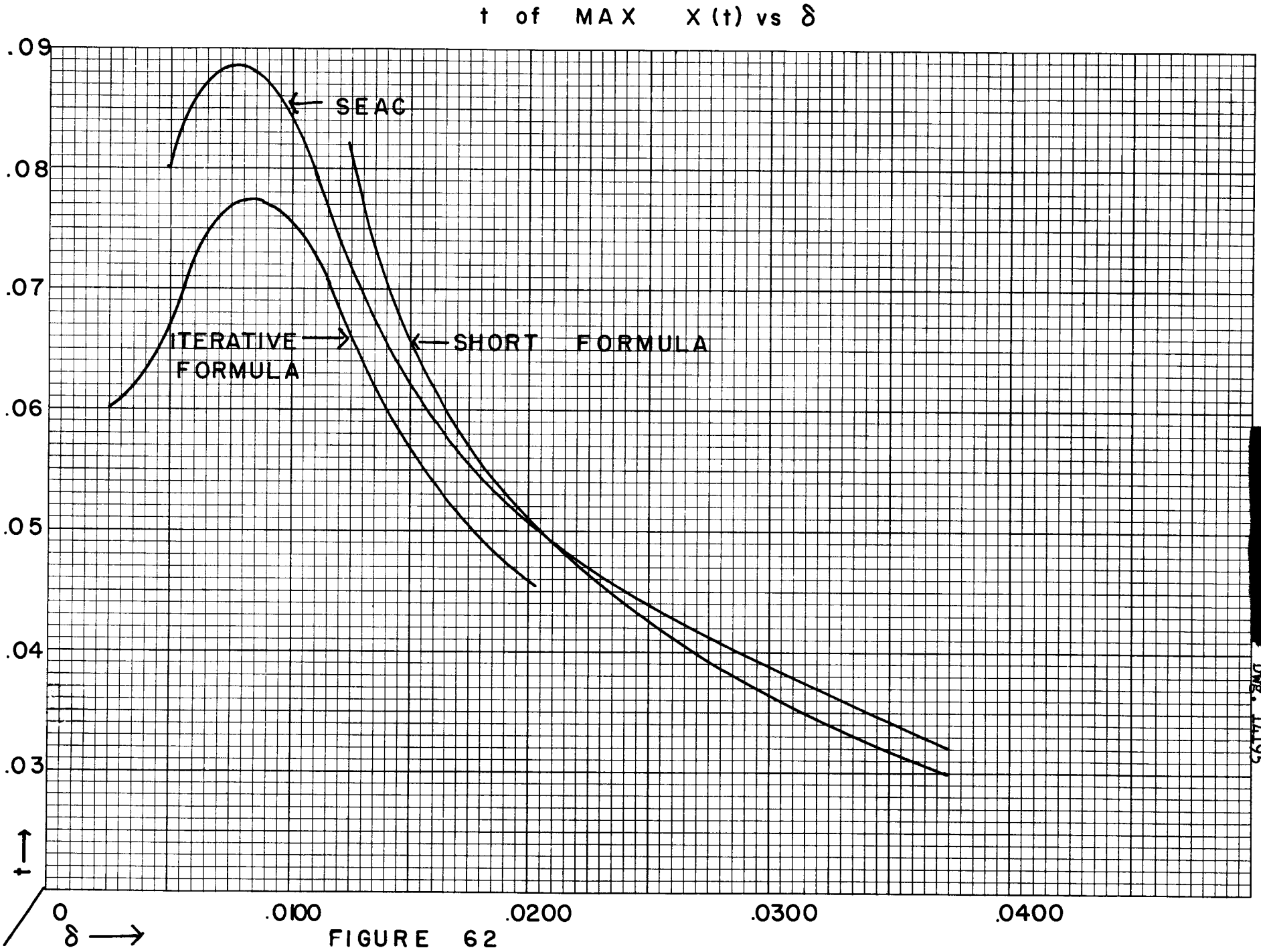

FIGURE 62

[redacted]

Y(t) and |Z(t)| at t=0.2 vs δ (Indistinguishab[redacted])

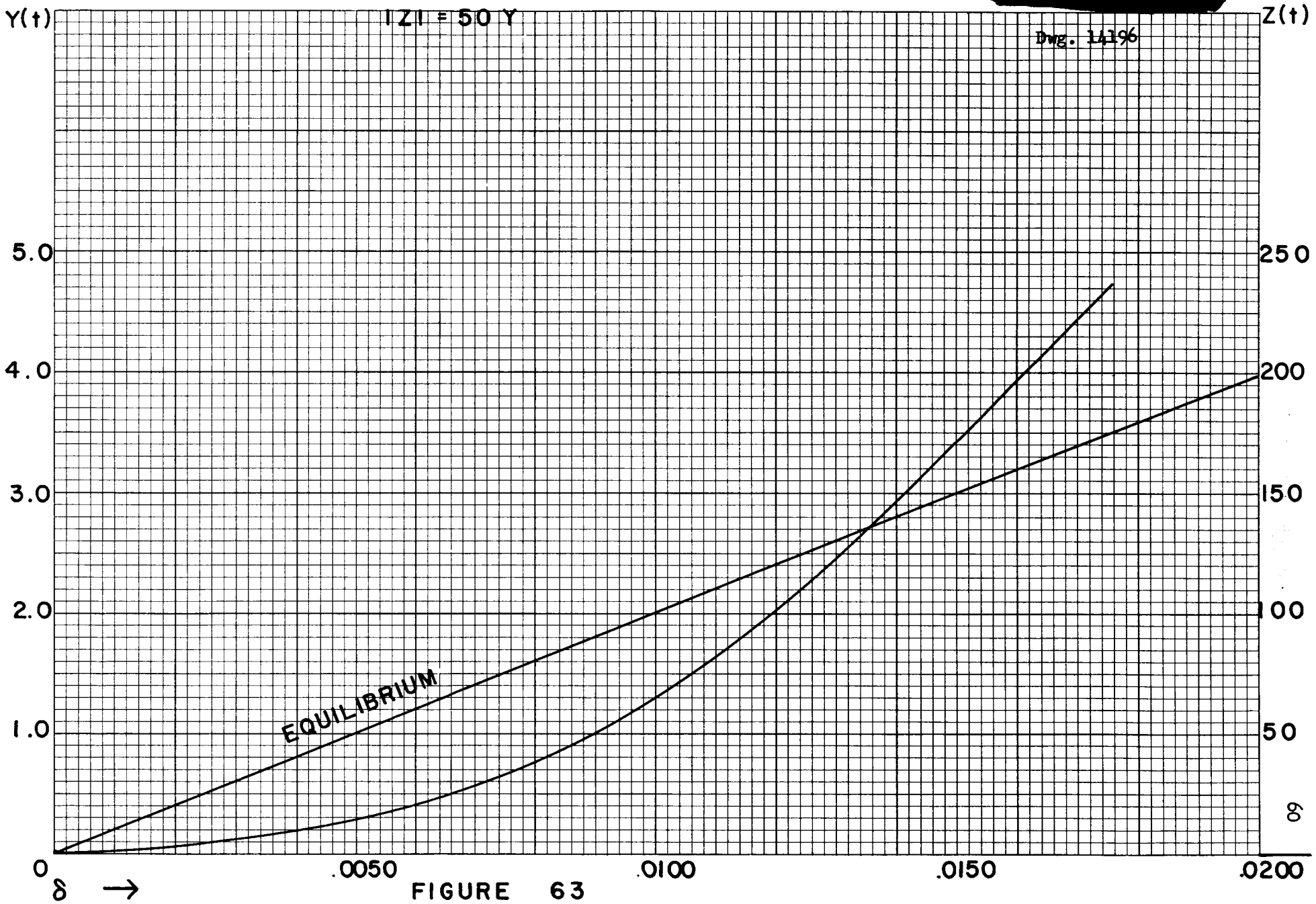

FIGURE 63

[redacted]

