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J. W. Webster

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J. W. Webster

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PREFACE

This text in elementary reactor theory is the outgrowth of a course given by the writer to a group of Phillips Petrolleum Co. employees in training to be operators of the MTR.

The author is indebted to the members of this class for their encouragement with respect to the lectures, as well as in the writing of this report which contains the subject matter of the course.

I am particularly indebted to Charles Cagle, who supplied me with a careful set of notes on the lectures. The notes were of great help in the preparation of the text.

The principal reports consulted regarding the various proofs were:

1. H. Soodak and E. C. Campbell, *Elementary Pile Theory*, AECD-2201.
2. S. Glasstone and M. C. Edlund, *The Elements of Nuclear Reactor Theory, Parts I, II, and III*, TID-384, -385, and -386.
3. C. R. McCullough *et al.*, *Summary Report on Design and Development of High Temperature Gas-Cooled Power Pile*, Appendix F, MonN-383 (Sept. 15, 1947).
4. M. J. Nielsen and J. W. Webster, *Solution of Kinetic Equations of Cylindrical Liquid Fuel Reactor*, ANP-68 (Sept. 18, 1950).

The aim throughout the writing was to present in as much detail as possible the derivations and basic formulas the physicist or nuclear engineer needs to do the necessary "nuclear calculation" for a reactor design. These necessary "nuclear calculations" seem to be (1) critical mass calculations, (2) flux and power distributions, (3) reactivity effects of temperature, depletion, and fission-product poisons, (4) control rod effectiveness, and (5) kinetics.

An attempt has been made to make clear the way in which the individual derivations of the fundamental quantities (for example, ξ, $\vec{j}_+$, $\vec{j}_-$, q) fit into the over-all plan of the development of the basic formulas for critical size, etc. It is very easy to become buried in the lengthy search for the formula for ξ, for instance, and lose sight of the reason for wanting such a quantity. It seems well to treat as incidental the fact that formulas for quantities like $\vec{j}_+$ and $\vec{j}_-$ are very useful in their own right.

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INTRODUCTION

The purpose of this writing is to provide a practical type of text in reactor theory for the use of nuclear engineers engaged in reactor design. Methods are derived for calculating critical mass; power distribution; reactivity effects of temperature, depletion, and fission-product poisons; the necessary number of control rods to compensate for these effects; and the kinetic behavior of flux and temperatures in a reactor with regard to reactor stability and self-regulation.

It is the opinion of the author that methods in the field of reactor theory must be short and simple in application if they are to be of real assistance to reactor design. An answer that can be obtained in a day or two with an accuracy of 50% is often very useful to a designer, whereas one that is accurate to within 10% but takes a month to obtain may be obsolete and utterly useless by the time it is presented to the designer by the physicist.

An understanding of fundamental ideas such as cross section and mean free path are assumed here. An excellent reference for such definitions is Part I of TID-384 by Glasstone and Edlund.[1]

The subjects discussed in this report, in the order listed, are:

1. *The one-group theory for the calculation of critical mass and power distribution.* The one-group model is very crude and is given partly for pedagogical reasons, but it does have practical usefulness.

2. *The Fermi age theory for calculating critical mass.* The Fermi age theory is one of the most useful approaches to critical mass determination, in spite of the fact that it can be applied only to unreflected reactors. The reflected reactor can be approximated by an equivalent unreflected reactor, as discussed in chap. 2. The formulas obtained by Fermi age theory are simple and relatively accurate for reactors moderated with beryllium or carbon, and a modification of the formulas can be applied to make the method reasonably accurate for hydrogen-moderated reactors.

3. *The two-group theory for calculating critical mass and power distribution.* The two-group model can be used in conjunction with the Fermi age method by determining the core constants for the former by means of the latter. This will be discussed in chap. 3.

4. *The effect of temperature, depletion, and fission-product poisons, such as xenon and samarium.* These effects act to decrease the effective neutron multiplication constant, and therefore the fuel inventory must be sufficient to ensure criticality when these effects are at their maximum.

5. *The necessary number of control rods (shim rods) to offset the virtual supercriticality that exists in a reactor at startup when the effects described in item 4 are at a minimum.*

6. *The kinetic behavior of flux and temperatures in a reactor.* The kinetic self-regulation, stability to accidents, and response to control rod (regulating rod) motion must be investigated for a reactor design proposal.

---

[1] S. Glasstone and M. C. Edlund, *The Elements of Nuclear Reactor Theory-Part I*, TID-384 (Nov. 1950).

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Chapter 1
CRITICALITY BY ONE-GROUP THEORY

The simplest formulas for obtaining critical mass or size and spatial power distribution are obtained from a mathematical model in which all neutrons are considered to be at one energy. The one-energy assumption is very crude, because reactor neutrons are actually spread in energy from about 10^7 ev down to around 0.25 ev (in a cold reactor). Since cross sections vary with energy and the energy distribution of the neutrons is not usually known, it is difficult to choose good, average values for the constants. Therefore the results obtained are not generally in good agreement with experiments. However, the one-group model does have application, and because of its simplicity it is an excellent subject to start with in discussing the methods of reactor physics.

The simplest geometrical arrangement is the bare (unreflected) reactor, which is a completely homogeneous, one-region mixture of fissionable material, moderator, and diluents.

BARE REACTOR

The fundamental equation of reactor theory is the neutron-balance equation. This is a differential equation which states that if a reactor is critical (that is, if there is just enough uranium concentration for a given size to sustain the chain reaction), the net flow of neutrons out of a volume element per second added to the number of neutrons absorbed per second in the element must equal the number of neutrons produced per second in the element.

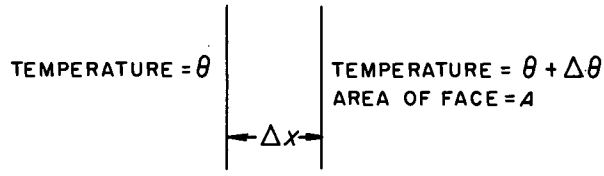
$$(\text{Leakage per sec per cm}^3) + (\text{Absorption per sec per cm}^3) - (\text{Production per sec per cm}^3) = 0 \quad .$$

The mathematical expression for the first term is by far the most difficult to obtain. The formulas for the three terms are discussed in the following sections of this chapter.

Leakage. The net flow of neutrons per second, a scaler quantity, from a cubical element of volume is easily obtained if the net neutron current, a vector quantity, is known at each face of the element. It is necessary to sum the differences between the net current in a face and the net current out of the opposite face for three pairs of parallel faces. The neutron current is a quantitative measure of the rate of diffusion of neutrons. The theory of neutron diffusion is the basis of reactor theory.

Diffusion of Neutrons. It is instructive and interesting to draw an analogy between heat conduction and diffusion of neutrons, provided the analogy is not carried too far, since both are examples of transport phenomena. In heat conduction, the kinetic energy of the molecules is transported because of the temperature difference between neighboring volume elements. Temperature is an index of the average kinetic energy of the molecules in a volume element; therefore in the volume element of high temperature, the faster moving molecules collide with the slower moving molecules of the adjacent cooler volume element and impart energy to them. Energy, or heat, is spatially transferred. In neutron diffusion, the quantity corresponding to temperature is neutron density, N. The quantity that corresponds to $\vec{Q}$, the Btu's of heat energy transported across an area per second, is $\vec{j}$, the net number of neutrons that move across an area per second.

Consider a slab of thickness Δx that has one face at temperature θ and the other face at temperature $\theta + \Delta\theta$.



The heat flow (Btu) in unit time is proportional to A and to $\Delta\theta/\Delta x$. That is

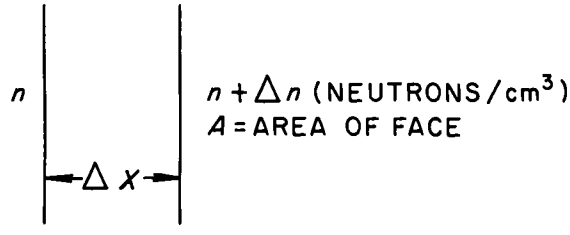
$$\vec{Q} \propto A \frac{\Delta\theta}{\Delta x} ,$$

or, in the limit at any point,

$$\vec{Q} = -kA \frac{d\theta}{dx} ,$$

where k is the conductivity, which depends on the material and on the temperature.

Now consider the section of a slab that lies between two parallel planes and contains neutrons of concentration n per cubic centimeter at one plane and $n + \Delta n$ at the other plane.



Then, the net number of neutrons flowing through an area A is proportional to A and to $\Delta n/\Delta x$:

$$\vec{j} \propto A \frac{\Delta n}{\Delta x} ,$$

or

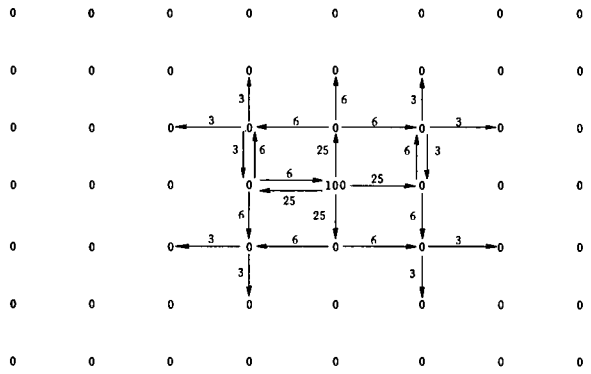
$$\vec{j} = -DA \frac{dn}{dx} ,$$

where $\vec{j}$ is the net number of neutrons that move across area A per second and D is a constant, called the diffusion constant, which plays the same role as the conductivity in heat transfer. The value of D depends on the material and the velocity of the diffusing neutrons.

Thus the expression for the diffusion of neutrons is precisely the same in form as that for the diffusion of heat. The attempt here is to make the formula for $\vec{j}$ seem reasonable in the light of a more familiar example of the transport phenomenon. In a subsequent section of this chapter the formula will be derived in a more rigorous manner.

In neutron diffusion, as opposed to, say, heat diffusion in solids, one is interested in the *physical movement* of neutrons from one place to another. It follows from the formula for $\vec{j}$ that there is a net flow of neutrons toward the left in the above diagram of the slab. This net flow results simply because there are more neutrons present at the right ($n + \Delta n$) than at the left (n). Since all neutrons have velocity (assumed the same for all neutrons, in this chapter) and are traveling randomly (in direction), a net flow exists in the direction of lower concentration. Diffusing neutrons frequently collide with the nuclei of the medium (about one collision in every inch in an average medium). The velocity of the nuclei can usually be considered negligible compared with that of the neutrons. To get a geometrical picture of diffusion, consider 100 neutrons, with velocity, released at a point in a two-dimensional stationary lattice of nuclei, and suppose that these neutrons are restricted to move only north, east, south, or west. Then the number of neutrons moving along any path is about as shown in the picture below. The stationary nuclei are represented by the regular array of circles. At the instant of release, there are 100 neutrons at the center of the picture. A moment later there are 25 neutrons traveling in the four different directions as shown. After each neutrons collides with the

nucleus in its path, each group of 25 divides into groups of approximately 6 neutrons moving away in the four directions from the point of collision, etc. It is clear that the 100 neutrons are rapidly spreading out from the point of their release. If the concentration of neutrons being maintained is higher at one point than at another, a net flow will exist toward the point of lower concentration.



The density of the nuclei is very much greater than the density of neutrons; therefore collision of neutrons with nuclei is the important event; collisions of neutrons with neutrons are so rare that they can be neglected. For example, in beryllium there are 10^{23} atoms per cubic centimeter. Suppose that the scaler flux (total centimeters traveled in one second by all neutrons in a cubic centimeter) is

$$nv = 10^{14} \frac{\text{neutron}\cdot\text{cm}}{\text{cm}^3\cdot\text{sec}}.$$

Then, since v, the velocity of neutrons at thermal energy, is equal to about 0.22×10^6 cm/sec, the neutron density corresponding to this flux is

$$n = \frac{10^{14} \dfrac{\text{neutron}\cdot\text{cm}}{\text{cm}^3\cdot\text{sec}}}{0.22 \times 10^6 \dfrac{\text{cm}}{\text{sec}}}$$

$$= 5 \times 10^8 \frac{\text{neutrons}}{\text{cm}^3}.$$

Thus for each neutron in an average reactor there are about

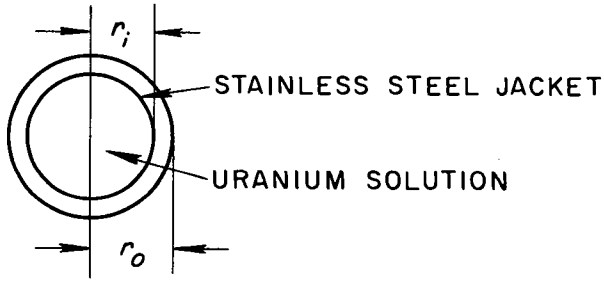
$$\frac{10^{23}}{5 \times 10^8} = 2 \times 10^{14}$$

nuclei against which the neutron can collide. A qualitative picture of the diffusion of neutrons may be obtained by visualizing these particles as moving in excess of 5000 mph in a medium containing 10^{23} obstacles per cubic centimeter with which they may collide and rebound randomly (in direction).

