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Ward C. Sangren

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DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

Ward C Sangren

Date Issued

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PREFACE

A primary objective of this report is to indicate that multi-region or composite boundary value problems can be solved by using the conventional methods The conventional methods are, of course, the method of separation of variables and the method of the Laplace transform

This report consists essentially of two chapters Chapter I is divided into three parts which deal with the general N + 1-region or the N-interface problem In Chapter I the emphasis has been placed on obtaining formal results without a great deal of attention to mathematical rigor Chapter II by contrast deals quite rigorously with a Sturm-Liouville system with two regions or one interface

The two appendices are concerned with expansion problems associated with composite cylinders and spheres Since Carslaw and Jaegar [2]* obtain some of these results and give reference to other results which have been obtained by using the Laplace transform, the method of separation of variables has been used here It was intended originally to have an appendix with the results associated with "contact resistance" type interfaces [2] Although the results have been obtained for finite and infinite cases it is felt that sufficient illustrations of expansions are given without them

It is worth pointing out that in addition to the conventional heat conduction, diffusion, potential and vibration boundary value problems which lead to multi-region problems there also exist some interesting problems connected with

nuclear reactors In reactor language this report deals with a one group, multi-region reactor problem An important extension would be to the multi-groups, multi-region reactor problem

* The number in the brackets refers to the Bibliography

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CHAPTER I

Part I

GENERAL THEORY OF M-th ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

<u>Problem</u> The general linear differential equation of the M-th order is of the type

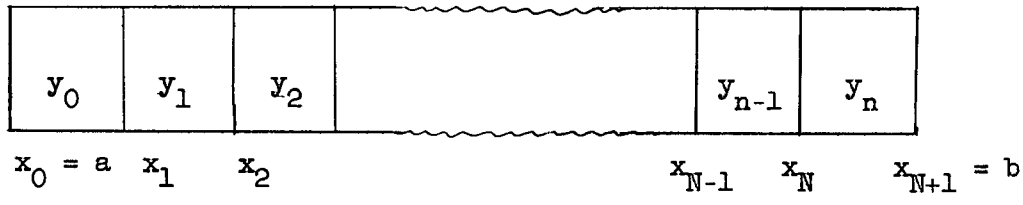
$$\sum_{i=0}^{M} g_i(x)\, y^{(M-i)}(x) = f(x)$$

In symbolic form this may be written as

$$L(y) = f(x) \quad , \tag{1 1}$$

where

$$L(y) \equiv \sum_{i=0}^{M} g_i(x)\, y^{(M-i)}(x)$$



The usual assumptions [9, p 114] are that the coefficients g_i ($i = 0, 1, \quad , M$) and the function $f(x)$ are real, continuous one-valued functions of x throughout an interval $a \leqslant x \leqslant b$ and that $g_0(x)$ does not

vanish at any point of the interval These assumptions are going to be relaxed here in the following fashion Let $a = x_0 < x_1 < x_2 < \quad < x_{N+1} = b$ and let $g_{ij} \equiv g_i$, $f_j \equiv f$ and $y_j \equiv y$ when $x_j < x < x_{j+1}$ $(j = 0, 1, \quad , N)$ In each subinterval g_{ij} and f_j are assumed to be continuous and one-valued and such that their limits when x tends to x_j or x_{j+1} exist and are finite We also assume that $g_{0j}(x)$ does not vanish at any point of the subinterval, and in general has the same sign in all the subintervals The final assumption is that at each "<u>interface</u>" or point of discontinuity x_k $(k = 1, 2, \quad , N)$ the y_j satisfy M homogeneous interface conditions of the form

$$(1\ 2) \qquad y_k^{(m)}(x_k) = \sum_{\ell=0}^{M-1} \beta_{\ell m}^{(k)} y_{k-1}^{(\ell)}(x_k) \qquad (m = 0, 1, \quad , M-1) ,$$

where the determinants, $\det(\beta_{\ell m}^{(k)})$, the superscript k indicating the particular matrix at the interface k , do not vanish Slightly more general interface conditions could be considered

The fundamental existence theorem of differential equations [9, p 73] proves that there exists a unique continuous solution $y_j(x)$ in each subinterval $x_j \leqslant x \leqslant x_{j+1}$ which assumes a given value $\hat{y}_j$ at a point $\hat{x}$ in the subinterval Further the first $M - 1$ derivatives of y_j are continuous and have respectively the given values $\hat{y}_j^{(m)}$ $(m = 1, \quad , M - 1)$ at $\hat{x}$ Further it follows from the interface conditions that there exists a unique solution $y(x)$, continuous in each subinterval, which assumes a given value $\hat{y}$ at a point $\hat{x}$ in a subinterval The first $M - 1$ derivatives of y are likewise unique and continuous in each subinterval and have respectively the given values $\hat{y}^{(m)}$ at the point $\hat{x}$

Fundamental Theorems We shall use the formal operator L^+ in the following sense If the equations $L(y) = 0$ and (1 2) are satisfied for y, then we shall write $L^+(y) = 0$ The following conventional elementary theorems can be easily shown

1 Let $y = v(x)$ be a solution of $L^+(y) = 0$, then $y = C\,v(x)$ is also a solution, where C is any arbitrary constant

2 Let $y = v_1(x)$, v_2, , v_ℓ be ℓ solutions of $L^+(y) = 0$ then $y = \sum_{m=1}^{\ell} c_m v_m$ is a solution of $L^+(y) = 0$, where the c's are arbitrary constants

3 Let $y = v_0(x)$ be any solution of equations (1 1) and (1 2), then if $v(x)$ is the complete primitive of $L^+(y) = 0$, $y = v_0(x) + v(x)$ is the general solution of equations (1 1) and (1 2) The complete primitive $v(x)$ of $L^+(y) = 0$ is given by the formula

$$v(x) = \sum_{m=1}^{M} c_m v_m(x) ,$$

where the $v_m(x)$ are M linearly independent solutions of $L^+(y) = 0$ and the c's are arbitrary constants

4 Let $y = v_1(x)$, v_2, , v_m be m $(m \leqslant M)$ solutions of $L^+(y) = 0$, then if the Wronskian vanishes at any point of the interval $a \leqslant x \leqslant b$ these solutions are linearly dependent The Wronskian is defined in the conventional manner as the determinant

$$\det(v_1 , v_2 , \quad , v_m) = \begin{vmatrix} v_1 , v_2 , & , v_m \\ v_1' , & \\ v_1^{(m-1)}, & , v_m^{(m-1)} \end{vmatrix}$$

In the subinterval j this notation signifies $\det(v_{1j}, v_{2j}, \quad , v_{mj})$, where $v_{ij} \equiv v_i$ in this interval At the interface x_k the notation applies to either the $v_{i\,k-1}$ or the v_{ik} , i e ,

$$\det(v_{1k}, v_{2k}, \quad , v_{mk}) = \det(\beta_{\ell h}^{(k)}) \det(v_{1\,k-1}, v_{2\,k-1}, \quad , v_{mk})$$

5 Let $\Delta_j(x)$ be the Wronskian for M linearly independent solutions of $L^+(y) = 0$ in the subinterval $x_j < x < x_{j+1}$, then

$$\Delta_j(x) = \Delta_j(\hat{x}) \exp\left[- \int_{\hat{x}}^{x} \frac{g_{1j}}{g_{0j}} dx \right] ,$$

where $x_j < \hat{x} < x_{j+1}$ In particular ,

$$\Delta_j(x_{j+1}) = \Delta_j(x_j) \exp\left[- \int_{x_j}^{x_{j+1}} \frac{g_{1j}}{g_{0j}} dx \right]$$

Since

$$\Delta_{j+1}(x_{j+1}) = \det(\beta_{\ell h}^{(j+1)}) \Delta_j(x_{j+1}) ,$$

it follows that

$$\Delta_{j+1}(x_{j+1}) = \det(\beta_{\ell h}^{(j+1)})\, \Delta_j(x_j) \exp\left[-\int_{x_j}^{x_{j+1}} \frac{g_{1j}}{g_{0j}}\, dx\right] ,$$

and therefore

$$\Delta_{j+1}(x) = \Delta_{j-m}(\hat{x}) \det(\beta_{\ell h}^{(j+1)}) \det(\beta_{\ell h}^{(j)}) \quad \det(\beta_{\ell h}^{(j-m)} \exp\left[-\int_{\hat{x}}^{x} \frac{g_1}{g_0}\, dx\right] ,$$

where $x_{j+1} \leqslant x \leqslant x_{j+2}$ and $x_{j-m} \leqslant \hat{x} \leqslant x_{j-m+1}$ This generalized <u>Abel identity</u> indicates that $\Delta_{j+1}(x)$ vanishes if and only if $\Delta_{j-m}(\hat{x})$ does

<u>Adjoint Operator</u> The formal linear differential operator L^+ can be associated with another formal operator $\bar{L}^+$ which can be called the formal or <u>Lagrange adjoint</u> of L^+ The notation $\bar{L}^+(z) = 0$ signifies that

$$\bar{L}(z) = \sum_{i=0}^{M} (-1)^i (g_i z)^{(M-i)} = 0$$

and that

$$z_k^{(m)}(x_k) = \sum_{\ell=0}^{M-1} \gamma_{\ell m}^{(k)} z_{k-1}^{(\ell)}(x_k) , \tag{1 3}$$

where the determinants, $\det(\gamma_{\ell m}^{(k)})$, do not vanish, and are such that a <u>generalized Lagrange identity</u> vanishes at each interface More specifically we can write in

each subinterval the Lagrange identity

$$z_j \, L(y_j) \; - \; y_j \; \bar{L}\,(z_j) \; = \; \frac{d}{dx} \; L^*(y_j \, , \, z_j) \quad ,$$

where

$$L^*(y_j \, , \, z_j) \; = \; \sum_{\ell=1}^{n} \; \sum_{p+q=\ell-1} \; (-1)^p \; y_j^{(q)} \; (f_\ell \; z_j)^{(p)}$$

The two integrations of the Lagrange identity or the Green's formulas which involve the interface at x_k are

$$\int_{x_{k-1}}^{x_k} \left[z_{k-1} \; L(y_{k-1}) \; - \; y_{k-1} \; \bar{L}(z_{k-1}) \right] dx \; = \; L^*(y_{k-1}, \, z_{k-1}) \; \Big|_{x_{k-1}}^{x_k}$$

and

$$\int_{x_k}^{x_{k+1}} \left[z_k \; L(y_k) \; - \; y_k \; \bar{L}\,(z_k) \right] \; dx \; = \; L^*(y_k \, , \, z_k) \; \Big|_{x_k}^{x_{k+1}}$$

Our requirement is that the matrices $(\gamma_{\ell m}^{(k)})$ be such that there exists non-zero constants A_k such that when the L*'s do not vanish at the interface

$$A_{k-1} \; L^* \left[y_{k-1} \, (x_k) \, , \, z_{k-1}(x_k) \right] \; = \; A_k \; L^* \left[y_k(x_k), \; z_k(x_k) \right]$$

for any y and z which satisfy the interface conditions

It appears desirable to show that the matrices $C_k \equiv (\gamma_{\ell m}^{(k)})$ not only exist but are unique up to an arbitrary scale factor We note that we may write the above equation in the form

$$(1\ 4) \qquad \gamma_k \; Y_{k-1}^t \; E_k \; Z_{k-1} = Y_k^t \; F_k \; Z_k \qquad ,$$

where the bilinear forms L^* are written in the form of a row vector times a matrix times a column vector and where $\gamma_k = \frac{A_{k-1}}{A_k}$ The vector Y_k^t indicates the vector $(y_k , y_k' , \quad , y_k^{(M-1)})$ where the superscript t indicates the transpose of a vector In similar fashion we may write

$$(1\ 5) \qquad Y_k = B_k \; Y_{k-1} \qquad \text{and} \qquad Z_k = C_k \; Z_{k-1} \qquad ,$$

in place of equations (1 2) and (1 3) By assumption the matrices B_k , C_k , E_k and F_k are non-singular Upon substituting from equations (1 5) in equation (1 4), we obtain,

$$\gamma_k \; Y_{k-1}^t \; E \; Z_{k-1} = Y_{k-1}^t \; B^t \; F \; C \; Z_{k-1}$$

or

$$Y_{k-1}^t \; (C_k \; E_k - B_k^t \; F_k \; C_k) \; Z_{k-1} = 0$$

Since this last statement is true for every vector Y_{k-1} and Z_{k-1}, it follows that

$$\gamma_k E_k = B_k^t F_k C_k ,$$

and consequently that

$$C_k = \gamma_k F_k^{-1} (B_k^t)^{-1} E_k$$

With the aid of the scale factor γ_k, this equation gives a recipe for obtaining the n matrices $C_k \equiv (\gamma_{\ell m}^{(k)})$

Equations (1 2) and (1 3) can be rewritten in the form

$$W_m \left[y_k(x_k)\right] \equiv y_k^{(m)} - \sum_{\ell=0}^{M-1} \beta_{\ell m}^{(k)} y_{k-1}^{(\ell)} (x_k) = 0$$

and

$$V_p \left[z_k(x_k)\right] \equiv z_k^{(p)} - \sum_{\ell=0}^{M-1} \gamma_{\ell p}^{(k)} z_{k-1}^{(\ell)} (x_k) = 0 ,$$

where $m , p = 0 , 1, \quad , M - 1$ The operator L^+ will be called self-adjoint if $L \equiv \bar{L}$ and if $V_p \left[y_k(x_k)\right] \equiv \sum_{m=0}^{M-1} \alpha_{pm}^{(k)} W_m \left[y_k(x_k)\right]$ where $\det (\alpha_{pm}^{(k)}) \neq 0$, is true for every k In this case the conditions $V_m = 0$ are said to be <u>equivalent</u> to the conditions $W_m = 0$

Boundary Conditions In addition to the interface conditions it is, of course, necessary to give a set of supplementary conditions in order to specify the operator (or problem) We shall be concerned with two-point boundary problems rather than one-point or initial conditions problems Our supplementary conditions have the form

$$(1\ 6) \qquad U_m(y) \equiv \sum_{\ell=0}^{M-1} \left[a_{\ell m}\, y^{(\ell)}(a) + b_{\ell m}\, y^{(\ell)}(b) \right] = 0 \ ,$$

$$(m = 1, 2, \quad , h)$$

For a particular m , in the cases to be considered, either all the $a_{\ell m} = 0$ or all the $b_{\ell m} = 0$ The h boundary conditions are assumed to be linearly independent

The equation $L_d(y) = 0$ signifies that $L^+(y) = 0$ and that $U_m(y) = 0$ L_d is then the "discontinuous" linear operator associated with the differential equations system defined by equations (1 1), (1 2) and (1 6) Let $\bar{L}_d(z) = 0$ signify that $\bar{L}^+(z) = 0$ and that $\bar{U}_m(z) = 0$, where

$$(1\ 7) \qquad \bar{U}_m(z) \equiv \sum_{\ell=0}^{M-1} \left[\bar{a}_{\ell m}\, z^{(\ell)}(a) + \bar{b}_{\ell m}\, z^{(\ell)}(b) \right] = 0$$

$$(m = 1, 2, \quad , p) \ ,$$

and the bars over the a's and b's indicate elements unrelated to the original a's and b's The operator L_d (or the associated differential system) is called

adjoint to the operator $\bar{L}_d$ (or the corresponding differential system), and conversely, if L^+ is adjoint to $\bar{L}^+$ and

$$A_0 L^* \left[y_0 (a) , z_0(a) \right] = A_N L^* \left[y_N(b), z_N(b) \right] ,$$

where A_0 and A_N are the constants previously used at the interfaces and where the relations (1 4) and (1 5) are used In the cases to be considered, the requirements at a and b are to be satisfied separately, i e ,

$$L^* \left[y_0(a), z_0(a) \right] = 0 = L^* \left[y_N (b), z_N (b) \right]$$

Further we shall consider from now on that $h = M$ in relation (1 4) and that $p = M$ in relation (1 5)

The operator L_d is called self-adjoint if L^+ is self-adjoint and if the conditions (1 6) and (1 7) are equivalent

Green's Function We now seek a generalized Green's function $G(x, \xi)$ for the differential system $L_d(y) = 0$ The Green's function satisfies the following conditions

(a) Each $G_j(x, \xi)$ is continuous and possesses continuous derivatives of orders up to and including $(M - 2)$ when $x_j \leqslant x \leqslant x_{j+1}$ $\left[G_j(x, \xi) \equiv G(x, \xi) \text{ when } x_j \leqslant x \leqslant x_{j+1} , (j = 0, 1, \quad , N) \right]$

(b) Each $G_j(x, \xi)$ possesses a derivative of order $M - 1$ which is discontinuous at $x = \xi$ Explicitly, the jump is given by

$$G_j^{(M-1)} (\xi^+ , \xi) - G_j^{(M-1)} (\xi^- , \xi) = \frac{1}{g_0 (\xi)} ,$$

where $x_j < \xi < x_{j+1}$

(c) The $G_j(x, \xi)$ formally satisfy $L_d(G_j) = 0$ except possibly at $x = \xi$

The "symmetry" property of the Green's function appears in the form

$$A(x) G(x, \xi) = A(\xi) H(\xi, x) ,$$

where $H(x, \xi)$ is the Green's function for the adjoint system $\bar{L}_d (y) = 0$ and where $A(x) \equiv A_j$, the A_j as previously used, when $x_j < x < x_{j+1}$ This may be verified in the following manner [9] Divide the interval from a to b into three parts (a, ξ_1) , (ξ_1 , ξ_2) , (ξ_2, b) and consider the two Green's functions $G(x, \xi_1)$ and $H(x, \xi_2)$ which correspond respectively to L_d and $\bar{L}_d$ The <u>generalized Green's formula</u> is

$$\int_a^b A(x) \left\{ z L (y) - y \bar{L} (z) \right\} dx = A(x) L^* (y, z) \Big|_a^b ,$$

where the range of integration may be regarded as the limiting case for the sum of the three intervals $(a, \xi_1 - \epsilon)$, $(\xi_1 + \epsilon , \xi_2 - \epsilon)$, $(\xi_2 + \epsilon , b)$ when ϵ tends

to zero Since $L^+(G) = 0$ and $\bar{L}^+(H) = 0$ in each of these ranges, it follows that

$$\lim_{\epsilon \to 0} \left\{ A(x)\, L^*(G, H) \Big|_{a}^{\xi_1 - \epsilon} + A(x)\, L^*(G, H) \Big|_{\xi_1 + \epsilon}^{\xi_1 - \epsilon} + A(x)\, L^*(G, H) \Big|_{\xi_2 + \epsilon}^{b} \right\} = 0$$

Upon applying the end conditions (1 6) and (7) and requirement (c) for the Green's functions this reduces to

$$A(\xi_1)\, H(\xi_1, \xi_2) = A(\xi_2)\, G(\xi_2, \xi_1) \quad ,$$

the result indicated above

This result has been proved when $\xi_2 > \xi_1$ and it may likewise be proved when $\xi_2 \leqslant \xi_1$ The above manipulations likewise were carried out for the ξ's lying within the subintervals If L_d is a self-adjoint operator then

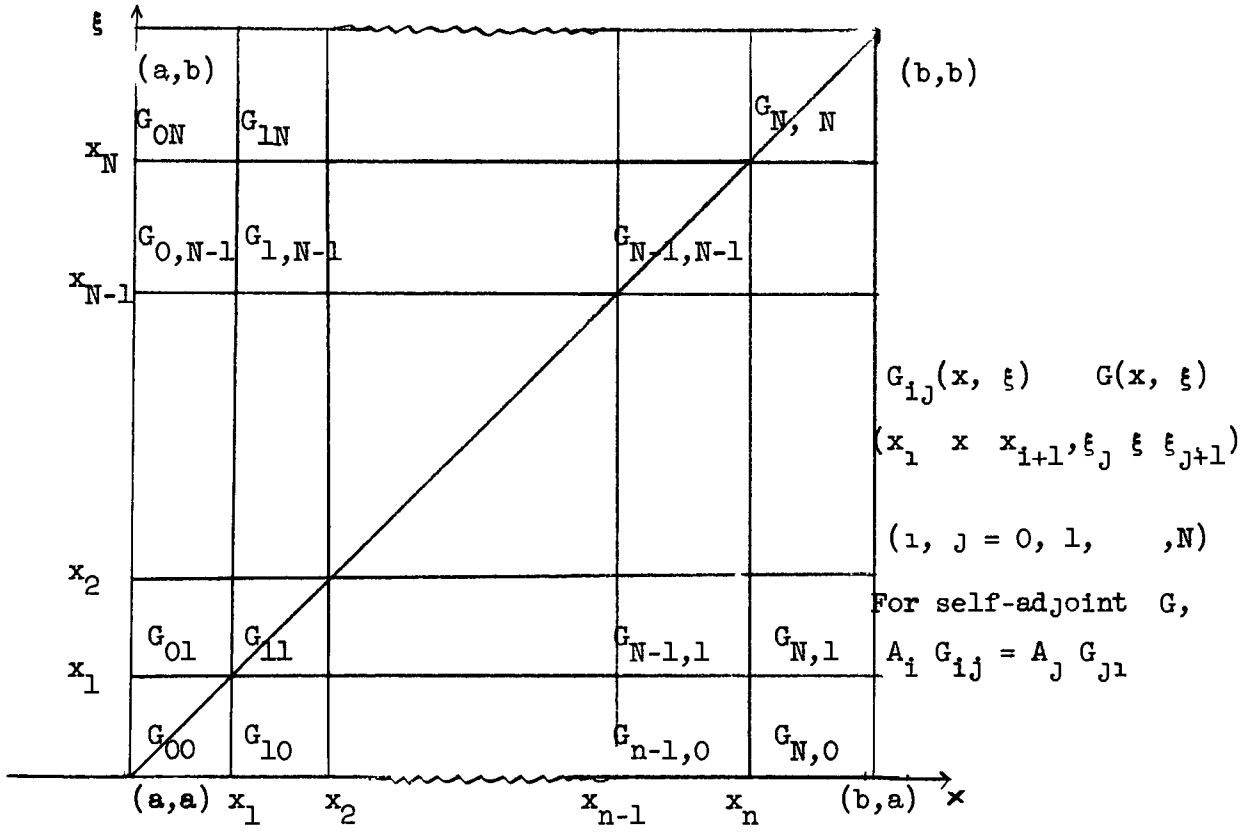
$$A(x)\, G(x, \xi) = A(\xi)\, G(\xi, x)$$

If therefore we define $S(x, \xi)$ by

$$S(x, \xi) = A(x)\, G(x, \xi) \quad ,$$

we see that $S(x, \xi)$ is symmetric when L_d is self-adjoint

The following diagram exhibits the character of the Green's function



<u>Non-homogeneous Solution</u> The non-homogeneous system

(1 8) $$L(y) = f(x) \quad ,$$

$$y_k^{(m)}(x_k) = \sum_{\ell=0}^{M-1} \beta_{\ell m}^{(k)} \, y_{k-1}^{(\ell)}(x_k) \qquad (m = 0, 1, \quad , M-1) \; ,$$

$$U_m(y) \equiv \sum_{\ell=0}^{M-1} \left[a_{\ell m} \, y^{(\ell)}(a) + b_{\ell m} \, y^{(\ell)}(b) \right] = 0 \quad (m = 1, 2, \quad , M) \; ,$$

can be solved with the aid of the generalized Green's function The solution for $y(x)$ is given by

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi$$

This can be verified by noting that

$$y^{(m)}(x) = \int_a^b \frac{\partial^m G(x, \xi)}{\partial x^m} f(\xi)\, d\xi \qquad (m = 1, 2, \quad , M - 1)$$

and that

$$y^{(M)}(x) = \int_a^b \frac{\partial^M G(x, \xi)}{\partial x^M} f(\xi)\, d\xi + \frac{f(x)}{g_0(x)}$$

The non-homogeneous system

(1 9) $$L(y) = f(x)$$

$$D_m \left[y(x_k) \right] \equiv y_k^{(m)}(x_k) - \sum_{\ell=0}^{M-1} \beta_{\ell m}^{(k)}\, y_{k-1}^{(\ell)}(x_k) = 0 \qquad (m = 0, 1, \quad , M-1)$$

$$U_m(y) \equiv \sum_{\ell=0}^{M-1} \left[a_{\ell m}\, y^{(\ell)}(a) + b_{\ell m}\, y^{(\ell)}(b) \right] = \nu_m \qquad (m = 1, 2, \quad , M) ,$$

has the solution

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi + \sum_{i=1}^{M} \nu_i\, G_i(x) ,$$

where

$$L(G_1) = 0 \quad , \quad D_m \left[G_i(x_k) \right] = 0$$

and

$$U_m(G_1) = \delta_{m1} \quad ,$$

where δ_{m1} is the Kronecker delta

Non-homogeneous systems involving non-homogeneous interface conditions could also be handled

CHAPTER I

PART II

THE EIGENVALUE PROBLEM FOR M-TH ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

Problem We shall now consider the differential system $L(y) = 0$, $U_m(y) = 0$ and $D_m\left[y(x_k)\right] = 0$, where the coefficients in the equation $L(y) = 0$ depend upon a parameter λ The notation remains the same as in Part I A primary problem is to find those values of λ for which the differential system has non-trivial solutions, i e , solutions other than $y \equiv 0$ These values of λ are called eigenvalues or characteristic numbers The solutions of the differential system corresponding to the eigenvalues are called eigenfunctions or characteristic functions

The eigenvalues are roots of an equation called the characteristic equation For our case we may look at this characteristic equation in two ways First if we let y_i $(i = 1, 2, \quad , M)$ be a fundamental set of real solutions, i e , $\sum_{i=1}^{M} C_i\, y_i\,(x)$ forms a complete primitive, of the system $L^+(y) = 0$, then the characteristic equation becomes

$$H(\lambda) \equiv \begin{vmatrix} U_i(y_i) & U_i(y_M) \\ U_M(y_i) & U_M(y_M) \end{vmatrix} = \det\left[U_m(y_i)\right] = 0$$

Second, if we let $y_{ij}(i = 1, 2, \quad , M, j = 0, 1, \quad , N)$ be the fundamental set of real solutions in the j-th interval for the equation $L(y) = 0$, then the characteristic equation becomes symbolically

$$h(\lambda) \equiv \det \left[E_{mn}(y_{ij}) \right] = 0,$$

where $E_{mn}(y_{ij})$ is defined as follows

$$E_{mo}(y_{1o}) = U_m^- \left[y_{1o}(a) \right] \quad , \qquad (m, i = 1, 2, \quad , M) ,$$

$$E_{mo}(y_{1j}) = 0 \quad , \qquad (j = 1, 2, \quad , N - 1) ,$$

$$E_{mo}(y_{1N}) = U_m^+ \left[y_{1N}(b) \right] \quad ,$$

$$E_{mn}(y_{in-1}) = D_m^- \left[y_{in-1}(x_n) \right] \quad , \qquad (n = 1, 2, \quad , N) \quad ,$$

$$E_{mn}(y_{1j}) = 0 \quad , \qquad (j \neq n - 1 \text{ or } n \quad ,$$

$$E_{mn}(y_{in}) = D_m^+ \left[y_{in}(x_n) \right] \quad ,$$

when

$$U_m(y) \equiv U_m^- \left[y(a) \right] + U_m^+ \left[y(b) \right] ,$$

and

$$D_m \left[y(x_k) \right] \equiv D_m^- \left[y_{k-1}(x_k) \right] + D_m^+ \left[y_k(x_k) \right]$$

In each subinterval j, the fundamental set of solutions y_{ij} and their derivatives with respect to x up to order M - 1 are continuous functions of the real variable pair x and λ [9, p 218] We are assuming here that in each subinterval the coefficients of the formal operator L are continuous functions of λ

At this stage it appears essential to restrict the dependence of the coefficients upon λ in the following fashion We shall consider the formal operator L to be defined either by

$$L = P - \lambda Q \quad ,$$

where

$$P(y) \equiv \sum_{i=0}^{p} (p_i y^{(i)})^{(i)}$$

and

$$Q(y) \equiv \sum_{i=0}^{q} (q_i y^{(i)})^{(i)} \quad ,$$

are self-adjoint expressions which do not depend upon λ or by

$$L = M - \lambda g(x) \quad ,$$

where

$$M(y) \equiv \sum_{i=0}^{M} g_i(x) \; y^{(M-i)}(x)$$

may be a self-adjoint expression and does not depend upon λ

General Problem We shall consider first therefore the problem defined by the equations

$$L(y) \equiv P(y) - \lambda Q(y) = 0 \qquad (2p = M,\ 0 \leqslant q < p) \quad ,$$

$$(2\ 1) \qquad U_m(y) = 0 \qquad (m = 1, 2, \quad , M) \quad ,$$

$$D_m\left[y(x_k) \right] = 0 \qquad (k = 1, 2, \quad , N) \quad ,$$

where $p_i = p_1(x)$ and $q_i = q_1(x)$ are i-differentiable continuous functions in each subinterval The eigenvalue problem given by the equations of (2 1) is called self-adjoint if the operator L_d is self-adjoint and is called adjoint if L_d is adjoint In particular this means that we shall define the eigenvalue problem to be self-adjoint if the relations

$$(2\ 2) \qquad \begin{aligned} \int_a^b A(x) \left[z\,P(y) - y\,P(z) \right] dx &= 0 \quad , \\ \int_a^b A(x) \left[z\,Q(y) - y\,Q(z) \right] dx &= 0 \quad , \end{aligned}$$

are satisfied for two arbitrary functions y and z which are M-differentiable in each subinterval and which satisfy either the end and interface conditions of problem (2 1) or satisfy linearly equivalent conditions If conditions (2 2) are

satisfied but non-linearly equivalent conditions were used, i e , adjoint conditions were used, at either the ends or interfaces for one of the arbitrary functions y and z , then the problem is called adjoint