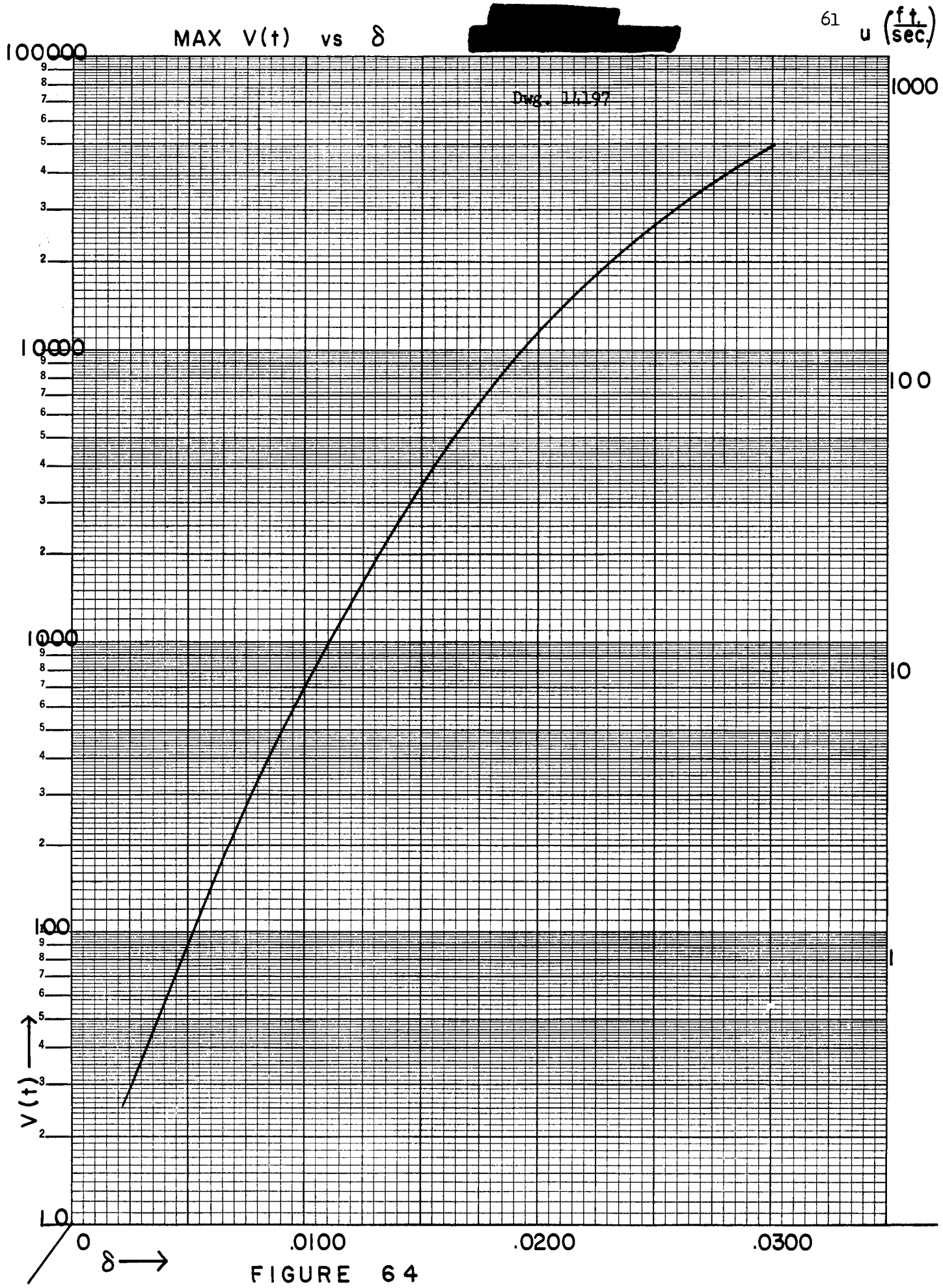

FIGURE 64

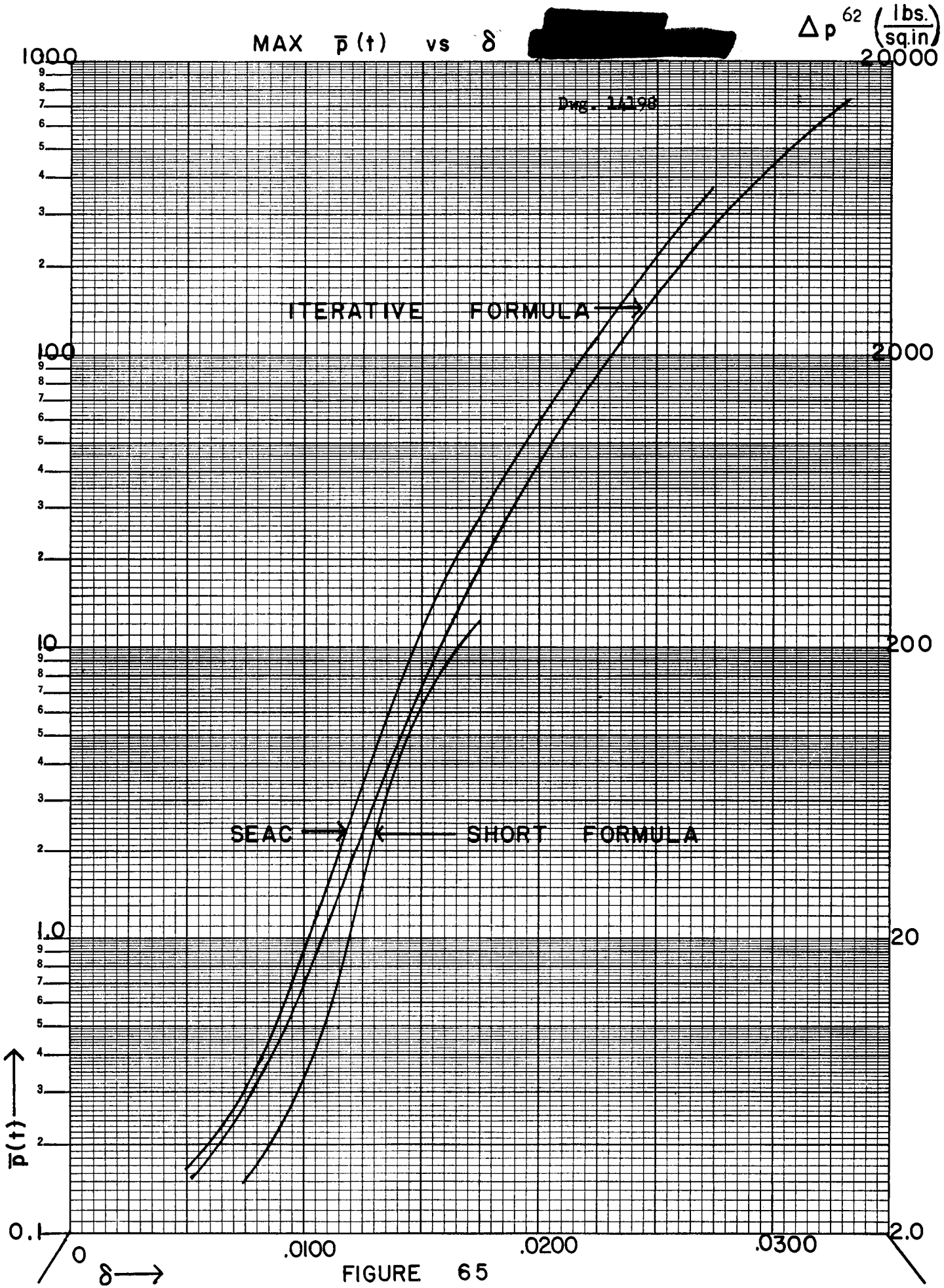
MAX $\bar{p}(t)$ vs δ
[redacted]
Δp^{62} $\left(\frac{lbs.}{sq.in}\right)$
Dwg. 14198
1000
100
10
1.0
0.1
20000
2000
200
20
2.0
ITERATIVE FORMULA
SEAC
SHORT FORMULA
$\bar{p}(t)$
0
.0100
.0200
.0300
δ

FIGURE 65

[redacted]

Dwg. 14199

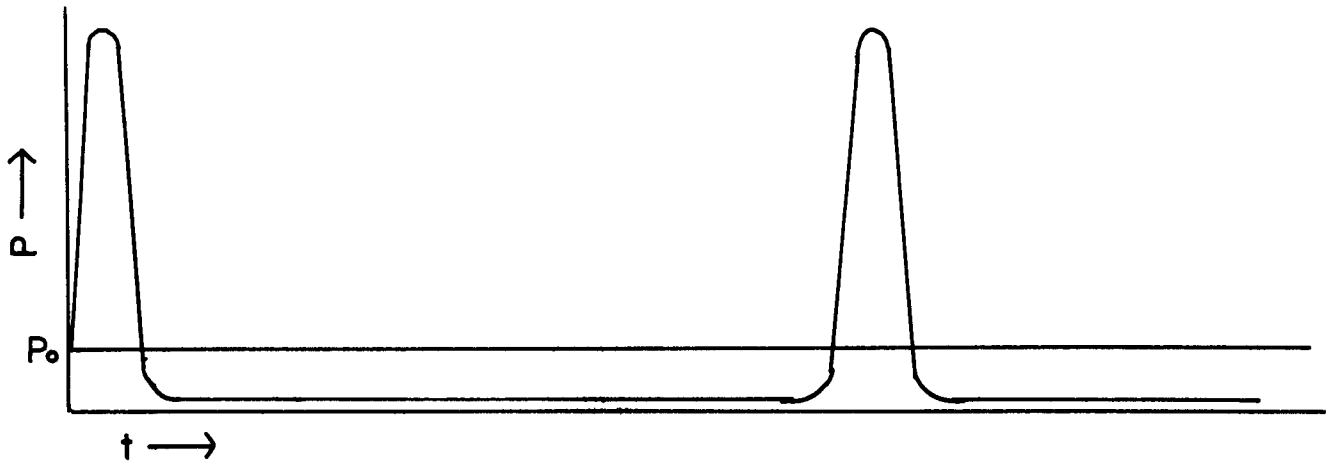

FIGURE 66

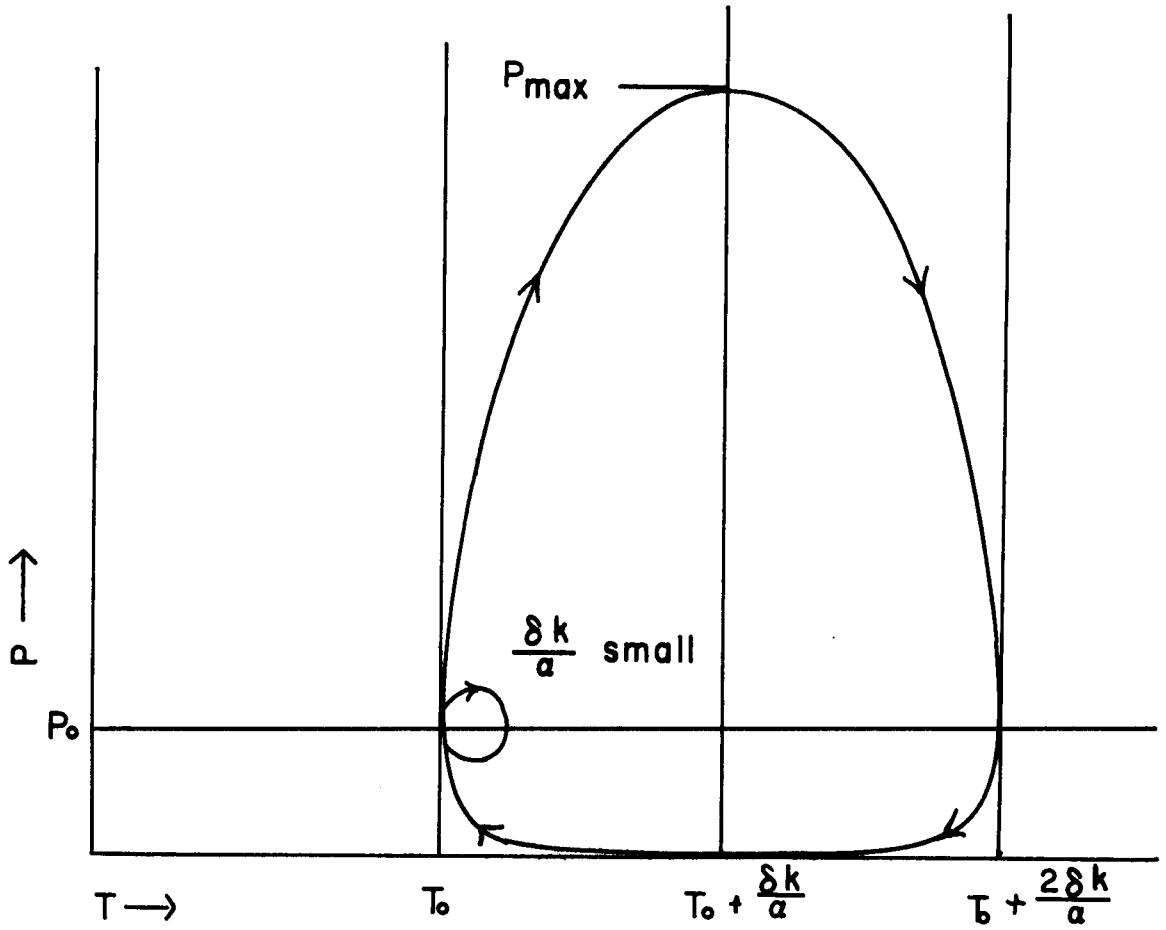

FIGURE 67

[redacted]