There are many useful problems in heat conduction that can be solved by using the formula of Fick's law

$$\vec{Q} = -kA\frac{dQ}{dx}.$$

The following problem is an example of the application of Fick's law. Consider a fuel tube in a reactor such as the ANP sodium-cooled reactor, which is designed for a power of 200 megawatts. There are 15,000 fuel tubes; each tube has the cross section pictured below, and each tube is surrounded by sodium coolant.



We wish to determine the drop in temperature through the stainless steel jacket of the fuel tube as part of an investigation of the peak fuel temperature. Each fuel tube must generate 13,333.33 watts (average), which is

$$\frac{200 \times 10^6 \text{ (watts)}}{15{,}000 \text{ (tubes)}}.$$

However, 1 joule/sec = 1 watt and there are 4.18 joules per calorie; so the heat generated per tube is

$$\frac{\frac{1}{4.18}\ \text{(cal/sec)}}{1\ \text{(watt)}} \times 13{,}333\ \text{(watts)}$$

$$= 3200\ \text{(cal/sec)}\ .$$

That is,

$$Q = 3200\ \text{cal/sec (for each tube)}$$

$$= -kA\frac{d\theta}{dr}$$

$$= -k\ 2\pi rL\ \frac{d\theta}{dr},$$

and integrating this differential equation we obtain

$$\theta_i - \theta_o = \frac{Q}{k2\pi L}\log\frac{r_o}{r_i}\ ,$$

where r_i is the inside radius of the stainless steel jacket, r_o is the outside radius, and L is the length of the tube. For the fuel tube being considered, k is 1 cal/sec·cm·°C (for stainless steel) and the dimensions of the fuel tubes are L = 90 cm, r_i = 0.1 cm, and r_o = 0.1524 cm. Thus,

$$\theta_i - \theta_o = \frac{3200\ \text{(cal/sec)}}{1\ \text{(cal/sec·cm·°C)} \times 2\pi \times 90\ \text{(cm)}} \times \log\frac{0.1524}{0.1} = 2.4°\text{C}\ ,$$

which is the temperature drop through the jacket.

A practical problem involving Fick's law for neutron diffusion is solved in the following illustrative example.

Example 1. It is desired to know the net flow of neutrons per unit area across the interface between core and reflector of a spherical reactor that has a solid core of uranium mixed homogeneously with beryllium and a solid beryllium reflector. Since only a very rough answer is desired, the flux can be calculated on a one-group model. The flux, ϕ, has been found to be

$$\phi(r) = 10^{13}\frac{\sin\frac{\pi}{60}r}{r}\ ,$$

where

$\phi = nv \equiv$ the product of the neutron velocity and the neutron density at a distance r centimeters from the center of the core.

The radius of the core is 45 cm.

The net flow of neutrons is given by

$$\vec{j} = -AD\frac{dn}{dr}\ .$$

It will be shown later that

$$D = \frac{v\lambda_{tr}}{3} = \frac{v}{3N_{Be}\sigma_{tr}}\ ,$$

where

N_{Be} = atoms of beryllium per cubic centimeter,

σ_{tr} = microscopic transport cross section.

λ_{tr} = transport mean free path,

Thus we can write

$$\vec{j} = -A\frac{v\lambda_{tr}}{3}\frac{dn}{dr} = -A\frac{\lambda_{tr}}{3}\frac{d(nv)}{dr} = -A\frac{\lambda_{tr}}{3}\frac{d\phi}{dr}\ .$$

Since the problem is to find the net flow through unit area, A = 1 cm^2. The density of beryllium is 2.7 g/cm^3 and σ_{tr} is 5.5 barns. Therefore,

$$N_{Be} = \frac{2.7}{9} \times 0.6 \times 10^{24}$$

$$= 0.18 \times 10^{24}\ \text{atoms/cm}^3\ ,$$

$$\frac{D}{v} = \frac{\lambda_{tr}}{3} = \frac{1}{3 \times 0.18 \times 10^{24} \times 5.5 \times 10^{-24}}$$

$$= 0.33\ \text{cm}\ ,$$

$$\frac{d\phi}{dr} = 10^{13}\frac{r\frac{\pi}{60}\cos\frac{\pi}{60}r - \sin\frac{\pi}{60}r}{r^2}\ ,$$

and

$$\left(\frac{d\phi}{dr}\right)_{r=45} = 10^{13}\frac{-\frac{3}{4} \times 0.707 - 0.707}{(45)^2}$$

$$= -1.17 \times 10^{10}\ .$$

Thus $\vec{j} = 0.33 \times 1.17 \times 10^{10} = 0.38 \times 10^{10}$ neutrons/cm^2·sec, which is the net flow per unit area through the interface.

Derivation of Fick's Law. It is hoped that the preceding section has

made plausible the formula for the net flow of monoenergetic neutrons in the z direction:

$$\vec{j} = -DA \frac{dn}{dz} \cdot$$

However, it is well to take a careful look at the mathematical assumptions made in the derivation. Also, a very useful formula arises in the derivation that gives the actual current of neutrons in the positive z direction, as well as that in the opposite direction, as opposed to just the *net* flow (which is the difference of the two) as given by Fick's law.

Consider an element of area ds at the origin in the x, y plane of a three-dimensional region (Fig. 1.1). We wish to find an expression in terms of the flux at the position of ds for the number of neutrons passing through ds per second in a downward direction. To do this, we pick an element of volume dv above the x, y plane, find the number of collisions taking place there per second, find what fraction of these deflected neutrons starts in the direction of ds, and finally find what remaining fraction survives the journey to ds without being deflected by collision. The number of collisions in dv depends on the flux at dv. However, the flux at the point of dv can be expressed by Taylor's expansion in terms of the

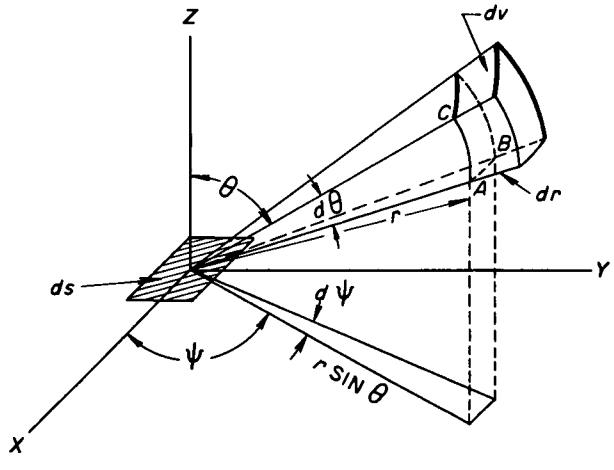


Fig. 1.1

flux at the origin (position of ds). Thus, integration can be carried out over all volume elements lying above the x, y plane, and the desired expression for the current through ds per second, in the downward direction, in terms of the flux at the position of ds is obtained.

It is convenient to use a spherical coordinate system, as shown in Fig. 1.1. The number of collisions per second in the volume element dv at (r, θ, ψ) is $\phi(r, \theta, \psi)\Sigma_s \, dv$. It is *assumed* at this point that *scattering is isotropic*; that is, neutrons coming out of collision in dv are spherically symmetric in angle. This is not a good assumption for light elements, and a correction will be made later.

If a sphere were drawn with center at the volume element dv with the point (0,0,0) lying on the surface, it is clear that the projection of ds on the sphere in the direction of r is $\cos\theta \, ds$, to within first-order differentials. From the assumption of isotropic scattering, it follows that the fraction of neutrons scattering in dv that start toward ds is equal to the ratio of the projected area of ds on the postulated sphere to the entire area of the sphere, that is,

$$\frac{\cos\theta \, ds}{4\pi r^2} .$$

The fraction of these that actually reach ds without collision is $e^{-\Sigma r}$, where $\Sigma = \Sigma_a + \Sigma_s$, in which Σ is the the total macroscopic cross section in cm^{-1} and is equal to the macroscopic absorption cross section (Σ_a) plus the macroscopic scattering cross section (Σ_s). The exponential form follows from the fact that Σ is the probability that a neutron will make a collision in 1 centimeter. In going an incremental distance dr, the change in beam strength, dn, would be

$$dn = -N \Sigma \, dr ,$$

and, hence, integrating from $r = 0$ to r,

$$\frac{N}{N_0} = e^{-\Sigma r}$$

is the fraction surviving collision in going a distance r.

Now, putting these facts together and noting that

$$dv = \overline{AB}\ \overline{AC}\ dr\ ,$$

where

$$\overline{AB} = r \sin\theta\ d\psi\ ,$$
$$\overline{AC} = r\ d\theta\ ,$$

and thus

$$dv = r^2 \sin\theta\ d\theta\ d\psi\ dr\ ,$$

it follows that the number of neutrons passing through ds owing to collisions in dv is

$$\frac{ds}{4\pi}\ \phi(r,\theta,\psi)\ \Sigma_s\ e^{-\Sigma r} \cos\theta \sin\theta\ dr\ d\psi\ d\theta\ .$$

This expression is then integrated over the half-space lying above the x,y plane. Thus, $\vec{j}_-$, the current per unit area in the negative z direction, is given by:

$$\vec{j}\ ds = \frac{ds}{4\pi}\ \Sigma_s \int_0^\infty \int_0^{\pi/2} \int_0^{2\pi} \phi(r,\theta,\psi)\ e^{-\Sigma r} \cos\theta \sin\theta\ d\psi\ d\theta\ dr\ . \qquad (1.1)$$

Up to this point, the formulation is rigorous (to within the approximation that the scattering is isotropic), since nothing has been said about $\phi(r,\theta,\psi)$. However, we are seeking a differential equation (the neutron-balance equation) that involves the flux at an arbitrary point. A general solution of this differential equation will be obtained, and when the boundary conditions of a particular geometrical assembly are applied, it will be possible to find ϕ as a general function of (r,θ,ψ). Thus it appears that to evaluate $\vec{j}_-$, it is necessary to know $\phi(r,\theta,\psi)$, but we are attempting to find j (Fick's law) so that we can determine $\phi(r,\theta,\psi)$.

To proceed further, then, $\phi(x,y,z)$ is expanded in a Taylor's series about $\phi(0,0,0) = \phi_0$, and *an approximation* is made that it is sufficient to keep only the *first-order terms of the expansion*; that is,

$$\phi(x,y,z) \cong \phi_0 + x\left(\frac{\partial\phi}{\partial x}\right)_0 + y\left(\frac{\partial\phi}{\partial y}\right)_0 + z\left(\frac{\partial\phi}{\partial z}\right)_0 . \qquad (1.2)$$

This is an important approximation to bear in mind; as a matter of fact, it is rather surprising, a priori, that it works at all. However, because of the factor $e^{-\Sigma r}$ in the integrand of Eq. 1.1, the magnitude of the integrand drops off sharply for values of r greater than two or three mean free paths, so that it is only necessary that the approximation of Eq. 1.2 be valid for a fairly short distance. The approximation says, then, that ϕ is changing slowly within a distance of two or three mean free paths (several inches). This is true if the position for which ϕ_0 is being evaluated is not close to a concentrated source, the control rods, or the external boundary of the reactor. It is only near such sources and sinks that the shape of the flux will vary substantially from that predicted by this formulation.

Although much is made of the fact that this so-called "asymptotic flux distribution" is not correct near sources and sinks, obtaining the correct flux at such points is so extremely difficult that it has been customary to use the asymptotic flux in engineering applications.