<u>Orthogonality</u> Let us assume that λ and μ are two eigenvalues with associated eigenfunctions y_λ and y_μ for a self-adjoint problem (2 1) Since the problem is self-adjoint we have

$$\int_a^b A(x) \left[y_\lambda \; P(y_\mu) \; - \; y_\mu \; P(y_\lambda) \right] dx \; = \; 0 \qquad ,$$

and

$$\int_a^b A(x) \left[y_\lambda \; Q(y_\mu) \; - \; y_\mu \; Q(y_\lambda) \right] dx \; = \; 0$$

From the differential equation we also have

$$P(y_\lambda) \; = \; \lambda \, Q(y_\lambda) \quad \text{and} \quad P(y_\mu) \; = \; \mu \, Q(y_\mu)$$

Consequently

$$0 \; = \; \int_a^b A(x) \left[\mu \, y_\lambda \; Q(y_\mu) \; - \; \lambda \, y_\mu \; Q(y_\lambda) \right] dx \; =$$

$$(\mu - \lambda) \int_a^b A(x) \; y_\lambda \; Q(y_\mu) \; dx$$

If then $\mu \neq \lambda$ the functions y_λ and y_μ are called orthogonal in the general sense, i e ,

$$\int_a^b A(x) \quad y_\lambda \; Q(y_\mu) \quad dx \;=\; 0$$

If further $\mu \neq 0$ then we also have

$$\int_a^b A(x) \quad y_\lambda \; P(y_\mu) \quad dx \;=\; 0$$

If we eliminate the requirements that the end or the interface conditions be self-adjoint and allow them to be adjoint conditions then we have

$$\int_a^b A(x) \left[z_\lambda \; P(y_\mu) \;-\; y_\mu \; P(z_\lambda) \right] dx \;=\; 0 \quad ,$$

and

$$\int_a^b A(x) \left[z_\lambda \; Q(y_\mu) \;-\; y_\mu \; Q(z_\lambda) \right] dx \;=\; 0$$

The differential equations are

$$P(y_\mu) \;=\; \mu \; Q(y_\mu)$$

and

$$P(z_\lambda) \;=\; \lambda \; Q(z_\lambda)$$

Therefore, we can obtain as above, that

$$\int_a^b A(x)\ z_\lambda\ Q(y_\mu)\ dx = 0 \qquad (\text{if } \mu \neq \lambda)$$

and

$$\int_a^b A(x)\ z_\lambda\ P(y_\mu)\ dx = 0 \qquad (\text{if } \mu \neq 0)$$

The y_μ and z_λ form a generalized biorthogonal system where the orthogonality is indicated and the y_μ and z_λ satisfy adjoint systems

<u>Reality of Eigenvalues</u> If $\lambda = s + i\,t$ is an eigenvalue for problem (2 1) then so is $\bar{\lambda} = s - i\,t$ For the self-adjoint problem we can obtain in the conventional manner that

$$2\,i\,t \int_a^b A(x)\ y_\lambda\ Q(\bar{y}_{\bar{\lambda}})\ dx = 0 \qquad ,$$

where the bar denotes a complex conjugate If then the integral

$$\int_a^b A(x)\ y_\lambda\ Q(\bar{y}_{\bar{\lambda}})\ dx \neq 0$$

it follows that $t = 0$ and that the roots are real If we let $y_\lambda = u + i\,v$, then

$$\int_a^b A(x)\ y_\lambda\ Q(\bar{y}_{\bar{\lambda}})\ dx = \int_a^b A(x)\ u\ Q(u)\ dx + \int_a^b A(x)\ v\ Q(v)\ dx$$

$$+\ i \int_a^b A(x) \left[v\ Q(u) - u\ Q(v) \right] dx$$

Now

$$\int_a^b A(x) \left[v\, Q(u) - u\, Q(v) \right] dx = 0 \quad ,$$

since the real and imaginary parts of y satisfy (2 2) It follows that if

$$\int_a^b A(x) \; u \; Q(u) \; dx \neq 0$$

for all functions u which are M-differentiable and satisfy the end and interface conditions then the self-adjoint problem (2 1) has only real eigenvalues

<u>Definite</u> An eigenvalue problem is called <u>definite</u> if all the eigenvalues are real and of one sign If all eigenvalues are positive the problem is called positive definite while if the eigenvalues are negative the problem is called negative definite Semi-definite implies that the problem is definite except that zero is also an eigenvalue Problem (2 1) is called completely definite if

$$\int_a^b A(x) \; y \; P(y) \; dx > 0$$

and

$$\int_a^b A(x) \; y \; Q(y) \; dx > 0$$

for all y which are M-differentiable and satisfy the end and interface conditions A completely definite problem is clearly positive definite

<u>Rayleigh Quotient</u> The Rayleigh quotient $R(y)$ is defined by the formula

$$R(y) = \frac{\int_a^b A(x)\ y\ P(y)\ dx}{\int_a^b A(x)\ y\ Q(y)\ dx}$$

For a completely definite problem $R(y) > 0$ If y_λ is an eigenfunction $R(y_\lambda) = \lambda$ Without proof it will be asserted that if the problem (2 1) is self-adjoint and completely definite, then the minimum value of $R(y)$, for the class of functions y which are M-differentiable and satisfy the end and interface conditions, is λ_1 , the smallest eigenvalue of problem (2 1)

<u>Multiple Eigenvalues</u> An eigenvalue is called an r-fold eigenvalue, or of multiplicity r, if just r linearly independent eigenfunctions belong to this eigenvalue If $\varphi_1,$, φ_r are r linearly independent eigenfunctions belonging to an eigenvalue λ of multiplicity r, then any eigenfunction Y associated with λ can be written in the form

$$Y = \sum_{s=1}^{r} \alpha_s\ \varphi_s$$

In particular it is possible to choose constant α_{st} so that if we define

$$Y_s = \sum_{t=1}^{r} \alpha_{st}\ \varphi_t \qquad (s = 1, 2, \quad , r) \quad ,$$

then

$$\int_a^b A(x)\ Y_s\ Q(Y_t)\ dx = \delta_{st}$$

In particular we can always choose a set of eigenfunctions for the self-adjoint

completely definite eigenvalue problem (2 1) which are orthonormal in the generalized sense For the adjoint problem we have two sets of eigenfunctions which are bi-orthonormal

<u>Expansions</u> For the self-adjoint complete definite eigenvalue problem (2 1), consider the set of orthonormal eigenfunctions $\left\{ Y_j \right\}$ Let $f(x)$ be an "arbitrary" function subject to certain restrictions which will not be considered Let us assume that it is possible to write

$$f(x) = \sum_{j=1}^{\infty} a_j \, Y_j(x)$$

If we multiply both sides of this equation by $A \, Q(Y_k)$ and integrate with respect to x, we obtain

$$\int_a^b A \, Q(Y_k) \, f \, dx = \int_a^b A \, Q(Y_k) \left[\sum_{j=1}^{\infty} a_j \, Y_j \right] dx$$

If on the right hand side of this equation we interchange the order of summation and integration and then apply the orthogonality conditions, we obtain

$$a_k = \int_a^b A(x) \quad f(x) \quad Q \left[Y_k(x) \right] \, dx$$

If the $\left\{ Y_j \right\}$ were not normalized, the formula for a_k would be

$$a_k = \frac{\int_a^b A(x) \, f(x) \quad Q \left[Y_k(x) \right] \, dx}{\int_a^b A(x) \, Y_k(x) \quad Q \left[Y_k(x) \right] \, dx}$$

There are a number of questions which will be left unanswered but nevertheless should be mentioned First, is the set of eigenfunctions complete? By a <u>complete set</u> of eigenfunctions $\left\{ Y_j \right\}$ one means that the only function which is orthogonal in the generalized sense to every $Y_j(x)$ is the zero function The second question is, does the series $\sum_{j=1}^{\infty} a_j Y_j(x)$ converge? Third, if the series converges, does it converge to $f(x)$?

<u>Special Problem</u> Although most of the above remarks apply, let us consider the following special system

$$
\begin{aligned}
L(y) &= M(y) - \lambda\, g(x)\, y = 0 \qquad , \\
U_m(y) &= 0 \qquad (m = 1, 2, \quad , M) \quad , \\
D_m \left[y(x_k) \right] &= 0 \qquad (k = 1, 2, \quad , N)
\end{aligned}
\tag{2 3}
$$

The corresponding adjoint system has the form

$$
\begin{aligned}
\bar{L}(z) &= \bar{M}(z) - \lambda\, g(x)\, z = 0 \qquad , \\
\bar{U}_m(z) &= 0 \qquad (m = 1, 2, \quad , M) \quad , \\
\bar{D}_m \left[z(x_k) \right] &= 0 \qquad (k = 1, 2, \quad , N) \quad ,
\end{aligned}
\tag{2 4}
$$

where the bars denote the adjoint conditions If $M \equiv \bar{M}$ and both the sets

$\bar{U}_m$ and U_m as well as $\bar{D}_m\left[y(x_k)\right]$ and $D_m\left[y(x_k)\right]$ are linearly equivalent, then the system (2 3) is self-adjoint

<u>Orthogonality</u> If we proceed in the same fashion as was followed for the general problem, we can show that

$$\int_a^b A(x)\ g(x)\ y_\lambda(x)\ z_\mu(x)\ dx = 0 \qquad ,$$

where $y_\lambda(x)$ is the eigenfunction for the system (2 3) with eigenvalue λ , $z_\mu(x)$ is the eigenfunction for the adjoint system (2 4) with eigenvalue μ and where $\lambda \neq \mu$ If the system (2 3) is self-adjoint the orthogonality condition becomes

$$\int_a^b A(x)\ g(x)\ y_\lambda(x)\ y_\mu(x)\ dx = 0 \qquad (\lambda \neq \mu)$$

<u>Expansions</u> If the set $\left\{y_j(x)\right\}$ and $\left\{z_k(x)\right\}$ are a countable complete bi-orthogonal system of eigenfunctions, then the formal expansion of $f(x)$ is given by

$$f(x) = \sum_{j=1}^{\infty} a_j\ y_j(x) \qquad ,$$

where

$$a_j = \frac{\int_a^b A(x)\ g(x)\ f(x)\ z_j(x)\ dx}{\int_a^b A(x)\ g(x)\ y_j(x)\ z_j(x)\ dx}$$

For the self-adjoint problem

$$a_j = \frac{\int_a^b A(x)\ g(x)\ f(x)\ y_j(x)\ dx}{\int_a^b A(x)\ g(x)\ y^2_j(x)\ dx}$$

CHAPTER I

PART III

STURM-LIOUVILLE SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

1 Results of Ordinary Sturm-Liouville Theory Systems of the following type,

$$(1\ 1) \qquad \left[r\, y'(x)\right]' + \left[\lambda\, p + g\right] y = 0 \ ,$$

$$\alpha_1\, y(a) + \alpha_2\, y'(a) + \alpha_3\, y(b) + \alpha_4\, y'(b) = 0 \ ,$$

$$\beta_1\, y(a) + \beta_2\, y'(a) + \beta_3\, y(b) + \beta_4\, y'(b) = 0 \ ,$$

where the ' denotes differentiation with respect to x , are known as Sturm-Liouville systems The coefficients $r(x) > 0$, $g(x)$ and $p(x)$ are continuous functions of x in the interval $a \leqslant x \leqslant b$ and λ is an arbitrary parameter The first systematic development of the theory of such systems was published, as the name indicates, by J C F Sturm and J Liouville Many mathematicians have added to or generalized the results of Sturm and Liouville It is the objective of this part to indicate some generalizations resulting from discontinuous coefficients which are believed to be new

The great interest in S - L (Sturm-Liouville) theory arises largely from the fact that many boundary value problems in physics and engineering lead to questions which can be answered by using S-L theory In addition certain problems involving

the expansion of an arbitrarily given function either in an infinite series whose terms are prescribed functions or in terms of an integral of prescribed functions can be resolved by using S-L theory Although the expansion problem is generally an auxiliary problem to the boundary value problem it is also of interest in other fields of mathematics The primary interest in this part will be centered upon either obtaining the expansions or indicating the character of the expansions Before considering the generalized situations some of the known results for S-L theory will be listed

Consider the general second order linear differential equation

$$f_2(x)\, y'' + f_1(x)\, y' + \left[f_0(x) + \lambda\, g_0(x)\right] y = 0 , \tag{1 2}$$

where λ is a parameter and where f_2 , f_1 , f_0 and $g_0(x)$ are continuous functions of x on the interval $a \leqslant x \leqslant b$ Further it is assumed that $f_2(x)$ is not equal to zero anywhere on the interval The general homogeneous boundary conditions

$$\alpha_1\, y(a) + \alpha_2\, y'(a) + \alpha_3\, y(b) + \alpha_4\, y'(b) = 0 \tag{1 3}$$

$$\beta_1\, y(a) + \beta_2\, y'(a) + \beta_3\, y(b) + \beta_4\, y'(b) = 0 ,$$

are taken to be linearly independent Upon dividing the differential equation (1.2) by $f_2(x)$ and then multiplying the result by $r = \exp \int \frac{f_1}{f_2}\, dx$ one

obtains the equation

$$\left[r\, y'(x)\right]' + \left[\lambda\, p + g\right] y = 0 ,$$

where $p = \frac{g_0}{f_2} r$ and $g = \frac{f_0}{f_2} r$ It follows since this equation and the boundary conditions (1 3) form a S-L system that such systems are quite general

The differential equation of a S-L system is self-adjoint The concept of self-adjoint has been discussed in Part I A S-L system is called self-adjoint when

$$(1\ 4) \qquad (\alpha_1 \beta_2 - \beta_1 \alpha_2)\, r(b) = (\alpha_4 \beta_3 - \alpha_3 \beta_4)\, r(a)$$

The most common type of self-adjoint S-L system has two additional conditions First $p(x)$ is assumed not to vanish for $a \leqslant x \leqslant b$ In this case both $r(x)$ and $p(x)$ can be assumed greater than zero The second condition is that the boundary conditions are of the Sturmian type, that is,

$$(1\ 5) \qquad \alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \quad \text{and} \quad \beta_1\, y(b) + \beta_2\, y'(b) = 0 ,$$

where

$$|\alpha_1| + |\alpha_2| > 0 \qquad \text{and} \qquad |\beta_1| + |\beta_2| > 0$$

The following two theorems summarize the primary results of ordinary S-L theory

TH 1 <u>Oscillation theorem</u> The system

$$\left[r\, y'(x)\right]' + \left[\lambda p + g\right] y = 0 ,$$

$$\alpha_1\, y(a) + \alpha_2\, y'(a) \qquad \text{and} \qquad \beta_1\, y(b) + \beta_2\, y'(b) = 0 ,$$

where

$$\left|\alpha_1\right| + \left|\alpha_2\right| > 0 \qquad \text{and} \qquad \left|\beta_1\right| + \left|\beta_2\right| > 0 ,$$

has an infinite number of real characteristic numbers which may be arranged in a monotone increasing sequence,

$$\lambda_0 < \lambda_1 < \lambda_2 <$$

which tends to $+\infty$ Corresponding to each simple eigenvalue λ_n , there exists an eigenfunction $\varphi_n(x)$, unique except for a multiplicative constant Each $\varphi_n(x)$ has exactly n zeros in the interval $a < x < b$

TH 2 <u>Expansion theorem</u> Given an arbitrary continuous and piecewise differential function $f(x)$ which vanishes at the end points of the interval when $\varphi_0(x)$ vanishes, then the series,

$$\sum_{n=0}^{\infty} C_n \varphi_n(x) \quad \text{where} \quad C_n = \int_a^b f(x)\, p(x)\, \varphi_n(x)\, dx \quad ,$$

converges uniformly and absolutely and has the sum $f(x)$

The eigenfunctions fulfill the orthogonality conditions,

$$\int_a^b p(x)\, \varphi_i(x)\, \varphi_j(x)\, dx = 0 \qquad \text{when } i \neq j$$

The $\varphi_n(x)$ form a complete set, that is,

$$\int_a^b p(x)\, f(x)\, \varphi_n(x)\, dx = 0 \qquad \text{for all } n \ ,$$

implies $f(x)$ is identically zero and is assumed that the $\varphi_n(x)$ are normal, that is,

$$\int_a^b p(x)\, \varphi_n^2(x)\, dx = 1$$

The normality condition can easily be attained by letting

$$\tilde{\varphi}_n(x) = \varphi_n(x) \Big/ \sqrt{\int_a^b p(x)\, \varphi_n^2(x)\, dx}$$

The general self-adjoint boundary conditions given by conditions (1 3) and (1 4) can be reduced easily to the form

(1 6)
$$y(b) = \alpha\, y(a) + \beta\, y'(a) \quad ,$$

$$y'(b) = \gamma\, y(a) + \delta\, y'(a) \quad ,$$

where

$$(\alpha\,\delta - \beta\,\gamma)\, r(b) = r(a)$$

In this case also the eigenvalues are all real The totality of eigenvalues can be arranged into two sequences converging to $+\infty$ These sequences

$$\lambda_1 < \lambda_2 < \lambda_3 < \qquad \text{and} \quad (\lambda_0' <)\ \lambda_1' < \lambda_2' < \qquad ,$$

where λ_0' is present if either $\beta < 0$ or $\beta = 0$ and $\alpha > 0$, have the property that $\lambda_n \leqslant \lambda_n' < \lambda_{n+1}$ If $\lambda_n = \lambda_n'$ then this number is a double eigenvalue If $\lambda_n < \lambda_n'$ then both numbers are simple eigenvalues The corresponding eigenfunctions have $2n$, $2n-1$ or $2n-2$ zeros in the interval $a < x \leqslant b$, depending upon α and β The expansion theorem and the properties of the eigenvalues and eigenfunctions for large n have been developed [12]

If $p(x)$ changes sign on the interval, two cases arises according to whether the S-L system is definite A S-L system is called definite when $\lambda = 0$ is not an eigenvalue and when

$$-\int_a^b \left[(r\,y')' + g\,y^2\right] dx \geqslant 0 \quad ,$$

for each twice differentiable function $y(x)$, which satisfies the given self-adjoint boundary conditions Both the definite and non-definite cases have been investigated [12]

Let $\frac{f_1}{f_2}$ be once and $\frac{g_0}{f_2}$ be twice differentiable then the transformation

$$Y(z) = S(x)\,y(x) \quad ,$$

where

$$z = \int_a^x \sqrt{\frac{g_0}{f_2}}\; dx$$

and

$$S(x) = \sqrt[4]{\frac{g_0}{f_2}}\; \exp \frac{1}{2} \int_a^x \frac{f_1}{f_2}\; dx \quad ,$$

takes equation (1 2) into the <u>Liouville normal form</u>

$$Y''(z) + \left[\lambda + g(z)\right] Y = 0 \quad , \tag{1 7}$$

where

$$q(z) = \frac{f_0}{g_0} + \frac{1}{2}\left(\frac{g_0}{f_2}\right)^2 \frac{S'}{S} - \frac{f_2}{g_0}\frac{S''}{S}$$

In this transformation it is necessary that f_2 and g_0 be positive (or, which is equivalent, non-zero) on the interval It may be noted that the above transformation takes Sturmian boundary conditions into Sturmian boundary conditions

The eigenvalue problem given by equations (1 5) and (1 7) is equivalent to the Volterra integral equation

$$Y(z) = -\alpha_2 \cos s(z - a) + \frac{\alpha_1}{s} \sin s(z - a) - \frac{1}{s} \int_a^x g(x)\, Y(x) \sin s(z-x)dx,$$

where $\lambda = s^2$ and s is determined so that $Y(z)$ satisfies

$$\beta_1 Y(b) + \beta_2 Y'(b) = 0$$

This integral equation form of the eigenvalue problem is most convenient for finding the eigenvalue and eigenfunction for large values of the parameter λ, i e , λ_n and n are large It is not difficult to show [12] for example, that if $\alpha_2 \neq 0$, $\delta \neq 0$ and

$$H(c, d) = \frac{1}{2} \int_c^d q(x)\, dx \quad ,$$

then

$$s_n = \frac{\pi n}{b - a} - \frac{1}{\pi n}\left[H(a, b) + \frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right] + O\left(\frac{1}{n^2}\right) ,$$

and

$$\varphi_n(x) = \cos \frac{\pi n(x - a)}{b - a} + \frac{1}{\pi n}\left[(x - a)\left\{H(x, b) - \frac{\gamma}{\delta}\right\} - (b - x)\left\{H(a, x) + \frac{\alpha}{\beta}\right\}\right] \sin \frac{\pi n(x - a)}{b - a} + O\left(\frac{1}{n^2}\right)$$

The final interesting result can be summarized in the following theorem The proof of this theorem involves of course, some of the above results

TH 3 <u>Equi-convergence theorem</u> Given $f(x)$ is integrable over the interval (a, b) then the S-L expansion behaves as regards convergence in the same way as an ordinary Fourier series

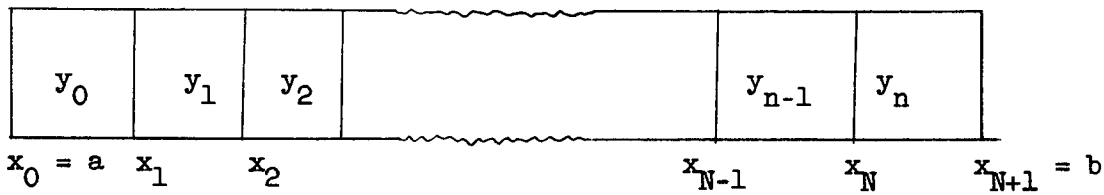
2 <u>S-L Theory with N-points of Discontinuity</u> <u>Problem</u> We shall now consider a S-L system with N points of discontinuity The differential equation is of the type

$$\left[r\, y'\right]' - \left[g + \lambda h\right] y = 0 ,$$

that is in each subinterval $x_i < x < x_{i+1}$ $(i = 0, \quad , n)$ the differential equation is given by

$$\left[r_i\, y_i'\right]' - \left[g_i + \lambda h_i\right] y_i = 0 \tag{2 1}$$

The assumptions which the coefficients of these equations will fulfill will be given later



The interface conditions are given by the equations

$$
\begin{aligned}
y_k(x_k) &= d_k\, y_{k-1}(x_k) + e_k\, y'_{k-1}(x_k) \\
y'_k(x_k) &= b_k\, y_{k-1}(x_k) + c_k\, y'_{k-1}(x_k) \quad ,
\end{aligned}
\tag{2 2}
$$

where we assume $d_k c_k - b_k e_k \neq 0$ The end conditions will be chosen as the Sturmian conditions

$$
\begin{aligned}
\alpha_1\, y(a) + \alpha_2\, y'(a) &= 0 \\
\beta_1\, y(b) + \beta_2\, y'(b) &= 0 \quad ,
\end{aligned}
\tag{2 3}
$$

where $|\alpha_1| + |\alpha_2| \neq 0$ and $|\beta_1| + |\beta_2| \neq 0$, although most of what follows applies to the more general self-adjoint end conditions Conditions (2 1), (2 2) and (2 3) therefore constitute the general S-L system we shall consider

<u>Orthogonality</u> Let the subscripts s and t denote association with the eigenvalues λ_s and λ_t If we multiply the equations

$$\left[r_i\, y'_{si}\right]' - \left[g_i + \lambda_s\, h_i\right] y_{si} = 0 \quad ,$$

by y_{ti} and the equations

$$\left[r_i\, y'_{ti}\right]' - \left[g_i + \lambda_t\, h_i\right] y_{ti} = 0 \quad ,$$

by y_{si} and then subtract the second set from the first set, we obtain

$$r_i \left[y_{ti}\, y'_{si} - y_{si}\, y'_{ti}\right]' = (\lambda_s - \lambda_t)\, h_i\, y_{si}\, y_{ti}$$

We now multiply this equation by A_i , integrate over the subinterval $x_i < x < x_{i+1}$ and sum from $i = 0$ to $i = N$ This gives

$$\sum_{i=0}^{N} A_i \Big\{ r_i(x_{i+1}) \left[y_{ti}(x_{i+1})\, y'_{si}(x_{i+1}) - y_{si}(x_{i+1})\, y'_{ti}(x_{i+1})\right] ,$$

$$- r_i(x_i) \left[y_{ti}(x_i)\, y'_{si}(x_i) - y_{si}(x_i)\, y'_{ti}(x_i)\right] \Big\}$$

$$= (\lambda_s - \lambda_t) \sum_{i=0}^{N} A_i \int_{x_i}^{x_{i+1}} h_i(x)\, y_{is}(x)\, y_{it}(x)\, dx$$

Now

$$\left[y_{to}(b)\, y'_{so}(b) - y_{so}(b)\, y'_{to}(b)\right] = 0$$

and

$$\left[y_{tN}(a)\, y'_{sN}(a) - y_{sN}(a)\, y'_{tN}(a)\right] = 0 \quad ,$$

because of condition (2 3) The other terms of the above sum can be written as

$$\sum_{i=1}^{N} \left[A_{i-1}\, r_{i-1}(x_i) - A_i\, r_i(x_i)\,(c_i\, d_i - b_i\, e_i)\right]$$

$$\left[y_{t\,i-1}(x_i)\, y'_{s\,i-1}(x_i) - y_{s\,i-1}(x_i)\, y'_{t\,i-1}(x_i)\right]$$

Consequently if

$$A_i = \frac{r_{i-1}(x_i)}{r_i(x_i)}\,(c_i\, d_i - b_i\, e_i)^{-1}\, A_{i-1} \qquad (i = 1, 2, \quad , N),$$

we have

$$(\lambda_s - \lambda_t) \sum_{i=0}^{N} A_i \int_{x_i}^{x_{i+1}} h_i(x)\, y_{is}\, y_{it}\, dx = (\lambda_s - \lambda_t) \int_a^b A(x)\, h(x)\, y_s(x)\, y_t(x) dx = 0$$

The orthogonality conditions therefore become

$$\int_a^b A(x)\, h(x)\, y_s(x)\, y_t(x) dx = 0 \qquad (\lambda_s \neq \lambda_t)$$

Self-adjoint Interface Conditions In the above derivation of the orthogonality conditions it was assumed implicitly that the general S-L system was self-adjoint Although this assumption proved successful it appears worthwhile to show that the system is actually self-adjoint We note that the differential operator in condition (2 1) is self-adjoint Further the Sturmian boundary conditions (2 2) are self-adjoint conditions In Part I the recipe

$$C_k = \frac{A_{k-1}}{A_k} F_k^{-1} (B_k^t)^{-1} E_k \quad ,$$

was obtained for finding the adjoint interface conditions In our case

$$F_k \equiv \begin{pmatrix} 0 & -r_k(x_k) \\ r(x_k) & 0 \end{pmatrix} \quad ,$$

$$B_k \equiv \begin{pmatrix} d_k & e_k \\ b_k & c_k \end{pmatrix} \quad ,$$

and

$$E_k \equiv \begin{pmatrix} 0 & -r_{k-1}(x_k) \\ r(x_k) & 0 \end{pmatrix}$$

Consequently

$$C_k = \frac{A_{k-1}}{A_k} \frac{r_{k-1}(x_k)}{r_k(x_k)} \frac{1}{(d_k c_k - b_k e_k)} B_k$$

Since we choose

$$A_k = \frac{r_{k-1}(x_k)}{r_k(x_k)} (d_k c_k - b_k e_k)^{-1} A_{k-1} ,$$

we see that

$$C_k = B_k ,$$

and that the interface conditions are self-adjoint

<u>Reality of Eigenvalues</u> If $\lambda = s + it$ is an eigenvalue with eigenfunction y then $\bar{\lambda} = s - it$ is an eigenvalue with eigenfunction $\bar{y}$, where the bar denotes a complex conjugate Proceeding in the manner used to find the orthogonality condition, it can be shown that

$$2 it \int_a^b A(x) h(x) \left| y(x) \right|^2 dx = 0$$

We now indicate the assumptions which the coefficients in condition (2 1) will satisfy As in the general case in each subinterval i $(i = 0, 1, \quad , N)$ the coefficients r_i, g_i and h_i are continuous and such that their limits when

x tends to x_i or x_{i+1} are finite Further we assume that r_i is positive for $x_i \leqslant x \leqslant x_{i+1}$ and that each h_i is positive or identically zero for $x_i \leqslant x \leqslant x_{i+1}$ For at least one subinterval h_i is positive The final assumption is that $g_j d_j - f_j e_j$ be positive for $j = 1, 2, \quad , N$ With these assumptions, we have since

$$\int_a^b A(x)\, h(x) \left| y(x) \right|^2 dx = \sum_{i=0}^{N} A_i \int_{x_i}^{x_{i+1}} h_i(x) \left| y_i(x) \right|^2 dx > 0 ,$$

that t must be zero and that all the eigenvalues must be real

Since $t = 0$ and

$$2it \left[h_i(x)\, y_i(x)\, \bar{y}_i(x) \right] = r_i \left[\bar{y}_i\, y_i' - y_i\, \bar{y}_i' \right]' ,$$

it follows that

$$\bar{y}_i\, y_i' - y_i\, \bar{y}_i' = \text{constant for} \quad i = 0, 1, \quad , N$$