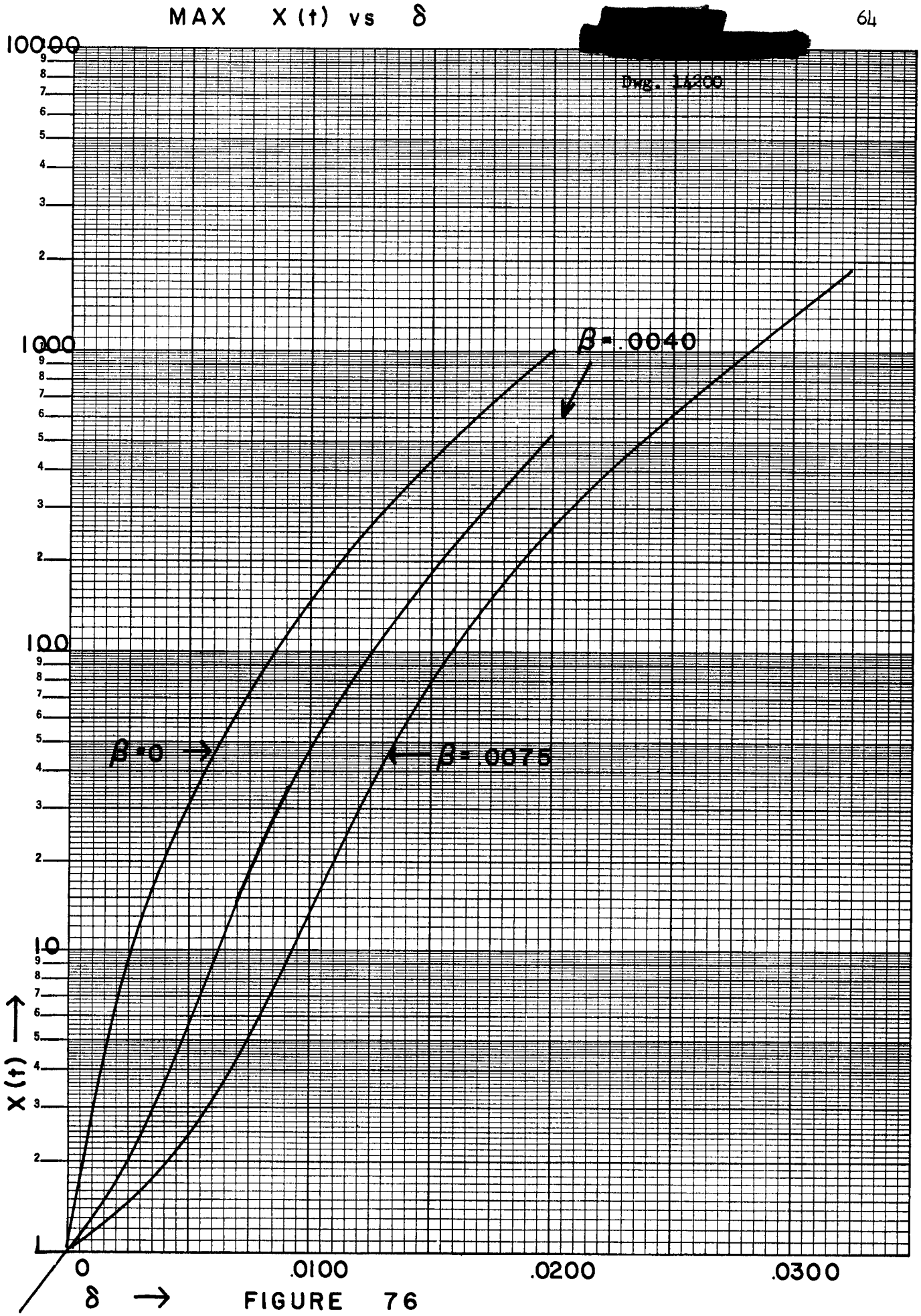

FIGURE 76

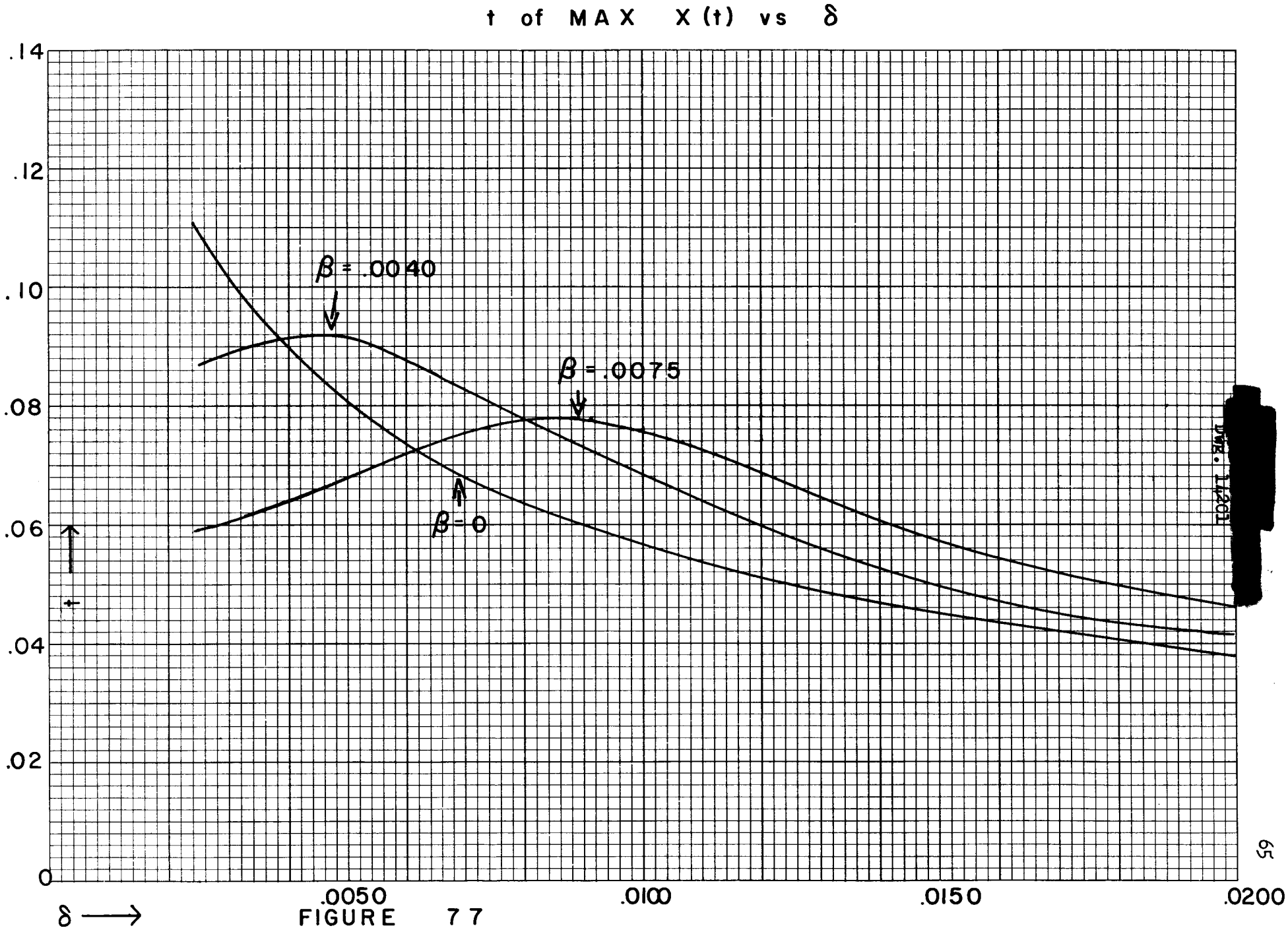

FIGURE 77

[redacted]

6 FT. MAX X(t) vs δ

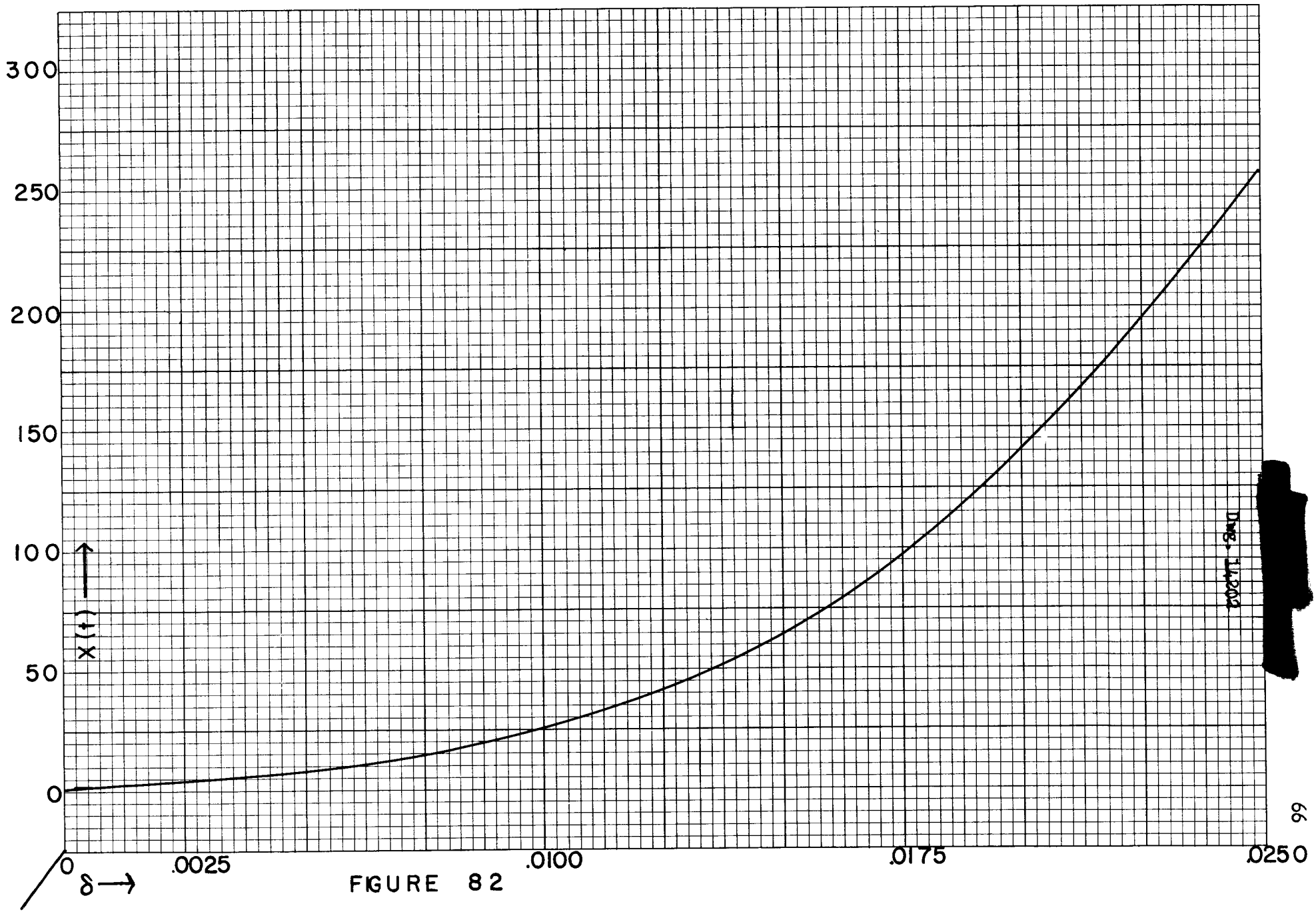

FIGURE 82

[redacted]