To get back to the derivation, it is necessary to put Eq. 1.2 in terms of spherical coordinates; that is,

$$x = r \sin\theta \cos\psi\ ,$$
$$y = r \sin\theta \sin\psi\ , \qquad (1.3)$$
$$z = r \cos\theta\ .$$

Upon substituting Eq. 1.2 in Eq. 1.1 and incorporating Eq. 1.3, it is found

that the terms in x and y drop out upon integration over ψ from 0 to 2π. Upon carrying through the remaining integration and canceling ds from both sides of Eq. 1.1, the result is

$$\vec{j}_- = \frac{\Sigma_s}{4\Sigma}\phi_0 + \frac{\Sigma_s}{6\Sigma^2}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.4)$$

If a similar derivation is carried through for the flow upward, $\vec{j}_+$, it is found that

$$\vec{j}_+ = \frac{\Sigma_s}{4\Sigma}\phi_0 - \frac{\Sigma_s}{6\Sigma^2}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.5)$$

The net flow upward, $\vec{j}$, then, is

$$\vec{j} = \vec{j}_+ - \vec{j}_- = -\frac{\Sigma_s}{3\Sigma^2}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.6)$$

For most assemblies, $\Sigma_s >> \Sigma_a$, $\Sigma \approx \Sigma_s$, and since $\Sigma_s = 1/\lambda_s$, where λ_s is the mean free path for scattering, Eqs. 1.4, 1.5, and 1.6 can be written, only slightly more approximately, as

$$\vec{j}_- = \frac{1}{4}\phi_0 + \frac{\lambda_s}{6}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.7)$$

$$\vec{j}_+ = \frac{\phi_0}{4} - \frac{\lambda_s}{6}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.8)$$

$$\vec{j} = \vec{j}_+ - \vec{j}_- = -\frac{\lambda_s}{3}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.9)$$

Since the assumption that scattering is isotropic is not good for most problems of interest, a corrected λ_s can be used to account for the anisotropy of the scattering. It has been shown[1] by transport theory, in which neutron direction is included as a variable in addition to position, that this corrected λ_s, called λ_{tr} (the transport mean free path), is given by

$$\lambda_{tr} = \lambda_s\left(\frac{1}{1 - \frac{2}{3A}}\right),$$

where A is the mass number of the scattering nucleus. Equations 1.7, 1.8, and 1.9 become

$$\vec{j}_- = \frac{\phi_0}{4} - \frac{\lambda_{tr}}{6}\left(\frac{d\phi}{dz}\right)_0 , \qquad (1.10)$$

$$\vec{j}_+ = \frac{\phi_0}{4} - \frac{\lambda_{tr}}{6}\left(\frac{d\phi}{dz}\right)_0 , \qquad (1.11)$$

$$\vec{j} = \vec{j}_+ - \vec{j}_- = -\frac{\lambda_{tr}}{3}\left(\frac{d\phi}{dz}\right)_0 . \qquad (1.12)$$

These important equations will be used in the work to follow.

The units of $\vec{j}_-$, $\vec{j}_+$, and $\vec{j}$ are neutrons/cm^2·sec. It is well to emphasize that these are vector quantities. The fact that the vector quantity $\vec{j}$ has the same units as the scaler quantity ϕ causes a good deal of confusion. The quantity ϕ is the total number of centimeters traveled per second by all the neutrons in a cubic centimeter regardless of direction. The units of ϕ are then neutrons times centimeters per second divided by cubic centimeters, and upon simplication (at the expense of clarity) this becomes neutrons/cm^2·sec, which unfortunately is the same as the units for $\vec{j}$. As a matter of fact, if a thin foil of area 1 cm^2 is placed in a reactor in which the flux (scaler flux, not current) is 10^{13} neutrons/cm^2·sec, the foil will feel $\frac{1}{2} \times 10^{13}$ neutrons/sec passing through it, counting both sides - the sum of Eqs. 1.10 and 1.11.

The following example is intended to illustrate the use of Eq. 1.10 (or Eq. 1.11).

Example 2. One design for the ARE (Aircraft Reactor Experiment) has liquid fuel, and the fuel tubes extend in a vertical direction through a sheet of boron carbide that lies

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[1] A. M. Weinberg and L. C. Noderer, *Theory of Neutron Chain Reactions*, Vol. I. *Diffusion and Slowing Down of Neutrons*, ORNL CF-51-5-98 (May 15, 1951).

horizontally at the top of the core. (The purpose of this is to make the reactor self-stablizing. When the power increases, the temperature of the fuel increases and some of the fuel goes into the boron carbide layer where it is ineffective in supporting the chain reaction; therefore the power tends to decrease, and thus gives a stablizing effect.)

The designer wants to know how much heating will occur in the boron carbide owing to the exothermic neutron-capture reaction in B^{10} so that he can decide whether the core coolant flowing through holes in the boron carbide will provide adequate cooling.

The neutron capture reaction in B^{10} is

$$_5B^{10} + {}_0n' = {}_3Li^7 + {}_2He^4 + (2.60 \text{ Mev})_{avg} ,$$

where the 2.60 Mev is the kinetic energy of the recoil particles and can be considered as heat release at the point of capture. The problem is, then, to find the number of neutrons captured in the rod.

Let us suppose that the boron carbide layer is thick enough so that it is "black" to thermal neutrons (captures all thermal neutrons that enter) but is transparent to above-thermal neutrons.

Equation 1.11 will give approximately the number of thermal neutrons entering per unit area per second if ϕ is the thermal flux at the face of the boron carbide. Only the asymptotic flux is known. Here is a case in which the answer obtained will clearly not be exact. As was pointed out, the approximation leading to Eq. 1.11 is not valid near a strong absorber. However, only a rough answer is wanted. An error of a factor of 2 is probably not serious inasmuch as the designer will probably overdesign by more than this to be on the safe side.

It is found, by methods to be described later, that at the face of the boron carbide the thermal flux and slope of the thermal flux have the values

$$\phi_{th} = 2 \times 10^{12} \frac{\text{neutrons cm}}{\text{sec cm}^3} ,$$

$$\frac{d\phi_{th}}{dz} = -0.4 \times 10^{12} \frac{\text{neutrons cm}}{\text{sec cm}^3 \text{ cm}} ,$$

based on the asymptotic flux. From the properties of the reactor, it is known that $\lambda_{tr} = 1.73$ cm. Thus the number of thermal neutrons entering the boron carbide per square centimeter is

$$\vec{j}_+ = \frac{2 \times 10^{12}}{4} - \left(\frac{1.73}{6}\right)(-0.4 \times 10^{12})$$
$$= 0.6 \times 10^{12} .$$

All the thermal neutrons that enter are captured and each results in a heat release of

$$2.60 \text{ Mev} = \frac{2.60 \times 10^6 \times 1.6 \times 10^{-12}}{10^7}$$
$$= 4.15 \times 10^{-13} \text{ joules} .$$

Thus the heat release in a portion of the boron carbide layer having a facial area of 1 cm^2 is $0.6 \times 10^{12} \times 4.15 \times 10^{-13} = 0.25$ watts. Since this is a rather small rate of heat release, the designer can thus be assured that the core coolant, which flows through holes in the boron carbide about 3 3/4 in. apart, will provide adequate cooling of the boron carbide.

Use of Fick's Law to Obtain the Leakage per Cubic Centimeter. To review briefly, we are attempting to obtain a differential equation that declares a balance between the creation and loss of neutrons at an arbitrary position in terms of the flux at this position:

$$\frac{\text{Leakage}}{\text{sec}\cdot\text{cm}^3} + \frac{\text{Absorption}}{\text{sec}\cdot\text{cm}^3} - \frac{\text{Production}}{\text{sec}\cdot\text{cm}^3} = 0 .$$

When boundary conditions of a particular problem are applied to the general solution of this differential equation, formulas are obtained for the critical size (or critical mass) and, in turn, the flux or power distribution.

We are seeking the expression for the first term, the leakage or net flow of neutrons out of a cubic centimeter per second. It is easily obtained now that the formula for the net current is known.

Consider an element of volume as shown in Fig. 1.2. According to Fick's

law, the net flow up through the bottom of the volume element is

$$\vec{j}_z \; dx \; dy = -\frac{\lambda_{tr}}{3}\left(\frac{d\phi}{dz}\right)_z dx \; dy \; .$$

The net flow up through the top is

$$\vec{j}_{z+dz} \; dx \; dy = -\frac{\lambda_{tr}}{3}\left(\frac{d\phi}{dz}\right)_{z+dz} dx \; dy \; .$$

The difference is the net loss of neutrons per second by leakage from the top and bottom of the volume element:

$$\left(\vec{j}_{z+dz} - \vec{j}_z\right) dx \; dy = -\frac{\lambda_{tr}}{3}\left[\left(\frac{d\phi}{dz}\right)_{z+dz} - \left(\frac{d\phi}{dz}\right)_z\right] dx \; dy. \tag{1.13}$$

The quantity $(d\phi/dz)_{z+dz}$ is now expanded by Taylor's series about z,

$$\left(\frac{d\phi}{dz}\right)_{z+dz} \approx \left(\frac{d\phi}{dz}\right)_z + \left(\frac{d^2\phi}{dz^2}\right)_z dz \; ,$$

with terms after the second discarded. Equation 1.13 becomes, with this substitution,

$$\left(\vec{j}_{z+dz} - \vec{j}_z\right) dx \; dy = -\frac{\lambda_{tr}}{3}\frac{d^2\phi}{dz^2} dx \; dy \; dz \, .$$

In a similar way, the net loss of neutrons per second by leakage from the front and back of the volume element is

$$\left(\vec{j}_{x+dx} - \vec{j}_x\right) dy \; dz = -\frac{\lambda_{tr}}{3}\frac{d^2\phi}{dx^2} dx \; dy \; dz \, ,$$

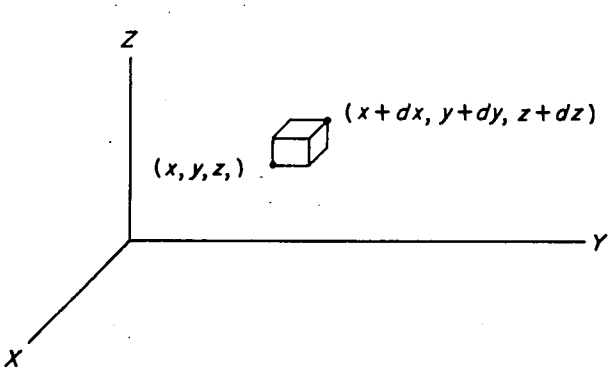


Fig. 1.2

and, from the left and right sides,

$$\left(\vec{j}_{y+dy} - \vec{j}_y\right) dx \; dz = -\frac{\lambda_{tr}}{3}\frac{d^2\phi}{dy^2} dx \, dy \, dz \; .$$

The total loss of neutrons per second by leakage from the volume element is the sum or

$$-\frac{\lambda_{tr}}{3}\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right) dx \; dy \; dz \; .$$

Since $dv = dx \; dy \; dz$, the leakage can be expressed on a unit volume basis as

$$-\frac{\lambda_{tr}}{3}\nabla^2\phi \; , \tag{1.14}$$

where

$$\nabla^2\phi \equiv \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} \; . \tag{1.15}$$

Formula 1.14 is the result we have been seeking – the first term of the neutron-balance equation. The rather lengthy derivation leading to this important expression uncovered some formulas that are very useful in their own right, such as the expression for $\vec{j}_-$, $\vec{j}_+$, and Fick's law $\vec{j}$.

Formula 1.14, with the coordinate system used in Eq. 1.15, is suitable only for assemblies of parallelepiped shape. However, it is not necessary to repeat the derivation for the two other useful geometries: spheres (in spherical coordinates) and cylinders (in cylindrical coordinates). Formula 1.14 is perfectly general, with the understanding that in spherical coordinates[2]

$$\nabla^2\phi \equiv \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} \; , \tag{1.16}$$

and in cylindrical coordinates

$$\nabla^2\phi \equiv \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dz^2} \; . \tag{1.17}$$

Absorption. The second term of the neutron-balance equation follows easily. The number of neutrons absorbed

[2] H. Margenau, and G. M. Murphy, *The Mathematics of Physics and Chemistry*, Chap. V, Van Nostrand, New York (1943).

per second per cubic centimeter is $\phi\Sigma_a$, where Σ_a is the macroscopic absorption cross section in cm^{-1}.