The constant for each subinterval may be taken as zero Therefore $\bar{y}_i$ is a constant times y_i and y_i can be considered as real since they can always be made real by multiplying by the appropriate constant The eigenfunctions will be considered henceforth to be real

<u>Characteristic Equations</u> Let the general solution of (2 1) for the i-th interval be

$$(2\ 4) \qquad y_i(x, \lambda) = E_i(\lambda)\, Y_i\,(x, \lambda) + F_i(\lambda)\, Z_i(x, \lambda) =$$

$$E_i\, Y_i(x) + F_i\, Z_i(x) \qquad (i = 0, 1, \quad , N) ,$$

where $Y_o(x)$ satisfies $\alpha_1\, Y_o(a) + \alpha_2\, Y_o'(a) = 0$, where $Z_N(x)$ satisfies $\beta_1\, Z_N(b) + \beta_2\, Z_N'(b) = 0$, where $Y_i(x)$ satisfies $Y_i(x_i) = 0$ and $Y_i'(x_i) = -1$ $(i = 1, 2, \quad , N-1)$, and where $Z_i(x)$ satisfies $Z_i(x_i) = 1$ and $Z_i'(x_i) = 0$ $(i = 1, 2, \quad , N-1)$ The existence and uniqueness of the $Y_i(x)$ and $Z_i(x)$ are assured by the fundamental existence theorem for differential equations

We now substitute from equations (2 4) in conditions (2 2) and obtain for $k = 1, 2, \quad , N-1$

$$F_k = F_k\, Z_k(x_k) = d_k \left[E_{k-1}\, Y_{k-1}(x_k) + F_{k-1}\, Z_{k-1}(x_k)\right] +$$

$$e_k \left[E_{k-1}\, Y_{k-1}'(x_k) + F_{k-1}\, Z_{k-1}'(x_k)\right]$$

$$= E_{k-1}\left[d_k\, Y_{k-1}(x_k) + e_k\, Y_{k-1}'(x_k)\right] + F_{k-1}\left[d_k\, Z_{k-1}(x_k) + e_k\, Z_{k-1}'(x_k)\right] ,$$

$$(2\ 5)$$

$$-E_k = E_k\, Y_k'(x_k) = E_{k-1}\left[b_k\, Y_{k-1}(x_k) + c_k\, Y_{k-1}'(x_k)\right] +$$

$$F_{k-1}\left[b_k\, Z_{k-1}(x_k) + c_k\, Z_{k-1}'(x_k)\right]$$

For $k = N$ we obtain

$$F_N Z_N(x_N) = E_{N-1}\left[d_N Y_{N-1}(x_N) + e_N Y'_{N-1}(x_N)\right] + F_{N-1}\left[d_N Z_{N-1}(x_N) + e_N Z'_{N-1}(x_N)\right] ,$$

$$F_N Z'_N(x_N) = E_{N-1}\left[b_N Y_{N-1}(x_N) + c_N Y'_{N-1}(x_N)\right] + F_{N-1}\left[b_N Z_{N-1}(x_N) + c_N Z'_{N-1}(x_N)\right]$$

Since we can consider $F_o \equiv 0$ and $E_N \equiv 0$ and satisfy conditions (2 3), the relations (2 5) represent $2N$ linear homogeneous equations in the $2N$ unknowns, E_o , F_N , E_i , F_i $(i = 1, 2, \quad , N-1)$ In order that the E's and F's be non-trivial it is necessary that the determinant Δ of the coefficients should vanish This requirement gives, of course, the characteristic equation The determinant has the following form

$$\Delta = \left|\begin{array}{ccccc|cccc|ccc}
E_o & E_1 & F_1 & E_2 & F_2 & E_{i-1} & F_{i-1} & E_i & F_i & E_{N-1} & F_{N-1} & F_N \\
v_1(Y) & 0 & -1 & 0 & 0 & & & & & & & \\
w_1(Y) & 1 & 0 & 0 & 0 & & \bigcirc & & & & \bigcirc & \\
0 & v_2(Y) & v_2(Z) & 0 & -1 & & & & & & & \\
0 & w_2(Y) & w_2(Z) & 1 & 0 & & & & & & & \\
\hline
 & & & & & v_i(Y) & v_i(Z) & 0 & -1 & & & \\
 & \bigcirc & & & & w_i(Y) & w_i(Z) & 1 & 0 & & \bigcirc & \\
\hline
 & & & & & & & & & v_N(Y) & v_N(Z) & -Z_N(x_N) \\
 & \bigcirc & & & & & \bigcirc & & & w_N(Y) & w_N(Z) & -Z'_N(x_N)
\end{array}\right|$$

where $v_i(X) = d_i X_{i-1}(x_i) + e_i X'_{i-1}(x_i)$ and $w_i(X) = b_i X_{i-1}(x_i) + c_i X'_{i-1}(x_i)$
The eigenvalues are therefore the solutions of the equation $\Delta(\lambda) = 0$

It is believed that with suitable conditions upon the d's, e's, b's and c's the eigenvalues can be shown to be simple

<u>Expansions</u> Let $f(x)$ be an "arbitrary" function subject to certain restrictions which will not be considered Further let $\left\{y(x, \lambda_j)\right\}$ be the set of orthogonal functions associated with the entire set of eigenvalues $\left\{\lambda_j\right\}$ If λ_j is a non-simple eigenvalue then the set of orthogonal functions includes two linearly independent functions associated with λ_j Now we assume that it is possible to write

$$f(x) = \sum_{j=1}^{\infty} a_j \, y(x, \lambda_j)$$

Proceeding as usual we multiply both sides of the equation by $A(x)\, y(x, \lambda_k)$ and integrate from a to b This gives

$$\int_a^b A(x)\, h(x)\, y(x, \lambda_k)\, f(x)dx = \int_a^b A(x)\, h(x)\, y(x, \lambda_k) \left[\sum_{j=1}^{\infty} a_j \, y(x, \lambda_j)\right] dx$$

Upon interchanging the order of integration and summation on the right hand side and then applying the orthogonality conditions, we obtain

$$\int_a^b A(x)\, h(x)\, y(x, \lambda_k)\, f(x)dx = a_k \int_a^b A(x)\, h(x)\, y^2(x, \lambda_k)dx$$

3 Singular S-L Theory with N-points of Discontinuity We now wish to consider the problem in Section 2 where either b is $+\infty$ or a is $-\infty$ and b is $+\infty$ These two cases, one on the semi-infinite interval and the other on the infinite interval, are called singular If on a finite interval, r vanishes at a or b, or if g or h become infinite at a or b, then the S-L system is also called singular These cases will not be considered but could be treated in a manner similar to that used for the infinite case

Semi-infinite Problem We first consider a semi-infinite problem The finite S-L system defined by the relations (2 1), (2 2) and (2 3) must be modified for the semi-infinite interval The differential equations (2 1) are modified in that we shall take for $i = N$ the equation

$$y_N'' - \left[\lambda + g_N\right] y_N = 0 \qquad (x > x_N)$$

As has been pointed out this is the normal form for a second order differential equation and is most convenient when obtaining asymptotic properties for large $|x|$ and $|\lambda|$ Although this form is not necessary for the y_N equation since the asymptotic properties can be obtained in any case, it is less work and just as instructive to consider this form

We will consider only one type of g_N g_N will be assumed to be greater than or equal to zero, to be continuous and to be absolutely integrable

Finally it is necessary to make a statement about the boundary condition at infinity We will not be careful or explicit here but merely comment that the solution y should be bounded as x tends to infinity

Expansions Without any justification outside of the results obtained in Chapter II, we shall consider expanding an arbitrary function $f(x)$ in the following fashion

$$(3\ 1) \qquad f(x) = \int_{-\infty}^{\infty} G(s)\, Y(x, s)\, ds \quad ,$$

where $\lambda = -s^2$ and $Y(x, s)$ is a solution to the modified relation (2 1) which satisfies the condition $\alpha_1 Y(a, s) + \alpha_2 Y'(a, s) = 0$ In order to obtain the formal $G(s)$ which is appropriate for the expansion, we will proceed in a manner similar to that used in the finite case

First we multiply both members of equation (3 1) by $A(x)\, h(x)\, Y(x, t)$ and integrate the reult, on both sides, from a to ∞ This gives

$$\int_a^{\infty} A(z)\, h(z)\, Y(z, t)\, f(z)dz = \int_a^{\infty} A(z)\, h(z)\, Y(z, t) \left[\int_{-\infty}^{\infty} G(s)\, Y(z, s)ds \right] dz$$

Recalling the procedure used in the finite case we now wish to interchange the order of integrations and carry out the inner integration This leads to the following

$$\int_a^{\infty} A(z)\, h(z)\, Y(z, t)\, f(z)dz = \lim_{\bar{x} \to \infty} \int_{-\infty}^{\infty} G(s) \left[\int_a^{\bar{x}} A(z)\, h(z)\, Y(z, t)\, Y(z, s)dz \right] ds$$

$$= \lim_{\bar{x} \to \infty} \int_{-\infty}^{\infty} G(s) \left[\sum_{i=0}^{N} A_i \int_{x_i}^{x_{i+1}} h(z)\, Y_i(z, t)\, Y_i(z, s)dz \right] ds \quad ,$$

where x_{N+1} here represents $\bar{x}$

In order to evaluate this final expression we note that the differential equations lead to the following result

$$(t^2 - s^2) \int_{x_i}^{x_{i+1}} h_i(z)\, Y_i(z, s)\, Y_i(z, t)dz = r_i \left[Y'_{si}\, Y_{ti} - Y'_{ti}\, Y_{si} \right] \Bigg|_{z=x_i}^{z=x_{i+1}}$$

At $x = a$ the quantity $\left[Y'_{so}\, Y_{to} - Y_{to}\, Y_{so} \right]$ vanish as a consequence of the initial condition which Y satisfies In the same fashion as used for the finite case, we have as a consequence of the interface conditions and the selection of the A_i $(i = 1, 2, \quad , N)$ that no contribution is given to the value of the integral

$$\int_a^{\bar{x}} A(z)\, h(z)\, Y(z, s)\, Y(z, t)dz \quad ,$$

by the discontinuities at the interfaces Consequently, we obtain

$$(t^2 - s^2) \int_a^{\bar{x}} A(z)\, h(z)\, Y(z, s)\, Y(z, t)dz = A_N \left[Y'_N(\bar{x}, s)\, Y_N(\bar{x},t) - Y'_N(\bar{x},t)\, Y_N(\bar{x},s) \right]$$

Therefore

$$\int_a^{\infty} A(z)\, h(z)\, Y(z, t)\, f(z)dz = \lim_{\bar{x} \to \infty} \int_{-\infty}^{\infty} G(s)\, A_N \left[Y'_N(\bar{x},s)\, Y_N(\bar{x},t) - Y'_N(\bar{x},t)\, Y_N(\bar{x},s) \right] (t^2 - s^2)^{-1}\, ds$$

At this point we need, in order to evaluate the right hand side of this last equation, the asymptotic properties of Y_N and Y_N' for large x We first note that Y_N and Y_N' can be written in the form

$$Y_N(x, s) = \theta_o(s)\, Y_{N1}(x, s) + \varphi_o(s)\, Y_{N2}(x, s) \quad ,$$

and

$$Y_N'(x, s) = \theta_o(s)\, Y_{N1}'(x, s) + \varphi_o(s)\, Y_{N2}'(x, s) \quad ,$$

where Y_{N1} and Y_{N2} are any two linearly independent solutions of the equation $y_N'' - \left[\lambda + g_N\right] y_N = 0$ It is convenient to let Y_{N1} and Y_{N2} be the particular solutions which satisfy

$$Y_{N1} = \cos s(x - x_N) - s^{-1} \int_x^\infty \sin s(x - z)\, Y_{N1}(z, s)\, g_N(z)dz \quad ,$$

and

$$Y_{N2} = s^{-1} \sin s(x - x_N) - s^{-1} \int_x^\infty \sin s(x - z)\, Y_{N2}(z, s)\, g_N(z)dz$$

It is easily verified by differentiation that these are solutions It can be shown, as it will be in Chapter II, that by using the above forms of Y_{N1} and Y_{N2} that

$$Y_{N1} = u_1(s) \sin s\, x + v_1(s) \cos s\, x + \delta(1) \quad ,$$

as x tends to ∞ and

$$Y_{N2} = u_2(s) \sin s\,x + v_2(s) \cos s\,x + o(1)$$

as x tends to ∞, where formulas for the v's and u's can be obtained The essential result is that Y_N and Y_N' have the forms

$$Y_N = \theta(s) \sin s\,x + \varphi(s) \cos s\,x + o(1)$$

and

$$Y_N' = \theta(s) \cos s\,x - s\,\varphi(s) \sin s\,x + o(1)$$

as x tends to infinity

We now have

$$\lim_{\bar{x}\to\infty} \int_{-\infty}^{\infty} G(s)\, A_N\, (t^2 - s^2)^{-1} \left[Y_N'(\bar{x}, s)\, Y_N(\bar{x}, t) - Y_N'(\bar{x},t) Y_N(\bar{x},s) \right] ds$$

$$= \lim_{\bar{x}\to\infty} \int_{-\infty}^{\infty} A_N \frac{G(s)}{2} \left\{ \frac{\sin \bar{x}(s+t)}{s+t} \left[\varphi(s)\,\varphi(t) - \theta(s)\,\theta(t) \right] \right.$$

$$+ \frac{\sin \bar{x}\,(s-t)}{s-t} \left[\varphi(s)\,\varphi(t) + \theta(s)\,\theta(t) \right] - \frac{\cos \bar{x}(s+t)}{s+t}$$

$$\left[\varphi(s)\,\theta(t) + \varphi(t)\,\theta(s) \right]$$

$$\left. - \frac{\cos x\,(s-t)}{s-t} \left[\theta(s)\,\varphi(t) - \theta(t)\,\varphi(s) \right] \right\} ds$$

At this point we apply formally the formulas

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} F(x) \cos k\,x\,dx = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z-x)}{z-x}\,dx = \pi F(z)$$

We now can obtain the desired result for we have

$$\int_a^\infty A(z)\ h(z)\ Y(z,\ t)\ f(z)dz \ = \ \frac{\pi}{2}\ \ A_N \left[\varphi^2(t) + \theta^2(t)\right] \left[G(t) + G(-t)\right] \ ,$$

since $\theta(s)$ is an odd function of s and $\varphi(s)$ is an even function of s

Since $Y(x,\ s)$ is an even function of s , we have

$$f(x) \ = \ \int_{-\infty}^{\infty} G(s)\ Y(x,\ s)ds \ = \ \int_0^\infty \left[G(s) + G(-s)\right]\ Y(x,\ s)ds$$

Consequently

$$(3\ 2) \qquad f(x) \ = \frac{2}{\pi} \int_0^\infty \ \left[\int_a^\infty A(z)\ h(z)\ Y(z,\ s)\ f(z)dz\right] \left[\varphi^2(s) + \theta^2(s)\right]^{-1}$$

$$Y(x,\ s)\ A_N^{-1}\ ds$$

<u>Infinite Problem</u> For the infinite interval the finite S-L system must be modified at both ends For $x > x_N$ we again take the differential equation to be

$$y_N'' \ - \left[\lambda + g_N\right] y_N \ = \ 0 \quad ,$$

while for $x < x_1$ we take the differential equation to be

$$y_o'' \ - \left[\lambda \ + \ g_o\right]\ y_o \ = \ 0$$

We assume that g_o and g_N are greater than zero, continuous and absolutely integrable

<u>Expansions</u> Again without justification, we shall consider expanding an arbitrary function $f(x)$ in the following fashion

$$(3\ 3) \qquad f(x) = \int_{-\infty}^{\infty} \left[H(s)\ Y(x,\ s) + K(s)\ Z(x,\ s) \right] ds \quad ,$$

where $\lambda = -s^2$ and $Y(x,\ s)$ and $Z(x,\ s)$ are linearly independent solutions of the differential equations (2 1) The procedure here is similar to that used for the semi-infinite case

In order to find $H(s)$ and $K(s)$ we first multiply both sides of equation (3 3) by $A(x)\ h(x)\ Y(x,\ t)$ and integrate between $-\infty$ and $+\infty$ The result is given by

$$\int_{-\infty}^{\infty} A(z)\ h(z)\ Y(z,t)\ f(z)dz = \int_{-\infty}^{\infty} A(z)\ h(z)\ Y(z,t) \left\{ \int_{-\infty}^{\infty} \left[H(s)\ Y(z,s) + K(s)\ Z(z,s) \right] ds \right\} dz$$

Proceeding formally we obtain

$$\int_{-\infty}^{\infty} A(z)\ h(z)\ Y(z,t)\ f(z)dz = \lim_{\bar{x} \to \infty} \int_{-\infty}^{\infty} \left[H(s) \int_{-\bar{x}}^{\bar{x}} A(z)\ h(z)\ Y(z,\ s)\ Y(z,\ t)dz \right.$$

$$\left. + \ K(s) \int_{-\bar{x}}^{\bar{x}} A(z)\ h(z)\ Y(z,t)\ Z(z,\ s)dz \right] ds$$

$$= \lim_{\bar{x}\to\infty} \int_{-\infty}^{\infty} H(s)\,(t^2 - s^2)^{-1} \Big\{ A_N \Big[Y_N'(\bar{x}, s)\, Y_N(\bar{x}, t) - Y_N'(\bar{x}, t)\, Y_N(\bar{x}, s) \Big] -$$

$$A_o \Big[Y_o'(\bar{x}, s)\, Y_o(\bar{x}, t) - Y_N'(\bar{x}, t)\, Y_N(\bar{x}, s) \Big] \Big\} ds$$

$$+ \lim_{\bar{x}\to\infty} \int_{-\infty}^{\infty} K(s)\,(t^2 - s^2)^{-1} \Big\{ A_N \Big[Z_N'(\bar{x}, s)\, Y_N(\bar{x}, t) - Y_N'(\bar{x}, t)\, Z_N(\bar{x}, s) \Big] -$$

$$A_o \Big[Y_o'(\bar{x}, s)\, Z_o(\bar{x}, t) - Z_o'(\bar{x}, t)\, Y_o(\bar{x}, s) \Big] \Big\} ds$$

At this stage it is necessary to have the order properties for large $|x|$ We shall assume that as x tends to infinity that

$$Y_N(x, s) = u_1(s) \cos s\,x + v_1(s) \sin s\,x + o(1) \quad ,$$

$$Y_N'(x, s) = -s\, u_1(s) \sin s\,x + s\, v_1(s) \cos s\,x + o(1) \quad ,$$

$$Z_N(x, s) = u_2(s) \cos s\,x + v_2(s) \sin s\,x + o(1) \quad ,$$

and

$$Z_N'(x, s) = -s\, u_2(s) \sin s\,x + s\, v_2(s) \cos s\,x + o(1)$$

We also assume that as $-x$ tends to infinity that

$$Y_o(x, s) = \widetilde{u}_1(s) \cos s\,x + \widetilde{v}_1(s) \sin s\,x + o(1) \quad ,$$

$$Y_o'(x, s) = -s\, \widetilde{u}_1(s) \sin s\,x + s\, \widetilde{v}_1(s) \cos s\,x + o(1) \quad ,$$

$$Z_o(x, s) = \tilde{u}_2(s) \cos s x + \tilde{v}_2(s) \sin s x + o(1) ,$$

and

$$Z_o'(x, s) = -s \tilde{u}_2(s) \sin s x + s \tilde{v}_2(s) \cos s x + o(1)$$

It should be noted that the u's are even in s and the v's are odd in s

We shall also use the formulas

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \cos k x \, dx = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z-x)}{z - x} dx = \pi F(z)$$

With the aid of the above properties and formulas, we obtain

$$\int_{-\infty}^{\infty} A(z) h(z) Y(z, t) f(z) dz = \pi H(t) u(t) + \pi K(t) v(t) ,$$

where

$$u(t) = A_N \left[u_1^2(t) + v_1^2(t) \right] + A_o \left[\tilde{u}_1^2(t) + \tilde{v}_1^2(t) \right]$$

and

$$v(t) = A_N \left[u_1(t) u_2(t) + v_1(t) v_2(t) \right] +$$

$$A_o \left[\tilde{u}_1(t) \tilde{u}_2(t) + \tilde{v}_1(t) \tilde{v}_2(t) \right]$$

Proceeding in a similar fashion it is possible to obtain

$$\int_{-\infty}^{\infty} A(z) h(z) Z(z, t) f(z) dz = \pi H(t) v(t) + \pi K(t) w(t) ,$$

where

$$w(t) = A_N \left[u_2^2(t) + v_2^2(t)\right] + A_o \left[\tilde{u}_2^2(t) + \tilde{v}_2^2(t)\right]$$

We can now solve for $H(t)$ and $K(t)$ and obtain

$$H(t) = \frac{1}{\pi} \frac{w(t) \int_{-\infty}^{\infty} A(z)\, h(z)\, f(z)\, Y(z, t)dz - v(t) \int_{-\infty}^{\infty} A(z)\, h(z)\, f(z)\, Z(z, t)dz}{u(t)\, w(t) - v^2(t)},$$

and

$$K(t) = \frac{1}{\pi} \frac{w(t) \int_{-\infty}^{\infty} A(z)\, h(z)\, f(z)\, Z(z, t)dz - v(t) \int_{-\infty}^{\infty} A(z)\, h(z)\, f(z)\, Z(z, t)dz}{u(t)\, w(t) - v^2(t)}$$

It should also be pointed out that $H(s)$, $K(s)$, $Y(x, s)$ and $Z(x, s)$ are all even functions of s Consequently,

$$f(x) = 2 \int_0^{\infty} \left[H(s)\, Y(x, s) + K(s)\, Z(x, s)\right] ds$$

Chapter II

SOME EXPANSIONS ASSOCIATED WITH SECOND ORDER DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

1 Introduction In this chapter, in addition to obtaining some expansions which apparently have not appeared in the literature, the method of using the Laplace transform in order to obtain expansions will be illustrated [2, 6] In order to conveniently use the Laplace transform we shall consider certain boundary value problems This has the advantage of also illustrating the well known connection between boundary value problems and expansion problems We are concerned primarily, of course, with the expansions, since the solutions to the boundary value problems may then be easily found, and shown unique and valid by the usual methods [5]

Another objective of this chapter is to develop the regular (finite) and singular (semi-infinite and infinite) cases in parallel rather than in sequence It is believed that this development in parallel will exhibit more clearly the similarities and differences between the regular and singular cases

The concept of equi-convergence plays a fundamental role in the following As previously noted the S - L expansion of an integrable function behaves as regards convergence in the same way as an ordinary Fourier series Similar situations are shown to exist in both the following regular and singular cases

It is interesting to note that the classical method of separation can be used to obtain the expansions

This chapter consists of eleven sections Since the details become somewhat involved it appears worthwhile to point out in advance the essentials

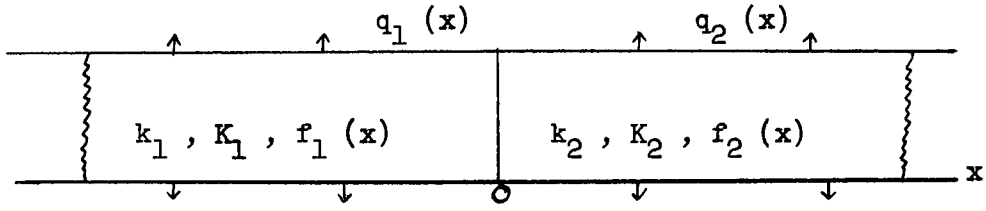
Section 2 contains a statement of the "composite" boundary value problems under consideration In Section 3 the corresponding Laplace transform problem is obtained and solved by using a "double" Green's function The formal inversions of the solution of the transform problem is carried out in Section 4 The formal expansions are given by equations (4 3), (4 4) and (4 5) In Section 5 the formal expansions, given by equations (5 4), (5 5) and (5 6), are obtained for the simplest composite cases

The asymptotic properties, which are needed to prove the equi-convergence of the general expansions (4 3), (4 4) and (4 5) and the special expansions (5 4), (5 5) and (5 6) respectively, are developed in Section 6 In Section 7 the characteristic equations are investigated and the properties of the spectrum are found Using primarily the results of Section 6 and 7 an equi-convergence theorem is proven in Section 8 The verification of the formal expansions is obtained in Section 9 This result is summarized in an expansion theorem In Section 10 the boundary value problems and expansions are found by using the method of separation of variables The equations (7 4), (10 11) and (10 14) which are derived in this fashion are equivalent to the equations (4 3), (4 4) and (4 5) The final section contains some remarks concerning some specializations and generalizations

In summary the formal results of this chapter may be found in the first five sections, p 1 through p 19, the Equi-Convergence Theorem, p 45, the Expansion Theorem, p 50, and the last two sections p 50 through p 70

2 The boundary value problems Consider the one dimensional distribution of temperature $U(x, t)$ in a slender rod or wire consisting of two sections composed of different materials, when the initial temperature in each section is an arbitrarily given function of the distance from one end and heat transfer,

with thermal emissivity $q(x)$, takes place at the surfaces Let $x = 0$ be taken as the surface of separation and let k_1 , k_2 and K_1 , K_2 be the thermal diffusivity and conductivity constants of the two materials There are three cases to consider corresponding to whether the x-interval is (a) finite, (b) semi-infinite or (c) infinite In the finite case (a) let the ends be insulated and let the length of the layers be a and b In the semi-infinite case (b) let one be the length of the finite section and let the end at $x = -l$ be insulated Case (a) leads to the regular case while cases (b) and (c) lead to singular cases The boundary value problems become



(2 1) $$U_t(x, t) = k_1 U_{xx} - q_1(x) U \quad (x < 0) ,$$

(2 2) $$U(x, 0^+) = f_1(x) \quad (x < 0) ,$$

(2 3) $$U_t = k_2 U_{xx} - q_2(x) U \quad (x > 0) ,$$

(2 4) $$U(x, 0^+) = f_2(x) \quad (x > 0) ,$$

(2 5) $$U(0^-, t) = U(0^+, t) ,$$ (2 6) $$K_1 U_x(0^-, t) = K_2 U_x(0^+, t) ,$$

(2 7a) $$U_x(-b^+, t) = 0 , \quad U_x(a^-, t) = 0 ,$$

(2 7b) $$U_x(-1^+, t) = 0 , \quad |U(x, t)| < M , \text{ where } M \text{ is a constant,}$$

(2 7c) $$|U(x, t)| < M , \quad \text{where } M \text{ is a constant,}$$

where $t > 0$ in relations (2 1), (2 3), (2 5), (2 6), (2 7a), (2 7b) and (2 7c) For convenience let $f(x) \equiv f_1(x)$ when $x < 0$ and $f(x) \equiv f_2(x)$ when $x > 0$ and let $q(x) \equiv q_1(x)$ when $x < 0$ and $q(x) \equiv q_2(x)$ when $x > 0$ $f(x)$ is, of course, assumed bounded

We shall assume that $q(x)$ satisfies the following three conditions First $q(x)$ is continuous except possibly at $x = 0$ Second the integral of $|q(x)|$ over the x-interval exists This condition is closely connected with the equi-convergence properties Third $q(x)$ is greater than or equal to zero This final condition is not a necessary one for the following methods However, the proofs and results do become less complicated if we use this condition

If we let $K_1 = K_2$, $k_1 = k_2 = 1$ and $q(x)$ be continuous at $x = 0$ the composite (or two materials) problems reduce to continuous (or one material) problems

3 <u>The transformed problems and their solutions</u> Let $w(x, \mu)$ denote the Laplace transform of $U(x, t)$ with respect to t , i e ,

$$L\left\{U(x, t)\right\} = \int_0^\infty e^{-\mu t} U(x, t)\, dt = w(x, \mu)$$

Applying this transformation formally to the boundary value problems, we obtain

(3 1) $(\mu + q_1)\, w(x, \mu) - f_1 = k_1\, w_{xx}$, (3 2) $(\mu + q_2)\, w - f_2 = k_2\, w_{xx}$,

(3 3) $w(0^-, \mu) = w(0^+, \mu)$, (3 4) $K_1\, w_x(0^-, \mu) = K_2\, w_x(0^+, \mu)$,

(3 5a) $w_x(-b, \mu) = 0$, $w_x(a, \mu) = 0$,

(3 5b) $w_x(-1, \mu) = 0$, $\left| w(x, \mu) \right| < M\, \mu_r^{-1}$, where μ_r is the real part of μ and is greater than zero ,

(3 5c) $$\left| w(x, \mu) \right| < M\, \mu_r^{-1}$$

If $\mu_r \geqslant \delta > 0$ then $\left| w(x, \mu) \right| < M\, \mu_r^{-1}$ can be written $\left| w(x, \mu) \right| < N_\delta$, where N_δ is a constant dependent only upon δ