6 FT. MAX V(t) vs δ [redacted]

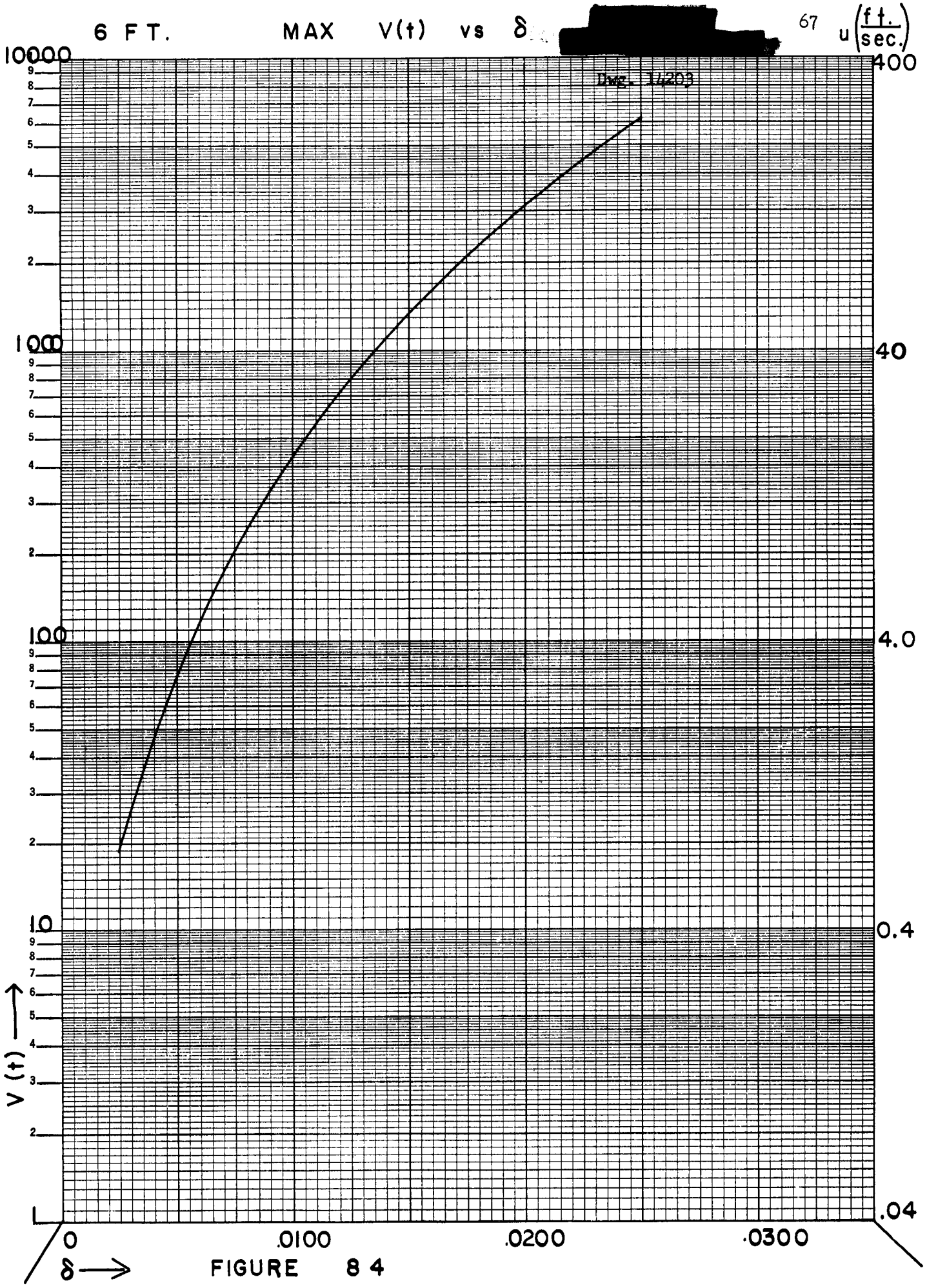

FIGURE 84

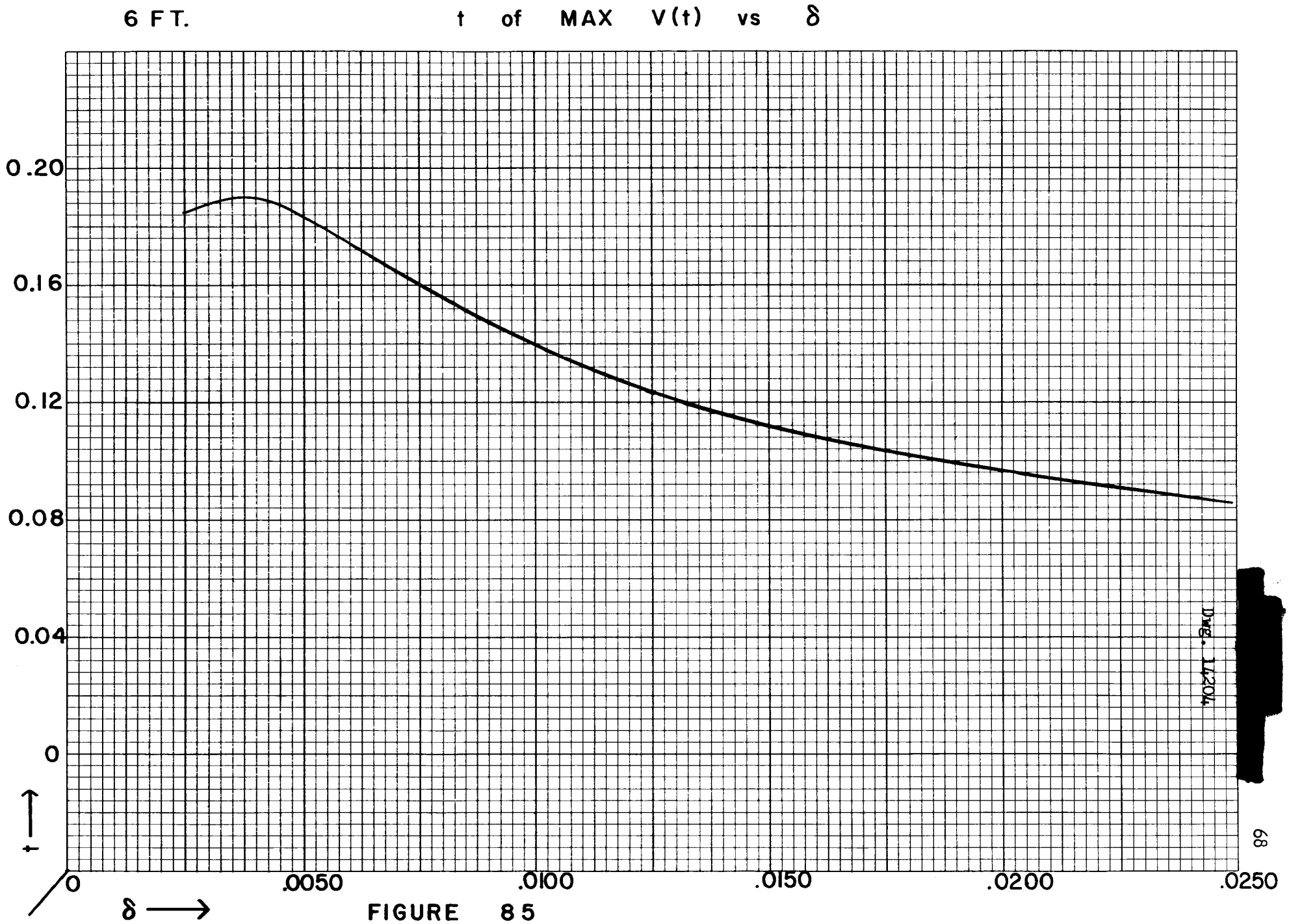
6 FT.
t of MAX V(t) vs δ
0.20
0.16
0.12
0.08
0.04
0
t
0
.0050
.0100
.0150
.0200
.0250
δ
Dwg. 1/4204