Production. The third term of the neutron-balance equation is also easily expressed. The number of neutrons produced on our one group model is $\phi\Sigma_a k$, where $k = \nu\Sigma_f/\Sigma_a$ and is the number of neutrons produced per neutron absorbed in any reactor material; ν is the number of neutrons per fission (2.5 for U^{235}) and Σ_f is the macroscopic fission cross section.

The definition of k here is different from that given in the older literature, for example, that given by the so-called "four-factor formula." The four-factor formula was originated with the theory of heterogeneous natural uranium reactors. The four-factor definition of k is not convenient in the theory of the enriched power reactors that are receiving the emphasis in the present military era in reactor development.

Balance Equation in Differential Form. The differential equation expressing the conservation of neutrons can now be written:

$$\frac{\text{Leakage}}{\text{sec}\cdot\text{cm}^3} + \frac{\text{Absorption}}{\text{sec}\cdot\text{cm}^3} - \frac{\text{Production}}{\text{sec}\cdot\text{cm}^3} = 0$$

$$-\frac{\lambda_{tr}}{3}\nabla^2\phi + \phi\Sigma_a - \phi\Sigma_a k = 0 \qquad (1.18)$$

where $\nabla^2\phi$ represents either Eq. 1.15, 1.16, or 1.17 according to whether the reactor is a parallelepiped, a sphere, or a cylinder.

Equation 1.18 can be rewritten as

$$\nabla^2\phi + B^2\phi = 0 , \qquad (1.19)$$

where

$$B^2 = \frac{\Sigma_a (k - 1)}{\frac{\lambda_{tr}}{3}} , \qquad (1.20)$$

which is of the form of the wave equation, or of Hemholtz' equation. The quantity B^2 is called the "buckling," because it determines the degree of curvature of the flux. The quantity $\lambda_{tr}\lambda_a/3$ is often denoted by L^2, and hence $B^2 = k - 1/L^2$, where L^2 is called the migration area. It will be shown in Chap. 2 that L^2 has a simple interpretation in terms of the crow-flight distance a neutron travels to the point of capture.

General Solution. The general solution will be written down for four geometries: slab, parallelepiped, sphere, and cylinder. In all cases, the buckling, B^2, has the definition given by Eq. 1.20.

Slab. Equation 1.19 for a slab takes the form

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0 , \qquad (1.21)$$

where $x = 0$ is the middle of the slab (Fig. 1.3), which is finite in the x direction and infinite in the y and z directions. The general solution of Eq. 1.21 is clearly

$$\phi = A \sin Bx + C \cos Bx . \qquad (1.22)$$

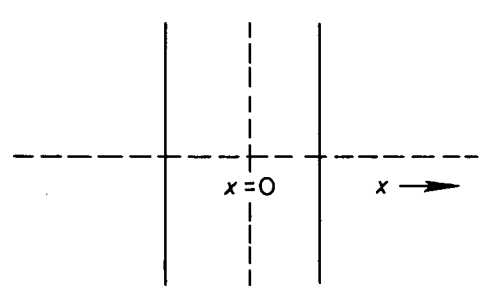


Fig. 1.3

Parallelepiped. Equation 1.19 for a parallelepiped takes the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0 , \qquad (1.23)$$

where the origin is at the middle of the parallelepiped (Fig. 1.4). The general solution is

$$\phi = (A \cos \alpha x + C \sin \alpha x)(E \cos \beta x + F \sin \beta x)(G \cos \gamma x + H \sin \gamma x) , \qquad (1.24)$$

where

$$\alpha^2 + \beta^2 + \gamma^2 = B^2 .$$

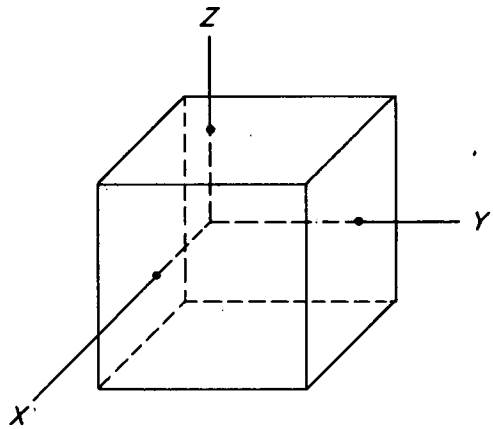


Fig. 1.4

Sphere. Equation 1.19 for a sphere takes the form

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + B^2\phi = 0 , \qquad (1.25)$$

with $r = 0$ at the center (Fig. 1.5). The general solution, as can be verified by differentiation, is

$$\phi = A\,\frac{\sin Br}{r} + C\,\frac{\cos Br}{r} . \qquad (1.26)$$

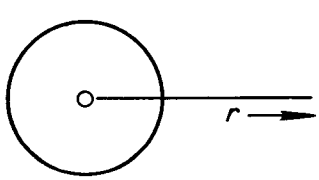


Fig. 1.5

Cylinder. Equation 1.19 for a cylinder takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0 ; \qquad (1.27)$$

with $r = 0$, $z = 0$ at the geometrical center (Fig. 1.6). Equation 1.27 is solved by the method of separation of variables. The substitution is made that

$$\phi(r,z) = R(r)\,Z(z) , \qquad (1.28)$$

where R is a function of r only and Z is a function of z only, as a trial to see whether the partial differential equation in r and z can be separated into an ordinary differential equation in r only and an ordinary differential equation in z only. This is a valuable method for solving partial differential equations, and if it happens to be the student's first knowledge of it, he should take time to understand clearly Eqs. 1.28 through 1.35.

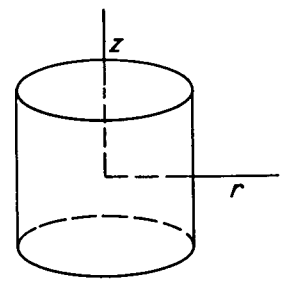


Fig. 1.6

By substituting Eq. 1.28 into Eq. 1.27, one obtains

$$Z(z)\,R''(r) + \frac{1}{r}Z(z)\,R'(r) + R(r)\,Z''(z) + B^2\,R(r)\,Z(z) = 0 , \qquad (1.29)$$

and by dividing through by $Z(z)\,R(r)$, the result is

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} - \frac{Z''(z)}{Z(z)} + B^2 = 0 . \qquad (1.30)$$

Equation 1.30 can be rewritten in the form

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + B^2 = -\frac{Z''(z)}{Z(z)} . \qquad (1.31)$$

In Eq. 1.31, the left side is a function of r only, and the right side is a function of z only. Now, this equation must hold for all points (r,z). Suppose it holds at a point, say (r_1,z_1). Then let r vary but hold z constant. The right side of Eq. 1.31 has not changed, and the left side must not change from the condition of equality. Since the new choice of r was perfectly general, it follows that the left side must equal a constant for all r; that is,

the left side equals a pure constant. It follows that the right side equals this same constant. Denote this constant by γ^2. Then,

$$-\frac{Z''(z)}{Z(z)} = \gamma^2 ,$$

or

$$Z''(z) + \gamma^2 Z(z) = 0 , \quad (1.32)$$

and

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + B^2 = \gamma^2 . \quad (1.33)$$

Let $\mu^2 = B^2 - \gamma^2$. Then Eq. 1.33 becomes

$$R''(r) + \frac{1}{r} R'(r) + \mu^2 R(r) = 0. \quad (1.34)$$

The attempted separation of variables is successful in this case. The solution of Eq. 1.32 is easy:

$$Z(z) = A \cos \gamma z + C \sin \gamma z . \quad (1.35)$$

The solution of Eq. 1.34 is not easy; however, this equation has been widely studied for many years by mathematicians. The properties of this equation – Bessel's equation of order zero – and related equations and the properties of the solutions have formed the subject matter for many large books.[3] The equation is solved by a series method, the terms of which are obtained by recursion formulas (the method of Frobinius). One series solution is the following:

$$R_1(r) = 1 - \frac{(\mu r)^2}{2^2} + \frac{(\mu r)^4}{2^4 (2!)^2} - \ldots + (-1)^k \frac{(\mu r)^{2k}}{2^{2k} (k!)^2} + \ldots . \quad (1.36)$$

This series is denoted by the notation $J_0(\mu r)$, much like the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

(which is a solution of $(d^2y/dx^2) + y = 0$) is denoted by $\sin x$.

Since Eq. 1.34 is an ordinary differential equation of order 2, it possesses two independent solutions. The other series solution is

$$R_2(r) = \frac{2}{\pi}\left[0.57722 + \ln\left(\frac{\mu r}{2}\right)\right] J_0(\mu r) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\frac{(\mu r)^{2n}}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) . \quad (1.37)$$

This series is denoted by $Y_0(\mu r)$ or, sometimes, $N_0(\mu r)$.

The values of $J_0(\mu r)$ and $Y_0(\mu r)$ are tabulated in various places[4] for values of the argument μr.

The general solution of Eq. 1.34 can be written, then, as

$$R(r) = E\, J_0(\mu r) + F\, Y_0(\mu r) , \quad (1.38)$$

where E and F are arbitrary constants. Since $\phi(r,z) = R(r)\, Z(z)$ by definition, it follows that the general solution of Eq. 1.27 is

$$\phi(r,z) = \left[A \cos \gamma z + C \sin \gamma z\right]\left[E\, J_0(\mu r) + F\, Y_0(\mu r)\right] . \quad (1.39)$$

Boundary Conditions. The boundary conditions that apply to the problem of determining the critical size of a bare reactor are as follows:

1. The flux must remain finite at all points.
2. The flux must be symmetrical around any points, lines, or planes of symmetry that exist in the geometrical setup.
3. The flux must be nonnegative at all points.
4. At the external surface, the return current must be zero.

The fourth boundary condition makes use of Eq. 1.10 (or Eq. 1.11). To

[3] G. N. Watson, *A Treatise on the Theory of Bessel Function*, Cambridge Univ. Press (1944).

[4] E. Jahnke and F. Emde, *Tables of Functions with Formulae and Curves*, Dover Publications, New York (1945).

illustrate, suppose we wish to write that the return current at the right face of the slab in Fig. 1.3 is zero. This says

$$\frac{\phi(x_0)}{4} + \frac{\lambda_{tr}}{6}\left(\frac{d\phi}{dx}\right)_{x_0} = 0 , \qquad (1.40)$$

where x_0 is the half thickness of the slab. By writing this in the form

$$-\left(\frac{d\phi}{dx}\right)_{x_0} = \frac{3\phi(x_0)}{2\lambda_{tr}} , \qquad (1.41)$$

it is seen that this is a condition on the slope of the flux at the surface such that the flux extrapolates to zero at a distance $2/3\ \lambda_{tr}$ outside the surface.

The condition 1.40 may seem to be somewhat artificial, since $\vec{j}_-$ was derived on the basis of a scattering medium and since condition 1.40 is supposed to be a statement regarding the current coming back from a void. It may be a little more palatable if one thinks of condition 1.40 as

$$\lim_{x \to x_0}\left[\frac{\phi(x)}{4} + \frac{\lambda_{tr}}{6}\frac{d\phi(x)}{dx}\right] = 0 . \qquad (1.42)$$

In any event, Eqs. 1.10 and 1.11 are based on the assumption that the flux is slowly varying (can be represented by a Taylor expansion to first-order terms only). This assumption is not good near the reactor surface. It is clear that the flux distribution obtained by applying these boundary conditions to the general solution of the preceding section will not be a true representation near the surface. It has been shown[1] by transport theory, however, that if a slight correction to the extrapolation distance is made, the flux distribution that is obtained *will* have the proper shape and magnitude a distance of a few mean free paths from the surface. This corrected condition is:

$$-\left(\frac{d\phi}{dx}\right)_{x_0} = \frac{\phi(x_0)}{0.71\ \lambda_{tr}} . \qquad (1.43)$$

The extrapolation distance $0.71\ \lambda_{tr}$ is approximately correct for all reactor geometries if the curvature of the outer surface is not too great. The fact that the predicted flux distribution is not correct close to the surface does not seriously affect the critical mass estimate if the reactor is not too small.