The solution of these nonhomogeneous problems can be written in terms of the solution of corresponding homogeneous problems Define the "double" Green's functions $G(x, y, \mu)$ by $J\, G(x, y, \mu) \equiv R(x, y, \mu)$ when $x < 0$ and $J\, G(x, y, \mu) \equiv S(x, y, \mu)$ when $x > 0$, where $J \equiv K_1$ when $y < 0$ and $J \equiv K_2$ when $y > 0$ and where $R(x, y, \mu) \equiv R(x, y)$ and $S(x, y, \mu) \equiv S(x, y)$ satisfy

$$k_1\, R_{xx} - (\mu + q_1)\, R = 0 \text{ when } x < y\ ,\ y < x < 0\ ,\quad k_2\, S_{xx} - (\mu + q_2)\, S = 0 \text{ when}$$

$$0 < x < y\ ,\ y < x\ ,$$

$R(x, y)$ is continuous for $x, y < 0$, $S(x, y)$ is continuous for

$$0 < x, y \quad ,$$

$$R_x(y^+, y) - R_x(y^-, y) = 1 \quad , \qquad S_x(y^+, y) - S_x(y^-, y) = 1 \quad ,$$

$$R(0, y) = S(0, y) \quad , \qquad K_1 R_x(0, y) = K_2 S_x(0, y) \quad ,$$

(a) $R_x(-b, y) = 0 \quad , \qquad S_x(a, y) = 0 \quad ,$

(b) $R_x(-1, y) = 0 \quad , \qquad S$ is bounded ,

(c) R and S are bounded

As might be expected then from the theory of Green's function the solution to the above transformed problem is given by

$$w(x, \mu) = - \int p(y) \; G(x, y, \mu) \; f(y) \; dy \quad , \tag{3 6}$$

where $p(y) \equiv \frac{K_1}{k_1}$ when $y < 0$ and $p(y) \equiv \frac{K_2}{k_2}$ when $y > 0$ The interval of integration is from $-b$ to a for case (a), from -1 to ∞ for case (b) and from $-\infty$ to ∞ for case (c) This solution may be verified in each of the three cases in the usual manner by using the properties of $G(x, y, \mu)$

$G(x, y, \mu)$ can in turn be expressed in terms of solutions of the second order R and S differential equations Let g's be solutions to the

R differential equation while h's are solutions to the S differential equation For case (a) let $g_1(x, \mu) \equiv g_1(x)$ and $g_2(x)$, and $h_1(x)$ and $h_2(x)$ satisfy $g_1'(-b) = 0$, $g_1(-b) = 1$ and $g_2(0) = 0$, and $h_1'(a) = 0$, $h_1(a) = 1$ and $h_2'(0) = 1$ For case (b) let $g_1(x)$ and $g_2(x)$, and $h_1(x)$ and $h_2(x)$ satisfy $g_1'(-1) = 0$, $g_1(-1) = 1$ and $g_2(-1) = 0$, $g_2'(-1) = -1$, and $h_1(0) = 0$, $h_1'(0) = 1$ and $h_2'(0) = 0$, $h_2(0) = -1$ For case (c) let $g_1(x)$ and $g_2(x)$, and $h_1(x)$ and $h_2(x)$ satisfy $g_1(0)$, $g_1'(0) = 1$ and $g_2'(0) = 0$, $g_2(0) = -1$, and $h_1(0) = 0$, $h_1'(0) = 1$ and $h_2'(0) = 0$, $h_2(0) = -1$

In case (b) and (c) we also need solutions of the differential equations which are bounded as x tends to infinity even though μ be complex When x tends to infinity let the bounded solution be $h(x) = A\, h_1(x) + B\, h_2(x)$ and when $-x$ tends to infinity let the bounded solution be $g(x) = E\, g_1(x) + F\, g_2(x)$ We will indicate how to obtain these particular solutions later

It is shown in the theory of differential equations [9 , p 72 and p 218] that solutions g and h exist which, together with their first derivatives with respect to x , are continuous in x and μ together and analytic in μ for each finite x region and bounded μ region

For case (a) if we define W_g , W_h and W_f by

$$W_g \equiv g_1(y)\, g_2'(y) - g_1'(y)\, g_2(y) \ , \quad W_h \equiv h_2(y)\, h_1'(y) - h_1(y)\, h_2'(y) \quad \text{and}$$

$$W_f \equiv K_1\, g_1'(0)\, h_1(0) - K_2\, h_1'(0)\, g_1(0) \ ,$$

we obtain

$$R(x, y) = \left[-K_2\, h_1(y)\, W_f^{-1} - K_1\, h_1(0)\, g_2'(0)\, g_1(y)\, W_g^{-1}\, W_f^{-1} \right] g_1(x) + g_2(y)\, g_1(x)\, W_g^{-1}$$

$$(-b \leqslant x \leqslant y)$$

$$= \left[-K_2\, h_1(y)\, W_f^{-1} - K_1\, h_1(0)\, g_2'(0)\, g_1(y)\, W_g^{-1}\, W_f^{-1} \right] g_1(x) + g_1(y)\, g_2(x)\, W_g^{-1}$$

$$(y \leqslant x \leqslant 0) \quad ,$$

$$S(x, y) = \left[-K_1\, g_1(y)\, W_f^{-1} - K_1\, h_2(0)\, g_1'(0)\, h_1(y)\, W_h^{-1}\, W_f^{-1} \right] h_1(x) + h_1(y)\, h_2(x)\, W_h^{-1}$$

$$(0 \leqslant x \leqslant y)$$

$$= \left[-K_1\, g_1(y)\, W_f^{-1} - K_1\, h_2(0)\, g_1'(0)\, h_1(y)\, W_h^{-1}\, W_f^{-1} \right] h_1(x) + h_2(y)\, h_1(x)\, W_h^{-1}$$

$$(y \leqslant x \leqslant a)$$

For case (b) if we define M and N by

$$M = \left[B\, K_1\, g_1'(0) + A\, K_2\, g_1(0) \right] \text{ and } N = -\left[B\, K_1\, g_2'(0) + A\, K_2\, g_2(0) \right] ,$$

we obtain

$$R(x, y) = \left[N\, g_1(y) + K_2\, h(y) \right] g_1(x)\, M^{-1} + g_2(y)\, g_1(x) \qquad (-1 \leqslant x \leqslant y)$$

$$= \left[N\, g_1(y) + K_2\, h(y) \right] g_1(x)\, M^{-1} + g_1(y)\, g_2(x) \qquad (y \leqslant x \leqslant 0) ,$$

$$S(x, y) = \left[A^{-1} K_1 g_1'(0) h(y) + K_1 g_1(y)\right] h(x) M^{-1} - A^{-1} h(y) h_2(x)$$

$$(0 \leqslant x \leqslant y)$$

$$= \left[A^{-1} K_1 g_1'(0) h(y) + K_1 g_1(y)\right] h(x) M^{-1} - A^{-1} h_2(y) h(x) \quad (y \leqslant x)$$

For case (c) if we define W by $W = K_1 B E - K_2 A F$, we obtain

$$R(x, y) = \left[K_2 A E^{-1} g(y) + K_2 h(y)\right] g(x) W^{-1} + E^{-1} g_2(y) g(x) \quad (x \leqslant y \leqslant 0)$$

$$= \left[K_2 A E^{-1} g(y) + K_2 h(y)\right] g(x) W^{-1} + E^{-1} g(y) g_2(x) \quad (y \leqslant x \leqslant 0) ,$$

$$S(x, y) = \left[K_1 E A^{-1} h(y) + K_1 g(y)\right] h(x) W^{-1} - A^{-1} h(y) h_2(x) \quad (0 \leqslant x \leqslant y)$$

$$= \left[K_1 E A^{-1} h(y) + K_1 g(y)\right] h(x) W^{-1} - A^{-1} h_2(y) h(x) \quad (y \leqslant x)$$

It may be noted that $G(x, y, \mu) = G(y, x, \mu)$, i e , $G(x, y, \mu)$ is a symmetric kernel in each case Consequently, with recourse to the theory of integral equations an expansion theory could be developed

Substituting in equation (3 6), we obtain for case (a)

$$(3\ 7) \quad w(x, \mu) = \left[\int_0^a \frac{K_2}{k_2} h_1(y) f_2(y)\, dy + \int_{-b}^0 \frac{K_1}{k_1} g_2'(0) h_1(0) g_1(y) f_1(y) W_g^{-1}\, dy\right] g_1(x) W_f^{-1}$$

$$- \int_{-b}^x \frac{1}{k_1} g_1(y) f_1(y)\, dy\, g_2(x) W_g^{-1} - \int_x^0 \frac{1}{k_1} g_2(y) f_1(y)\, dy\, g_1(x) W_g^{-1}$$

$$= g_1(x) \int_0^a \frac{K_2}{k_2} h_1(y) f_2(y)\, dy\, W_f^{-1} + \left[K_1 g_2'(0) h_1(0) g_1(x) - W_f g_2(x)\right]$$

$$\int_{-b}^{x} \frac{1}{k_1} g_1(y) f_1(y)\, dy\, W_g^{-1} W_f^{-1}$$

$$+ g_1(x) \int_x^0 \frac{1}{k_1} \left[K_1 g_2'(0) h_1(0) g_1(y) - W_f g_2(y)\right] f_1(y)\, dy\, W_g^{-1} W_f^{-1}$$

$$(-b \leqslant x \leqslant 0) \quad ,$$

$$w(x, \mu) = \left[\int_{-b}^{0} \frac{K_1}{k_1} g_1(y) f_1(y)\, dy + \int_0^a \frac{K_1}{k_2} g_1'(0) h_2(0) h_1(y) f_2(y) W_h^{-1}\, dy\right]$$

$$h_1(x) W_f^{-1}$$

$$- \int_0^x \frac{1}{k_2} h_2(y) f_2(y)\, dy\, h_1(x) W_h^{-1} - \int_x^a \frac{1}{k_2} h_1(y) f_2(y)\, dy\, h_2(x) W_h^{-1}$$

$$= h_1(x) \int_{-b}^{0} \frac{K_1}{k_1} g_1(y) f_1(y)\, dy\, W_f^{-1} + \left[K_1 g_1'(0) h_2(0) h_1(x) - W_f h_2(x)\right]$$

$$\int_x^a \frac{1}{k_2} h_1(y) f_2(y)\, dy\, W_h^{-1} W_f^{-1}$$

$$+ h_1(x) \int_0^x \frac{1}{k_2} \left[K_1 g_1'(0) h_2(0) h_1(y) - h_2(y) W_f\right] f_2(y)\, dy\, W_h^{-1} W_f^{-1}$$

$$(0 \leqslant x \leqslant a)$$

while we obtain for case (b)

$$(3\ 8)\quad w(x,\mu) = \left[- \int_0^\infty \frac{K_2}{k_2}\, h(y)\, f_2(y)\, dy - N \int_{-1}^{0} \frac{1}{k_1}\, g_1(y)\, f_1(y)\, dy \right] g_1(x)\, M^{-1}$$

$$- \int_x^0 \frac{1}{k_1}\, g_2(y)\, f_1(y)\, dy\, g_1(x) - \int_x^{-1} \frac{1}{k_1}\, g_1(y)\, f_1(y)\, dy\, g_2(x)$$

$$= - \int_0^\infty \frac{K_2}{k_2}\, h(y)\, f_2(y)\, dy\, g_1(x)\, M^{-1} - \left[N\, g_1(x) + M\, g_2(x) \right] \int_{-1}^{x} \frac{1}{k_1}\, g_1(y)\, f_1(y)\, dy\, M^{-1}$$

$$- \int_x^0 \frac{1}{k_1}\, f_1(y) \left[N\, g_1(y) + M\, g_2(y) \right] dy\, g_1(x)\, M^{-1} \qquad (-1 \leqslant x \leqslant 0)$$

$$w(x,\mu) = \left[- \int_{-1}^{0} \frac{K_1}{k_1}\, g_1(y)\, f_1(y)\, dy - K_1\, g_1'(0) \int_0^\infty \frac{1}{k_2}\, h(y)\, f_2(y) dy\, A^{-1} \right] h(x)\, M^{-1}$$

$$+ A^{-1} \int_0^x \frac{1}{k_2}\, h_2(y)\, f_2(y)\, dy\, h(x) + A^{-1} \int_x^\infty \frac{1}{k_2}\, h(y)\, f_2(y)\, dy\, h_2(x)$$

$$= - \int_{-1}^{0} \frac{K_1}{k_1}\, g_1(y)\, f_1(y) dy\, h(x)\, M^{-1} - \int_0^x \frac{1}{k_2} \left[K_1\, g_1'(0)\, h_1(y) - K_2\, g_1(0)\, h_2(y) \right]$$

$$f_2(y)\, dy\, h(x)\, M^{-1}$$

$$- \int_x^\infty \frac{1}{k_2}\, h(y)\, f_2(y)\, dy \left[K_1\, g_1'(0)\, h_1(x) - K_2\, g_1(0)\, h_2(x) \right] M^{-1} \qquad (x \geqslant 0)$$

and we obtain for case (c)

$$(3\ 9)\quad w(x,\mu) = \left[- \int_0^\infty \frac{K_2}{k_2}\, h(y)\, f_2(y)\, dy - A\, E^{-1} \int_0^\infty \frac{K_2}{k_1}\, g(y)\, f_1(y)\, dy \right] g(x)\, W^{-1}$$

$$- E^{-1} \int_x^0 \frac{1}{k_1} g_2(y)\, f_1(y)\, dy\, g(x) - E^{-1} \int_{-\infty}^x \frac{1}{k_1} g(y)\, f_1(y)\, dy\, g_2(x)$$

$$= - \int_0^\infty \frac{K_2}{k_2} h(y)\, f_2(y)\, dy\, g(x)\, W^{-1} - \int_{-\infty}^x \frac{1}{k_1} g(y)\, f_1(y)\, dy\, W^{-1} \left[K_1\, B\, g_2(x) + K_2\, A\, g_1(x) \right]$$

$$- \int_x^0 \frac{1}{k_1} f_1(y) \left[K_1\, B\, g_2(y) + K_2\, A\, g_1(y) \right] dy\, g(x)\, W^{-1} \qquad (x \leqslant 0) ,$$

$$w(x, \mu) = \left[- \int_{-\infty}^0 \frac{K_1}{k_1} g(y)\, f_1(y)\, dy - E\, A^{-1} \int_0^\infty \frac{K_1}{k_2} h(y)\, f_2(y)\, dy \right] h(x)\, W^{-1}$$

$$+ A^{-1} \int_0^x \frac{1}{k_2} h_2(y)\, f_2(y)\, dy\, h(x) + A^{-1} \int_x^\infty \frac{1}{k_2} h(y)\, f_2(y)\, dy\, h_2(x)$$

$$= - \int_{-\infty}^0 \frac{K_1}{k_1} g(y)\, f_1(y)\, dy\, h(x)\, W^{-1} - \int_x^\infty \frac{1}{k_2} h(y)\, f_2(y)\, dy \left[K_1\, E\, h_1(x) + K_2\, F\, h_2(x) \right] W^{-1}$$

$$- \int_0^x \frac{1}{k_2} \left[K_1\, E\, h_1(y) + K_2\, F\, h_2(y) \right] f_2(y)\, dy\, h(x)\, W^{-1} \qquad (x \geqslant 0)$$

4 <u>The formal inversions</u> Formula (3 6) expresses $w(x, \mu)$ in terms of a double Green's function, a step function and the arbitrary function $f(x)$ Roughly speaking we obtain the desired expansions by inverting formula (3 6) and expressing $f(x)$ in terms of $w(x, \mu)$ For case (a) we shall now suppose, and will prove later, that $w(x, \mu)$ has a countably infinite number of simple real poles, the zeros of W_f In this case we formally let

(4 1) $$f(x) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{(C_n)} w(x, \mu)\, d\mu ,$$

where (C_n) are large contours in the μ-plane which enclose all the poles in the limit For cases (b) and (c) we shall now suppose, and will prove later, that $w(x, \mu)$ has as its only singular point a branch point at the origin For these cases we formally let

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{0} I \left\{ w(x, \mu) \right\} d\mu \quad , \tag{4 2}$$

where $I \left\{ z \right\}$ is the imaginary part of z

Using formula (3 7) in equation (4 1) we obtain formally

$$f(x) = \sum_{n=1}^{\infty} \int_{-b}^{a} p(y)\, X_n(y)\, f(y)\, dy\, X_n(x) \left[\varphi_n(0)\, \psi_n(0)\, W_n \right]^{-1} \tag{4 3}$$

where $p(y)$ has the previous meaning, where W_n is the derivative of W_f with respect to μ evaluated at the n-th zero of W_f , $\mu = \mu_n$, where either $\varphi_n(0) \equiv h_n(0) \equiv h(0, \mu_n)$ and $\psi_n(0) \equiv g_n(0)$ or $\varphi_n(0) \equiv K_2\, h_n'(0)$ and $\psi_n(0) \equiv K_1\, g_n'(0)$, either choice being equivalent for each n , unless for a particular n both $h_n(0)$ and $g_n(0)$ are zero or both $h_n'(0)$ and $g_n'(0)$ are zero, in which case the non-zero pair is chosen and where

$$X_n(x) \equiv \varphi_n(0)\, g_n(x) \equiv \varphi_n(0)\, g_1(x, \mu_n) \qquad (-b \leqslant x \leqslant 0)$$

$$\equiv \psi_n(0)\, h_n(x) \equiv \psi_n(0)\, h_1(x, \mu_n) \qquad (0 \leqslant x \leqslant a)$$

When μ is real in case (b), $g_1(x, \mu) = g_1(x)$, $g_2(x)$, $h_1(x)$ and $h_2(x)$ are real while $A = A(\mu)$ and B may be complex Using formula (3 8) in equation (4 2), we obtain formally

$$(4\ 4) \qquad f_1(x) = -\frac{1}{\pi} \int_{-\infty}^{0} \left[\int_{-1}^{0} \frac{1}{k_1} f_1(y)\, g_1(y)\ I\left\{\bar{M}N\right\} dy + \int_{0}^{\infty} \frac{K_2}{k_2} f_2(y) I \left\{h(y)\bar{M}\right\} dy \right]$$

$$g_1(x)\ M^{-1}\ \bar{M}^{-1}\ d\mu \quad ,$$

$$f_2(x) = -\frac{1}{\pi} \int_{-\infty}^{0} \left[\int_{-1}^{0} \frac{K_1}{k_1} f_1(y)\ g_1(y) dy\ I \left\{h(x)\bar{M}\right\} + \int_{0}^{x} \frac{1}{k_2} \right.$$

$$\left\{K_1\ g_1'(0)\ h_1(y) - K_2\ g_1(0)\ h_2(y)\right\}\ f_2(y)\ dy\ I\ \left\{h(x)\ \bar{M}\right\}$$

$$\left. + \int_{x}^{\infty} \frac{1}{k_2} f_2(y)\ I \left\{h(y)\ \bar{M}\right\} dy\ \left\{K_1\ g_1'(0)\ h_1(x) - K_2\ g_1(0)\ h_2(x)\right\} \right]$$

$$M^{-1}\ \bar{M}^{-1}\ d\mu \quad ,$$

where $\bar{M}$ is the complex conjugate of M

When μ is real in case (c), $g_1(x, \mu) = g_1(x)$, $g_2(x)$, $h_1(x)$ and $h_2(x)$ are real while $A = A(\mu)$, B , E and F may be complex Using formula (3 9) in equation (4 2), we obtain formally

$$(4\ 5) \qquad f_1(x) = -\frac{1}{\pi} \int_{-\infty}^{0} \left[\int_{0}^{\infty} \frac{K_2}{k_2} f_2(y)\ I \left\{h(y)\bar{W}\ g(x)\right\} dy + \int_{-\infty}^{x} \frac{1}{k_1} f_1(y)\ \ I \right.$$

$$\left. \left\{g(y)\ \bar{W}\ \left[K_1\ B\ g_2(x) + K_2\ A\ g_1(x)\right]\right\} dy \right.$$

$$+ \int_x^0 \frac{1}{k_1} f_1(y) \; I \left\{ g(x) \; \bar{W} \left[K_1 \; B \; g_2(y) + K_2 \; A \; g_1(y) \right] \right\} dy \Bigg] \; W^{-1} \; \bar{W}^{-1} \; d\mu \quad ,$$

$$f_2(x) = -\frac{1}{\pi} \int_{-\infty}^0 \Bigg[\int_{-\infty}^0 \frac{K_1}{k_1} f_1(y) \; I \left\{ g(y) \; \bar{W} \; h(x) \right\} dy + \int_x^\infty \frac{1}{k_2} f_2(y) \; I$$

$$\left\{ h(y) \; \bar{W} \left[K_1 \; E \; h_1(x) + K_2 \; F \; h_2(x) \right] \right\} dy$$

$$+ \int_0^x \frac{1}{k_2} f_2(y) \; I \left\{ h(x) \; \bar{W} \; \left[K_1 \; E \; h_1(y) + K_2 \; F \; h_2(y) \right] \right\} dy$$

5 <u>The special cases $q(x)$ identically zero</u> When $q(x)$ is identically zero equations (3 1) and (3 2) become differential equations with constant coefficients Consequently, they can be solved along with the auxiliary conditions by using variation of parameters or by using the Laplace transform with respect to x For case (a), we can obtain

$$(5\ 1) \qquad w(x, \mu) = - \Big[\cos s_1(x + b) \; p(s) + (\cos s_2 \; a \cos s_1 \; x + K^{-1} \sin s_2 \; a \sin s_1 \; x) q(s)$$

$$W_o^{-1} \; s^{-1} - s_1^{-1} \int_0^x \frac{1}{k_1} f_1(y) \sin s_1 \; (x - y) \; dy \qquad (-b \leqslant x \leqslant 0)$$

$$w(x, \mu) = - \Big[\cos s_1(a - x) \; q(s) + (\cos s_1 \; b \cos s_2 \; x - K \sin s_1 \; b \sin s_2 \; x) \; p(s) \Big]$$

$$W_o^{-1} \; s^{-1} - s_2^{-1} \int_0^x \frac{1}{k_2} f_2(y) \sin s_2 \; (x - y) \; dy \qquad (0 \leqslant x \leqslant a) \; ,$$

where

$$K \equiv K_1 \; K_2^{-1} \; k_1^{-\frac{1}{2}} \; k_2^{\frac{1}{2}} \quad , \quad \mu \equiv -s^2 \; , \; s_1 \equiv s \; k_1^{-\frac{1}{2}} \quad , \quad s_2 \equiv s \; k_2^{-\frac{1}{2}} \quad ,$$

$$p(s) \equiv \int_0^a \frac{K_2}{k_2} f_2(y) \cos s_2 (a - y) dy \quad , \quad q(s) \equiv \int_{-b}^0 \frac{K_1}{k_1} f_1(y) \cos s_1 (b + y)\, dy \quad \text{and}$$

$$W_o \equiv K_1 \; k_1^{-\frac{1}{2}} \cos s_2 a \sin s_1 b + K_2 \; k_2^{-\frac{1}{2}} \cos s_1 b \sin s_2 a$$

We shall always select the branch of $\mu^{\frac{1}{2}}$ where the real part of $\mu^{\frac{1}{2}}$ is greater than zero, i e , we choose $-\pi < \theta \leqslant \pi$ when $\mu = r\, e^{i\theta}$ and $\mu^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ Let $s = s_r + i\, s_i$ Since $s = i\sqrt{\mu}$, $s_i \geqslant 0$

For case (b), we can obtain

$$(5\ 2) \quad w(x, \mu) = - \int_0^\infty \frac{K_2}{k_2} \frac{1}{2\, i\, s_2} e^{i\, s_2\, y} f_2(y)\, dy \cos s_1 (1 + x)\, M_o^{-1}$$

$$- \int_{-1}^x \frac{1}{k_1} f_1(y) \cos s_1 (1 + y)\, dy\, M_o^{-1} \left[\frac{K_1}{2\, i\, s_2} \cos s_1 x + \frac{K_2}{2\, s_1} \sin s_1 x \right]$$

$$- \int_x^0 \frac{1}{k_1} f_1(y) \left[\frac{K_1}{2\, i\, s_2} \cos s_1 y + \frac{K_2}{2\, s_1} \sin s_1 x \right] dy \cos s_1 (1+x) M_o^{-1}$$

$$(-1 \leqslant x \leqslant 0)$$

$$w(x, \mu) = - \int_{-1}^0 \frac{K_1}{k_1} \cos s_1 (1 + y) f_1(y)\, dy \frac{1}{2\, i\, s_2} e^{i\, s_2\, x} M_o^{-1}$$

$$- \int_0^x \frac{1}{k_2} \left[K_2 \cos s_1 \cos s_2 y - K_1 \sqrt{\frac{k_2}{k_1}} \sin s_1 \sin s_2 y \right] f_2(y) dy \frac{1}{2 i s_2} e^{i s_1 x} M_o^{-1}$$

$$- \int_x^\infty \frac{1}{k_2} \frac{1}{2\, i\, s_2} e^{i\, s_2\, y} f_2(y) dy \left[K_2 \cos s_1 \cos s_2 x - K_1 \sqrt{\frac{k_2}{k_1}} \sin s_1 \sin s_2 x \right]$$

$$M_o^{-1} \qquad (x \geqslant 0) ,$$

where

$$M_o \equiv \frac{K_1 \sqrt{k_2}}{2i \sqrt{k_1}} \sin s_1 + \frac{K_2}{2} \cos s_1$$

For case (c), we can obtain

$$(5\ 3) \qquad w(x, \mu) = \Bigg[\int_0^\infty \frac{K_2}{k_2} f_2(y) \frac{e^{i s_2 y}}{2 i s_2} \frac{e^{-i s_1 x}}{2 i s_1} dy + \int_{-\infty}^{x} \frac{1}{k_1} \frac{1}{2 i s_1} e^{-i s_1 y}$$

$$f_1(y)\, dy \left(\frac{K_1}{2 i s_2} \cos s_1 x + \frac{K_2}{2 s_1} \sin s_1 x \right)$$

$$+ \frac{e^{-i s_1 x}}{2 i s_1} \int_x^0 \frac{1}{k_1} f_1(y) \left(\frac{K_1}{2 i s_2} \cos s_1 y + \frac{K_2}{2 s_1} \sin s_1 y \right) dy \Bigg]$$

$$(-4 i s) \left(K_1 \sqrt{k_2} + K_2 \sqrt{k_1}\right)^{-1} \qquad (x \leqslant 0) \quad ,$$

$$w(x, \mu) = \Bigg[\int_{-\infty}^{0} \frac{K_1}{k_1} f_1(y) \frac{e^{-i s_1 y}}{2 i s_1} \frac{e^{i s_2 x}}{2 i s_2} dy + \int_x^\infty \frac{1}{k_2} \frac{e^{i s_2 y}}{2 i s_2} f_2(y)\, dy$$

$$\left(\frac{K_2}{2 i s_1} \cos s_2 x - \frac{K_1}{2 s_2} \sin s_2 x \right)$$

$$+ \frac{e^{i s_2 x}}{2 i s_2} \int_0^x \frac{1}{k_2} \left(\frac{K_2}{2 i s_1} \cos s_2 y - \frac{K_1}{2 s_2} \sin s_2 y \right) f_2(y) dy \Bigg]$$

$$(-4 i s)\left(K_1 \sqrt{k_2} + K_2 \sqrt{k_1}\right)^{-1} \qquad (x \geqslant 0)$$

Using these last formulas and equations (4 1) and (4 2) we can obtain the formal expansions For case (a), we obtain

$$(5\ 4) \qquad f(x) = \sum_{n=0}^{\infty} \int_{-b}^{a} p(y)\ Z_n(y)\ f(y)\ dy\ Z_n(x) \left[\varphi_{on}(0)\ \psi_{on}(0)\ W_{on}\right]^{-1}$$

where W_{on} is the derivative of W_o with respect to s evaluated at the n-th zero of W_o, $\mu = \mu_n$, where either $\varphi_{on}(0) \equiv \cos s_n \frac{a}{\sqrt{k_2}}$ and $\psi_{on}(0) \equiv \cos s_n \frac{b}{\sqrt{k_1}}$ or $\varphi_{on}(0) \equiv + \frac{K_2}{\sqrt{k_2}} s_n \sin s_n \frac{a}{\sqrt{k_2}}$ and $\psi_{on}(0) \equiv \frac{K_1}{\sqrt{k_1}} s_n \sin s_n \frac{b}{\sqrt{k_1}}$, either choice being equivalent for each n, unless for particular n both $\cos s_n \frac{a}{\sqrt{k_2}}$ and $\cos s_n \frac{b}{\sqrt{k_1}}$ are zero or both $\sin s_n \frac{b}{\sqrt{k_1}}$ and $\sin s_n \frac{a}{\sqrt{k_2}}$ are zero, in which case the non-zero pair is chosen and where

$$Z_n(x) \equiv \varphi_{on}(0) \cos s_n \frac{1}{\sqrt{k_1}} (x + b) \qquad (-b \leqslant x \leqslant 0)$$

$$\equiv \psi_{on}(0) \cos s_n \frac{1}{\sqrt{k_2}} (a - x) \qquad (0 \leqslant x \leqslant a)$$