FIGURE 85

6 FT. MAX $\bar{p}$ (t) vs δ [redacted]

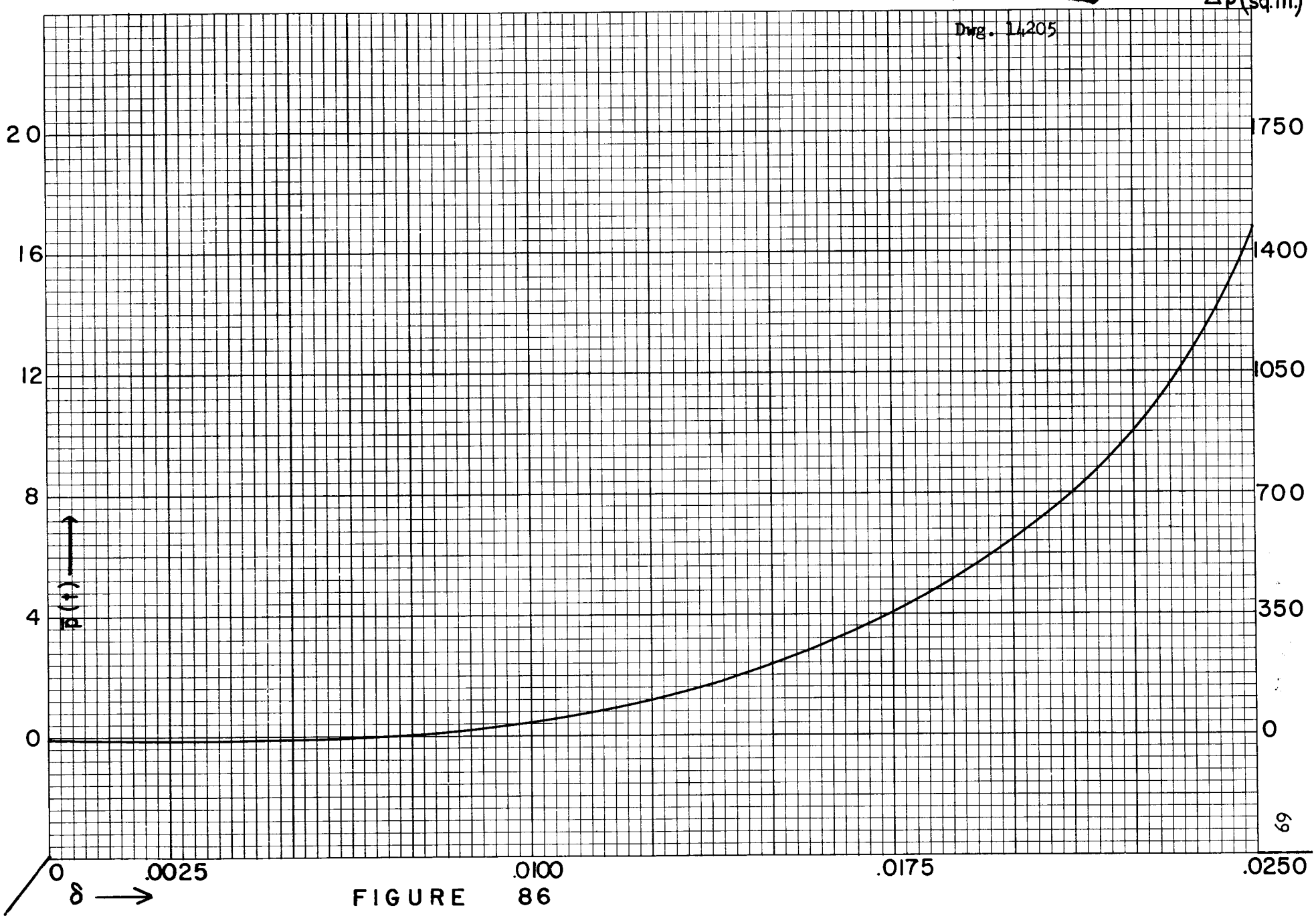

FIGURE 86

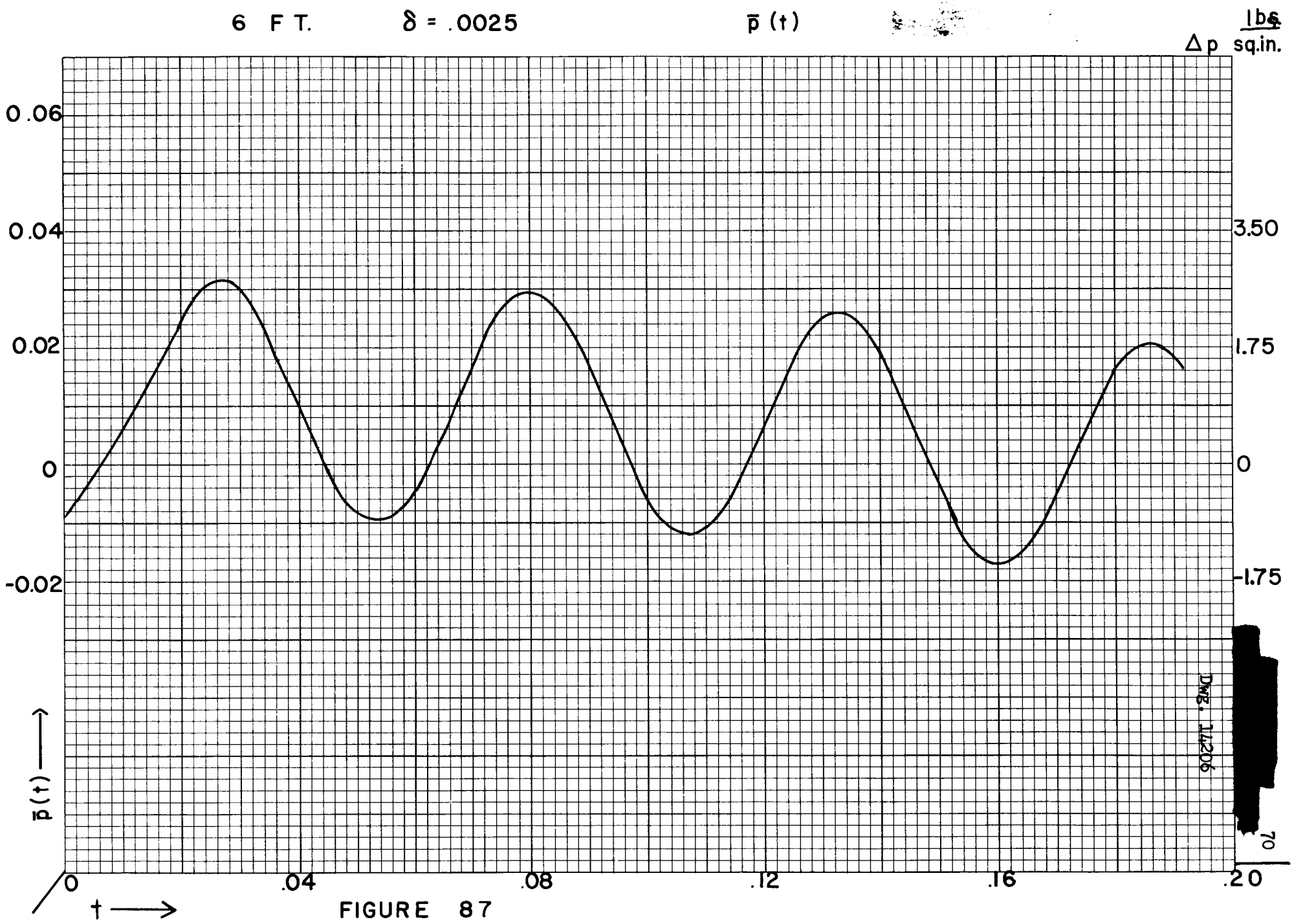
6 F T.
δ = .0025
p̄ (t)
Δp lbs/sq.in.
0.06
0.04
0.02
0
-0.02
3.50
1.75
0
-1.75
p̄ (t) ⟶
0
.04
.08
.12
.16
.20
t ⟶

FIGURE 87

6 F T.
δ = .0100
p̄ (t)
[redacted]
Δ p (lbs. / sq. in.)
Dwg. 14207
0.5
0.4
0.3
0.2
0.1
0
-0.1
p̄ (t) ⟶
35.0
17.5
0
0
.04
.08
.12
.16
.20
t ⟶

FIGURE 88

[redacted]