An approximation to Eq. 1.43 can usually be made without serious error; that is,

$$\phi\left(x_0 + 0.71\ \lambda_{tr}\right) = 0 . \qquad (1.44)$$

In summary, then, the boundary conditions for the bare reactor problem are

1. finiteness,
2. symmetry,
3. nonnegativeness,
4. $\phi(x_0 + 0.71\ \lambda_{tr}) = 0$

for slabs,

$$\phi(r_0 + 0.71\ \lambda_{tr}) = 0$$

for a sphere,

$$\phi(R + 0.71\ \lambda_{tr},\ z) = 0$$

and

$$\phi(r, \frac{H}{2} + 0.71\ \lambda_{tr}) = 0$$

for a cylinder, where x_0 is the critical half thickness of the slab, r_0 is the critical radius of the sphere, R is the critical radius of the cylinder, and H is the critical height of the cylinder.

Application of Boundary Conditions to General Solutions. The boundary conditions will be applied to the general solutions in two geometries – slabs and cylinders – and the spherical geometry will be left as an exercise.

Slab. The general solution for the flux in the slab was

$$\phi(x) = A \sin Bx + C \cos Bx ,$$

where

$$B = \sqrt{\frac{3(k-1)}{\lambda_a\ \lambda_{tr}}} .$$

The first boundary condition that the flux must be everywhere finite is not necessary in the slab geometry, since the sine and cosine are everywhere

bounded functions. The second boundary condition that the flux must be symmetrical about any planes of geometrical symmetry, however, requires that we drop the sine term; that is, set $A = 0$, because $x = 0$ is at the middle of the slab, the cosine is an even function, and the sine is an odd function. No combination of the sine and cosine can ever be symmetrical about the origin. The solution reduces, then, to

$$\phi(x) = A \cos Bx \ . \qquad (1.40)$$

The constant A depends on the power level at which the reactor is operating and will be left as an arbitrary constant. It is not necessary to specify A in order to find the critical dimension. This checks with physical reasoning, because the critical size and critical mass do not depend on the power level at which it is desired to operate the reactor.

The flux distribution is already determined completely by Eq. 1.40. It remains only to decide at what value of x to terminate the cosine function as being the critical half thickness, x_0. The third boundary condition states that the flux cannot be negative, and, hence,

$$x_0 \leq \frac{\pi}{2B} \ , \qquad (1.41)$$

since $\pi/2B$ is the first root of $\cos Bx$. On the other hand, the fourth boundary condition says that the flux goes to zero at the extrapolated boundary; that is,

$$\phi(x_0 + \delta) = 0 \ , \qquad (1.42)$$

where δ is the extrapolation distance, $0.71\ \lambda_{tr}$. Since there are no roots of $\cos Bx$ less than $\pi/2B$, it follows that

$$x_0 + \delta \equiv \frac{\pi}{2B} = \frac{\pi}{2}\sqrt{\frac{\lambda_a \lambda_{tr}}{3(k-1)}}$$

or (1.43)

$$x_0 = \frac{\pi}{2}\sqrt{\frac{\lambda_a \lambda_{tr}}{3(k-1)}} - 0.71\ \lambda_{tr} \ ,$$

where

x_0 = the half thickness of the critical slab,

λ_a = the mean free path for absorption $= 1/N\sigma_a$,

N = number of nuclei per cubic centimeter of reactor mixture,

σ_a = an average microscopic cross section for elements in the reactor mixture,

λ_{tr} = the transport mean free path $= 1/N\sigma_{tr}$,

σ_{tr} = the average microscopic transport cross section,

$k = \nu\Sigma_f/\Sigma_a$,

ν = 2.5 for U^{235},

$\Sigma_a = 1/\lambda_a$,

$\Sigma_f = N_U\sigma_f$,

N_U = atoms of U^{235} per cubic centimeter,

σ_f = microscopic fission cross section of U^{235}.

When the reactor is made up of a mixture of several elements, the quantities are best evaluated as follows:

$$\frac{1}{\lambda_a} = \Sigma_a = N_x\sigma_{ax} + N_y\sigma_{ay} + N_z\sigma_{az} \ ,$$

where

N_x = atoms per cubic centimeter of element x, etc.,

σ_{ax} = microscopic cross section of element x, etc.,

and

$$\frac{1}{\lambda_{tr}} = \Sigma_{tr} = N_x\sigma_{sx}\left(1 - \frac{2}{3A_x}\right) + N_y\sigma_{sy}\left(1 - \frac{2}{3A_y}\right) + N_z\sigma_{sz}\left(1 - \frac{2}{3A_z}\right) ,$$

where A_x is the atomic mass number of element x, etc.

Cylinder. The general solution for the flux in the bare cylinder was

$$\phi(r,z) = \left[A\ J_0(\mu r) + C\ Y_0(\mu r)\right]\left[E \cos \gamma z + F \sin \gamma z\right] \ , \qquad (1.44)$$

where

$$B^2 = \mu^2 + \gamma^2 .$$

From boundary condition 1 that the flux is finite everywhere, it follows that C must be zero, since

$$\lim_{r \to 0} Y_0(\mu r) = \infty .$$

From boundary condition 2 that the flux must be symmetrical, it follows that $F = 0$, since $z = 0$ was chosen to be at the center of the reactor (Fig. 1.6). The expression for the flux distribution reduces then to:

$$\phi(r, z) = A\, J_0(\mu r) \cos \gamma z \quad , (1.45)$$

where AE has been replaced by simply A, since the coefficients are arbitrary.

The radial component, μ^2, of the buckling, and the longitudinal component, γ^2, are not determined independently by the reactor composition. Only the sum is determined

$$\mu^2 + \gamma^2 = B^2 = \frac{3(k - 1)}{\lambda_a \lambda_{tr}} . \qquad (1.46)$$

Thus there is an infinite set of pairs (μ^2, γ^2) that determines flux distributions in the infinite set of critical cylindrical reactors of given composition and buckling. These reactors all have the same uranium concentration, by hypothesis, but not the same critical mass (which depends on the volume). Suppose for a given composition, and hence for B^2, we choose a value γ^2 $(\gamma^2 < B^2)$ and thus determine a corresponding $\mu^2 = B^2 - \gamma^2$.

As in the slab geometry, the half length $H/2$ must be such that $\gamma(H/2) \leq \pi/2$, since the flux must be everywhere nonnegative. From boundary condition 4,

$$\phi\left(r, \frac{H}{2} + \gamma\right) = 0 ,$$

and since the cosine has no roots less than $\pi/2$, it follows that

$$\gamma \frac{H}{2} + \delta = \frac{\pi}{2}$$

or (1.47)

$$\frac{H}{2} = \frac{\pi}{2\gamma} - 0.71\, \lambda_{tr} .$$

Similarly, since $\mu\gamma = 2.405$ is the first root of $J_0(\mu r)$, it follows that

$$\mu(R + \delta) = 2.405 ,$$

$$R = \frac{2.405}{\mu} - 0.71\, \lambda_{tr} . \qquad (1.48)$$

From Eqs. 1.46, 1.47, and 1.48,

$$\left(\frac{2.405}{R + \delta}\right)^2 + \left[\frac{\pi}{2\left(\frac{H}{2} + \delta\right)}\right]^2 = B^2 = \frac{3(k - 1)}{\lambda_a \lambda_{tr}} , \qquad (1.49)$$

which determines the infinite set of critical cylindrical reactors of given composition. The square cylinder is the cylindrical geometry of most applications. If $R \to \infty$, it is seen that Eq. 1.49 becomes Eq. 1.43, that is, the cylinder becomes a slab.

REFLECTED REACTOR

A reflector is by definition a layer of material surrounding the active core of the reactor that contains no fissionable material. Its purpose is to "reflect" neutrons back into the core by single or multiple scatterings. Since no fissioning takes place in the reflector, the balance equation reduces to

$$\frac{\text{Leakage}}{\text{cm}^3 \cdot \text{sec}} + \frac{\text{Absorption}}{\text{cm}^3 \cdot \text{sec}} = 0$$

where here the leakage is negative; that is, there is a net inward flow into the volume element. The differential equation is

$$-\frac{\lambda_{tr} R}{3} \nabla^2 \phi_R + \phi_R \Sigma_{aR} = 0 , \qquad (1.50)$$

or

$$\nabla^2 \phi_R - B_R^2 \phi_R = 0 , \qquad (1.51)$$

where

$$B_R^2 = \frac{3}{\lambda_{trR} \lambda_{aR}} ,$$

and the quantities with an R subscript

refer to the reflector. The reflector buckling is $-B_R^2$.

The differential equation in the core is, as in the bare reactor,

$$-\frac{\lambda_{trc}}{3}\nabla^2\phi_c + \phi_c\Sigma_{ac} - \phi_c\Sigma_{ac}k = 0 .$$

The same plan is followed for obtaining the formulas for critical size of reflected reactors as is followed for bare reactors. The general solutions are obtained, the boundary conditions are set down, and the boundary conditions are applied to the general solution. The last step leads to the critical size formulas. As before, the procedure is carried through for the slab and cylinder geometry and the sphere is left as an exercise.

General Solutions

Slab. The general solution in the core is the same as in the bare slab.

$$\phi_c(x) = A \sin B_c x + C \cos B_c x , \quad (1.52)$$

where

$$B_c^2 = \frac{3(k-1)}{\lambda_{ac}\lambda_{trc}} , \quad (1.53)$$

and the origin is chosen as before, at the center of the core (Fig. 1.7).

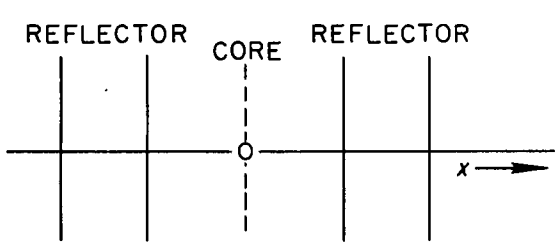


Fig. 1.7

The differential balance equation in the reflector is

$$\frac{d^2\phi_R(x)}{dx^2} - B_R^2\phi_R(x) = 0 , \quad (1.54)$$

and the general solution is

$$\phi_R(x) = E \sinh B_R x + F \cosh B_R x , \quad (1.55)$$

where the hyperbolic functions arise because the buckling in the reflector is $-B_R^2$ instead of being positive as in the core. It will be recalled that

$$\sinh B_R x \equiv -i \sin iB_R x \equiv \frac{e^{B_R x} - e^{-B_R x}}{2},$$

and

$$\cosh B_R x \equiv \cos iB_R x \equiv \frac{e^{B_R x} + e^{-B_R x}}{2} .$$

Cylinder. The problem of a reactor with cylindrical core and all-around reflector cannot be solved analytically, but the reactor with cylindrical core, jacket reflector, and no end reflectors is soluble. An approximation to the former case will be discussed later. The assembly to be solved is shown in Fig. 1.8.