For case (b), we obtain

$$(5\ 5) \qquad f(x) = \frac{2}{\pi} \int_0^{\infty} Z(x, s) \left[\int_{-1}^{\infty} p(y)\ Z(y, s)\ f(y)\ dy\right] \left[G(s)\right]^{-1} ds\ ,$$

where

$$G(s) \equiv \frac{K_2}{\sqrt{k_2}} (\cos^2 s_1 + K^2 \sin^2 s_1)$$

and where

$$Z(x, s) \equiv \cos s_1 (1 + x) \qquad (-1 \leqslant x \leqslant 0)$$

$$\equiv \cos s_1 \cos s_2 x - K \sin s_1 \sin s_2 x \qquad (x \geqslant 0)$$

For case (c), we obtain

$$(5\ 6) \qquad f(x) = \frac{2}{\pi} \int_0^\infty \frac{\int_{-\infty}^{\infty} p(y)f(y) \left[Z_1(x,s)\ Z_1(y,s) + K_1K_2\sqrt{k_1k_2}\ Z_2(x,s)Z_2(y,s)\right]dyds}{K_1\ K_2 \left[K_1\sqrt{k_2} + K_2\sqrt{k_1}\right]}$$

where

$$Z_1(x, s) = K_2\sqrt{k_1} \sin s_1 x\ , \quad Z_2(x, s) = -\cos s_1 x \qquad (x \leqslant 0)$$

$$= K_1\sqrt{k_2} \sin s_2 x\ , \qquad = -\cos s_2 x \qquad (x \geqslant 0)$$

6 <u>Asymptotic properties</u> As previously indicated we shall show that the general expansions (4 3), (4 4) and (4 5) are equi-convergent respectively with the special expansions (5 4), (5 5) and (5 6) In order to prove the equi-convergence property we need asymptotic or order properties for the two cases, $|x|$ is large and $|s|$ (or $|\mu|$) is large The methods of obtaining these properties are found largely in Titchmarsh [18] Consequently only some of the results and a few words of explanation will be displayed

Using the differential equations and the initial conditions which the solutions satisfy it is easily verified that [18, p 9] the g's and h's satisfy integral equations similar to the following particular cases

(6 1a) $$g(x) = \cos s_1 (x + b) + \frac{1}{s_1 k_1} \int_{-b}^{x} \sin s_1(x - y)\, q_1(y)\, g_1(y)\, dy$$

(6 1c) $$g_1(x) = \frac{1}{s_1} \sin s_1 x + \frac{1}{s_1 k_1} \int_{0}^{x} \sin s_1(x - y)\, q_1(y)\, g_1(y)\, dy$$

(6 2a) $$g_1'(x) = - s_1 \sin s_1(x + b) + \frac{1}{k_1} \int_{-b}^{x} \cos s_1(x - y)\, q_1(y)\, g_1(y)\, dy$$

Using equations (6 1a) through (6 2a), it is possible to obtain the following representative order properties Let $\sigma = \frac{s_i}{\sqrt{k_1}}$ and $\tau = \frac{s_i}{\sqrt{k_2}}$, where $s = s_r + i\, s_i$ *

(6 3a) $$g_1(x) = \cos s_1(x + b) + 0 \left\{ |s|^{-1} e^{\sigma(b+x)} \right\} \text{ uniformly for } -b \leqslant x \leqslant 0$$

Equation (6 3a), for example, may be verified in the following manner Let

$$G(x) = g_1(x)\, e^{-\sigma(b+x)} \qquad \text{and} \qquad C = \max_{-b \leqslant x \leqslant 0} |G(x)|$$

Therefore by equation (6 1a)

$$G(x) = \cos s_1(x + b)\, e^{-\sigma(b+x)} + \frac{1}{s_1 k_1} \int_{-b}^{x} \sin s_1(x - y)\, G(y)\, q_1(y)\, e^{-\sigma(x-y)}\, dy$$

---

* We shall write $g_1(x) = 0 \left\{ e^{\sigma(b+x)} \right\}$ uniformly for $-b \leqslant x \leqslant 0$ to signify that for large $|s|$, $\left| g_1(x)\, e^{-\sigma(b+x)} \right| < M$, where M is independent of x in the interval $-b \leqslant x \leqslant 0$

By taking the absolute value, we have

$$C \leqslant 1 + \frac{1}{|s_1|\, k_1} \int_{-b}^{x} C \left| q_1(y) \right| dy$$

and consequently

$$C \leqslant \left[1 - \frac{1}{|s_1|\, k_1} \int_{-b}^{0} \left| q_1(y) \right| dy \right]^{-1} ,$$

provided the denominator is positive This is certainly true if $|s|$ is large Consequently $\left| g_1(x)\, e^{-\sigma(b+x)} \right| < M$, where M is independent of x Using this in equation (6 1a) we have equation (6 3a)

The following order properties for large $|x|$ can be obtained in a manner similar to that used by Titchmarsh [18, p 98]

(6 4c) $$g_1(x) = e^{i s_1 x} \left\{ R(s) + o(1) \right\}$$

as $-x$ tends to infinity (complex s) , where

$$R(s) = \frac{1}{2\, i\, s_1} - \frac{1}{2\, i\, s_1\, k_1} \int_{-\infty}^{0} e^{-i s_1 y}\, q_1(y)\, g_1(y)\, dy ,$$

(6 5c) $$g_1(x) = \tilde{u}_1(s) \cos s_1 x + \tilde{v}_1(s) \sin s_1 x + o(1)$$

as $-x$ tends to infinity (real s) , where

$$\tilde{u}_1(s) = \frac{1}{s_1 k_1} \int_{-\infty}^{0} \sin s_1 y \, q_1(y) \, g_1(y) dy \quad \text{and} \quad \tilde{v}_1(s) = \frac{1}{s_1} - \frac{1}{s_1 k_1} \int_{-\infty}^{0}$$

$$\cos s_1 y \, q_1(y) \, g_1(y) dy \quad ,$$

(6 6c)
$$g_2(x) = e^{i s_1 x} \left\{ S(s) + o(1) \right\}$$

as $-x$ tends to infinity (complex s) , where

$$S(s) = -\frac{1}{2} - \frac{1}{2i \, s_1 k_1} \int_{-\infty}^{0} e^{-i s_1 y} \, q_1(y) \, g_2(y) \, dy \quad ,$$

(6 7c)
$$g_2(x) = \tilde{u}_2(s) \cos s_1 x + \tilde{v}_2(s) \sin s_1 x + o(1)$$

as $-x$ tends to infinity (complex s), where

$$\tilde{u}_2(s) = -1 + \frac{1}{s_1 k_1} \int_{-\infty}^{0} \sin s_1 y \, q_1(y) \, g_2(y) \, dy$$

and

$$\tilde{v}_2(s) = \frac{1}{s_1 k_1} \int_{-\infty}^{0} \cos s_1 y \, q_1(y) \, g_2(y) \, dy \quad ,$$

(6 8b, c)
$$h_1(x) = e^{-i s_2 x} \left\{ P(s) + o(1) \right\}$$

as x tends to infinity (complex s) , where

$$P(s) = -\frac{1}{2 i s_2} - \frac{1}{2 i s_2 k_2} \int_{0}^{\infty} e^{i s_2 y} \, q_2(y) \, h_1(y) \, dy \, ,$$

(6 9b, c) $$h_1(x) = u_1(s) \cos s_2 x + v_1(s) \sin s_1 x + o(1)$$

as x tends to infinity (real s), where

$$u_1(s) = - \frac{1}{s_2 k_2} \int_0^\infty \sin s_2 y \, q_2(y) \, h_1(y) dy$$

and

$$v_1(s) = \frac{1}{s_2} + \frac{1}{s_2 k_2} \int_0^\infty \cos s_2 y \, q_2(y) \, h_1(y) \, dy \quad ,$$

(6 10b, c) $$h_2(x) = e^{-i s_2 x} \left\{ Q(s) + o(1) \right\}$$

as x tends to infinity (complex s), where

$$Q(s) = -\frac{1}{2} - \frac{1}{2 i s_2 k_2} \int_0^\infty e^{i s_2 y} \, g_2(y) \, h_2(y) \, dy \quad ,$$

(6 11b, c) $$h_2(x) = u_2(s) \cos s_2 x + v_2(s) \sin s_2 x + o(1)$$

as x tends to infinity (real s), where

$$u_2(s) = -1 - \frac{1}{s_2 k_2} \int_0^\infty \sin s_2 y \, q_2(y) \, h_2(y) \, dy$$

and

$$v_2(s) = \frac{1}{s_2 k_2} \int_0^\infty \cos s_2 y \, q_2(y) \, h_2(y) \, dy$$

The following equations can be derived in a straight forward fashion from the above definition and order properties

(6 12c) $$R(s) = \frac{1}{2}\left[\tilde{u}_1(s) - i\,\tilde{v}_1(s)\right] = \left[\frac{1}{2\,i\,s_1} + O\left\{|s|^{-2}\right\}\right]$$

(6 13c) $$S(s) = \frac{1}{2}\left[\tilde{u}_2(s) - i\,\tilde{v}_2(s)\right] = \left[-\frac{1}{2} + O\left\{|s|^{-1}\right\}\right]$$

(6 14b, c) $$P(s) = \frac{1}{2}\left[u_1(s) + i\,v_1(s)\right] = \left[-\frac{1}{2\,i\,s_2} + O\left\{|s|^{-2}\right\}\right]$$

(6 15b, c) $$Q(s) = \frac{1}{2}\left[u_2(s) + i\,v_2(s)\right] = \left[-\frac{1}{2} + O\left\{|s|^{-1}\right\}\right]$$

(6 16c) $$\tilde{u}_1(s)\,\tilde{v}_2(s) - \tilde{v}_1(s)\,\tilde{u}_2(s) = \frac{1}{s_1}$$

(6 17b, c) $$u_1(s)\,v_2(s) - v_1(s)\,u_2(s) = \frac{1}{s_2}$$

Let us now consider the existence of solutions to the R and S differential equations which are bounded even when s (or μ) is complex Since the methods are similar consider only the case of a solution bounded as x tends to infinity We previously let $h(x) = A\,h_1(x) + B\,h_2(x)$ Using equations (6 8b, c) and (6 10b, c), we have

$$h(x) = A\,e^{-i\,s_2\,x}\left\{P(s) + o(1)\right\} + B\,e^{-i\,s_2\,x}\left\{Q(s) + o(1)\right\}$$

as x tends to infinity When $s_i \geqslant \delta > 0$, it is apparent that $h(x)$ can not be bounded as x tends to infinity unless $A(s)\,P(s) + B(s)\,Q(s) = 0$ Therefore

we put $A = Q(s)$ and $B = -P(s)$ Consequently, we have

(6 18) $$A = Q(s) \text{ and } B = -P(s), \quad h(x) = Q(s)\, h_1(x) - P(s)\, h_2(x)$$

$$= e^{i s_2 x} \left\{ Q(s) \frac{1}{2 i s_2} + P(s) \frac{1}{2} + \frac{1}{2 i s_2 k_2} \int_0^x q_2(y)\, h(y)\, e^{-i s_2 y}\, dy \right.$$

$$\left. + e^{-2i s_2 x} \frac{1}{2 i s_2 k_2} \int_x^\infty e^{i s_2 y}\, q_2(y)\, h(y)\, dy \right\}$$

It is possible to verify directly that the right hand side of this equation satisfies the S differential equation

Now let $H(x) = s_2 e^{-i s_2 x} h(x)$ and let $C = \underset{x \geqslant 0}{\text{LUB}} \left| H(x) \right|$ Taking absolute values, we obtain from equations (6 18)

$$\left| H(x) \right| \leqslant \left| Q(s) \frac{1}{2i} + P(s) \frac{s_2}{2} \right| + \frac{1}{2|s_2| k_2} \int_0^x q_2(y) \left| H(y) \right| dy +$$

$$\frac{1}{2|s_2| k_2} \int_x^\infty q_2(y) \left| H(y) \right| dy$$

Therefore

$$C \leqslant \left| Q(s) \frac{1}{2i} + \frac{s_2}{2} P(s) \right| + \frac{1}{2 k_2 |s_2|} C \int_0^\infty q_2(y)\, dy ,$$

and

$$C \leqslant \left| Q(s) \frac{1}{2i} + \frac{s_2}{2} P(s) \right| \left[1 - \frac{1}{2|s_2|k_2} \int_0^\infty q_2(y) \right]^{-1} ,$$

provided the denominator is positive This is certainly the case if $|s|$ is big enough By equations (6 14b, c) and (6 15b, c) we can see that $\left| Q(s) \frac{1}{2i} + \frac{s_2}{2} P(s) \right|$ is bounded for large $|s|$ Consequently

$$h(x) = 0\left\{ |s|^{-1} e^{-\tau x} \right\} \text{ uniformly for } x \geqslant 0$$

Using this order property along with equations (6 14b, c) and (6 15b, c), we obtain

$$\text{(6 19b, c)} \qquad h(x) = \left(\left[-\frac{1}{2} + 0\left\{ |s|^{-1} \right\} \right] \frac{1}{2i\, s_2} + \left[-\frac{1}{2i\, s_2} + 0\left\{ |s|^{-2} \right\} \right] \frac{1}{2} + 0\left\{ |s|^{-2} \right\} \right) e^{i s_2 x}$$

$$= -\frac{1}{2 i s_2} e^{i s_2 x} \left[1 + 0\left\{ |s|^{-1} \right\} \right]$$

In a similar fashion, the following equations can be obtained Let $E = S(s)$ and $F = -R(s)$

$$\text{(6 20c)} \qquad g(x) = S(s)\, g_1(x) - R(s)\, g_2(x) = e^{-i s_1 x}$$

$$\left\{ -S(s) \frac{1}{2i\, s_1} + R(s) \frac{1}{2} + \frac{1}{2i\, s_1 k_1} \int_x^0 q_1(y)\, g(y)\, e^{i s_1 y}\, dy + e^{2i s_1 x} \frac{1}{2i\, s_1 k_1} \int_{-\infty}^x e^{-2i s_1 y} q_1(y)\, g(y)\, dy \right\} ,$$

(6 21c) $$g(x) = \frac{1}{2i\, s_1} e^{-i\, s_1\, x} \left[1 + 0 \left\{ |s|^{-1} \right\} \right]$$

Equations (6 19b, c) and (6 21c) indicate that $h(x)$ and $g(x)$ not only are bounded but also tend to zero respectively as x and $-x$ tend to infinity, provided $|s|$ is large enough In the case of complex s it would be desirable to eliminate the requirement that $|s|$ be large The writer has been unable to do this directly Using Titchmarsh's results [18, p 19] this may be accomplished in the following indirect fashion

Now $h(x) = Q(s) \left[h_1(x) - \frac{P(s)}{Q(s)} h_2(x) \right]$ It is easily seen from the previous order properties that

$$\frac{P(s)}{Q(s)} = \lim_{b \to \infty} \frac{h_1(b)}{h_2(b)} = \lim_{b \to \infty} \frac{h_1'(b)}{h_2'(b)} = \lim_{b \to \infty} \frac{h_1(b) \cos \beta + h_1'(b) \sin \beta}{h_2(b) \cos \beta + h_2'(b) \sin \beta}$$

If we let $\ell(b, s) = - \dfrac{h_1(b) \cos \beta + h_1'(b) \sin \beta}{h_2(b) \cos \beta + h_2'(b) \sin \beta}$ then $h_1(x) + \ell(b, s)\, h_2(x)$ satisfies the boundary condition

$$\left[h_1(b) + \ell(b, s)\, h_2(b) \right] \cos \beta + \left[h_1'(b) + \ell(b, s)\, h_2'(b) \right] \sin \beta = 0$$

Using Titchmarsh's results [18, p 21] we can conclude that for every value of $\mu = -s^2$ other than real values $|h(x)|^2$ is Lebesgue integrable Further as Titchmarsh has shown [18, p 22] for any fixed complex μ and λ

$$\lim_{x \to \infty} \left[h(x, \mu)\, h'(x, \beta) - h'(x, \mu)\, h(x, \beta) \right] = 0$$

The last result is the one which will actually be used in investigating the properties of the roots of the characteristic equation

Likewise $g(x)$ satisfies

$$\lim_{x \to -\infty} \left[g(x, \mu)\, g'(x, \beta) - g(x, \beta)\, g'(x, \mu) \right] = 0$$

In the sections dealing with equi-convergence proofs we need the order properties of the characteristic equations For case (a) we can obtain by using order equations of the type (6 3a) the relation

$$W_f = -s\, W_o + 0 \left\{ e^{\tau a + \sigma b} \right\} \quad \text{uniformly for } -b \leqslant x \leqslant a , \tag{6 22a}$$

where W_o is defined in connection with equation (5 1) For case (b) we can obtain the relation

$$M = -\left(\frac{K_1 \sqrt{k_2}}{2i \sqrt{k_1}} \sin s_1 + \frac{K_2}{2} \cos s_1 \right) + 0 \left\{ |s|^{-1} e^{\sigma} \right\} \tag{6 22b}$$

uniformly for $x \geqslant -1$ For case (c) we can obtain the relation

$$W = -\frac{1}{4is} \left[K_1 \sqrt{k_2} + K_2 \sqrt{k_1} \right] \left[1 + 0 \frac{1}{|s|} \right] \tag{6 22c}$$

uniformly for all x

In case (b) we also need the order property for N

$$(6\ 23b) \qquad N = -\frac{K_1\sqrt{k_2}}{2\,i\,s}\cos s_1 + \frac{K_2\sqrt{k_1}}{2s}\sin s_1 + O\left\{|s|^{-2}\,e^{\sigma}\right\}$$

7 <u>The characteristic equations</u> Before considering the equi-convergence proof we need to consider whether the characteristic equations have any zeros And, if there are zeros we need to investigate their number and location

First let us show that there are no complex zeros μ for any of the three cases Consider the following "Sturm-Liouville" type systems which determine the characteristic equations

$$(7\ 1) \qquad k_1\,u_{xx} - \left[\mu + q_1(x)\right] u(x,\mu) = 0 \qquad (x < 0)\ ,$$

$$(7\ 2) \qquad k_2\,u_{xx} - (\mu + q_2)\,u = 0 \qquad (x > 0)\ ,$$

$$(7\ 3) \qquad u(0^-,\mu) = u(0^+,\mu)\ , \qquad (7\ 4) \quad K_1\,u_x(0^-,\mu) = K_2\,u_x(0^+,\mu)\ ,$$

$$(7\ 5a) \qquad u_x(-b) = 0 = u_x(a)\ , \qquad (7\ 5b) \quad u_x(-1) = 0\ ,\ u \text{ bounded}\ ,$$

$$(7\ 5c) \qquad u \text{ bounded}$$

Taking complex conjugates we obtain also

$$k_1 \bar{u}_{xx} - (\bar{\mu} + q_1)\,\bar{u} = 0 \;, \quad k_2 \bar{u}_{xx} - (\bar{\mu} + q_2)\,\bar{u} = 0 \;,$$

$$\bar{u}(0^-, \mu) = \bar{u}(0^+, \mu) \;, \quad K_1\,\bar{u}(0^-, \mu) = K_2\,\bar{u}(0^+, \mu) \;,$$

$$\bar{u}_x(-b) = 0 \;, \quad \bar{u}_x(a) = 0 \;, \quad \bar{u}_x(-1) = 0$$

Eliminating $q_1(x)$ and $q_2(x)$ between corresponding differential equations, we obtain

$$\left[u''\,\bar{u} - \bar{u}''\,u\right] = -\frac{2}{k_1}\,\mu_1\,u\,\bar{u} \quad (x < 0) \;, \quad \left[u''\,\bar{u} - \bar{u}''\,u\right] = \frac{2}{k_2}\,\mu_i\,u\,\bar{u} \quad (x > 0),$$

where μ_i is the imaginary part of μ Integrating these two equations, we have

$$\left[u'\,\bar{u} - \bar{u}'\,u\right]_{-b}^{0} = -2\mu_i \int_{-b}^{0} \frac{K_1}{k_1}\,u(y)\,\bar{u}(y)\,dy \;, \quad \left[u'\,u - u'\,u\right]_{0}^{a} =$$

$$-2\mu_i \int_{0}^{a} \frac{K_2}{k_2}\,u\,\bar{u}\;dy \;,$$

$$\left[u'\,\bar{u} - \bar{u}'\,u\right]_{-1}^{0} = -2\mu_i \int_{-1}^{0} \frac{K_1}{k_1}\,u\,\bar{u}\,dy \;, \quad \left[u'\,\bar{u} - \bar{u}'\,u\right]_{0}^{\infty} = -2\mu_i \int_{0}^{\infty} \frac{K_2}{k_2}\,u\,\bar{u}\,dy \;,$$

$$\left[u'\,\bar{u} - \bar{u}'\,u\right]_{-\infty}^{0} = -2\mu_i \int_{-\infty}^{0} \frac{K_1}{k_1}\,u\,\bar{u}\,dy \;, \quad \left[u'\,u - u'\,u\right]_{0}^{\infty} = -2\mu_i \int_{0}^{\infty} \frac{K_2}{k_2}\,u\,\bar{u}\,dy$$

In case (a) if we apply equations (7 3), (7 4), and (7 5a) and their conjugates we have

$$-2\mu_i \left[\int_{-b}^{0} \frac{K_1}{k_1} u \bar{u} \, dy + \int_{0}^{a} \frac{K_2}{k_2} u \bar{u} \, dy \right] = 0$$

In case (b) we note that the only possible bounded solution of equation (7 2) are given by constants times $h(x)$ Using equations (7 3), (7 4), and (7 5b) and the result

$$\lim_{x \to \infty} \left[h(x, \mu) \, h'(x, \lambda) - h'(x, \mu) \, h(x, \lambda) \right] = 0 ,$$

we have

$$-2\mu_i \left[\int_{-1}^{0} \frac{K_1}{k_1} u \bar{u} \, dy + \int_{0}^{\infty} \frac{K_2}{k_2} u \bar{u} \, dy \right] = 0$$

In case (c) in a similar fashion

$$-2\mu_i \left[\int_{-\infty}^{0} \frac{K_1}{k_1} u \bar{u} \, dy + \int_{0}^{\infty} \frac{K_2}{k_2} u \bar{u} \, dy \right] = 0$$

If $\frac{K_1}{k_1} \frac{K_2}{k_2} > 0$, which is certainly true in the physical case, then for case (a), (b), and (c), $\mu_i = 0$

Next let us show that there are no positive roots μ of the characteristic equations Since $q(x) \geqslant 0$, equations (7 1) and (7 2) indicate that u and u_{xx} always have the same sign In case (a) it is easily seen that $u'(x)$ is unable to satisfy the requirement of vanishing at both a and $-b$ This is

apparent since both $u(x)$ and $u'(x)$ are monotone functions on the interval $-b \leqslant x \leqslant a$ In case (b) it is again apparent that $u(x)$ and $u'(x)$ are monotone functions with the same sign Consequently $u(x)$ can not satisfy the condition of being bounded as x tends to infinity A similar argument shows that there are no positive roots of the characteristic equation in case (c)

In case (b) and (c) we can show that when μ is negative, i e , s is real, that the characteristic equations have no zeros In case (b) let us suppose that $M \equiv K_2\, Q(s)\, g_1(0, s) - K_1\, P(s)\, g_1'(0, s)$ has a real zero σ Then for this value σ , $K_2\, Q\, g_1(0) = K_1\, P\, g_1'(0)$ Multiplying by $\bar{P}$, the complex conjugate of P , we obtain $K_2\, Q\, \bar{P}\, g_1(0) = K_1\, P\, \bar{P}\, g_1'(0)$ Since the right hand side of this last equation is real for real σ , the imaginary part of $Q\, \bar{P}$ must equal zero Now $4I\left\{Q\, \bar{P}\right\} = I\left\{(u_2 + i\, v_2)\,(u_1 + i\, v_1)\right\} = u_1\, v_2 - u_2\, v_1$ This latter quantity $u_1\, v_2 - u_2\, v_1$, however, can not equal zero since by equation (6 17) it is equal to $\frac{k_2}{\sigma}$

In case (c) let us suppose that $W = -K_2\, A\, F + K_1\, B\, E = K_2\, Q(s)\, R(s) - K_1\, P(s)\, S(s)$ has a real zero σ For this value of σ , $K_2\, Q\, R = K_1\, P\, S$ Consequently $K_2\, Q\, R\, \bar{P}\, \bar{S} = K_1\, P\, S\, \bar{P}\, \bar{S}$ and $K_2\, Q\, R\, \bar{P}\, \bar{S}$ is positive In particular $I\left\{K_2\, Q\, R\, \bar{P}\, \bar{S}\right\} = 0$ Making use of equations (6 12c) through (6 17), we have

$$8\, Q\, R\, \bar{P}\, \bar{S} = (u_2 + i\, v_2)(u_1 - i\, v_1)(\tilde{u}_1 - i\, \tilde{v}_1)(\tilde{u}_2 + i\, \tilde{v}_2) =$$

$$\left[(u_1\, u_2 + v_1\, v_2)(\tilde{u}_1\, \tilde{u}_2 + \tilde{v}_1\, \tilde{v}_2) - \frac{\sqrt{k_1\, k_2}}{\sigma^2}\right] + i\left[\frac{\sqrt{k_1}}{\sigma}\,(u_1\, u_2 + v_1\, v_2) + \frac{\sqrt{k_2}}{\sigma}\,(\tilde{u}_1\, \tilde{u}_2 + \tilde{v}_1\, \tilde{v}_2)\right]$$

By equations (6 16) and (6 17) P , Q R and S can not be zero for finite σ Since $I\left\{K_2\ Q\ R\ \bar{P}\ \bar{S}\right\} = 0$ and $K_2\ Q\ R\ \bar{P}\ \bar{S}$ is positive,

$$u_1\ u_2 + v_1\ v_2 = -\sqrt{\frac{k_2}{k_1}}\ (\tilde{u}_1\ \tilde{u}_2 + \tilde{v}_1\ \tilde{v}_2)$$

and

$$(u_1\ u_2 + v_1\ v_2)\ (\tilde{u}_1\ \tilde{u}_2 + \tilde{v}_1\ \tilde{v}_2) > \frac{\sqrt{k_1\ k_2}}{\sigma^2}$$

These last two relations give

$$-\sqrt{\frac{k_2}{k_1}}\ (\tilde{u}_1\ \tilde{u}_2 + \tilde{v}_1\ \tilde{v}_2)^2 > \frac{\sqrt{k_1\ k_2}}{\sigma^2}$$

This is obviously a contradiction We can conclude there exist no zeros of W when s is real

We still must consider case (a) when μ is negative, i e , s is real From equation (6 22) we have

$$W_f \equiv K_1\ g_1'(o)\ h_1(o) - K_2\ h_1'(o)\ g_1(o) = -\ s\ W_o(s) + O\left\{e^{\sigma b+\tau a}\right\}$$

uniformly for $-b \leqslant x \leqslant a$, where again

$$W_o(s) \equiv \frac{K_1}{\sqrt{k_1}}\ \cos s_2\ a\ \sin s_1\ b + \frac{K_2}{\sqrt{k_2}}\ \cos s_1\ b\ \sin s_2\ a$$

Now when μ is negative s_i, and consequently σ and τ, vanish The characteristic equation becomes

$$W_f = - s_r \left[W_o(s_r) + 0 \left\{ |s_r|^{-1} \right\} \right] = 0$$

For large $|s_r|$ the zeros of W_f are therefore approximately those of $W_o(s_r)$

The equation $W_o(s_r) = 0$ may be written in the simpler form $\sin \delta = K \sin C\delta$, where

$$\delta = s_r \left(\frac{b}{\sqrt{k_1}} + \frac{a}{\sqrt{k_2}} \right), \quad K = \left(\frac{K_1}{\sqrt{k_1}} - \frac{K_2}{\sqrt{k_2}} \right) \left(\frac{K_1}{\sqrt{k_1}} + \frac{K_2}{\sqrt{k_2}} \right)^{-1} \quad \text{and}$$

$$C = \left(\frac{a}{\sqrt{k_1}} - \frac{b}{\sqrt{k_2}} \right) \left(\frac{a}{\sqrt{k_1}} + \frac{b}{\sqrt{k_2}} \right)^{-1}$$

It is clear that the value $\delta = \frac{1}{2}(2n + 1)\pi$, n an integer, does not satisfy this last equation It is also clear that there exists one and only one δ between $\delta = \frac{1}{2}(2n + 1)\pi$ and $\delta = \frac{1}{2}(2n + 3)\pi$ which does satisfy the equation $W_o(s_r) = 0$

We next shall show that the zeros of $W_o(s)$ are simple Let s_n be a root of $W_o(s) = 0$ It is easily seen from the differential equations involved that

$$\left[g_1'(x, s)\, g_1(x, s_n) - g_1'(x, s_n)\, g_1(x, s) \right]' = (s - s_n)\, g_1(x, s_n)\, g_1(x, s)$$

and

$$\left[h_1'(x, s)\, h_1(x, s_n) - h_1'(x, s_n)\, h_1(x, s)\right]' = (s - s_n)\, h_1(x, s_n)\, h_1(x, s)$$

Integrating these last expressions and using the end and interface conditions, we can obtain using the notation of section 5 ,

$$(s - s_n)^{-1}\, \varphi_{on}(o)\, \psi_{on}(o)\, W_o(s) = \int_{-b}^{a} X(y, s)\, p(y)\, X(y, s_n)\, dy$$