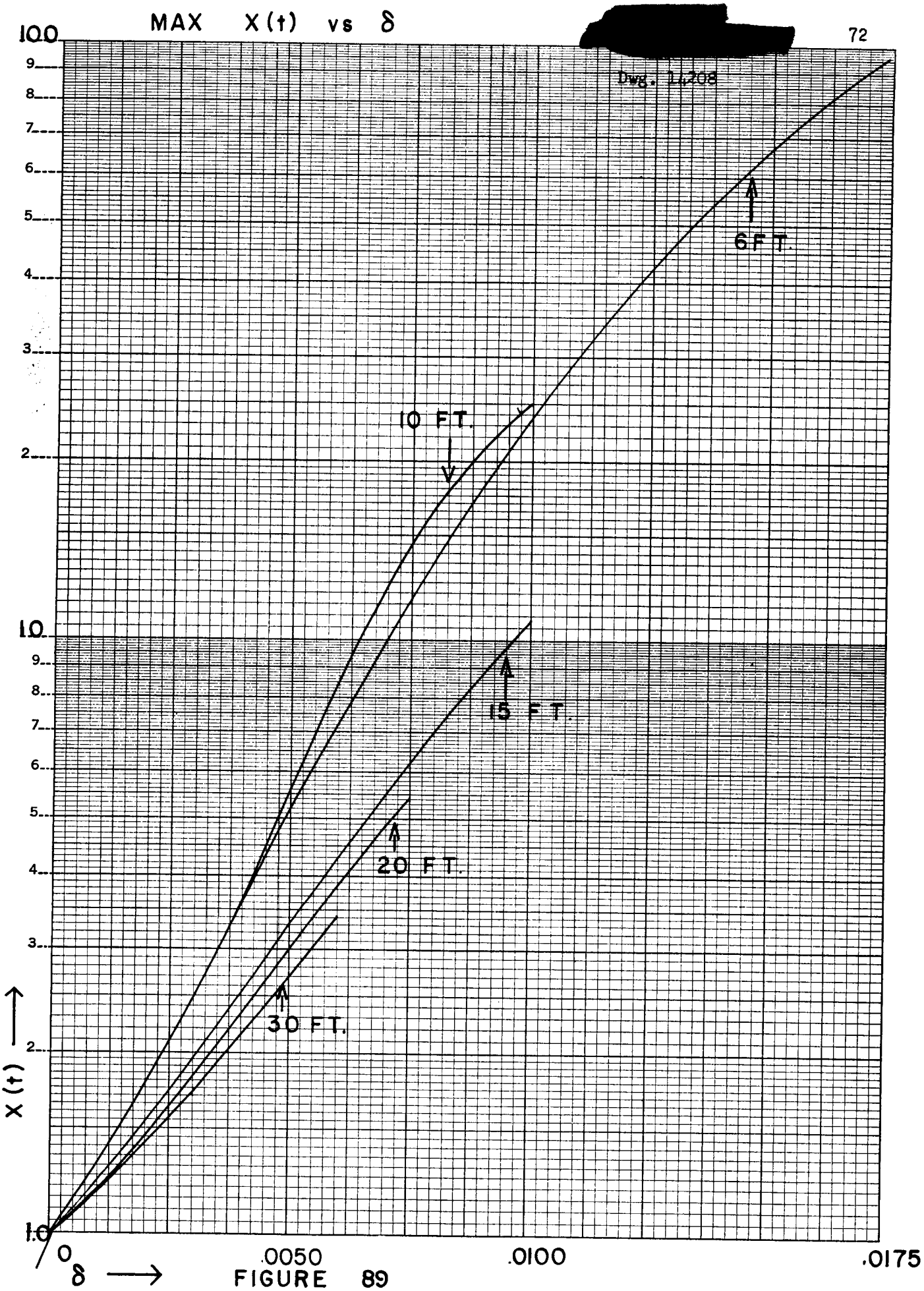

FIGURE 89

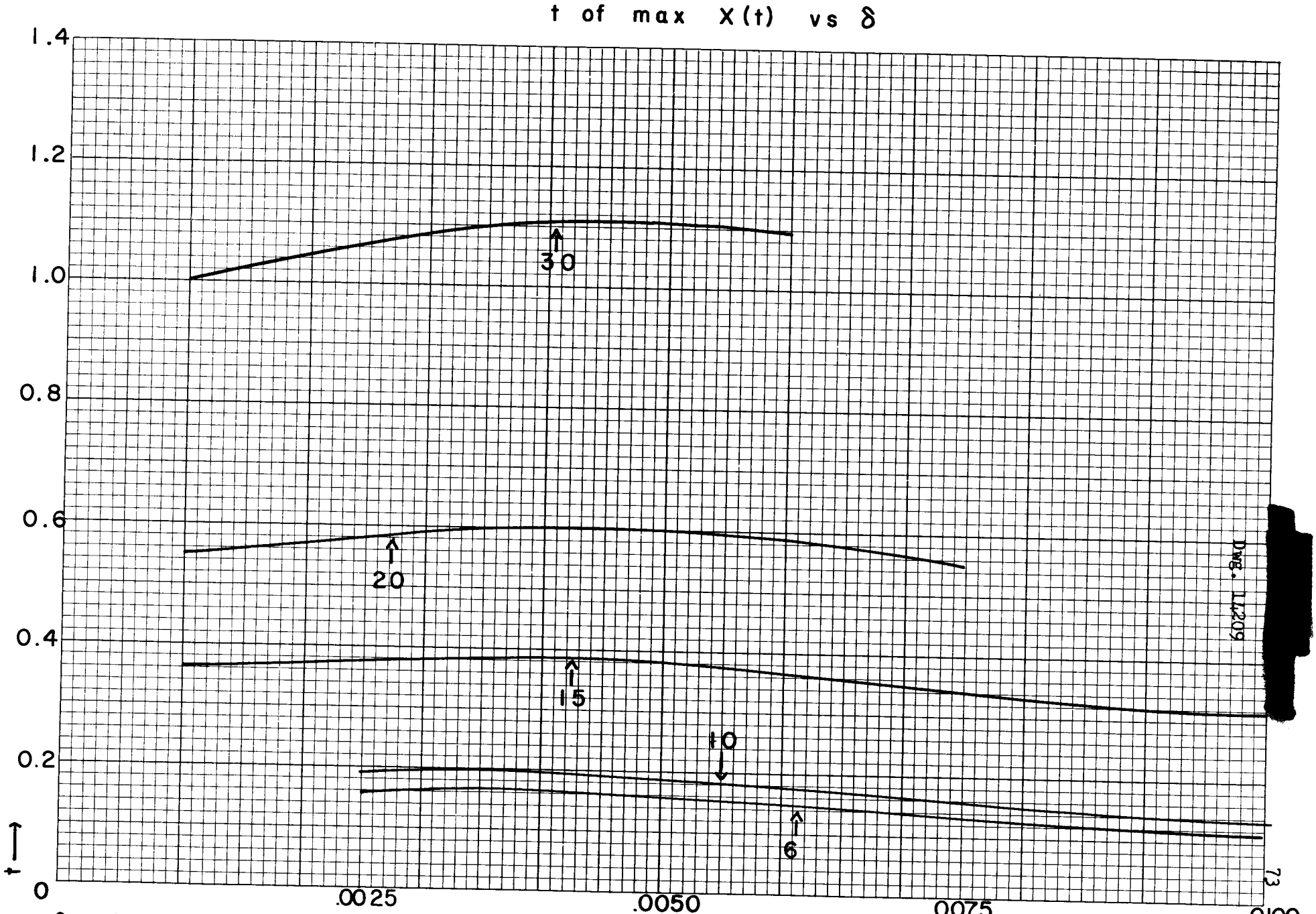

FIGURE 90

Dwg. 14209

[redacted]

[redacted]

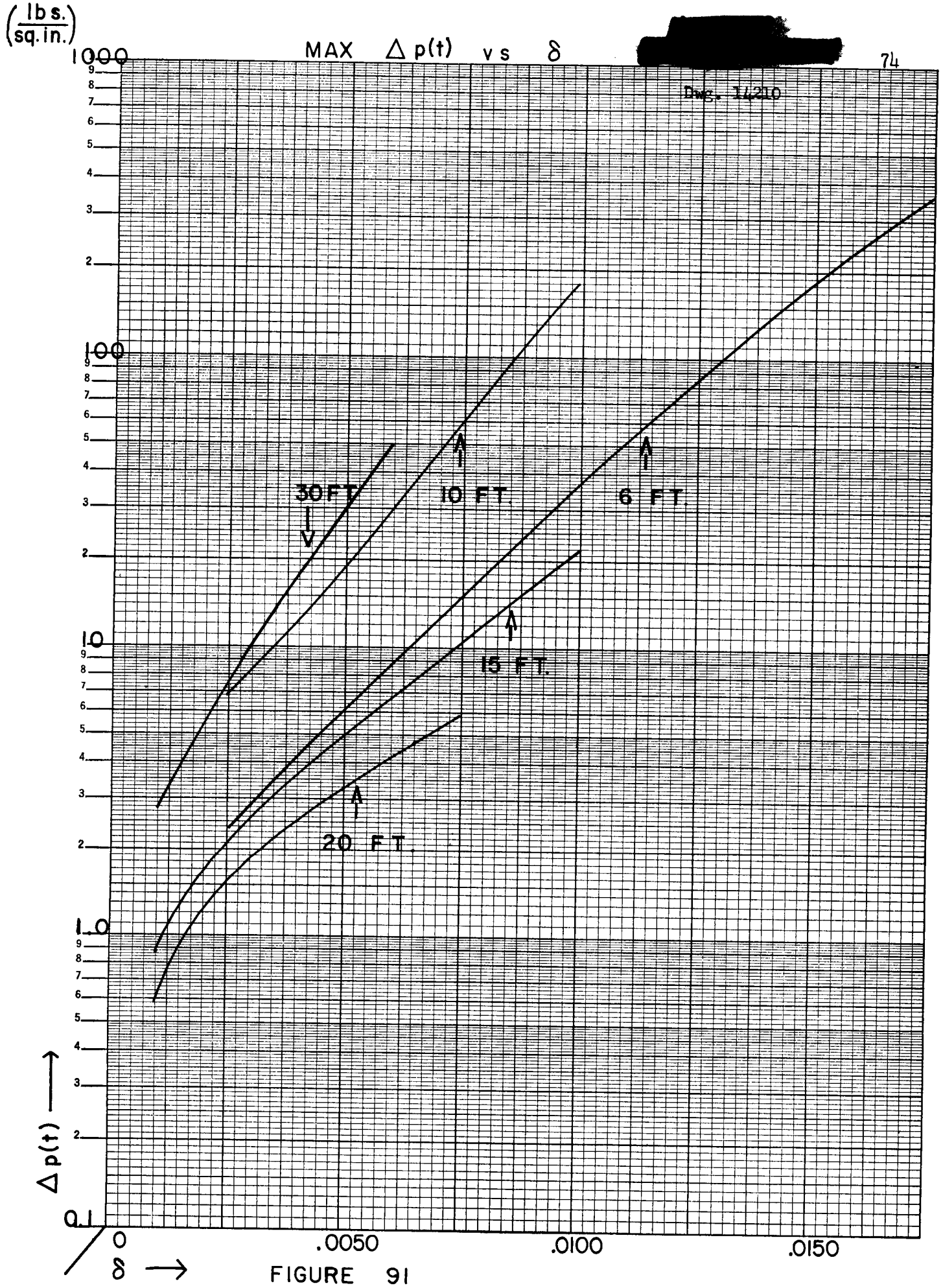

FIGURE 91

[redacted]

Dwg. 14211

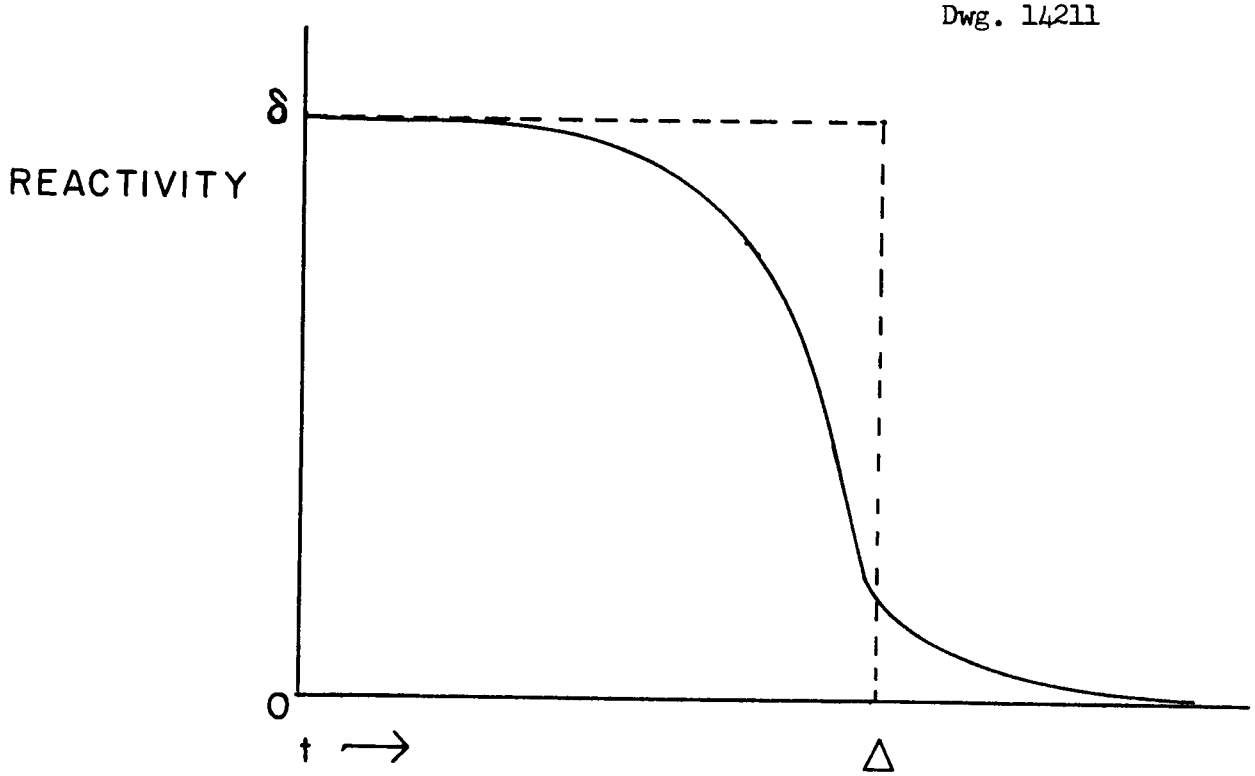

FIGURE 92

TABLE OF TABLES

SEAC	1	Maximum of x, y, z, v and $\bar{p}$ with time
Formulas	2	List of iterative formulas for x, y, z, $\bar{p}$
	2A	Iterative formula for v
	3	List of short formulas for x_{max}, t of x_{max} and $\bar{p}$
Comparisons	4	Maximum x and Δp from SEAC and formulas (HRE)
	5	Time of maximum x from SEAC and formulas (HRE)
	6	Maximum x and Δp from formulas (larger cores)
	7	Time of maximum x from formulas (larger cores)
	8	Maximum $\dot{x}$ from SEAC and formulas (HRE)
Symbols	9	Symbols