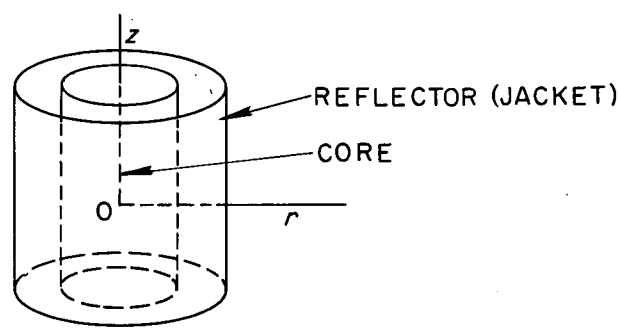


Fig. 1.8

The general solution in the core is the same as for the bare cylinder:

$$\phi_c(r,z) = [A_c J_0(\mu_c r) + C_c Y_0(\mu_c r)][E_c \sin \gamma_c z + F_c \cos \gamma_c z] , \quad (1.56)$$

where

$$\mu_c^2 + \gamma_c^2 = B_c^2 .$$

The differential balance equation in the reflector is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} - B_R^2\phi = 0 , \quad (1.57)$$

where

$$B_R^2 = \frac{3}{\lambda_{aR}\lambda_{trR}} .$$

Equation 1.57 is solved by separation of variables, as in the case of the bare cylinder equation. It is found that the solution is

$$\phi R = [A_R\ I_0(\mu_R r) + C_R\ K_0(\mu_R r)][E_R \sin \gamma_R z + F_R \cos \gamma_R z] , \qquad (1.58)$$

where $-\mu_R^2 + \gamma_R^2 = -B_R^2$ (radial buckling + longitudinal buckling = total buckling). It is to be noted that the radial component of the buckling has been denoted by $-\mu_R^2$, rather than $+\mu_R^2$ thus making μ_R^2 positive. The differential equation in the variable r that results from the separation-of-variables technique therefore has the *modified Bessel's functions*, I_0 and K_0 as solutions, where

$$I_0(\mu_R r) = 1 + \frac{(\mu_R r)^2}{2^2(1!)^2} + \frac{(\mu_R r)^4}{2^4(2!)^2} + \frac{(\mu_R r)^6}{2^6(3!)^2} + \ldots , \qquad (1.59)$$

$$K_0(\mu_R r) = -\left[0.5772 + \ln\left(\frac{\mu_R r}{2}\right)\right] I_0(\mu_R r) + \sum_{n=1}^{\infty} \frac{1}{(n!)^2}\left(\frac{\mu_R r}{2}\right)^{2n}\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right). \qquad (1.60)$$

The functions I_0 and K_0 are tabulated in various places.[5]

Boundary Conditions. The first three boundary conditions are the same as for the bare reactor:

1. The flux is everywhere finite.
2. The flux distribution is symmetrical about any points, lines, or planes of geometrical symmetry.
3. The flux is nonnegative.
4. The fourth boundary condition is the surface condition, which for *cylinders* is:

a. Jacket

$$\frac{\left[\dfrac{d\phi_R(r,z)}{dr}\right]_{(r=R+T)}}{\phi(R+T,z)} = \frac{1}{\delta} , \qquad (1.61)$$

where

R = radius of core,

T = reflector thickness,

δ = extrapolation distance,

$= 0.71\ \lambda_{tr}$,

or

$$\phi_R(R + T + \delta,\ z) = 0 . \qquad (1.62)$$

b. Reflector ends

$$\frac{\left[\dfrac{d\phi_R(r,z)}{dr}\right]_{(z=H/2)}}{\phi_R(r,H/2)} = \frac{1}{\delta}, \qquad (1.63)$$

where $H/2$ is the half length of the reactor, or

$$\phi_R[r,(H/2) + \delta] = 0 . \qquad (1.64)$$

c. Core ends

$$\frac{\left[\dfrac{d\phi_c(r,z)}{dr}\right]_{(z=H/2)}}{\phi_c(r,H/2)} = \frac{1}{\delta} \qquad (1.65)$$

or

$$\phi_c[r,(H/2) + \delta] = 0 . \qquad (1.66)$$

(5)British Association for the Advancement of Science Mathematical Tables, Vol. II, Bessel Functions, Cambridge Univ. Press.

These equations are derived in the same manner as for the bare-reactor extrapolation conditions. The value $\delta = 0.71\ \lambda_{tr}$ is approximately correct for any surface that does not have too much curvature and is satisfactory for thermal and intermediate energy reactors.

For *slabs* the fourth boundary condition is

$$\frac{\left[\dfrac{d\phi_R(x)}{dx}\right]_{(x=x_0+T)}}{\phi_R(x_0 + T)} = \frac{1}{\delta}, \qquad (1.67)$$

where

x_0 = the half thickness of the core,
T = the thickness of reflector,

or

$$\phi_R(x_0 + T + \delta) = 0 . \qquad (1.68)$$

5. The fifth boundary condition is the continuity condition. The flux density and the current must be continuous across the interface between core and reflector.

For *slabs* this condition is:

$$a.\quad \phi_c(x_0) = \phi_R(x_0) . \qquad (1.69)$$

$$b.\quad \frac{\lambda_{trc}}{3}\left[\frac{d\phi_c(x)}{dx}\right]_{(x=x_0)} = \frac{\lambda_{trR}}{3}\left[\frac{d\phi_R(x)}{dx}\right]_{(x=x_0)} . \qquad (1.70)$$

For *cylinders* this condition is:

$$a.\quad \phi_c(R) = \phi_R(R) \qquad (1.71)$$

$$b.\quad \frac{\lambda_{trc}}{3}\left[\frac{\partial\phi_c(r,z)}{\partial r}\right]_{(r=R)} = \frac{\lambda_{trR}}{3}\left[\frac{\partial\phi_R(r,z)}{\partial r}\right]_{(r=R)} . \qquad (1.72)$$

Application of Boundary Conditions to General Solutions. It now remains to apply the boundary conditions to the general solutions. Formulas for the critical size for a given composition or the critical uranium concentration for a given size will result.

Slabs. The general solution in the core was

$$\phi_c(x) = A \sin B_c x + C \cos B_c x . \qquad (1.73)$$

The second boundary condition requires that $A = 0$, so that we have

$$\phi_c(x) = C \cos B_c x . \qquad (1.74)$$

The general solution in the reflector was

$$\phi_R(x) = E \sinh B_R x + F \cosh B_R x . \qquad (1.75)$$

Applying boundary condition 4 to Eq. 1.75,

$$0 = E \sinh B_R(x_0 + T + \delta) + F \cosh B_R(x_0 + T + \delta) , \qquad (1.76)$$

and thus

$$F = -\frac{E \sinh B_R(x_0 + T + \delta)}{\cosh B_R(x_0 + T + \delta)} . \qquad (1.77)$$

The reflector flux becomes

$$\phi_R(x) = E\left\{\sinh B_R x - \frac{\sinh\,[B_R(x_0 + T + \delta)]\,\cosh B_R x}{\cosh\,[B_R(x_0 + T + \delta)]}\right\} . \qquad (1.78)$$

Boundary condition 5*a* requires that

$$A \cos B_c x_0 = E\left[\sinh B_R x_0 - \frac{\sinh B_R(x_0 + T + \delta)}{\cosh B_R(x_0 + T + \delta)} \cosh B_R x_0\right] , \qquad (1.79)$$

and boundary condition 5*b* requires that

$$-A\frac{\lambda_{trc}}{3}B_c \sin B_c x_0 = E\frac{\lambda_{trR}}{3}\left[B_R \cosh B_R x_0 - \frac{\sinh B_R(x_0 + T + \delta)}{\cosh B_R(x_0 + T + \delta)} B_R \sinh B_R x_0\right] . \quad (1.80)$$

Equations 1.79 and 1.80 can be regarded as two simultaneous homogeneous equations in A and E. These two constants can thus be eliminated by solving for A/E in one and substituting in the other to yield an equation that is the final critical size relationship. It is somewhat more general, however, to use the mathematical theorem that, if there are n homogeneous equations in n unknowns, then in order for a nontrivial solution to exist for the unknowns, the determinant of the coefficient must be zero. When applied to Eqs. 1.79 and 1.80, this yields the final equation.

$$\begin{vmatrix} \cos B_c x_0 & \sinh B_R x_0 - \dfrac{\sinh B_R(x_0 + T + \delta)}{\cosh B_R(x_0 + T + \delta)} \cosh B_R x_0 \\ -\dfrac{\lambda_{trc}}{3} B_c \sin B_c x_0 & B_R \dfrac{\lambda_{trR}}{3}\left[\cosh B_R x_0 - \dfrac{\sinh B_R(x_0 + T + \delta)}{\cosh B_R(x_0 + T + \delta)} \sinh B_R x_0\right] \end{vmatrix} = 0 , \quad (1.81)$$

where

$$B_c^2 = \frac{3(k-1)}{\lambda_{ac}\lambda_{trc}},$$

$$B_R^2 = \frac{3}{\lambda_{aR}\lambda_{trR}}.$$

For the reflected slab in which the composition is known, it is necessary to specify the x_0 and determine the T or to specify the T and determine the x_0. On the other hand, it is possible to specify both x_0 and T and solve for the uranium concentration by trying various concentrations until the associated B_c^2 is found that satisfies Eq. 1.81. (The critical mass is, of course, infinite in the slab geometry.)

Once the criticality is established, the expressions for the flux are given by Eqs. 1.74 and 1.78, wherein Eq. 1.78 E is replaced by its value in terms of A from Eq. 1.79 (or Eq. 1.80). This determines ϕ to within the multiplicative constant A, which depends on the power level at which the reactor is operating.

Cylinders. The general solution in the core was

$$\phi_c(r,z) = [A_c J_0(\mu_c r) + C_c Y_0(\mu_c r)][E_c \sin \gamma_c z + F_c \cos \gamma_c z] . \quad (1.82)$$

The first boundary condition that the flux be everywhere finite requires that $C_c = 0$. The second boundary condition regarding symmetry requires that $E_c = 0$. Thus the expression for core flux reduces, then, to

$$\phi_c(r, z) = A_c \, J_0(\mu_c r) \cos \gamma_c z \ , \quad (1.83)$$

where $A_c E_c$ is replaced by just A_c, since the constants are arbitrary.

The general solution in the reflector was

$$\phi_R(r, z) = [A_R \, I_0(\mu_R r) + C_R \, K_0(\mu_R r)] \times [E_R \sin \gamma_R z + F_R \cos \gamma_R z] . \quad (1.84)$$

The second boundary condition regarding symmetry requires that $E_R = 0$. The boundary condition 4*a* requires that

$$0 = \{A_R \, I_0[\mu_R(R + T + \delta)] + C_R \, K_0[\mu_R(R + T + \delta)]\} \, F_R \cos \gamma_R z \quad (1.85)$$

or

$$C_R = - A_R \frac{I_0[\mu_R(R + T + \delta)]}{K_0[\mu_R(R + T + \delta)]} , \quad (1.86)$$

where
R = radius of core,
T = reflector thickness.
The reflector flux is, then,

$$\phi_R(r, z) = A_R \left\{ I_0(\mu_R r) - \frac{I_0[\mu_R(R + T + \delta)]}{K_0[\mu_R(R + T + \delta)]} K_0[\mu_R(r + T + \delta)] \right\} \cos \gamma_R z \ , \quad (1.87)$$

where $A_R E_R$ has been replaced by just A_R, since the coefficients are arbitrary.

In Eq. 1.83, there is the additional condition that

$$\mu_c^2 + \gamma_c^2 = B_c^2 \ , \quad (1.88)$$

and, similarly, in Eq. 1.87

$$-\mu_R^2 + \gamma_R^2 = -B_R^2 \ . \quad (1.89)$$

As in the bare cylinder, only the sum of the radial and longitudinal components of the buckling is determined. If the composition is specified (and hence B_c^2), some value is chosen for γ_c^2, and μ_c^2 is thus determined from Eq. 1.88. With γ_c fixed, the longitudinal core flux distribution is fixed as $\cos \gamma_c z$, and from the boundary conditions that the flux cannot be negative and that $\phi_c(r, (H/2) + \delta) = 0$, it follows that

$$\gamma_c \left(\frac{H}{2} + \delta \right) = \frac{\pi}{2} \ , \quad (1.90)$$

the first root of the cosine. Thus

$$\gamma_c = \frac{\pi}{2\left(\frac{H}{2} + \delta\right)} ,$$

where δ = extrapolation distance $= 0.71 \, \lambda_{trc}$. The assumption is made that the extrapolation distance at the ends of the reflector is equal to that at the ends of the core. Since by hypothesis the reflector is of the same length as the core, it follows that

$$\gamma_R = \gamma_c = \frac{\pi}{2\left(\frac{H}{2} + \delta\right)} \ . \quad (1.91)$$

After a length is chosen for a reactor of given composition, the continuity conditions are used to determine a relation between R and T for criticality. The jacket reflector thickness is then usually specified and the critical core radius determined.