Consider now the result of taking the limit on both sides of this equation as s tends to s_n Since s_n is a zero of $W_o(s)$ and $W_o(s)$ is an entire function of s because $g_1(x, s)$, $g_1'(x, s)$, $h_1(x, s)$ and $h_1'(x, s)$ are, the left hand side of the equation becomes $\varphi_{on}(o)\, \psi_{on}(o)\, W_{on}$ From function theory we have, since $g_1(x, s)$, $g_1'(x, s)$, $h_1(x, s)$ and $h_1'(x, s)$ are continuous in x and s together and therefore bounded provided s is bounded, that $X(x, s)$ converges uniformly to $X(x, s_n)$ Consequently we can interchange the order of taking the limit and integrating on the right hand side and obtain $\int_{-b}^{a} p(y)\, X_n^2(y)\, dy$ It follows that $\varphi_{on}(o)\, \psi_{on}(o)\, W_{on} = \int_{-b}^{a} p(y)\, X_n^2(y)\, dy$ and $W_{on} \neq 0$ The zeros of $W_o(s)$ are therefore simple

8 <u>The equi-convergence proofs</u> In rough language the equi-convergence properties for case (a), (b) and (c) result from using limiting processes on certain integrals For case (a) we shall start with the contour-integral $(2\pi i)^{-1} \int_{(Pn)} w(x, \mu)\, d\mu$, where the path (Pn) is the simple closed curve determined by the two large parabolas $\mu_1^2 = a_n^2\, (a_n^2 - \mu_r)$ and $\mu_1^2 = a_n^2 (a_n^2 + \mu_r)$, where

$$a_n = \frac{2M + 1}{2} \left(\frac{b}{\sqrt{k_1}} + \frac{a}{\sqrt{k_2}} \right)^{-1}$$

For case (b) and (c) we shall start with the integral

$$\frac{1}{\pi} \int_{-r+i\epsilon}^{r+i\epsilon} w(x, \mu) d\mu \quad , \qquad \text{where } \epsilon > 0$$

Since our order properties are expressed in terms of s, we desire integrals in the s-plane corresponding to the above two integrals in the μ-plane Recalling that $\mu = -s^2$, we have as an equivalent integral for case (a)

$$-(\pi i)^{-1} \int_{(Hn)} w(x, -s^2) s ds \ ,$$

where the path (Hn) consists of the three lines $s_i = a_n$ where $-a_n \leqslant s_r \leqslant a_n$, $s_r = a_n$ where $0 \leqslant s_i \leqslant a_n$ and $s_r = -a_n$ where $0 \leqslant s_i \leqslant a_n$ For case (b) and (c) an equivalent integral is

$$- \frac{2}{\pi} \int_{(H)} w(x, -s^2) s ds \ ,$$

where (H) is the path made up of the lines $s_i = (R + \delta)$ where $-(R + \delta) \leqslant s_r \leqslant -\delta$ and $s_r = -(R + \delta)$ where $\delta \leqslant s_i \leqslant R + \delta$, with $\delta = \frac{\epsilon}{2(R + \delta)} > 0$

Since in case (a) the equivalent integral results from merely a change of complex variables, we can write

$$(8\ 1) \qquad -(\pi i)^{-1} \int_{(Hn)} w(x, -s^2)\, s\, ds = (2\pi i)^{-1} \int_{(Pn)} w(x, \mu)\, d\mu ,$$

provided $w(x, \mu)$ is analytic along the path (Pn) This last condition is fulfilled for large n since (Pn) was chosen to pass between the zeros of W_o, i e , the zeros of W_f

In case (b) and (c) the lines making up path (H) map into portions of the two parabolas $\mu_i^2 = 4(R + \delta)^2 \left[(R + \delta)^2 - \mu_r\right]$ and $\mu_i^2 = 4(R + \delta)^2 \left[(R + \delta)^2 + \mu_r\right]$ Considering large R and small δ, put $r = (R + \delta)^2 - \delta^2$ The parabolas in the μ-plane thus contain the points $\mu = -r + i\epsilon$, $\mu = 2i(r + \delta^2)$ and $\mu = r + i\epsilon$ This path in the μ-plane is equivalent relative to the integral

$$\frac{1}{\pi} \int w(x, \mu)\, d\mu ,$$

to the original straight line path of the integral between $\mu = -r + i\epsilon$ and $\mu = r + i\epsilon$ This follows directly from the Cauchy-Goursat integral theorem provided that $w(x, \mu)$ is single-valued and analytic within and on the simple closed curve formed by portions of the above two parabolas and the straight line path between $\mu = -r + i\epsilon$ and $r + i\epsilon$ This will be the case if $w(x, \mu)$ is analytic in the upper half of the μ-plane

Now $g_1(x, \mu) = g_1(x)$, $g_2(x)$, $h_1(x)$ and $h_2(x)$, along with their first derivatives with respect to x, were previously noted to be continuous in x and s together and analytic in s for each finite x region and bounded μ region Using equation (3 8) it follows that in case (b) $w(x, \mu)$ may have singularities only

at the singularities of $P(s)$ or $Q(s)$ and the zeros of $M(s)$ It is apparent from equations (6 8b, c) and (6 10b, c) that $P(s)$ and $Q(s)$ have at most branch points at $\mu = 0$ We previously showed that $M(s)$ has no zeros Using equation (3 9) it follows that in case (c) $w(x, \mu)$ may have singularities only at the singularities of $P(s)$, $Q(s)$, $R(s)$ or $S(s)$ and the zeros of $W(s)$ From equations (6 4c), (6 6c), (6 8b, c) and (6 10b, c) it follows that $P(s)$, $Q(s)$, $R(s)$ and $S(s)$ have at most branch points at $\mu = 0$ $W(s)$ was previously shown to have no zeros Consequently $w(x, \mu)$ is analytic in the upper half μ-plane for both case (b) and (c) Therefore

$$(8\ 2) \qquad \frac{1}{\pi} \int_{-r+i\epsilon}^{r+i\epsilon} w(x, \mu)d\mu = - \frac{2}{\pi} \int_{(H)} w(x, -s^2)s\, ds$$

Before substituting in the right hand sides of equations (8 1) and (8 2) we may make use of the order properties for large $|s|$ of the functions which make up $w(x, -s^2)$ In case (b), using euqation (3 8) we can obtain

$$(8\ 3) \qquad w(x, \mu) = - \int_0^\infty \frac{K_2}{k_2} \frac{1}{2i\ s_2} e^{i\ s_2\ y} f_2(y)dy \cos s_1(1 + x)\ M_0^{-1}$$

$$- \int_{-1}^{x} \frac{1}{k_1} f_1(y) \cos s_1(1 + y)dy\ M_0^{-1} \left[\frac{K_1}{2i\ s_2} \cos s_1\ x + \frac{K_2}{2\ s_1} \sin s_1\ x \right]$$

$$- \int_x^0 \frac{1}{k_1} f_1(y) \left[\frac{K_1}{2i\ s_2} \cos s_1\ y + \frac{K_2}{2s_1} \sin s_1\ y\ dy \right] \cos s_1(1 + x)\ M_0^{-1}$$

$$+ \ 0 \left\{ |s|^{-2} \int_0^\infty e^{\sigma x - \tau y}\ dy \right\} + \ 0 \left\{ |s|^{-2} \int_{-1}^{x} e^{\sigma(y-x)} dy \right\} +$$

$$0 \left\{ |s|^{-2} \int_0^x e^{\sigma(x-y)}\ dy \right\} \qquad (-1 \leqslant x \leqslant 0)$$

$$w(x, \mu) = -\int_{-1}^{0} \frac{K_1}{k_1} \cos s_1(1 + y)\, f_1(y)\, dy \frac{1}{2i\, s_2} e^{i\, s_2 x} M_0^{-1}$$

$$-\int_0^x \frac{1}{k_2} \left[K_2 \cos s_1 \cos s_2\, y - K_1 \sqrt{\frac{k_2}{k_1}} \sin s_1 \sin s_2\, y\right] f_2(y) dy \frac{1}{2i\, s_2} e^{is_2 x} M_0^{-1}$$

$$-\int_x^\infty \frac{1}{k_2} \frac{1}{2i\, s_2} e^{is_2 y} f_2(y) dy \left[K_2 \cos s_1 \cos s_2\, x - K_1 \sqrt{\frac{k_2}{k_1}} \sin s_1 \sin s_2\, x\right] M_0^{-1}$$

$$+ \; 0 \left\{ |s|^{-2} \int_{-1}^{0} e^{\sigma y - \tau x}\, dy \right\} + \; 0 \left\{ |s|^{-2} \int_0^x e^{\tau(x-y)}\, dy \right\} +$$

$$0 \left\{ |s|^{-2} \int_x^\infty e^{\tau(y-x)}\, dy \right\} \qquad (x \geqslant 0)$$

In case (c), using equation (3 9), we can obtain

$$(8\ 4) \qquad w(x, \mu) = \left[\int_0^\infty \frac{K_2}{k_2} f_2(y) \frac{e^{is_2 y}}{2i\, s_2} \frac{e^{-is_1 x}}{2i\, s_1} dy + \right.$$

$$\int_{-\infty}^x \frac{1}{k_1} \frac{e^{-is_1 y}}{2i\, s_1} f_1(y) dy \left(\frac{K_1}{2i\, s_2} \cos s_1\, x + \frac{K_2}{2s_1} \sin s_1\, x \right)$$

$$\left. + \frac{e^{-is_1 x}}{2i\, s_1} \int_x^0 \frac{1}{k_1} f_1(y) \left(\frac{K_1}{2i\, s_2} \cos s_1\, y + \frac{K_2}{2s_1} \sin s_1\, y \right) dy \right]$$

$$\left(-4is\right) \left(K_1 \sqrt{k_2} + K_2 \sqrt{k_1} \right)^{-1}$$

$$+ \; 0 \left\{ |s|^{-2} \int_0^\infty e^{\sigma x - \tau y}\, dy \right\} + \; 0 \left\{ |s|^{-2} \int_x^0 e^{-\sigma(y-x)}\, dy \right\}$$

$$+ \; 0 \left\{ |s|^{-2} \int_{-\infty}^x e^{-\sigma(x-y)} dy \right\} \qquad (x \leqslant 0)$$

$$w(x, \mu) = \left[\int_{-\infty}^{0} \frac{K_1}{k_1} f_1(y) \frac{e^{-is_1 y}}{2i\, s_1} \frac{e^{is_2 y}}{2i\, s_2} dy + \int_{x}^{\infty} \frac{1}{k_2} \frac{e^{is_2 y}}{2i\, s_2} f_2(y) dy \right.$$

$$\left(\frac{K_2}{2i\, s_1} \cos s_2 x - \frac{K_1}{2s_2} \sin s_2 x \right) + \int_{0}^{x} \frac{1}{k_2} f_2(y) \left(\frac{K_2}{2is_1} \cos s_2 y - \frac{K_1}{2s_2} \sin s_2 y \right)$$

$$\left. dy \frac{e^{is_2 x}}{2i\, s_2} \right] (-4is) \left(K_1 \sqrt{k_2} + K_2 \sqrt{k_1}\right)^{-1} + 0 \left\{ |s|^{-2} \int_{-\infty}^{0} e^{\sigma y - \tau x} dy \right\} +$$

$$0 \left\{ |s|^{-2} \int_{0}^{x} e^{\sigma(y-x)} dy \right\} + 0 \left\{ |s|^{-2} \int_{x}^{\infty} e^{\sigma(x-y)} dy \right\} \qquad (x \geqslant 0)$$

In case (a), using equation (3 7) we can obtain

$$(8\ 5) \qquad w(x, \mu) = - \left[\int_{0}^{a} \frac{K_2}{k_2} f_2(y) \cos s_2(a-y) dy \cos s_1(b + x) \right.$$

$$+ \int_{x}^{0} \frac{K_1}{k_1} f_1(y) \left(\cos s_2 a \cos s_1 y + K^{-1} \sin s_2 a \sin s_1 y\right) dy \cos s_1(b+x)$$

$$\left. + \int_{-b}^{x} \frac{K_1}{k_1} f_1(y) \cos s_1(b+y) dy \left(\cos s_2 a \cos s_1 x + K^{-1} \sin s_2 a \sin s_2 x\right) \right]$$

$$W_0^{-1} s^{-1} + 0 \left\{ |s|^{-2} \int_{0}^{a} e^{\sigma x - \tau y} dy \right\} + 0 \left\{ |s|^{-2} \int_{-b}^{x} e^{\sigma(y-x)} dy \right\} +$$

$$+ 0 \left\{ |s|^{-2} \int_{x}^{0} e^{\sigma(x-y)} dy \right\} \qquad (-b \leqslant x \leqslant 0)$$

$$w(x, \mu) = - \left[\int_{-b}^{0} \frac{K_1}{k_1} f_1(y) \cos s_1(b + y) dy \cos s_2(a - x) \right.$$

$$+ \int_{0}^{x} \frac{K_2}{k_2} f_2(y) \left(\cos s_1 b \cos s_2 y - K \sin s_1 b \sin s_2 y\right) dy \cos s_2(a - x)$$

$$+ \int_x^{\infty} \frac{K_2}{k_2} f_2(y) \cos s_2(a - y)dy \,(\cos s_1 b \cos s_2 x - K \sin s_1 b \sin s_2 x)\Big]$$

$$+ \; 0\left\{|s|^{-2} \int_{-b}^{0} e^{\sigma y - \tau x}\, dy\right\} + 0\left\{|s|^{-2} \int_0^x e^{\tau(y-x)}\, dy\right\} +$$

$$0\left\{|s|^{-2} \int_x^b e^{\tau(x-y)}\, dy\right\} \qquad (0 \leqslant x \leqslant a) \quad ,$$

where again $K = K_1 K_2^{-1} k_1^{-\frac{1}{2}} k_2^{\frac{1}{2}}$

The relations (8 3), (8 4) and (8 5) hold under the assumption that $f(x)$ is bounded and integrable They can be easily verified by using the various order properties

We see that the first three terms on the right hand sides of each equation in relation (8 3), (8 4) and (8 5) are exactly what we obtained for the cases where $q(x)$ is identically zero, i e , the right hand sides of equations (5 1), (5 2), and (5 3)

Consider the contribution to $-(\pi i)^{-1} \int_{(Hn)} w(x, -s^2)s\, ds$ as n tends to infinity of the $0\{\;\}$ terms in relation (8 5) Taking absolute values and using the $0\{\;\}$ terms for $-b \leqslant x \leqslant 0$, we obtain

$$\Bigg| -(\pi i)^{-1} \int_{(Hn)} \Bigg[0\left\{|s|^{-2} \int_0^a e^{\sigma x - \tau y}\, dy\right\} + 0\left\{|s|^{-2} \int_{-b}^{x} e^{\sigma(y-x)}\, dy\right\} +$$

$$0\left\{|s|^{-2} \int_x^0 e^{\sigma(x-y)}\, dy\right\}\Bigg] \quad s\, ds \Bigg|$$

$$\leqslant \pi^{-1} \int_{(Hn)} \left[0 \left\{ |s|^{-1} s_i^{-1} (1 - e^{-\tau a}) \right\} + 0 \left\{ |s|^{-1} s_i^{-1} (1 - e^{-\sigma(b+x)}) \right\} \right.$$

$$\left. + 0 \left\{ |s|^{-1} s_i^{-1} (1 - e^{\sigma x}) \right\} \right] |ds|$$

Each of these last terms three $0 \{ \}$ terms gives a contribution of order $0 \left\{ a_n^{-1} \right\}$ For example,

$$\int_{(Hn)} 0 \left\{ |s|^{-1} s_i^{-1} (1 - e^{-\tau a}) \right\} |ds| = 0 \left\{ a_n^{-2} \int_{-a_n}^{a_n} d \; s_r \right\} +$$

$$0 \left\{ a_n^{-1} \int_0^{a_n} s_i^{-1} (i - e^{-\tau a}) \, ds_1 \right\} = 0 \left\{ a_n^{-1} \right\}$$

Since $a_n = (\frac{2n+1}{2}) \left(\frac{b}{\sqrt{k_1}} + \frac{a}{\sqrt{k_2}} \right)^{-1}$, it follows that the three original $0 \{ \}$ terms converge uniformly with respect to x to 0 as n tends to infinity

If we substitute the $0 \{ \}$ terms of either equation (8 3) or (8 4) in the right hand side of equation (8 2), we can show in a similar way that the $0 \{ \}$ terms converge uniformly with respect to x as r (or R) tends to infinity

The equi-convergence properties now follow directly In case (a) for each large n the right hand side of equation (8 1) is equal to the sum of a finite number of terms of the right hand side of the equations in the relation (4 3) This follows from the residue theorem At the same time upon substituting from equation (8 5) the right hand side of equation (8 1) is equal to the corresponding sum of a finite number of terms of the right hand side of equation (5 4)

plus a term of order $O\left\{a_n^{-1}\right\}$ Consequently the expansions given by relations (4 3) and (5 4) behave in the same way as regards convergence

In case (b) and (c) let us formally consider

$$I\left\{\lim_{\epsilon \to 0} \lim_{r \to \infty} \frac{1}{\pi} \int_{-r+i\epsilon}^{r+i\epsilon} w(x, \mu)\, d\mu\right\}$$

Assuming the limits exist and recalling that the $O\{\ \}$ terms of equations (8 3) and (8 4) contribute nothing when r tends to infinity, it is also clear that no contribution is made by using either equation (8 3) or (8 4) to the above expression when μ is positive The above expression therefore gives formally the right hand sides of equations (5 5) and (5 6) respectively when we use equations (8 3) and (8 4)

Again assuming the limits exist the above expression is equal to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} I\left\{w(x, \mu)\right\} d\mu$$

Using equations (3 8) and (3 9), which define $w(x, \mu)$, and the equations which define $P(s) = -B$, $Q(s) = A$, $R(s) = -F$ and $S(s) = E$, it is not difficult to show that $I\left\{w(x, \mu)\right\} = 0$ when $\mu > 0$ Therefore formally in both case (b) and (c)

$$I\left\{\lim_{\epsilon \to 0} \lim_{r \to \infty} \frac{1}{\pi} \int_{-r+i\epsilon}^{r+i\epsilon} w(x, \mu)\, d\mu\right\} = \frac{1}{\pi} \int_{-\infty}^{0} I\left\{w(x, \mu)\right\} d\mu$$

The right hand side of this equation was used in obtaining the right hand sides of the equations in the relations (4 4) and (4 5) Consequently the expansions given by equations (4 4) and (4 5) behave respectively as regards convergence in the same way as the expansions given in equations (5 5) and (5 6) Here of course the equi-convergence results from letting r tend to infinity

Using equations (6 14b, c), (6 16b, c) and (6 17), we can obtain

$$I\left\{N\,\bar{M}\right\} = -\frac{1}{4}\,K_1\,K_2\,s_2$$

$$I\left\{h(x)\bar{M}\right\} = -\frac{1}{4}\,s_2\left[K_1\,g_1'(0)\,h_1(y) - K_2\,g_1(0)\,h_2(y)\right]$$

$$M\,\bar{M} = \frac{1}{4}\left\{\left[K_1\,g_1'(0)\,u_1 - K_2\,g_1(0)\,u_2\right]^2 + \left[K_1\,g_1'(0)\,v_1 - K_2\,g_1(0)\,v_2\right]^2\right\}$$

Using equations (6 12c) through (6 17b, c), we can obtain

$$I\;\bar{W}\,g(y)\,h(x) = I\left\{\left[K_2\,\bar{Q}\,\bar{R} - K_1\,\bar{P}\,\bar{S}\right]\left[S\,g_1(y) - R\,g_2(y)\right]\left[Q\,h_1(x) - P\,h_2(x)\right]\right\}$$

$$= -\frac{1}{8}\frac{\sqrt{k_1\,k_2}}{s}\left[\left\{3\right\}h_1(x)\,g_1(y) + \frac{1}{K_1\,K_2}\left\{1\right\}h_2(x)\,g_2(y) - \frac{1}{K_1}\left\{2\right\}h_2(x)\,g_1(y)\right.$$

$$\left. - \frac{1}{K_2}\left\{2\right\}h_1(x)\,g_2(y)\right],$$

$$I\left\{\bar{W}\,g(y)\left[K_1\,B\,g_2(x) + K_2\,A\,g_1(x)\right]\right\} =$$

$$-\frac{\sqrt{k_1k_2}}{8s}\left[\left\{3\right\}K_2\,g_1(x)\,g_1(y) + \frac{1}{K_2}\left\{1\right\}g_2(x)\,g_2(y) - \left\{2\right\}g_2(x)g_1(y) - \left\{2\right\}g_1(x)g_2(y)\right]$$

$$I\left\{\overline{W}\, h(x) \left[K_1\, E\, h_1(y) + K_2\, F\, h_2(y)\right]\right\} =$$

$$\frac{-\sqrt{k_1 k_2}}{8s}\left[\{3\}\, K_1\, h_1(x)\, h_1(y) + \frac{1}{K_1}\{1\}\, h_2(x)\, h_2(y) - \{2\}\, h_1(y) h_2(x) - \{2\}\, h_1(x) h_2(y)\right]$$

and

$$W\,\overline{W} = \left[K_2\, Q\, R - K_1\, P\, S\right]\left[K_2\, \overline{Q}\,\overline{R} - K_1\, \overline{P}\,\overline{S}\right] = \frac{1}{4}\,\frac{\sqrt{k_1\, k_2}}{K_1\, K_2}\left[\{1\}\,\{3\} - \{2\}\,\{2\}\right],$$

where

$$\{1\} = \frac{K_2\, K_1^{\,2}}{\sqrt{k_2}}\left[u_1^2\,(s) + v_1^2\,(s)\right] + \frac{K_1\, K_2^{\,2}}{\sqrt{k_1}}\left[\tilde{u}_1^2\,(s) + \tilde{v}_1^2\,(s)\right],$$

$$\{2\} = \frac{K_1\, K_2}{\sqrt{k_1}}\left[\tilde{u}_1\,\tilde{u}_2 + \tilde{v}_1\,\tilde{v}_2\right] + \frac{K_1\, K_2}{\sqrt{k_2}}\left[u_1\, u_2 + v_1\, v_2\right]$$

and

$$\{3\} = \frac{K_2}{\sqrt{k_2}}\left[u_2^2 + v_2^2\right] + \frac{K_1}{\sqrt{k_2}}\left[\tilde{u}_2^2 + \tilde{v}_2^2\right]$$

These results may be used to write equations (4 4) and (4 5) in a form which will be obtained later by the method of separation of variables

The results may be summarized in the following theorem

<u>EQUI-CONVERGENCE THEOREM</u> Let $f(x)$ be bounded and integrable and let $q(x)$ be a function greater than or equal to zero which is continuous except possibly at $x = 0$ and whose integral exists then

(a) the right hand sides of equations (4 3) and (5 4) are equi-convergent,

(b) the right hand sides of equations (4 4) and (5 5) are equi-convergent,

(c) the right hand sides of equations (4 5) and (5 6) are equi-convergent

In order to verify the relations (4 3), (4 4) and (4 5) it thus suffices, subject to the above hypothesis, to verify the relations (5 4), (5 5) and (5 6) The situation then corresponds to verifying that a function is equal to its ordinary Fourier series expansion and thus is equal to its ordinary Sturm-Liouville series expansion

9 <u>A verification of the expansions in the case $q(x) \equiv 0$</u> In order to verify the expansions given by equations (5 4), (5 5) and (5 6) we may make use of the standard procedures [6] involving the inversion of the Laplace transform Specifically we shall use one of Churchill's theorems [6, p 159] In essence this theorem allows us to conclude that a certain inversion integral gives the inverse Laplace transform even at $t = 0$, provided the Laplace transform satisfied a certain order property Since the procedure is standard only a condensation of the verification will be given The details of the procedure may be found in an earlier paper [16]

Suppose $U(x, t)$ is equal to the inversion integral of $u(x, \mu)$, i e ,

$$(9\ 1) \qquad U(x, t) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{\mu t}\, w(x, \mu)\, d\mu ,$$

where $\gamma > 0$ is a fixed constant In case (a) the contour of the inversion integral may be completed by the arc C_n of the parabola $r = a_n^2 \csc^2 \frac{1}{2}\theta$, where as usual $\mu = r e^{i\theta}$ and $a_n = \frac{2n+1}{2}\left(\frac{b}{\sqrt{k_1}} + \frac{a}{\sqrt{k_2}}\right)^{-1}$ For large n

we know that C_n passes between the singularities of $w(x, \mu)$, i e , the zeros of $W_0(s)$ The right hand side of equation (5 4) can be obtained formally by using the residue theorem, letting n tend to infinity and letting t equal zero The inversion integral when t equals zero will thus be equal to the inversion series, i e , the right hand side of equation (5 4) if the integral over the added contour tends to zero as n tends to infinity

In case (b) and (c) because of the branch point at $\mu = 0$ we can complete the contour of the inversion integral by adding a curve A' B' C' D' E D C B A The curve A' B' C' D' E D C B A consists of the arc A' B' C' of a circle of large radius r_L and center at $\mu = 0$ starting at $\mu = \gamma - i\beta$ and ending at $\mu = r_L e^{-i(\pi-\epsilon)}$ where ϵ is a small positive angle, the straight line C' D' starting at $\mu = r_L e^{-i(\pi-\epsilon)}$ and ending at $\mu = r_S e^{-i(\pi-\epsilon)}$, the arc D' E of a circle of small radius r_S and center at $\mu = 0$ starting at $\mu = r_S e^{-i(\pi-\epsilon)}$ and ending at $\mu = r_S$, and the reflection of each of these in the real axis By the Cauchy-Goursat theorem it is apparent that

$$\int_{\gamma-i\beta}^{\gamma+i\beta} e^{\mu t} w(x, \mu) d\mu = \int_{(A'B'C'D'EDCBA)} e^{\mu t} w(x, \mu) d\mu$$

In the usual procedure for the Laplace transform when $\mu = 0$ is a branch point the integral of $e^{\mu t} w(x, \mu)$ over the circular arcs A' B' C' , D' E D and C B A tends to zero as the radius r_L tends to infinity and r_S tends to zero Granted that this is true for every fixed $\epsilon > 0$, then when we take the limit as ϵ tends to zero of the integral of $e^{\mu t} w(x, \mu)$ over the lines C' D' and D C , we formally obtain from equation (9 1)

$$U(x, t) = \frac{1}{2\pi i} \int_0^\infty e^{-rt} \left[w(x, re^{-i\pi}) - w(x, re^{i\pi}) \right] dr$$

Upon setting $t = 0$, we have

$$f(x) = \frac{1}{2\pi i} \int_0^\infty \left[w(x, re^{-i\pi}) - w(x, re^{i\pi}) \right] dr$$

Letting $r = s^2$ it is easily confirmed that this is equivalent to equation (4 2) from which we derived the expansions given by equations (4 4) and (4 5)

The first step in verifying the above formal procedure consists in showing that the inversion integral is equal to $U(x, t)$ in all three cases It proves advantageous in obtaining the order properties needed in order to apply Churchill's theorem to use integration by parts for the integrals on the right hand sides of equations (5 1), (5 2) and (5 3) In order to write down the resulting equations and to insure convergence of all integrals, we shall assume that $f(x)$ is continuous and absolutely integrable for the range of integration and that $f'(x)$ is bounded and integrable for the range of integration By assuming that $f(x)$ is continuous, it also follows that $f_1(0) = f_2(0)$

Upon carrying out the details for each case one can show that $v(x, \mu) = w(x, \mu) - \mu^{-1} f(x)$ is of order $0\left\{\mu^{-3/2}\right\}$ uniformly with respect to x for the x-interval for the region $r \geqslant \rho$ and $-\pi + \epsilon \leqslant \theta \leqslant \pi - \epsilon$, where ρ is a fixed positive number and where ϵ is a fixed positive angle which is chosen as small as desired With this order property and Churchill's theorem it follows that the inversion integral of $v(x, \mu)$ is equal to zero at $t = 0$ in each of

the three cases Consequently the inversion integral of $w(x, \mu)$ is equal to $f(x)$ when $t = 0$

Next it is necessary to investigate the relationship between the inversion integral of $v(x, s)$ at $t = 0$ and in case (a) the inversion series and in case (b) and (c)

$$(2\pi i)^{-1} \int_0^{\infty} \left[w(x, re^{-i\pi}) - w(x, re^{i\pi}) \right] dr$$

For case (a) this consists of showing that the integral of $v(x, \mu)$ over the path C_n tends to zero as n tends to infinity With proper care using previously obtained order properties for $v(x, \mu)$ and its components [16] this can be shown to converge to zero uniformly in x

For case (b) and (c) one needs to investigate the value of the integral of $v(x, \mu)$ over the previously indicated curve A' B' C' D' E D C B A By using $v(x, \mu) = w(x, \mu) - \mu^{-1} f(x)$ it is not difficult to show that the integral of $v(x, \mu)$ over D' E D is equal to $(2\pi - 2\epsilon) i\, f(x)$ when r_S tends to zero Choose $\cos \frac{\pi - \epsilon}{2} = r_L^{-1/4}$ Using the obtained order properties it can be shown that the integral of $v(x, \mu)$ over A' B' C' and C B A tends to zero when r_L tends to infinity As r_L tends to infinity ϵ tends to zero It follows that

$$L_i^{-1} \left\{ v(x, \mu) \right\} = - f(x) + \frac{1}{2\pi i} \int_0^{\infty} \left[v(x, re^{-i\pi}) - v(x, re^{i\pi}) \right] dr$$

Since $L_i^{-1} \left\{ v(x, \mu) \right\} = 0$ and $v(x, re^{-i\pi}) - v(x, re^{i\pi}) = w(x, re^{-i\pi}) - w(x, re^{i\pi})$, one obtains the desired relationship

$$f(x) = \frac{1}{2\pi i} \int_0^{\infty} \left[w(x, re^{-i\pi}) - w(x, re^{i\pi}) \right] dr$$

The results may be summarized in the following theorem

<u>EXPANSION THEOREM</u> If $f(x)$ is continuous, is absolutely integrable and has a bounded and integrable derivative then the expansions given by the right hand sides of relations (4 3), (4 4) and (4 5) converge uniformly to $f(x)$

In light of the Equi-convergence Theorem the right hand sides of the relations (5 4), (5 5) and (5 6) converge uniformly to $f(x)$ under the same hypothesis

10 <u>The method of separation of variables</u> We shall now obtain by the classical method of separation of variables the solutions and expansions for the boundary value problems given by the equations (2 1) and (2 7c) Let $U(x, t) = X(x)\, T(t)$ where $X(x) = Z(x)$ when $x < 0$ and $X(x) = W(x)$ when $x > 0$ We obtain

(10 1) $$-q_1(x) + k_1 \frac{Z_{xx}}{Z} = \frac{T_t}{T} \quad ,$$ (10 2) $$-q_2(x) + k_2 \frac{W_{xx}}{W} = \frac{T_t}{T} \quad ,$$

(10 3) $$Z(0) = W(0) \quad ,$$ (10 4) $$K_1 Z_x(0) = K_2 W_x(0) \quad ,$$

(10 5a) $$Z_x(-b) = 0 \quad , \quad W_x(a) = 0 \quad ,$$

(10 5b) $$Z_x(-1) = 0 \quad ,$$ $Z(x)$ bounded , (10 5c) $Z(x)$ bounded .