TABLE 1

System**	Maximum x	Time	Y	y_E	Z	Z_E	Maximum v	Time	Maximum P	Time
4	4560[*]	.032	55.0	7.5	-2800.*	-375	67,300*	.039	1860*	.039
9	14.9	.084	1.49	2.0	-74.5	-100	717.3*	.080	.73*	.060
12	36.2[*]	.072	2.27	2.5	-114.*	-125	1920*	.074	2.77*	.062
13	145.4[*]	.055	4.70	3.5	-252.*	-175	7567*	.064	28.0 *	.055
10	2.53	.084	.42	1.0	-21.0	-50	90*	.070	.156*	.018
11	4.71	.091	.76	1.4	-38.0	-70	200*	.070	.250*	.018
1	7470.	.026	58.1	7.5	-5400*	-375	400,000*	.036	2500*	.065
5	128.8	.058	4.1	2.0	-236*	-100	8,550*	.064	32.0*	.098
2	3360[*]	.032	32.8	7.5	-2938*	-375	205,000*	.039	1275*	.067
3	4560[*]	.032	55.0	7.5	-2800*	-375	67,300*	.039	1860*	.039
7	14.9	.084	1.49	2.0	-74.5	-100	720*	.078	.64*	.058
8	14.9	.084	1.49	2.0	-74.5	-100	720*	.082	.73*	.060
6	14.9	.084	1.49	2.0	-74.5	-100	720*	.078	.64*	.058

** $\delta k = \frac{-Z_E}{100}$

* oscillation observable

[*] oscillation occurs but not observable on curve

TABLE 2

Iterative Formulas

$$x = f_1 + f_2 t + f_3 t^2 + f_5 e^{\Psi t} + f_6 e^{2\Psi t} + f_7 t e^{\Psi t}$$

$$y = a_1 + a_2 t + a_3 t^2 + a_4 t^3 + a_5 e^{\Psi t} + a_6 e^{2\Psi t} + a_7 t e^{\Psi t}$$

$$z = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{\Psi t} + c_6 e^{2\Psi t} + c_7 t e^{\Psi t} + A\cos w_H t + B \sin w_H t$$

$$\overline{p} = z + w_n^2 y = E_1 + E_2 t + E_5 e^{\Psi t} + E_6 e^{2\Psi t} + E_7 t e^{\Psi t} + A\cos w_H t + B \sin w_H t$$

$$\Psi = \frac{\delta - \beta}{\tau}$$

where:

$$f_1 = -\left[f_5 + f_6 - 1\right]$$

$$f_2 = -\frac{\beta}{\delta} f_7$$

$$f_3 = \frac{f_2 \Psi}{2}$$

$$f_5 = \frac{1}{\Psi}\left[\frac{\delta}{\tau} + \frac{\beta\delta}{\tau^2\Psi^3} \cdot \frac{w_H^2 w_n^2}{\Psi^2 + w_H^2}\right]$$

$$f_6 = \frac{-w_n^2 w_H^2 \delta^2}{2\tau^2\Psi^4\left[\Psi^2 + w_H^2\right]}$$

$$f_7 = \frac{w_n^2 \delta^2}{\tau^2 \Psi^3}$$

$$a_1 = -\left[a_5 + a_6\right]$$

$$a_2 = f_1 - 1$$

$$a_3 = f_2/2$$

$$a_4 = f_3/3$$

$$a_5 = \frac{f_5}{\Psi} - \frac{f_7}{\Psi^2}$$

$$a_6 = \frac{f_6}{2\Psi}$$

$$a_7 = \frac{f_7}{\Psi}$$

$$c_1 = \left[\frac{2}{w_H^2} a_3 - a_1\right] w_n^2 = \left[\frac{f_2}{w_H^2} + \frac{2\Psi f_5 - 2f_7 + \Psi f_6}{2\Psi^2}\right] w_n^2$$

$$c_2 = \left[\frac{6}{w_H^2} a_4 - a_2\right] w_n^2 = \left[\frac{\Psi f_2}{w_H^2} - f_1 + 1\right] w_n^2$$

$$c_3 = -a_3 w_n^2 = \frac{-w_n^2 f_2}{2}$$

$$c_4 = -a_4 w_n^2 = \frac{-w_n^2 f_3}{3}$$

$$c_5 = \left[\frac{-w_H^2}{w_H^2 + \Psi^2} a_5 + \frac{2\Psi w_H^2}{(w_H^2 + \Psi^2)^2} a_7\right] w_n^2 = \frac{w_H^2 \, w_n^2}{\Psi\left[w_H^2 + \Psi^2\right]}\left[- f_5 + \frac{f_7}{\Psi}\left(\frac{w_H^2 + 3\Psi^2}{w_H^2 + \Psi^2}\right)\right]$$

$$c_6 = \left[\frac{-w_H^2}{w_H^2 + 4\Psi^2} a_6\right] w_n^2 = \frac{-w_H^2 \, w_n^2 f_6}{2\Psi\left[w_H^2 + 4\Psi^2\right]}$$

$$c_7 = \left[\frac{-w_H^2}{w_H^2 + \Psi^2} a_7\right] w_n^2 = \frac{-w_H^2 \, w_n^2 f_7}{\Psi\left[w_H^2 + \Psi^2\right]}$$

$$A = -\left[c_1 + c_5 + c_6\right]$$

$$B = -\frac{1}{w_H}\left[c_2 + \Psi c_5 + 2\Psi c_6 + c_7\right]$$

$$E_1 = c_1 + w_n^2 a_1 = \frac{2w_n^2}{w_H^2} a_3 = \frac{w_n^2}{w_H^2} f_2$$

$$E_2 = c_2 + w_n^2 a_2 = 6\frac{w_n^2}{w_H^2} a_4 = \frac{2w_n^2}{w_H^2} f_3 = \Psi E_1$$

$$E_3 = c_3 + w_n^2 a_3 = 0$$

$$E_4 = c_4 + w_n^2 a_4 = 0$$

$$E_5 = c_5 + w_n^2 a_5 = \frac{w_n^2 \Psi^2}{w_H^2 + \Psi^2} a_5 + 2\Psi\frac{w_H^2 w_n^2}{\left(w_H^2 + \Psi^2\right)} a_7 = \frac{w_n^2}{w_H^2 + \Psi^2}\left[\Psi f_5 + \frac{w_H^2 - \Psi^2}{w_H^2 + \Psi^2} f_7\right]$$

$$E_6 = c_6 + w_n^2 a_6 = w_n^2\left[1 - \frac{w_H^2}{w_H^2 + 4\Psi^2}\right] a_6 = \frac{2\Psi w_n^2}{w_H^2 + 4\Psi^2} f_6$$

$$E_7 = c_7 + w_n^2 a_7 = w_n^2\left[1 - \frac{w_H^2}{w_H^2 + \Psi^2}\right] a_7 = \frac{\Psi w_n^2}{w_H^2 + \Psi^2} f_7$$

TABLE 2A

Additional Iterative Formula

$$v = g_1 + g_2 t + g_3 t^2 + g_5 e^{\gamma t} + g_6 e^{2\gamma t} + g_7 t e^{\gamma t} + \frac{1}{w_H}\left[- A\sin w_H t + B\cos w_H t\right]$$

where

$g_1 = - c_2$

$g_2 = - 2c_3$

$g_3 = - 3c_4$

$g_5 = - \left[\gamma c_5 + c_7\right]$

$g_6 = - 2\gamma c_6$

$g_7 = - \gamma c_7$

TABLE 3

Short Formulas

$$*x_{max} = 1 + \frac{\Psi^2}{2w_n^2}\left[1 + \frac{\Psi^2}{w_H^2}\right] + \frac{\delta}{\delta - \beta}\log\left[\frac{\delta - \beta}{\delta} x_{max} + \frac{\beta}{\delta}\right] + \frac{\beta\Psi}{2w_n^2\tau}\left[1 + \frac{\Psi^2}{w_H^2}\right]\frac{1}{x_{max}}$$

For $\beta > \delta$ $\quad x_{max} = \frac{\beta}{\beta - \delta}$

$$t \text{ of } x_{max} = \frac{1}{|\Psi|}\log\left[1 + \frac{\Psi^2}{w_n^2}\left(1 + \frac{\Psi^2}{w_H^2}\right) + \log\left\{1 + \frac{\Psi^2}{w_n^2}\left(1 + \frac{\Psi^2}{w_H^2}\right)\right\}\right]$$

$$\dot{x}_{max} = .42\,\Psi\, x_{max}$$

For $\beta > \delta$ $\quad \dot{x}_{max} = \frac{\delta}{\tau}$

$$\bar{P}_i = \frac{w_n^2}{w_H^2}\,\dot{x}_{max}$$

$$Pf = \alpha_1 \frac{w_n^2}{w_H^2}(x_{max} - 1) + \alpha_2 \frac{w_n^4}{w_H^2}(x_{max} - 1)^2$$

Scaling Formula For p: $\quad \Delta p = \bar{p}\,\dfrac{v_5^2\,\tau}{\frac{\partial k}{\partial p}(144)(32.2)}$

$$* \; r = a + b\log r + c\frac{1}{r}$$

for large r and a: $\quad r_1 = a$

$$r_2 = a + b\log r_1 + c\frac{1}{r_1},$$

$$r_3 = a + b\log r_2 + c\frac{1}{r_2}$$

etc.