Boundary condition 5*a* requires that

$$A_c\ J_0(\mu_c R) = A_R\left\{ I_0(\mu_R R) - \frac{I_0[\mu_R\ (R + T + \delta)]}{K_0[\mu_R(R + T + \delta\ ]}\ K_0(\mu_R R)\right\}, \qquad (1.92)$$

in which the longitudinal components cancel because they are the same on both sides. Boundary condition 5*b* requires taking the derivatives with respect to *r* of the expressions for the core and reflector fluxes given by Eqs. 1.83 and 1.87. If the series $J_0(\mu r)$ is differentiated term by term, the series that results is denoted by

$$J_1(\mu r) \equiv \frac{\mu r}{2} - \frac{(\mu r)^3}{2^3 2!} + \frac{(\mu r)^5}{2^5 2!\,3!} - \frac{(\mu r)^7}{2^7 3!\,4!} + \cdots .$$

Similarly,

$$\frac{dI_0(\mu r)}{dr} = +\ \mu I_1(\mu r)\ ,$$

where

$$I_1(\mu r) = \frac{\mu r}{2} + \frac{(\mu r)^3}{2^3 2!} + \frac{(\mu r)^5}{2^5 2!\,3!} + \cdots ,$$

and

$$\frac{dK_0(\mu r)}{dr} = -\ \mu K_1(\mu r)\ ,$$

in which

$$K_1(\mu r) = -\sum_{n=0}^{\infty} \frac{\left(\frac{\mu r}{2}\right)^{1+2n}}{n!(1 + n)!}\left\{ \log\frac{\mu r}{2} + 0.577 - \frac{1}{2}\sum_{m=1}^{1+n} m^{-1} - \frac{1}{2}\sum_{m=1}^{n} m^{-1}\right\}.$$

Thus boundary condition 5*b* results in the equation

$$-A_c\frac{\lambda_{trc}}{3}\mu_c J_1(\mu_c R) = A_R\frac{\lambda_{trR}}{3}\mu_R\left\{ I_0(\mu_R R) + \frac{I_0[\mu_R(R + T + \delta)]}{K_0[\mu_R(R + T + \delta)]}K_1[\mu_R(R + T + \delta)]\right\}. \qquad (1.93)$$

As in the slab core, there are two homogeneous equations in two unknowns: A_c and A_R. If solutions are to exist for A_c and A_R, it is necessary that the determinant of the coefficients vanish

$$\begin{vmatrix} J_0(\mu_c R) & I_0(\mu_R R) - \dfrac{I_0[\mu_R(R+T+\delta)]}{K_0[\mu_R(R+T+\delta)]} K_0(\mu_R R) \\ -\dfrac{\lambda_{tr}}{3}\mu_c J_1(\mu_c R) & \dfrac{\lambda_{trR}}{3}\mu_R \left\{ I_1(\mu_R R) + \dfrac{I_0[\mu_R(R+T+\delta)]}{K_0[\mu_R(R+T+\delta)]} K_1[\mu_R(R+T+\delta)] \right\} \end{vmatrix} = 0; \tag{1.94}$$

this is the criticality equation.

To summarize, the procedure in the case in which the composition is specified is to fix the length of the core (= reflector length) and then evaluate

$$\mu_c^2 = B_c^2 - \gamma_c^2 = \frac{3(k-1)}{\lambda_{ac}\lambda_{trc}} - \left[\frac{\pi}{2\left(\frac{H}{2} + 0.71\,\lambda_{tr}\right)}\right]^2 \tag{1.95}$$

$$\mu_R^2 = B_R^2 + \gamma_R^2 = \frac{3}{\lambda_{aR}\lambda_{trR}} + \left[\frac{\pi}{2\left(\frac{H}{2} + 0.71\,\lambda_{tr}\right)}\right]^2 . \tag{1.96}$$

By using these values in the elements of the determinant, one can specify R (the core radius) and solve for T (the reflector thickness) by trial and error or specify T and solve for R by trial and error. It is also possible, of course, to specify R, T, and $H/2$ (fix the size and shape completely) and solve for the critical uranium concentration (and hence critical mass) by trying various concentrations until the corresponding μ_c^2, as given by Eq. 1.95, is found that satisfies Eq. 1.94.

Once criticality has been established, the expressions for the flux are given by Eqs. 1.83 and 1.87. In Eq. 1.87, A_R is replaced by its values in terms of A_c from Eq. 1.92 (or Eq. 1.93). This determines ϕ to within the multiplicative constant A_c, which depends on the power level at which the reactor is operating.

An additional remark is in order concerning the solution for the critical mass of a square cylinder or cubical reactor that is completely surrounded by a reflector. Neither of these important cases is analytically soluble. An approximation can be made, however, based on two facts: (1) a critical, bare, square cylinder of the same composition as a bare sphere has about 15% more volume and uranium mass than the latter; (2) it has been determined experimentally that the square cylinder surrounded by a very thick reflector has about 6% more volume and uranium mass than the sphere of same composition with a thick reflector. Thus with the usual reflector a figure of 10% difference between cylinders (or cubes) and spheres is reasonable. Usually the size of the reactor is specified by conditions of heat transfer and shielding. The procedure is then to reduce the volume by 10% and solve for criticality with the reflected sphere having this reduced volume. This critical sphere will have approximately the same uranium concentration as the reflected cylinder (or cube) would have. The critical mass of the actual reactor is then 10% greater than that of this critical sphere.

PROBLEMS

1. Consider a 0.1-in.-dia spherical ball of uranium surrounded by a 0.03-in.-thick

layer of copper. If neutron bombardment is causing a heat release of 40 cal/sec, what is the drop in temperature through the copper such that this heat is carried away?

2. A solid spherical reactor with graphite core and graphite reflector has a core flux, calculated on a one-group model, of

$$\phi(r) = 10^{13} \frac{\sin \frac{\pi r}{80}}{r} .$$

The radius of the core is 60 cm; the σ_{tr} used was 4 barns; the density of the graphite is 1.7 g/cm^3. Find the total net number of neutrons escaping from the entire core per second.

3. A bare, spherical reactor is considered that is composed of beryllium, which has a microscopic cross section of 4 barns at thermal energy, and a low concentration of uranium. The critical diameter is 3 feet. Find a rough answer for the leakage of thermal neutrons per second from the surface of the reactor. The properties of the asymptotic flux at the surface are

$$\phi_{th} = 10^{13} ,$$

$$\frac{d\phi_{th}}{dr} = -0.2 \times 10^{13} .$$

4. Derive the formula for the radius of the critical, bare, spherical reactor.

5. For a given composition, what is the ratio of extrapolated radius to length for the critical cylinder of least critical mass?

6. What is the critical mass of a square cylinder of pure U^{235}? Take $\sigma_{au} = \sigma_f = 1.34$ barns, $\sigma_s = 6.4$ barns, $\rho = 18.6$ g/cm^3. For these cross sections it is assumed that the neutrons remain at fission energy. (Note: The one-group model is roughly applicable here, but diffusion theory is not very accurate for small assemblies. The answer is roughly too large by a factor of 2.)

7. Derive the determinant condition for criticality in a reflected sphere of given composition.

Chapter 2
CRITICALITY BY FERMI AGE THEORY

In chapter 1, a method was developed for getting the critical size or critical mass of a fissionable material assembly by making the simplifying assumption that all neutrons were at the same energy. This was an admittedly crude model, since neutron energies are actually spread from about 10^7 ev to about 0.025 ev (for a room-temperature reactor). Since cross sections vary widely with energy and since the energy distribution of the neutrons is not generally known, it is difficult to choose good average values for the constants. In this chapter, a method is derived for calculating critical masses (or sizes) of *unreflected reactors* that accounts for the fact that neutrons exist in a reactor at all energies from fission to thermal.

The formulas take an especially simple form when most of the fissioning is caused by thermal neutrons (thermal reactors); this case is discussed first.

THERMAL REACTORS WITH NO APPRECIABLE ABOVE-THERMAL ABSORPTION

Reactors are considered for which the simplifying assumption can be made that there is no above-thermal absorption, that is, neither nonproductive capture in any reactor material nor absorption to produce fission. The plan of the derivation of the criticality formula for thermal reactors is the following:

1. Derive the formula for ξ, the average loss in the logarithm of the neutron energy per collision.
2. Establish the relationship between the slowing-down density, $q(E)$, the number of neutrons slowing down past energy E per second per cubic centimeter, and the flux per unit energy. This involves ξ.
3. Derive the differential equation (called the age equation) for $q(E)$, which involves the quantity defined as the age, $\tau(E)$.
4. Solve the age equation for q_{th}, the number of neutrons becoming thermal per second per cubic centimeter.
5. Set up the balance equation for thermal neutrons. The source term is q_{th}.
6. Apply the boundary conditions to obtain the criticality formula expressed in terms of the age-to-thermal, τ_{th}, and the thermal diffusion "area," L^2.

Following the derivation, a discussion of the physical significance of τ_{th} and L^2 is given.

Formula for ξ. It will be shown in this section that the average loss in log energy per collision, ξ, is a constant (independent of the incident energy of the neutron) that can be expressed simply in terms of the atomic mass number of the nucleus with which the neutron collides. The plan of the derivation is:

1. Denote by v_0, the velocity with which the neutron is approaching the nucleus in the laboratory system of coordinates. The nucleus is considered to be stationary.
2. Find expressions for the velocity with which the neutron and nucleus approach the center of mass in terms of v_0.
3. Prove that the receding velocities after collision in the center-of-mass system are the same as the incident velocities of (2).
4. Find, in terms of v_0, the velocity of the neutron after collision in the laboratory system by adding vectorially the velocity of the neutron in the center-of-mass system and the velocity of the center of mass. The velocity after collision is found to be a function of the scattering angle in the center-of-mass system, θ.
5. Find the ratio of the energy of the neutron after collision to its energy before collision, from (1) and

(4). The v_0 cancels out of this expression.

6. Find $P(\theta)d\theta$, the probability that the neutron will scatter at an angle between θ and $\theta + d\theta$, in the center-of-mass system.

7. Find $P(E_0,E)$, the probability that a neutron of energy E_0 before collision will have energy in the unit energy range around E after collision. Since E is a function of θ, from (5), it is possible to write immediately that $P(E_0,E)dE = P(\theta)d\theta$, and the expression for $P(E_0,E)$ follows.

8. Find ξ by integrating $P(E_0,E)$ $(\ln E_0 - \ln E)$ over all energies below E_0.

Incident Neutron Energy, Laboratory System. The definition is made that v_0 is the velocity with which the neutron is approaching the nucleus in the laboratory system of coordinates. The velocity of the nucleus is assumed to be negligible, compared with that of the neutron.

Incident Velocities, Center-of-Mass System. In the *laboratory system,* the picture is as shown in Fig. 2.1. The neutron of atomic mass number 1 and the center of mass are moving to the right toward the nucleus of atomic mass number A.

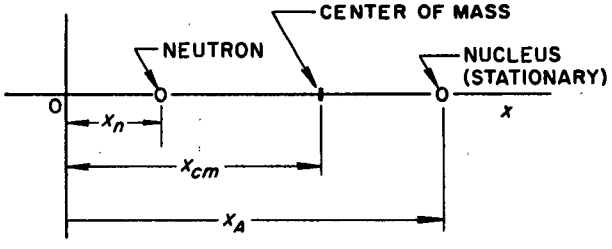


Fig. 2.1

From the definition of the center of mass it follows that

$$(A + 1)\, x_{cm} = A\, x_A + 1\, x_n$$

or

$$x_{cm} = \frac{A}{A + 1}\, x_A + \frac{1}{A + 1}\, x_n \ .$$

By taking the derivative with respect to time to find the velocity of the center of mass, it is found that

$$\frac{dx_{cm}}{dt} = \frac{1}{A + 1}\frac{dx_n}{dt} = \frac{1}{A + 1}\, v_0 \ ,$$

since the nucleus is stationary, which makes $dx_A/dt = 0$.

On the other hand, in the center-of-mass system the picture is as shown in Fig. 2.2.

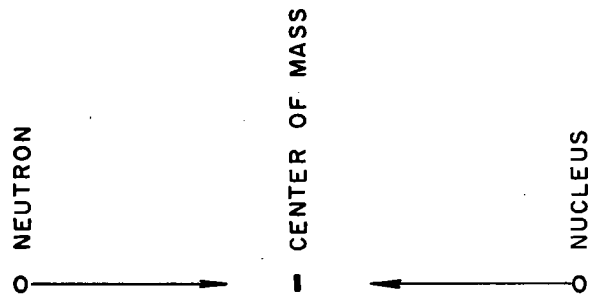


Fig. 2.2

The velocity with which the nucleus moves toward the stationary center of mass in this system is the same as the velocity with which the moving center of mass moves toward the nucleus in the laboratory system. Thus,

$$\text{Incident velocity of nucleus} = -\frac{1}{A + 1}\, v_0$$

$$\text{Incident velocity of neutron} = v_0 - \frac{1}{A + 1}\, v_0 = \frac{A}{A + 1}\, v_0 \ .$$

Receding Velocities. The proof that in the center-of-mass system the velocities after collision are the same as the velocities before collision follows from the principle of conservation of energy and the principle of momentum.