The members of equations (10 1) and (10 2) may be put equal to a constant $-s^2$ Let the solutions of $k_1 Z_{xx} = \left[q_1(x) - s^2 \right] Z$ and $k_2 W_{xx} = \left[q_2(x) - s^2 \right] W$ be

designated in the same fashion as they were in Section 3, i e , let $g(x)$'s be solutions of the Z differential equation, let $h(x)$'s be solutions of the W differential equation, for case (a) let $g_1(-b) = 1$, $g_1'(-b) = 0$, $h_1(a) = 1$ and $h'(a) = 0$, for case (b) let $g_1(-1) = 1$, $g_1'(-1) = 0$, $h_1'(0) = 1$, $h_2(0) = -1$ and $h_2'(0) = 0$, and for case (c) let $g_1(0) = 0$, $g_1'(0) = 1$, $g_2(0) = -1$, $g_2'(0) = 0$, $h_1(0) = 0$, $h_1'(0) = 1$, $h_2(0) = -1$ and $h_2'(0) = 0$

The solutions to the above differential equations subject to the finite end conditions can be written as

$$
\begin{aligned}
&(a) \qquad X(x) = Z(x) = A\, g_1(x) && (-b \leqslant x < 0) \\
&\qquad\qquad = W(x) = B\, h_1(x) && (0 < x \leqslant a) \quad , \\
&(b) \qquad X(x) = C\, g_1(x) && (-1 \leqslant x < 0) \\
&\qquad\qquad = D\, h_1(x) + E\, h_2(x) && (x > 0) \quad , \\
&(c) \qquad X(x) = F\, g_1(x) + G\, g_2(x) && (x < 0) \\
&\qquad\qquad = H\, h_1(x) + J\, h_2(x) && (x > 0) \quad ,
\end{aligned}
$$

where A, B, C, D, E, F, G, H and J are independent of x , although they as well as the g's and h's may be functions of s Writing $X(x) = X(x, s)$ and using the interface conditions (10 3) and (10 4) we obtain

(a) $$A\, g_1(0, s) = B\, h_1(0, s) \quad , \quad K_1\, A\, g_1'(0, s) = K_2\, B\, h_1'(0, s) \quad ,$$

(b) $$C\, g_1(0, s) = -E \quad , \quad K_1\, C\, g_1'(0, s) = K_2\, D \quad ,$$

(c) $$G = J \quad , \quad K_1\, F = K_2\, H$$

It is now necessary to treat each case in a separate but parallel fashion For case (a) A and B are non-trivial only if s satisfies the characteristic equation

$K_1\, g_1'(0, s)\, h(0, s) - K_2\, g_1(0, s)\, h'(0, s) = 0$ We can attach positive integers n $(n = 1, 2, 3, \quad)$ to the assumed simple positive roots The characteristic function corresponding to the characteristic number s_n is

$$X(x, s_n) = X_n = Z_n = \varphi_n(0)\, g_n(x) = \varphi_n(0)\, g_1(x, -s_n^2) \qquad (-b \leqslant x \leqslant 0)$$

$$= W_n = \psi_n(0)\, h_n(x) = \psi_n(0)\, h_1(x, -s_n^2) \qquad (0 \leqslant x \leqslant a),$$

where either $\varphi_n(0) = h_n(0)$ and $\psi_n(0) = g_n(0)$ or $\varphi_n(0) = K_2\, h_n'(0)$ and $\psi_n(0) = K_1\, g_n'(0)$, either choice being equivalent for each n , unless for a particular n both $h_n(0)$ and $g_n(0)$ are zero or both $h_n'(0)$ and $g_n'(0)$ are zero, in which case the non-zero pair is chosen

Paralleling Sturm-Liouville theory we now obtain orthogonality conditions for the characteristic functions Eliminating $q_1(x)$ and $q_2(x)$ from

$$Z_n'' - (q_1 - s_n^2)\, Z_n = 0 \qquad \text{and} \qquad Z_m'' - (q_1 - s_m^2)\, Z_m = 0 \,,$$

and

$$W_n'' - (q_2 - s_n^2)\, W_n = 0 \qquad \text{and} \qquad W_m'' - (q_2 - s_m^2)\, W_m = 0 \,,$$

and then integrating, we obtain

$$(s_m^2 - s_n^2) \int_{-b}^{0} \frac{1}{k_1}\, Z_n(z)\,dz = \Big[Z_n'\, Z_m - Z_n\, Z_m' \Big]_{-b}^{0} \,,$$

and

$$(s_m^2 - s_n^2) \int_{0}^{a} \frac{1}{k_2}\, W_n\, W_m\, dz = \Big[W_n'\, W_m - W_n\, W_m' \Big]_{0}^{a}$$

Applying the end and interface conditions (10 3), (10 4) and (10 5a), it follows that

$$(s_m^2 - s_n^2) \left[\int_{-b}^{0} \frac{K_1}{k_1}\, Z_n\, Z_m\, dz + \int_{0}^{a} \frac{K_2}{k_2}\, W_n\, W_m\, dz \right] = 0$$

Now if $n \neq m$, we obtain the orthogonality condition

(10 6) $$\int_{-b}^{a} p(z)\, X_m(z)\, X_n(z)\, dz = 0 \,,$$

where $p(z) = \dfrac{k_1}{K_1}$ when $-b < z < 0$ and $p(z) = \dfrac{k_2}{K_2}$ when $0 < z < a$

We assume that $U(x, t)$ and $f(x)$ can be expressed as

$$U(x, t) = \sum_{n=1}^{\infty} C_n X_n(x) \; e^{-s_n^2 t} \qquad \text{and} \qquad f(x) = \sum_{n=1}^{\infty} C_n X_n(x)$$

Still proceeding formally multiply the members of this last equation by $p X_m$ and integrate We obtain

$$\int_{-b}^{a} p X_m f \, dz = \int_{-b}^{a} p X_m \left(\sum_{n=1}^{\infty} C_n X_n \right) dz$$

If we interchange the order of integration and summation and use the orthogonality condition (10 6), the right hand side of this last equation reduces to

$$C_m \int_{-b}^{a} p X_m^2 \, dz$$

The solution and expansion therefore become

$$\text{(10 7)} \qquad U(x, t) = \sum_{n=1}^{\infty} \left[\int_{-b}^{a} p(z) X_n(z) f(z) dz \right] \left[\int_{-b}^{a} p(z) X_n^2(z) \, dz \right]^{-1} X_n(x) \; e^{-s_n^2 t} ,$$

$$\text{(10 8)} \qquad f(x) = \sum_{n=1}^{\infty} \left[\int_{-b}^{a} p(z) X_n(z) f(z) dz \right] \left[\int_{-b}^{a} p(z) X_n^2(z) \, dz \right]^{-1} X_n(x)$$

This last equation is seen to be the same as equation (4 3) if we use the result at the end of section 7

For case (b) we obtain using the interface conditions the characteristic functions

$$X(x, s) = C(s)\, g_1(x, s) \qquad (-1 \leqslant x \leqslant 0)$$

$$= C(s) \left[\frac{K_1}{K_2}\, g_1'(0, s)\, h_1(x, s) - g_1(0, s)\, h_2(x, s) \right]$$

$$(x \geqslant 0)$$

We shall therefore assume that $U(x, t)$ and $f(x)$ can be expressed as

$$U(x, t) = \int_{-\infty}^{\infty} C(s)\, X(x, s)\, e^{-s^2 t}\, ds \quad \text{and} \quad f(x) = \int_{-\infty}^{\infty} C(s)\, X(x, s) ds$$

Since the parameter s occurs as $-s^2$, $X(x, s) = X(x, -s)$, and we are therefore assuming that

$$f(x) = \int_{0}^{\infty} \left[C(s) + C(-s) \right] X(x, s) ds$$

In order to find $C(s)$ formally, first multiply both sides of the equation $f(x) = \int_{-\infty}^{\infty} C(s)\, X(x, s) ds$ by $p(x)\, X(x, r)$, where p has the usual meaning Upon integrating the result, we obtain

$$(10\ 9) \qquad \int_{-1}^{\infty} p(z)\ X(z,\ r)\ f(z)dz = \int_{-1}^{\infty} p(z)\ X(z,\ r) \left[\int_{-\infty}^{\infty} C(s)\ X(z,\ s)ds \right] dz$$

$$= \int_{-1}^{0} \frac{K_1}{k_1} \left[\int_{-\infty}^{\infty} C(s)\ g_1(y,\ s)\ g_1(y,\ r)ds \right] dz \quad +$$

$$\lim_{z_o \to \infty} \int_{0}^{z_o} \frac{K_2}{k_2} \left\{ \left[\frac{K_1}{K_2}\ g_1'(0,\ s)\ h_1(z,\ s) \ - \ g_1(0,\ s)\ h_2(z,\ s) \right] \right.$$

$$\left. \left[\frac{K_1}{K_2}\ g_1'(0,\ r)\ h_1(z,\ r) \ - \ g_1(0,\ r)\ h_2(z,\ r) \right] \right\} dz$$

It is easy to verify using the Z differential equation that

$$\frac{1}{k_1}\ g_1(z,\ s)\ g_1(z,\ r) = (s^2 - r^2)^{-1}\ g_1''(z,\ r)\ g_1(z,\ s) \ - \ g_1''(z,\ s)\ g_1(z,\ r)$$

Upon integrating and using the end condition at $x = -1$ we have

$$\int_{-1}^{0} \frac{1}{k_1}\ g_1(z,\ s)\ g_1(z,\ r)dz = (s^2 - r^2)^{-1} \left[g_1'(0,r)\ g_1(0,s) \ - \ g_1'(0,s)\ g_1(0,r) \right]$$

Similarly we have

$$\int_{0}^{z_o} \frac{1}{k_2}\ h_1(z,s)\ h_1(z,r)dz = (s^2 - r^2)^{-1} \left[h_1'(z_o,r)\ h_1(z_o,s) \ - \ h_1'(z_o,s)\ h_1(z_o,r) \right] ,$$

$$\int_{0}^{z_o} \frac{1}{k_2}\ h_1(z,s)\ h_2(z,r)dz = (s^2 - r^2)^{-1} \left[h_2'(z_o,r)\ h_1(z_o,s) \ - \ h_1'(z_o,s) h_2(z_o,r) + 1 \right]$$

and

$$\int_0^{z_o} \frac{1}{k_2} \; h_2(z,r) \; h_2(z,r)dz = (s^2 - r^2)^{-1} \left[h_2'(z_o,r) \; h_2(z_o,s) - h_2'(z_o,s) h_2(z_o,r) \right]$$

Upon interchanging the order of integration on the right hand side of equation (10 9) and simplifying we obtain

$$\int_{-1}^{\infty} p(z) \; X(y, r)dz = \lim_{z_o \to \infty} \int_{-\infty}^{\infty} C(s) \; K_2 \left\{ g_1(0, s) \; g_1(0, r) \right.$$

$$\left[h_2'(z_o, r) \; h_2(z_o, s) - h_2'(z_o, s) \; h_2(z_o, r) \right] + \left(\frac{K_1}{K_2} \right)^2 g_1'(0, s) \; g_1'(0, r)$$

$$\left[h_1'(z_o, r) \; h_1(z_o, s) - h_1'(z_o, s) \; h_1(z_o, r) \right] - \frac{K_1}{K_2} \; g_1'(0, s) \; g_1(0, r)$$

$$\left[h_2'(z_o, r) \; h_1(z_o, s) - h_1'(z_o, s) \; h_2(z_o, r) \right] - \frac{K_1}{K_2} \; g_1'(0, r) \; g_1(0, s)$$

$$\left. \left[h_1'(z_o, r) \; h_2(z_o, s) - h_2'(z_o, s) \; h_1(z_o, r) \right] \right\} \; (s^2 - r^2)^{-1} \; ds$$

In order to evaluate the above limit we may substitute various order properties of the g's and h's for large z_o For example, using equation (6 11) and the corresponding equation for $h_2'(x)$ we obtain

$$\left[h_2'(z_o, r) \; h_2(z_o, s) - h_2'(z_o, s) \; h_2(z_o, r) \right] =$$

$$r_2 \left[-u_2(r) \sin r_2 z_o + v_2(r) \cos r_2 z_o + o(1)\right] \left[u_2(s) \cos s_2 z_o + v_2(s) \sin s_2 z_o + o(1)\right]$$

$$-s_2 \left[-u_2(s) \sin s_2 z_o + v_2(s) \cos s_2 z_o + o(1)\right] \left[u_2(r) \cos r_2 z_o + v_2(r) \sin r_2 z_o + o(1)\right],$$

where $s_2 = \dfrac{s}{\sqrt{k_2}}$ and $r_2 = \dfrac{r}{\sqrt{k_2}}$

Rearranging terms, we have

$$\left[h_2'(z_o,r)\, h_2(z_o,s) - h_2'(z_o,s) h_2(z_o,r)\right] = \frac{s_2 - r_2}{2} \left[u_2(s) u_2(r) - v_2(s) v_2(r)\right]$$

$$\sin z_o(s_2 + r_2) + \frac{s_2 + r_2}{2} \left[u_2(s)\, u_2(r) + v_2(s)\, v_2(r)\right] \sin z_o(s_2 - r_2) +$$

$$\frac{s_2 + r_2}{2} \left[u_2(s)\, v_2(r) - u_2(r)\, v_2(s)\right] \cos z_o(s_2 - r_2) -$$

$$\frac{s_2 - r_2}{2} \left[u_2(s)\, v_2(r) + u_2(r)\, v_2(s)\right] \cos z_o(s_2 + r_2) + o(1)$$

The other bracketed quantities above can be written in a similar fashion

We now may make formal usage of two closely related formulas which involve Dirichlet type integrals These are

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \cos kx\, dx = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z - x)}{z - x} dx = \pi F(z)$$

Noting that g_1 , g_1' , u_1 and u_2 are even in s , while v_1 and v_2 are odd in s , we obtain the formal result,

$$\int_{-1}^{\infty} p(z)\, f(z)\, X(z, r)dz = \frac{\pi}{2}\left[C(r) + C(-r)\right] \frac{K_2}{\sqrt{k_2}} \left\{ g_1^2\,(0, r)\left[u_2^2(r) + v_2^2(r)\right] + \left(\frac{K_1}{K_2}\right)^2 g_1'^2\,(0, r)\left[u_1^2(r) + v_1^2(r)\right] - 2\,\frac{K_1}{K_2}\, g_1(0,r)\, g_1'(0,r)\left[u_1(r)u_2(r) + v_1(r)v_2(r)\right]\right\}$$

$$= \frac{\pi}{2}\,\frac{K_2}{\sqrt{k_2}}\left[C(r) + C(-r)\right]\left\{\left[\frac{K_1}{K_2}\, g_1'(0, r)\, u_1(r) - g_1(0, r)\, u_2(r)\right]^2 + \left[\frac{K_1}{K_2}\, g_1'(0, r)\, v_1(r) - g_1(0, r)\, v_2(r)\right]^2\right\}$$

It may be noted that this result may be obtained in a speedier fashion from equation (10 9) if we interchange the order of integration on the right hand side and use the order properties before performing the inner integration The solution and expansion therefore become

$$\text{(10 10)} \qquad U(x, t) = \frac{2}{\pi}\int_0^{\infty}\left[\int_{-1}^{\infty} p(z)\, f(z)\, X(z, s)dz\right] \frac{\sqrt{k_2}}{K_2}$$

$$\left\{\left[\frac{K_1}{K_2}\, g_1'(0, s)\, u_1(s) - g_1(0, s)\, u_2(s)\right]^2 + \left[\frac{K_1}{K_2}\, g_1'(0, s)\, v_1(s) - g_1(0, s)\, v_2(s)\right]^2\right\}^{-1} X(x, s)\; e^{-s^2 t}\; ds \quad ,$$

$$(10\ 11) \qquad f(x) = \frac{2}{\pi} \int_0^\infty \left[\int_{-1}^\infty p(z)\ f(z)\ X(z,\ s)dz \right] \frac{\sqrt{k_2}}{K_2}$$

$$\left\{ \left[\frac{K_1}{K_2}\ g_1'(0,\ s)\ u_1(s) - g_1(0,\ s)\ u_2(s) \right]^2 + \right.$$

$$\left. \left[\frac{K_1}{K_2}\ g_1'(0,\ s)\ v_1(s) - g_1(0,\ s)\ v_2(s) \right]^2 \right\}^{-1} X(x,\ s)ds$$

This last equation is seen to be the same as equation (4 4) if we use the results obtained in Section 8

For case (c) the interface conditions give the characteristic functions

$$X(x,\ s) = F(s)\ K_2\ g_1(x,\ s) + G(s)\ g_2(x,\ s) \qquad (x \leqslant 0)$$

$$= F(s)\ K_1\ h_1(x,\ s) + G(s)\ h_2(x,\ s) \qquad (x \geqslant 0)$$

We shall therefore assume that it is possible to express $U(x,\ t)$ and $f(x)$ as

$$U(x,\ t) = \int_{-\infty}^\infty \left[F(s)\ X_1(x,\ s) + G(s)\ X_2(x,\ s) \right] e^{-s^2 t}\ ds$$

and

$$f(x) = \int_{-\infty}^\infty \left[F(s)\ X_1(x,\ s) + G(s)\ X_2(x,\ s) \right] ds\ ,$$

where $X_1(x, s) = K_2\, g_1(x, s)$ when $x \leqslant 0$ and $X_1(x, s) = K_1\, h_1(x, s)$ when $x \geqslant 0$, and where $X_2(x, s) = g_2(x, s)$ when $x \leqslant 0$ and $X_2(x, s) = h_2(x, s)$ when $x \geqslant 0$

In order to find $F(s)$ and $G(s)$ formally we may proceed in a fashion similar to case (b) First multiply both sides of

$$f(x) = \int_{-\infty}^{\infty} \left[F(s)\, X_1(x, s) + G(s)\, X_2(x, s)\right] ds$$

by $p(x)\, X_1(x, r)$, where $p(x)$ has the usual meaning Upon integrating the result, we obtain

$$(10\ 12) \qquad \int_{-\infty}^{\infty} p(z)\, X_1(z, r)\, f(z)dz = \int_{-\infty}^{\infty} p(z)\, X_1(z, r) \left\{ \int_{-\infty}^{\infty} \left[F(x)X_1(z,s) + G(s)\, X_2(z, s)\right] ds \right\} dz$$

$$= \lim_{z_o \to \infty} \int_{-z_o}^{z_o} p(z) \left\{ \int_{-\infty}^{\infty} \left[F(s)\, X_1(z, s) + G(s)\, X_2(z, s)\right] X_1(z, r)ds \right\} dz$$

Interchanging the order of integration on the right hand side of the equation leads to

$$\int_{-z_o}^{z_o} p(z)\, X_1(z, r)\, f(z)dz =$$

$$\lim_{z_o \to \infty} \int_{-\infty}^{\infty} \left\{ F(s) \int_{-z_o}^{z_o} p(z)\, X_1(z,s)X_1(z,r)dz + G(s) \int_{-z_o}^{z_o} p(z)X_2(z,s)X_1(z,r)dz \right\} ds$$

Using the Z and W differential equations it is easy to show that

$$\int_{-z_o}^{z_o} p(z)\, X_1(z, r)\, X_1(z, s)\, dz =$$

$$K_1 K_2^2 \frac{1}{s^2 - r^2} \left[-g_1'(-z_o, r)\, g_1(-z_o, s) + g_1'(-z_o, s)\, g_1(-z_o, r) \right]$$

$$+ K_2 K_1^2 \frac{1}{s^2 - r^2} \left[h_1'(z_o, r)\, h_1(z_o, s) - h_1'(z_o, s)\, h_1(z_o, r) \right] ,$$

$$\int_{-z_o}^{z_o} p(z)\, X_2(z, s)\, X_1(z, r) dz =$$

$$K_1 K_2 \frac{1}{s^2 - r^2} \left[- g_1'(-z_o, r)\, g_2(-z_o, s) + g_2'(-z_o, s)\, g_1(-z_o, r) \right]$$

$$+ K_1 K_2 \frac{1}{s^2 - r^2} \left[h_1'(z_o, r)\, h_2(z_o, s) - h_2'(z_o, s)\, h_1(z_o, r) \right]$$

Similarly to case (b) we may substitute the various order properties for large z_o and make use of the two formulas

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \cos kx\, dx = 0$$

and

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z - x)}{z - x}\, dx = \pi F(z)$$

Now the u's and v's obtained by substituting the order properties have the feature of the u's being even in s and the v's being odd in s Using these properties and formulas, we obtain formally

$$\int_{-\infty}^{\infty} p(z)\, X_1(z, r)\, f(z)dz = F(r)\, \pi \left\{ \frac{K_2\, K_1^2}{\sqrt{k_2}} \left[u_1^2(r) + v_1^2(r) \right] + \right.$$

$$\left. \frac{K_1\, K_2^2}{\sqrt{k_1}} \left[\tilde{u}_1^2(r) + \tilde{v}_1^2(r) \right] \right\} + G(r)\, \pi \left\{ \frac{K_1\, K_2}{\sqrt{k_1}} \left[\tilde{u}_1(r)\, \tilde{u}_2(r) + \tilde{v}_1(r)\, \tilde{v}_2(r) \right] + \right.$$

$$\left. \frac{K_2\, K_1}{\sqrt{k_2}} \left[u_1(r)\, u_2(r) + v_1(r)\, v_2(r) \right] \right\} = F(r)\, \pi \left\{ 1 \right\} + G(r)\, \pi \left\{ 2 \right\} ,$$

where the meanings of $\left\{ 1 \right\}$ and $\left\{ 2 \right\}$ are the same as those given in Section 8 Proceeding in a similar fashion it is possible to show that

$$\int_{-\infty}^{\infty} p(z)\, X_2(z, r)\, f(z)dz =$$

$$G(r)\pi \left\{ \frac{K_2}{\sqrt{k_2}} \left[u_2^2(r) + v_2^2(r) \right] + \frac{K_1}{\sqrt{k_1}} \left[\tilde{u}_2^2(r) + \tilde{v}_2^2(r) \right] \right\}$$

$$+ F(r)\, \pi \left\{ \frac{K_1\, K_2}{\sqrt{k_1}} \left[\tilde{u}_1(r)\, \tilde{u}_2(r) + \tilde{v}_1(r)\, \tilde{v}_2(r) \right] + \frac{K_1\, K_2}{\sqrt{k_2}} \left[u_1(r)\, u_2(r) + v_1(r) v_2(r) \right] \right\}$$

$$= F(r)\, \pi \left\{ 2 \right\} + G(r)\, \pi \left\{ 3 \right\}$$

Solving for $F(r)$ and $G(r)$, we obtain

$$F(r) = \frac{1}{\pi} \frac{\{3\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_1(z, r)dz - \{2\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_2(z, r)dz}{\{1\}\{3\} - \{2\}\{2\}} ,$$

$$G(r) = \frac{1}{\pi} \frac{\{1\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_2(z, r)dz - \{2\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_1(z, r)dz}{\{1\}\{3\} - \{2\}\{2\}}$$

Since $\{1\}$, $\{2\}$, $\{3\}$, $F(r)$, $G(r)$, $X_1(x, r)$ and $X_2(x, r)$ are all even in r, the assumption that

$$f(x) = \int_{-\infty}^{\infty} \left[F(s)\, X_1(x, s) + G(s)\, X_2(x, s) \right] ds ,$$

is equivalent to

$$f(x) = 2 \int_{0}^{\infty} \left[F(s)\, X_1(x, s) + G(s)\, X_2(x, s) \right] ds$$

The solution and expansion therefore may be written as

(10 13) $$U(x, t) = 2 \int_{0}^{\infty} \left[F(s)\, X_1(x, s) + G(s)\, X_2(x, s) \right] e^{-s^2 t}\, ds ,$$

(10 14) $$f(x) = 2 \int_{0}^{\infty} \left[F(s)\, X_1(x, s) + G(s)\, X_2(x, s) \right] ds ,$$

where

$$F(s) = \frac{1}{\pi} \frac{\{3\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_1(z, s)dz - \{2\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_2(z, s)dz}{\{1\}\ \{3\} - \{2\}\ \{2\}},$$

and

$$G(s) = \frac{1}{\pi} \frac{\{1\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_2(z, s)dz - \{2\} \int_{-\infty}^{\infty} p(z)\, f(z)\, X_1(z, s)dz}{\{1\}\ \{3\} - \{2\}\ \{2\}}$$

Equation (10 14) is seen to be the same as equation (4 5) when we take into account the results in Section 8

11 Observations The Expansion Theorem takes care of the major step in the procedure for verifying that the solutions obtained really satisfy the original boundary value problems The remaining steps, e g , uniqueness of the solutions, may be verified using the standard procedures

The conditions given to insure the validity of the expansions are clearly sufficient rather than necessary ones It is obvious therefore that the expansions hold under weaker restrictions and that weaker restrictions could be obtained without much more work

It is possible, of course, to use the Laplace transform inversion procedures indicated in Section 9 to establish directly the general expansions given by equations (10 8), (10 11) and (10 14)

It seems likely that one can show quite directly that the expansions converge uniformly to $f(x)$, provided sufficient restrictions are placed upon $f(x)$ and $q(x)$ [9, pp 273-276]

When we let $K_1 = K_2$, $k_1 = k_2 = 1$ and $q(x)$ be continuous at $x = 0$ we obtain expansions for the continuous cases For case (a) we obtain

$$f(x) = \sum_{n=1}^{\infty} \left[\int_{-b}^{a} X_n(z)\, f(z)dz \right] \left[\int_{-b}^{a} X_n^2(z)dz \right]^{-1} X_n(x) ,$$

where

$$X_n(z) = g_n(z) = h_n(z) \qquad \text{for} \qquad -b \leqslant z \leqslant a ,$$

where n $(n = 1, 2, 3, \quad)$ is, as usual, the integer attached to the n-th simple positive root s_n of the characteristic equation $g_1'(o, s)\, h_1(o, s) - h_1'(o, s)\, g_1(o, s) = 0$, and where $g_1(x, s)$ and $h_1(x, s)$ are solutions of the equation $X_{xx} - \left[q(x) - s^2 \right] X(x) = 0$ which satisfy respectively $X_x(-b) = 0$ and $X_x(a) = 0$ Since $g_1'(x, s)\, h_1(x, s) - h_1'(x, s)\, g_1(x, s)$ can be shown to be constant by using the differential equations it follows that $g_1'(a, s) = 0 = h_1'(-b, s)$ are other ways of writing the characteristic equation The above expansion is, of course, a conventional Sturm-Liouville expansion When $q(x) \equiv 0$ we have

$$f(x) = \frac{1}{a + b} \int_{-b}^{a} f(z)dz + \frac{2}{a + b} \sum_{n=1}^{\infty} \int_{-b}^{a} f(z) \cos \frac{n\pi}{a + b} (z + b)dz$$

$$\cos \frac{n\pi}{a + b} (x + b) ,$$

the ordinary Fourier cosine expansion of $f(x)$

For case (b) the expansion in the continuous case is easily seen to give known results [19, p 101] It may be noted that $g_1(x, s) = g_1'(o, s)\, h_1(x, s) - g_1(o, s)\, h_1(x, s)$ for $x \geqslant -1$ For real valued s, $g_1(x, s)$ has the form $g_1(x, s) = u(s) \cos s\, x + v(s) \sin s\, x + o(1)$ as x tends to infinity Consequently,

$$u(s) = g_1'(o, s)\, u_1(s) - g_1(o, s)\, u_2(s)$$

and

$$v(s) = g_1'(o, s)\, v_1(s) - g_1(o, s)\, v_2(s)$$

The expansion therefore becomes

$$f(x) = \frac{2}{\pi} \int_0^\infty \left[\int_{-1}^\infty f(z)\, g_1(z, s)dz \right] \left[u^2(s) + v^2(s) \right]^{-1} g_1(x,s)ds$$