Second short formula $w_n^2 = 0$

$$x_{max} = 1 + w_n^2\frac{\delta}{\tau^2\Psi^2}\left[\frac{\beta}{\Psi}t + \frac{\beta}{2}t^2 + \frac{\delta}{2\Psi^2}e^{2\Psi t} - \frac{\delta}{\Psi}te^{\Psi t} - \left(\frac{\tau\Psi}{w_n^2} + \frac{\beta}{\Psi^2}\right)e^{\Psi t} + \left(\frac{\tau\Psi}{w_n^2} + \frac{\beta}{\Psi^2} - \frac{\delta}{2\Psi^2}\right)\right]$$

TABLE 4

$w_n^2 = 50$
$w_H^2 = 25000$
$\tau = .0001$

System	SEAC $\frac{P}{P_o}$	Iterative Formula $\frac{P}{P_o}$	Short Formula $\frac{P}{P_o}$	SEAC Δp #/Sq. in.	Iterative Formula Δp #/Sq.in.	Short Formula Δp_i #/Sq.in.	Short Formula Δp_2 #/Sq.in.	δ	α_1	α_2
1*	7470.		9326.5	34000 50000		58756.	0	.0375	0	0
2	3360		4151.4	15000 25500		20923.	49.80	.0375	0.3	0
3	4560		4151.4	37200		20923.	344520.	.0375	0	0.01
4	4560	4250	4151.4	37200	37400.	20923.	344520.	.0375	0.3	0.01
5*	128.8		145.98	280 600		245.24	0	.0100	0	0
6	14.9		14.542	12.8		6.1077	.05417	.0100	0.1	0
7	14.9		14.542	12.8		6.1077	.1625	.0100	0.3	0
8	14.9		14.542	14.6		6.1077	3.668	.0100	0	0.01
9	14.9		14.542	14.6		6.1077	3.830	.0100	0.3	0.01
10	2.53	2.43	3.0	3.12	3.20	2.1875	.1040	.0050	0.3	0.01
11	4.71	4.425	15.0	5.0	4.76	3.0625	4.088	.0070	0.3	0.01
12	36.2	36.88	36.421	56.0	50.0	30.594	25.52	.0125	0.3	0.01
13	145.4		149.49	560.		251.15	442.8	.0175	0.3	0.01
14		1.441			1.32			.0025	0.3	0.01
15		78.5			157.4			.0150	0.3	0.01
16		268.3			8440.			.0200	0.3	0.01

*$\beta = 0$; all others $\beta = .0075$

TABLE 5

System	SEAC t of x_{max}	Iterative Formula t of x_{max}	Short Formula t of x_{max}	
1*	.026			.0375
2	.033		.030	.0375
3	.032		.030	.0375
4	.032	.058	.030	.0375
5*	.058	.057		.0100
6	.084		.11	.0100
7	.084		.11	.0100
8	.084		.11	.0100
9	.084		.11	.0100
10	.084	.068		.0050
11	.091	.076		.0070
12	.072	.066	.082	.0125
13	.055		.057	.0175
14		.060		.0025
15		.058		.0150
16		.046		.0200

* $\beta = 0$; all others $\beta = .0075$

TABLE 6

Diam. of Core			δ	Iterative Formula $\frac{P}{P_o}$	Short Formula $\frac{P}{P_o}$	Iterative Formula ΔP #/Sq.in.	Short Formula ΔP_i #/Sq.in.	Short Formula ΔP_f #/Sq.in.
6	β = .0030	k_ρ = .00714	.0025	2.085	6.00	2.796	1.014	.07382
	τ = .00028	v_s^2 = 1.065 X 10^7	.0050	5.114	4.1614	5.242	1.418	.04667
	w_n^2 = 18.2	S = 87.36	.0075	11.84	11.477	14.85	8.798	.1023
	w_H^2 = 14,000	α_1 = .130	.0100	23.0	22.915	35.56	27.33	.3235
			.0175	95.0	94.052	353.8	232.3	1.374
			.0250	255.	251.53	1512.	942.7	3.699
10	β = .0034	k_ρ = .00217	.0025	2.06	3.7778	6.847	3.695	.1692
	τ = .00021	v_s^2 = .825 X 10^7	.0050	5.50	5.0771	18.87	5.042	.2366
	w_n^2 = 22.3	S = 167.0	.0075	14.2	13.531	59.29	34.44	1.706
	w_H^2 = 12,000	α_1 = .050	.0100	30.0	29.486	182.0	120.8	4.472
15	β = .0038	k_ρ = .00227	.0010	1.216	1.3571	.8652	.4518	.00385
	τ = .00077	v_s^2 = 1.083 X 10^7	.0025	1.70	2.9231	2.148	1.129	.02074
	w_n^2 = 4.98	S = 768.4	.0050	3.20	1.7067	4.882	.3886	.00762
	w_H^2 = 11,000	α_1 = .031	.0075	6.13	5.4938	10.30	3.857	.03115
			.0100	10.9	10.615	21.55	12.49	.1037
20	β = .0041	k_ρ = .00194	.0010	1.20	1.3226	.5524	.2833	.00729
	τ = .00127	v_s^2 = 1.182 X 10^7	.0025	1.64	2.5625	1.489	.7082	.00267
	w_n^2 = 2.00	S = 1619.	.0050	2.98	1.3746	1.3746	1.525	.01293
	w_H^2 = 9,000	α_1 = .023	.0075	5.50	4.5367	5.887	1.835	.02927
30	β = .0049	k_ρ = .00152	.0010	1.178	1.2564	2.715	1.507	.01282
	τ = .00237	v_s^2 = 1.271 X 10^7	.0025	1.560	2.0417	7.059	3.768	.05210
	w_n^2 = .517	S = 4145.	.0060	3.46	1.6096	49.74	1.121	.03047
	w_H^2 = 600	α_1 = .014						

TABLE 7

Diameter of Core	δ	Iterative Formula t of x_{max}	Short Formula
6	.0025	.19	.16
	.0050	.18	.23
	.0075	.16	.18
	.0100	.13	.15
	.0175	.10	.10
	.0250	.08	.08
10	.0025	.16	.21
	.0050	.15	.21
	.0075	.14	.16
	.0100	.12	.13
15	.0010	.36	.44
	.0025	.37	.42
	.0050	.37	.41
	.0075	.34	.42
	.0100	.31	.35
20	.0010	.55	.69
	.0025	.58	.69
	.0050	.60	.55
	.0075	.55	.68
30	.0010	1.00	1.27
	.0025	1.06	1.39
	.0060	1.09	1.22

TABLE 8

System	SEAC $\dot{x}_{max}$	SEAC x_{max}	SEAC c	Short Formula** $\dot{x}_{max}$	Short Formula** x_{max}	Second Short Formula*** $w_H^2 = 0$ x_{max}
1	1047700	7470	.374	1468908	9326.5	
2	401767	3360	.399	523068	4151.4	
3	478467	4560	.350	523068	4151.4	
4	478900	4560	.350	523068	4151.4	
5	5122.5	128.8	.398	6131.2	145.98	
6	248.03	14.9	.666	152.69	14.542	13.9
7	248.07	14.9	.666	152.69	14.542	13.9
8	248.53	14.9	.667	152.69	14.542	13.9
9	248.59	14.9	.667	152.69	14.542	13.9
10	41.673	2.53	*	50	3.0	2.42
11	67.270	4.71	*	70	15.0	
12	823.33	36.2	.455	764.86	36.421	32.3
13	6482.8	145.4	.446	6278.6	149.49	

* $\beta > \delta$ $\quad \dot{x}_{max} = \frac{\delta}{\tau}$

** $\beta < \delta$ $\quad \dot{x}_{max} = .424\ x_{max}$

*** second short formula for x_{max}

TABLE 9

Symbols

P = power level of reactor, kw

P_o = steady state power level (1000 kw)

P_i = initial power level

$x = P/P_o$

$Q = \log P/P_o$

T = temperature, °C

T_o = steady state temperature (240°C)

T_E = equilibrium temperature

S = reciprocal heat capacity, °C/kw-sec. (0.005 °C/kw-sec.)

$y = \frac{1}{SP_o}(T - T_o), \left[\frac{1}{5}(T - T_o)\right]$

$\alpha = \frac{\partial k}{\partial T}$ = temperature coefficient of reactivity (10^{-3}/°C)

τ = lifetime of prompt neutrons (10^{-4} sec.)

$w_n^2 = \frac{1}{\tau}\alpha SP_o$, square of nuclear angular frequency (50/sec.^{2})

β = bulk or average fraction of delayed neutron in core (0.0075)

λ = delay constant for the assumed single group of delayed neutrons emitted (0.08)

c = lumped concentration of delayed neutron emitters

k = material multiplication constant

ρ = density of core fluid, lbs./cu.ft.

ρ_o = steady state density (50 lbs./cu.ft.)

ρ_E = equilibrium density

$\frac{\partial k}{\partial \rho}$ = density coefficient of reactivity (0.0074 cu.ft./lbs.)

$z = \frac{1}{\tau}\left(\frac{\partial k}{\partial \rho}\right)(\rho - \rho_o), \quad 74\left[(\rho - \rho_o)\right]$

δ = fractional step change in reactivity

p = pressure in core, lbs./sq.in.

p_o = steady state pressure (1000 lbs./sq.in.)

v_s = velocity of sound in core (2560 ft./sec.)

$\overline{p} = z + w_n^2 y = \frac{1}{\tau v_s^2}\left(\frac{\partial k}{\partial \rho}\right)(p - p_o), \left[\frac{1}{20}(p - p_o)\right]$

U = excess velocity of fluid in relief pipe

V = volume of core (1.75 cu.ft.)

A = average cross-sectional area of relief pipe (0.037 sq. ft.)

M = mass of coolant in core

M_r = mass of coolant in relief pipe

$v = \frac{1}{\tau}\left(\frac{\partial k}{\partial \rho}\right)\frac{A\rho_o}{V} U$, (79U)

B, E = constants determined by pipe flow loss

$\alpha_1 = \frac{B}{M}$ = frictional constant in relief pipe

$\alpha_2 = \frac{E}{M} \quad \frac{A\rho_o}{\tau V}\left(\frac{\partial k}{\partial \rho}\right)^{-1}$ turbulent constant in relief pipe

L = length of pressurizer pipe (5.9 ft.)

$w_H^2 = \frac{A^2\rho_o}{VM} v_s^2$, square of pressurizer angular frequency (25,000/sec.^{2})

$\Psi = \frac{\delta - \beta}{\tau}$

d = rate of change of reactivity

π = step change in power

Δ = time of maximum power

ϕ = thermal flux

γ = proportionality factor between thermal flux and power

λ_o = reciprocal lifetime of prompt neutrons (10^4/sec.)

[redacted]

c_i = concentration of delayed neutron emitters

λ_i = time constant of delayed neutron emitter

k_c = multiplication constant required for criticality at temperature T

k_{co} = multiplication constant required for criticality at temperature T_o

k = material multiplication constant at time t

k_o = material multiplication constant at time t = 0

$\delta k = e = k_o - k_{co}$ = fractional step change in reactivity

β_i = fraction of neutrons from fission having time characteristic λ_i

$$\gamma_1 = \frac{1}{\rho_o}\left(\frac{\partial\rho}{\partial T}\right)^{-1}$$

$$\gamma_2 = \frac{1}{\rho_o}\left(\frac{\partial\rho}{\partial p}\right)^{-1}$$

$$\eta = \sum_i \beta_i \frac{1}{P - P_o}\int_{P_o}^{P} e^{\lambda_i\left[t(s)-t(p)\right]}ds$$

$$Y = \frac{\lambda_o}{wn}\left[e - \alpha\,(T - T_o)\right]$$

$$X = \frac{P}{P_o} - 1 = X - 1$$

$$w = k - k_c$$

$$K = \frac{\lambda_o}{wn}k\eta$$

c_t = rate of change of reactivity (external)

v_g = volume of gas as a fraction of core volume

v_{go} = steady state volume of gas as a fraction of core volume

ϵ = gas coefficient of reactivity

r = gas generation per unit core volume

V_o = steady state volume of core

$$\gamma_3 = \frac{1}{V_o}\left(\frac{\partial V}{\partial p}\right)$$

E_1 = constant determined by pipe flow loss

(HRE constants)

[redacted]