In case $q(x) = 0$ this reduces to the ordinary Fourier cosine integral representation

$$f(x) = \frac{2}{\pi} \int_0^\infty \left[\int_{-1}^\infty f(z) \cos s\,(1 + z)dz \right] \cos s\,(1 + x)ds$$

For case (c) in the continuous case, it is clear that $g_1(x, s) = h_1(x, s)$ and $g_2(x, s) = h_2(x, s)$ for all x The expansion for $f(x)$ may therefore be written as

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\left[g_1(x,s)\{3\}-g_2(x,s)\{2\}\right]\int_{-\infty}^{\infty} g_1(z,s)f(z)dz+\left[g_2(x,s)\{1\}-g_1(x,s)\{2\}\right]\int_{-\infty}^{\infty} g_2(z,s)f(z)dz}{\{1\}\ \{3\}\ -\ \{2\}\ \{2\}}ds$$

where now

$$\{1\} = u_1^2(s) + v_1^2(s) + \tilde{u}_1^2(s) + \tilde{v}_1^2(s) \quad ,$$

$$\{2\} = \tilde{u}_1(s)\ \tilde{u}_2(s) + \tilde{v}_1(s)\ \tilde{v}_2(s) + u_1(s)\ u_2(s) + v_1(s)\ v_2(s) \quad ,$$

and

$$\{3\} = u_2^2(s) + v_2^2(s) + \tilde{u}_2^2(s) + \tilde{v}_2^2(s)$$

When $q(x) = 0$ this reduces to an ordinary Fourier integral representation

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(z) \cos s\ (z - x)dz\right] ds$$

The Equi-convergence Theorem also applies to the continuous cases

It would be desirable to find conditions under which the expansions given by equations (5 4), (5 5) and (5 6) are equi-convergent respectively with the ordinary Fourier series, Fourier cosine integral and Fourier integral type expansions

The conditions $q(x) > 0$ is clearly not necessary for the application of the method used in this chapter When $q(x)$ is allowed to be negative new characteristic numbers and functions may arise

Proceeding in the manner of Section 10, it is not difficult to obtain formally expansions where the interface conditions take the form

$$b_1\, U(o^-, t) \;+\; b_2\, U_x(o^-, t) \;+\; b_3\, U(o^+, t) \;+\; b_4\, U_x(o^+, t) \;=\; 0$$

and

$$b_5\, U(o^-, t) \;+\; b_6\, U_x(o^-, t) \;+\; b_7\, U(o^+, t) \;+\; b_8\, U_x(o^+, t) \;=\; 0\ ,$$

provided the b's satisfy certain moderate conditions Likewise expansions can be obtained when the end conditions for case (a) take the form

$$a_1\, U(-b, t) \;+\; a_2\, U_x(-b, t) \;=\; 0$$

and

$$a_3\, U(a, t) \;+\; a_4\, U_x(a, t) \;=\; 0 \quad ,$$

and for case (b) the form

$$a_1\, U(-1, t) \;+\; a_2\, U_x(-1, t) \;=\; 0$$

It is also possible to obtain formally expansions which involve a finite number of sets of interface conditions It seems certain that this type of expansion, along with equi-convergence theorems, can be established in the manner of this chapter

APPENDIX A

CYLINDER

Composite Cylinder with Finite Cross Section Let us consider the temperature distribution $U(r, t)$ in an infinite cylinder with a circular cross section made up of two materials The boundary value problem under consideration is defined by the following equations

(1 1) $$U_t(r, t) = k_1(U_{rr} + \frac{1}{r} U_r) \qquad (0 < r < a , t > 0) ,$$

(1 2) $$U(r, 0^+) = f_1(r) \qquad (0 < r < a) ,$$

(1 3) $$U_t(r, t) = k_2 (U_{rr} + \frac{1}{r} U_r) \qquad (a < r < b , t > 0) ,$$

(1 4) $$U(r, 0^+) = f_2(r) \qquad (a < r < b) ,$$

(1 5) $$U(a^+, t) = U(a^-, t) \qquad (t > 0) ,$$

(1 6) $$K_1 U_r(a^-, t) = K_2 U_r(a^+, t) \qquad (t > 0) ,$$

(1 7) $$U(b^-, t) = 0 \qquad (t > 0)$$

Let $U(r, t) = R(r)\,T(t)$, where $R(r) \equiv X(r)$ when $0 < r < a$ and $R(r) \equiv Z(r)$ when $a < r < b$ The substitution for U in equations (1 1), (1 3), (1 5), (1 6) and (1 7) leads to the following equations

(1 8) $$k_1 \frac{(X_{rr} + \frac{1}{r} X_r)}{X} = \frac{T_t}{T} ,$$

(1 9) $$k_2 \frac{(Z_{rr} + \frac{1}{r} Z_r)}{Z} = \frac{T_t}{T} ,$$

(1 10) $$X(a) = Z(a) ,$$ (1 11) $$K_1 X_r(a) = K_2 Z_r(a) ,$$

(1 12) $$Z(b) = 0$$

The members of equations (1 8) and (1 9) must equal some constant $-s^2$ The solution which satisfy in r equations (1 8), (1 9) and (1 12), and a boundedness condition at the origin are given by

(1 13) $$R(r) \equiv X(r) = A\, J_o \left(\frac{s}{\sqrt{k_1}} r\right)$$

$$\equiv Z(r) = B\, L(r, s) \equiv B \left[Y_o \left(\frac{s}{\sqrt{k_2}} b\right) J_o\left(\frac{s}{\sqrt{k_2}} r\right) - J_o \left(\frac{s}{\sqrt{k_2}} b\right) Y_o\left(\frac{s}{\sqrt{k_2}} r\right)\right]$$

In order that A and B be non-trivial when conditions (1 10) and (1 11) are applied, s must satisfy the characteristic equation

$$\frac{K_2}{\sqrt{k_2}} J_o\left(\frac{s}{\sqrt{k_1}} a\right)\left[Y_o\left(\frac{s}{\sqrt{k_2}} b\right) J_1\left(\frac{s}{\sqrt{k_2}} a\right) - J_o\left(\frac{s}{\sqrt{k_2}} b\right) Y_1\left(\frac{s}{\sqrt{k_2}} a\right)\right] =$$

$$\frac{K_1}{\sqrt{k_1}} J_1\left(\frac{s}{\sqrt{k_1}} a\right) L(a, s)$$

We can attach positive integers n $(n = 1, 2, 3, \quad ,)$ to the assumed positive roots Corresponding then to the eigenvalues s_n we have the characteristic function

(1 14)
$$R_n \equiv R(r, s_n) = A\,\varphi_n\, J_o\left(\frac{s_n}{\sqrt{k_1}} r\right) \qquad (0 \leqslant r \leqslant a) \quad ,$$

$$= A\,\psi_n\, L_o(r, s_n) \qquad (a \leqslant r \leqslant b) \quad ,$$

where either $\varphi_n = L_o(a, s_n)$ and $\psi_n = J_o\left(\frac{s_n}{\sqrt{k_1}} a\right)$ or

$$\varphi_n = \frac{K_2}{\sqrt{k_2}}\left[Y_o\left(\frac{s_n}{\sqrt{k_2}} b\right) J_1\left(\frac{s_n}{\sqrt{k_2}} a\right) - J_o\left(\frac{s_n}{\sqrt{k_2}} b\right) Y_1\left(\frac{s_n}{\sqrt{k_2}} a\right)\right]$$

and

$$\psi_n = \frac{K_1}{\sqrt{k_1}} \quad J_1 \left(\frac{s_n}{\sqrt{k_1}} a \right) \qquad ,$$

either choice being equivalent for each n , unless for a particular n both of a pair are zero, in which case the non-zero pair is chosen

It is easily verified in the usual manner that the orthogonality condition becomes

$$\int_0^b p(r)\, r\, R_m(r)\, R_n(r)\, dr = 0 \qquad (s_m \neq s_n) \quad ,$$

where $p = \frac{K_1}{k_1}$ when $0 < r < a$ and $p = \frac{K_2}{k_2}$ when $a < r < b$ If then we assume that

$$f(r) = \sum_{n=1}^{\infty} a_n R_n(r) \qquad , \tag{1 15}$$

we can obtain formally that

$$a_n = \frac{\int_0^b p\, r\, R_n\, f\, dr}{\int_0^b p\, r\, R_n^2\, dr} \tag{1 16}$$

Consequently, we have

$$U(r, t) = \sum_{n=1}^{\infty} a_n \, e^{-s_n^2 t} \, R_n(r) \qquad , \tag{1 17}$$

where the a_n are defined by equation (1 16) and the $R_n(r)$ are defined in equation (1 14)

Composite Cylinder with Infinite Cross Section Consider equations (1 1) through (1 6) and again let $U(r, t) = R(r)\,T(t)$, where $R(r) \equiv X(r)$ when $0 < r < a$ and $R(r) \equiv Z(r)$ when $a < r$ This substitution leads again to equations (1 8) through (1 11) When we set the members of equations (1 8) and (1 9) equal to the constant $-s^2$, we can obtain using a boundedness condition at the origin

$$R(r) \equiv X(r) = A\,J_o\left(\frac{s}{\sqrt{k_1}}\,r\right)$$

$$\equiv Z(r) = B\,J_o\left(\frac{s}{\sqrt{k_2}}\,r\right) + C\,Y_o\left(\frac{s}{\sqrt{k_2}}\,r\right)$$

The interface conditions require that

$$A\,J_o\left(\frac{s}{\sqrt{k_1}}\,a\right) = B\,J_o\left(\frac{s}{\sqrt{k_2}}\,a\right) + C\,Y_o\left(\frac{s}{\sqrt{k_2}}\,a\right)$$

and

$$A\,\frac{K_1}{\sqrt{k_1}}\,J_1\left(\frac{s}{\sqrt{k_1}}\,a\right) = \frac{K_2}{\sqrt{k_2}}\left[B\,J_1\left(\frac{s}{\sqrt{k_2}}\,a\right) + C\,Y_1\left(\frac{s}{\sqrt{k_2}}\,a\right)\right]$$

These conditions are satisfied if we let

$$A = \frac{K_2}{\sqrt{k_2}}\left[Y_o\left(\frac{s}{\sqrt{k_2}}\,a\right)J_1\left(\frac{s}{\sqrt{k_2}}\,a\right) - J_o\left(\frac{s}{\sqrt{k_2}}\,a\right)Y_1\left(\frac{s}{\sqrt{k_2}}\,a\right)\right] =$$

$$\frac{2\,K_2}{\pi\,a\,s} \equiv \chi(s) \quad ,$$

$$B = \frac{K_1}{\sqrt{k_1}} J_1\left(\frac{s}{\sqrt{k_1}} a\right) Y_o\left(\frac{s}{\sqrt{k_2}} a\right) - \frac{K_2}{\sqrt{k_2}} J_o\left(\frac{s}{\sqrt{k_1}} a\right) Y_1\left(\frac{s}{\sqrt{k_2}} a\right) \equiv \phi(s)$$

and

$$C = \frac{K_2}{\sqrt{k_2}} J_o\left(\frac{s}{\sqrt{k_1}} a\right) J_1\left(\frac{s}{\sqrt{k_2}} a\right) - \frac{K_1}{\sqrt{k_1}} J_o\left(\frac{s}{\sqrt{k_2}} a\right) J_1\left(\frac{s}{\sqrt{k_1}} a\right) \equiv \psi(s)$$

The characteristic function is given by

$$R_s(r) \equiv R(r, s) \equiv X(r, s) = \chi(s) J_o\left(\frac{s}{\sqrt{k_1}} r\right)$$

(1 18)

$$\equiv Z(r, s) = \phi(s) J_o\left(\frac{s}{\sqrt{k_2}} r\right) + \psi(s) Y_o\left(\frac{s}{\sqrt{k_2}} r\right)$$

In this case we assume that

$$f(r) = \int_{-\infty}^{\infty} a(s) R(r, s) ds = \int_0^{\infty} b(s) R(r, s) ds$$

where $b(s) = a(s) - a(-s)$ Proceeding in the customary fashion, we have

$$\int_0^\infty p(r)\, r\, f(r)\, R(r, \sigma)dr = \lim_{r_o \to \infty} \int_0^{r_o} p(r)\, r\, R(r, \sigma) \left[\int_0^\infty b(s)\, R(r, s)ds \right] dr$$

$$= \lim_{r_o \to \infty} \int_0^\infty b(s) \left[\int_0^{r_o} p(r)\, r\, R(r, \sigma)\, R(r, s)dr \right] ds$$

$$= \lim_{r_o \to \infty} \int_0^\infty b(s)\, K_2 \frac{r_o}{s^2 - \sigma^2} \Big[Z(r_o, s)\, Z'(r_o, \sigma) -$$

$$Z'(r_o, s)\, Z(r_o, \sigma) \Big] ds$$

$$= \lim_{r_o \to \infty} \int_0^\infty b(s) \frac{K_2}{\sqrt{k_2}} \frac{r_o}{s^2 - \sigma^2} \Big[- \sigma Z(r_o, s)\, Z_1(r_o, \sigma) +$$

$$s\, Z_1(r_o, s)\, Z(r_o, \sigma) \Big] ds \quad ,$$

where $Z_1(r_o, s) = \phi(s)\, J_1 \left(\frac{s}{\sqrt{k_2}} r_o \right) + \psi(s)\, Y_1 \left(\frac{s}{\sqrt{k_2}} r_o \right)$ and $\sigma > 0$

In order to evaluate this last expression it is possible to use the following asymptotic formulas of the Bessel functions for large values of the arguments

$$J_p(x) \approx \frac{\cos \varphi}{\sqrt{\frac{1}{2} \pi x}} \quad \text{and} \quad Y_p(x) \approx \frac{\sin \varphi}{\sqrt{\frac{1}{2} \pi x}} \quad ,$$

where $\varphi = x - (p + \frac{1}{2}) \frac{\pi}{2}$, $-\pi < \arg x < \pi$

Using these asymptotic formulas, we can obtain

$$\left[-\sigma\, Z(r_o,\, s)\, Z_1(r_o,\, \sigma) + s\, Z_1(r_o,\, s)\, Z(r_o,\, \sigma)\right] \approx \frac{2}{\pi\, r_o} \sqrt{\frac{k_2}{s\,\sigma}}$$

$$\Big\{ s \Big[\phi(s)\, \phi(\sigma) \cos N(\sigma) \sin N(s) - \psi(s)\, \psi(\sigma) \sin N(\sigma) \cos N(s)$$

$$- \phi(\sigma)\, \psi(s) \cos N(\sigma) \cos N(s) + \phi(s)\, \psi(\sigma) \sin N(s) \sin N(\sigma) \Big]$$

$$- \sigma \Big[\phi(s)\, \phi(\sigma) \cos N(s) \sin N(\sigma) - \psi(s)\, \psi(\sigma) \sin N(s) \cos N(\sigma)$$

$$- \phi(s)\, \psi(\sigma) \cos N(s) \cos N(\sigma) + \phi(\sigma)\, \psi(s) \sin N(\sigma) \sin N(s) \Big] \Big\}$$

$$= \frac{1}{\pi\, r_o} \sqrt{\frac{k_2}{s\sigma}} \Big\{ \cos \left[N(\sigma) + N(s) \right] \left[\phi(\sigma)\, \psi(s) + \phi(s)\, \psi(\sigma) \right] (\sigma - s)$$

$$+ \cos \left[N(s) - N(\sigma) \right] \left[-\phi(\sigma)\, \psi(s) + \phi(s)\, \psi(\sigma) \right] (\sigma + s)$$

$$+ \sin \left[N(\sigma) + N(s) \right] \left[-\phi(s)\, \phi(\sigma) + \psi(s)\, \psi(\sigma) \right] (\sigma - s)$$

$$+ \sin \left[N(s) - N(\sigma) \right] \left[\phi(s)\, \phi(\sigma) + \psi(s)\, \psi(\sigma) \right] (\sigma + s) \Big\} ,$$

where $N(s) = \dfrac{s}{\sqrt{k_2}}\, r_o - \dfrac{\pi}{4}$

We now apply formally the formulas

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \cos k\, x\, dx = 0$$

and

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z - x)}{z - x} dx = \pi F(z)$$

We obtain using the above results that

$$\int_0^{\infty} p(r)\, r\, f(r)\, R(r, \sigma)\, dr = b(\sigma) \frac{K_2}{\sqrt{k_2}} \pi \left[\phi^2(\sigma) + \psi^2(\sigma) \right]$$

The solution of the boundary value problem is therefore given by

$$U(r, t) = \int_0^{\infty} e^{-s^2 t} b(s)\, R(r, s) ds \quad , \tag{1 19}$$

where $R(r, s)$ is defined by equation (1 18) and where

$$b(s) = \frac{1}{\pi} \frac{\sqrt{k_2}}{K_2} \int_0^{\infty} p(r)\, r\, R(r, s) dr \left[\phi^2(s) + \psi^2(s) \right]^{-1}$$

APPENDIX B

SPHERE

<u>Finite Composite Sphere</u> We now wish to consider the temperature distribution $V(r, t)$ in a sphere of radius b made up of two materials The problem being considered is defined by the following equations

(2 1) $$V_t = k_1 (V_{rr} + \frac{2}{r} V_r) \qquad (0 < r < a, t > 0) ,$$

(2 2) $$V(r, 0^+) = g_1(r) \qquad (0 < r < a) ,$$

(2 3) $$V_t = k_2(V_{rr} + \frac{2}{r} V_r) \qquad (a < r < b , t > 0) ,$$

(2 4) $$V(r, 0^+) = g_2(r) \qquad (a < r < b) ,$$

(2 5) $$V(a^+, t) = V(a^-, t) \qquad (t > 0) ,$$

(2 6) $$K_1 V_r(a^-, t) = K_2 V_r(a^+, t) \qquad (t > 0) ,$$

(2 7) $$V(b^-, t) = 0 \qquad (t > 0)$$

First let $U(r, t) = r\,V(r, t)$ Equations (2 1) through (2 7) then become

(2 8) $$U_t = k_1 U_{rr} \quad ,$$

(2 9) $$U(r, 0) = r\,g_1(r) = f_1(r) \quad ,$$

(2 10) $$U_t = k_2 U_{rr} \quad ,$$

(2 11) $$U(r, 0) = r\,g_2(r) = f_2(r) \quad ,$$

(2 12) $$U(a^+, t) = U(a^-, t) \quad ,$$

(2 13) $$K_1 \left[a\,U_r(a^-, t) - U(a^-, t) \right] = K_2 \left[a\,U_r(a^+, t) - U(a^+, t) \right]$$

(2 14) $$U(b^-, t) = 0$$

Next let $U = R(r)\,T(t)$, where $R(r) \equiv X(r)$ when $0 < r < a$ and $R(r) \equiv Z(r)$ when $a < r < b$ This substitution in the above homogeneous equations gives

(2 15) $$k_1 \frac{X_{rr}}{X} = \frac{T_t}{T} \quad ,$$ (2 16) $$k_2 \frac{Z_{rr}}{Z} = \frac{T_t}{T} \quad ,$$

(2 17) $X(a) = Z(a)$, (2 18) $K_1 \left[a\, X'(a) - X(a) \right] =$

$$K_2 \left[a\, Z'(a) - Z(a) \right] ,$$

(2 19) $Z(b) = 0$

We set the members of equations (2 15) and (2 16) equal to the constant $-s^2$. Applying condition (2 19) and the boundedness condition for V at $r = 0$, we can obtain

$$R(r) \equiv X(r) = A \sin \frac{s}{\sqrt{k_1}}\, r ,$$

$$\equiv Z(r) = B \sin \frac{s}{\sqrt{k_2}} (b - r)$$

In order that A and B be non-trivial when conditions (2 17) and (2 18) are applied, s must satisfy the characteristic equation

$$K_1 \sin \frac{s}{\sqrt{k_2}} (b - a) \left[a \frac{s}{\sqrt{k_1}} \cos \frac{s}{\sqrt{k_1}} a - \sin \frac{s}{\sqrt{k_1}} a \right] +$$

$$K_2 \sin \frac{s}{\sqrt{k_1}} a \left[a \frac{s}{\sqrt{k_2}} \cos \frac{s}{\sqrt{k_2}} (b - a) + \sin \frac{s}{\sqrt{k_2}} (b - a) \right]$$

$$= \sin \frac{s}{\sqrt{k_2}} (b - a) \left[(K_2 - K_1) \sin \frac{s}{\sqrt{k_1}} a + K_1 a \frac{s}{\sqrt{k_1}} \cos \frac{s}{\sqrt{k_1}} a \right] +$$

$$\cos \frac{s}{\sqrt{k_2}} (b - a) \left[K_2 a \frac{s}{\sqrt{k_2}} \sin \frac{s}{\sqrt{k_1}} a \right] = 0$$

We attach positive integers n $(n = 1, 2, 3, \quad ,)$ to the assumed positive roots Associated with the eigenvalue s_n is the eigenfunction

$$(2\ 20) \qquad R_n(r) \equiv R(r, s_n) = A\,\varphi_n \sin \frac{s_n}{\sqrt{k_1}}\, r$$

$$= A\,\psi_n \sin \frac{s_n}{\sqrt{k_2}}\,(b - r) \quad ,$$

where either $\varphi_n = \sin \frac{s_n}{\sqrt{k_2}}\,(b - r)$ and $\psi_n = \sin \frac{s_n}{\sqrt{k_1}}\, r$ or

$$\varphi_n = K_2 \left[a \frac{s_n}{\sqrt{k_2}} \cos \frac{s_n}{\sqrt{k_2}}\,(b - a) + \sin \frac{s_n}{\sqrt{k_2}}\,(b - a) \right]$$

and

$$\psi_n = K_1 \left[-a \frac{s_n}{\sqrt{k_1}} \cos \frac{s_n}{\sqrt{k_1}}\, a + \sin \frac{s_n}{\sqrt{k_1}}\, a \right] \quad ,$$

either choice being equivalent for each n , unless for a particular n both of a pair are zero, in which case the non-zero pair is chosen

In the usual manner the orthogonality condition can be shown to be

$$\int_0^b p(r)\, R_m(r)\, R_n(r)\, dr = 0 \qquad (s_m \neq s_n) \quad ,$$

where p has the customary meaning The assumption,

(2 21) $$f(r) = \sum_{n=1}^{\infty} a_n R_n(r) \quad ,$$

therefore leads formally to

(2 22) $$a_n = \frac{\int_0^b p R_n f \, dr}{\int_0^b p R_n^2 \, dr}$$

Consequently the solution becomes

$$U(r, t) = \sum_{n=1}^{\infty} a_n e^{-s_n^2 t} R_n(r) \quad ,$$

where the a_n are defined by equations (2 22) and the $R_n(r)$ are defined in equation (2 20)

<u>Infinite Composite Sphere</u> Consider equations (2 15) through (2 18) where the members of equations (2 15) and (2 16) are set equal to the constant $-s^2$ This time when we apply the boundedness at the origin but do not apply condition (2 19) we obtain

$$R(r) \equiv X(r) = A \sin \frac{s}{\sqrt{k_1}} r$$

$$\equiv Z(r) = B \sin \frac{s}{\sqrt{k_2}} r + C \cos \frac{s}{\sqrt{k_2}} r$$

From the interface condition we have

$$A \sin \frac{s}{\sqrt{k_1}} a = B \sin \frac{s}{\sqrt{k_2}} a + C \cos \frac{s}{\sqrt{k_2}} a$$

and

$$A K_1 \left[\frac{s}{\sqrt{k_1}} a \cos \frac{s}{\sqrt{k_1}} a - \sin \frac{s}{\sqrt{k_1}} a \right] =$$

$$K_2 \left[B \left(\frac{s}{\sqrt{k_2}} a \cos \frac{s}{\sqrt{k_2}} a - \sin \frac{s}{\sqrt{k_2}} a \right) - C \left(\frac{s}{\sqrt{k_2}} a \sin \frac{s}{\sqrt{k_2}} a + \cos \frac{s}{\sqrt{k_2}} a \right) \right]$$

These conditions can be satisfied by letting

$$A = \frac{K_2}{\sqrt{k_2}} s \equiv \chi(s)$$

$$C = - K_2 \left(\frac{s}{\sqrt{k_2}} a \cos \frac{s}{\sqrt{k_2}} a - \sin \frac{s}{\sqrt{k_2}} a \right) \sin \frac{s}{\sqrt{k_1}} a +$$

$$K_1 \sin \frac{s}{\sqrt{k_2}} a \left(\frac{s}{\sqrt{k_2}} a \cos \frac{s}{\sqrt{k_1}} a - \sin \frac{s}{\sqrt{k_2}} a \right) \equiv \psi(s) \quad ,$$

$$B = K_2 \left(\frac{s}{\sqrt{k_2}} a \sin \frac{s}{\sqrt{k_2}} a + \cos \frac{s}{\sqrt{k_2}} a \right) \sin \frac{s}{\sqrt{k_1}} a +$$

$$K_1 \cos \frac{s}{\sqrt{k_2}} a \left(\frac{s}{\sqrt{k_1}} a \cos \frac{s}{\sqrt{k_1}} a - \sin \frac{s}{\sqrt{k_1}} a \right) \equiv \phi(s) \quad .$$

The characteristic function is given by

$$(2\ 23) \qquad R(r, s) \equiv X(r, s) = \chi(s) \sin \frac{s}{\sqrt{k_1}} r$$

$$\equiv Z(r, s) = \phi(s) \sin \frac{s}{\sqrt{k_2}} r + \psi(s) \cos \frac{s}{\sqrt{k_2}} r$$

We assume that

$$f(r) = \int_{-\infty}^{\infty} a(s)\, R(r, s)\, ds = \int_0^{\infty} \left[a(s) + a(-s) \right] R(r, s) ds$$

If we proceed in the standard fashion we can obtain

$$\int_0^{\infty} p(r)\, f(r)\, R(r, t)\, dr = \lim_{r_o \to \infty} \int_0^{r_o} p(r)\, R(r, \sigma) \left[\int_{-\infty}^{\infty} a(s)\, R(r, s) ds \right] dr$$

$$= \lim_{r_o \to \infty} \int_{-\infty}^{\infty} a(s) \left[\int_0^{r_o} p(r)\, R(r, \sigma)\, R(r, s)\, dr \right] ds$$

$$= \lim_{r_o \to \infty} \int_{-\infty}^{\infty} a(s)\, K_2 \frac{1}{s^2 - \sigma^2} \left[Z(r_o, s)\, Z'(r_o, \sigma) - Z'(r_o, s) Z(r_o, \sigma) \right] ds$$

$$= \lim_{r_o \to \infty} \int_{-\infty}^{\infty} \frac{a(s)}{2} \frac{K_2}{\sqrt{k_2}} \left\{ \frac{\sin \frac{r_o}{\sqrt{k_2}} (s + \sigma)}{s + \sigma} \left[\psi(s)\ \psi(\sigma) - \phi(s)\ \phi(\sigma)\right] \right.$$

$$+ \frac{\sin \frac{r_o}{\sqrt{k_2}} (s - \sigma)}{s - \sigma} \left[\phi(s)\ \phi(\sigma) + \psi(s)\ \psi(\sigma)\right] - \frac{\cos \frac{r_o}{\sqrt{k_2}} (s+\sigma)}{s + \sigma}$$

$$\left[\phi(s)\ \psi(\sigma) + \phi(\sigma)\ \psi(s)\right]$$

$$\left. - \frac{\cos \frac{r_o}{\sqrt{k_2}} (s - \sigma)}{s - \sigma} \left[\phi(s)\ \psi(\sigma) - \phi(\sigma)\ \psi(s)\right] \right\} ds$$

At this stage, we apply formally the formulas

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \cos k\, x\, dx = 0$$

and

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} F(x) \frac{\sin k(z - x)}{z - x} dx = \pi F(z)$$

We then have

$$\int_0^{\infty} p(r)\ f(r)\ R(r, \sigma) dr = \frac{\pi}{2} \frac{K_2}{\sqrt{k_2}} \left[\psi^2(\sigma) + \phi^2(\sigma)\right] \left[a(\sigma) + a(-\sigma)\right] ,$$

where we have made use of the fact that ψ is an even function in t and ϕ is an odd function of t

The solution of the boundary value problem is therefore given by

$$V(r, t) = \frac{1}{r} \int_0^\infty e^{-s^2 t} \left[a(s) + a(-s)\right] R(r, s)ds \quad , \tag{2 24}$$

where $R(r, s)$ is defined by equation (23) and where

$$\left[a(s) + a(-s)\right] = \frac{2}{\pi} \frac{\sqrt{k_2}}{K_2} \int_0^\infty p(r)\, r\, g(r)\, R(r, s)dr \left[\psi^2(s) + \phi^2(s)\right]^{-1}$$

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