The principle of momentum states that the rate of change of momentum is equal to the force, or, as a special case, if there are no external forces, the momentum of a system remains unchanged. The momentum of the system

consisting of the neutron and nucleus is, before collision,

$$-\frac{A}{A+1}v_0 + \frac{A}{A+1}v_0 = 0 ,$$

and since there are no external forces, the momentum after collision must likewise be zero.

By defining v_n as the velocity of the neutron after collision in the center-of-mass system and v_a as the velocity of the nucleus after collision in the center-of-mass system, it is found that

$$1\, v_n + A\, v_a = 0 . \qquad (2.1)$$

From the principle of conservation of energy,

$$\frac{1}{2}\cdot 1 \cdot v_n^2 + \frac{1}{2}A\, v_A^2$$

$$= \frac{1}{2}\cdot 1\, \frac{A^2}{(A+1)^2}\, v_0^2$$

$$+ \frac{1}{2}A\, \frac{1}{(A+1)^2}\, v_0^2 . \qquad (2.2)$$

Upon solving for v_n and v_a in Eqs. 2.1 and 2.2, it is found that

$$v_n = -\frac{A}{A+1}\, v_0$$

and

$$v_a = \frac{v_0}{A+1} ,$$

which in absolute value are the same as the incident velocities.

From the principle of angular momentum and the fact that there is no torque, it is seen that the angular momentum must be zero after collision since it is zero before collision; that is, the neutron and nucleus move in exactly opposite directions after collision in the center-of-mass system. Thus the picture is as shown in Fig. 2.3.

Velocity of Neutron After Collision, Laboratory System. The definitions are made that θ is the scattering angle of the neutron in the center-of-mass system and ψ is the scattering angle

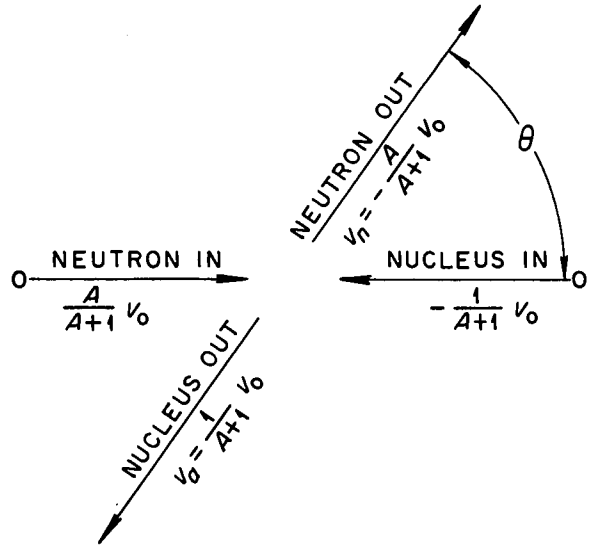


Fig. 2.3

of the neutron in the laboratory system.

The velocity of the neutron after collision in the laboratory system, v, is found by adding vectorially the velocity of the neutron relative to the center of mass and the velocity of the center of mass, as shown in Fig. 2.4. It follows from the law of cosines that

$$v^2 = \frac{v_0^2}{(A+1)^2} + \frac{A^2}{(A+1)^2}\, v_0^2$$

$$+ \frac{2v_0^2}{(A+1)^2}\, A \cos\theta . \qquad (2.3)$$

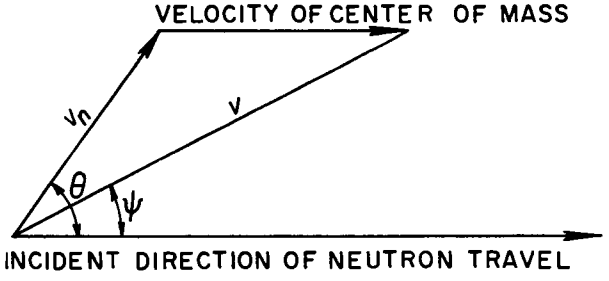


Fig. 2.4

Ratio of Energy After Collision To Energy Before Collision. It follows

from Eq. 2.3 that the ratio of the energy of the neutron after collision to the energy before collision is

$$\frac{E}{E_0} = \frac{A^2 + 1}{(A + 1)^2} + \frac{2A}{(A + 1)^2} \cos \theta \ . \quad (2.4)$$

The definition is made that

$$\alpha \equiv \left(\frac{A - 1}{A + 1}\right)^2 ,$$

and Eq. 2.4 becomes

$$\frac{E}{E_0} = \frac{1}{2}\left[(1 + \alpha) + (1 - \alpha) \cos \theta\right] . \quad (2.5)$$

Thus the more the neutron is deflected from its original direction, the more the energy loss. If $\theta = 0$, Eq. 2.5 gives $E/E_0 = 1$; that is, there is no energy loss. If $\theta = 180$ deg, the neutron suffers the maximum energy loss, that is,

$$\frac{E_{min}}{E_0} = \alpha = \left(\frac{A - 1}{A + 1}\right)^2 . \quad (2.6)$$

Equation 2.6 is an important formula that is worth remembering.

For neutron collision with a hydrogen atom, Eq. 2.6 says the neutron can lose all its energy; for beryllium collisions, it can lose a maximum of

$$1 - \left(\frac{9 - 1}{9 + 1}\right)^2 = 1 - 0.64 = 36\%$$

of its energy; for graphite, 28% of its energy.

Expression for $P(\theta)\ d\theta$. To get an expression for $P(\theta)\ d\theta$, the probability that a neutron will scatter at an angle in the center-of-mass system between θ and $\theta + d\theta$, the assumption is made that scattering is isotropic (spherically symmetric) in the center-of-mass system (a very good assumption). Therefore $P(\theta)\ d\theta$ is simply the fractional area of the incremental band between θ and $\theta + d\theta$ on the sphere drawn around the nucleus, as shown in Fig. 2.5. The area of the band is

$$2\pi r \sin \theta \ r \ d\theta \ .$$

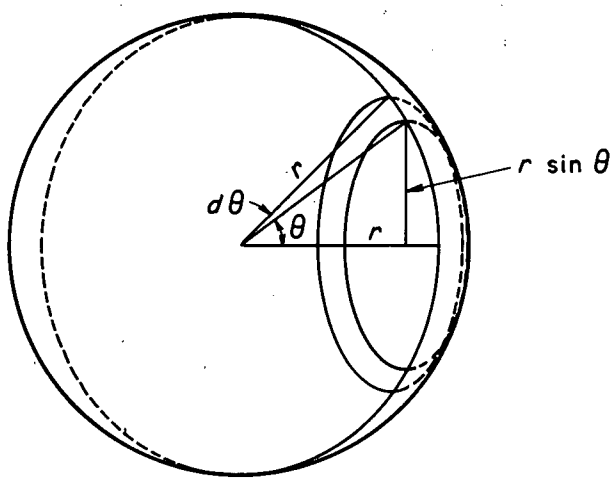


Fig. 2.5

Then the fractional area of the incremental band between θ and $\theta + d\theta$ is

$$P(\theta)\ d\theta = \frac{2\pi r^2 \sin \theta \ d\theta}{4\pi r^2} = \frac{1}{2} \sin \theta \ d\theta \ . \quad (2.7)$$

Expression for $P(E_0,E)$. From Eqs. 2.7 and 2.5, an expression can be obtained for $P(E_0,E)$, the probability that a neutron of incident energy E_0 will have energy in the unit energy band around E after collision.

Since $P(E_0,E)$ and $P(\theta)$ are both distribution functions, they are of the nature of derivatives, dn/dE and $dn/d\theta$, where dn would be the fraction of neutrons scattering into dE and $d\theta$, respectively. It can then be written immediately that

$$-P(E_0,E)\ dE = P(\theta)\ d\theta \ ,$$

where E and θ are related by Eq. 2.5. The negative sign is used because E decreases as θ increases.

It follows that

$$P(E_0,E) = P(\theta) \left(-\frac{d\theta}{dE}\right) ,$$

and from Eq. 2.5

$$\frac{d\theta}{dE} = -\frac{2}{E_0\ (1 - \alpha) \sin \theta} \ . \quad (2.8)$$

From Eqs. 2.7 and 2.8,

$$P(E_0,E) = \frac{1}{E_0\ (1 - \alpha)} \ . \qquad (2.9)$$

It is important to note that $P(E_0,E)$ is independent of E. From Eq. 2.6, it is clear that the denominator of Eq. 2.9, $E_0 - \alpha E_0$, is the range of possible energy loss (see Fig. 2.6), and, since

$$P(E_0,E)\ dE = \frac{dE}{E_0 - \alpha E_0} \ ,$$

the fraction of neutrons degraded into dE is independent of E, it follows that the *neutrons are spread uniformly in energy between* E_0 *and* αE_0. The fraction of neutrons being degraded into any subinterval of energy in E_0 to αE_0 is just the width of the subinterval divided by the total interval of possible energy loss, $E_0 - \alpha E_0$, regardless of where the subinterval is located in the total interval. For hydrogen nuclei, $\alpha = 0$, and it follows that neutrons are spread uniformly from the incident energy E_0 to zero energy. Hence, on the average, a neutron loses one-half its energy upon a collision with a hydrogen atom.

Formula for ξ. The formula for ξ, the average loss in log energy, follows easily from Eqs. 2.9 and 2.5.

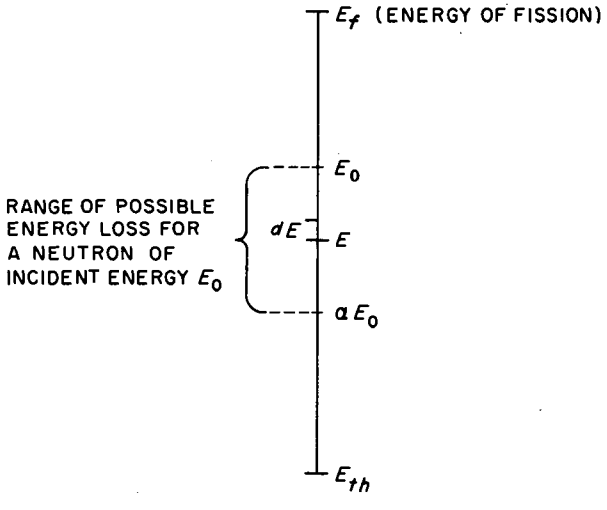


Fig. 2.6

$$\xi = \frac{\displaystyle\int_{\alpha E_0}^{E} (\ln E_0 - \ln E)\ P(E_0,E)\ dE}{\displaystyle\int_{\alpha E_0}^{E_0} P(E_0,E)\ dE}$$

$$= \int_{\alpha E_0}^{E_0} (\ln E_0 - \ln E) \cdot \frac{dE}{E_0\ (1 - \alpha)} \ .$$

If the integration is carried out, this becomes

$$\xi = 1 + \frac{\alpha}{1 - \alpha} \ln \alpha$$

$$= 1 + \frac{(A - 1)^2}{2A} \ln \left(\frac{A - 1}{A + 1}\right) , \qquad (2.10)$$

which was the formula to be derived.

When ξ is evaluated for some common moderating elements, it is found that

$$\xi_H = \lim_{\alpha \to 0} \left(1 + \frac{\alpha}{1 - \alpha} \ln \alpha\right)$$

$$= + \lim_{\alpha \to 0} (\alpha \ln \alpha)$$

$$= 1 + \lim_{\alpha \to 0} \frac{\frac{1}{\alpha}}{-\frac{1}{\alpha^2}}$$

$$= 1 + \lim_{\alpha \to 0} (-\alpha) = 1,$$

for hydrogen;

$$\xi_{Be} = 1 + \frac{8^2}{16} \ln \frac{8}{10} = 0.206,$$

for beryllium;

$$\xi_C = 1 + \frac{11^2}{24} \ln \frac{11}{13} = 0.158,$$

for carbon.

It is to be emphasized that ξ is defined as the average of the log and

not the log of the average. Students sometimes erroneously conclude that, since $\xi_H = 1$, then the ratio of neutron energy after collision with hydrogen to neutron energy before collision must, on the average, be $1/e$. As previously stated, the neutron loses, on the average, one-half its energy upon hydrogen collision.

Relationship of Flux and Slowing-Down Density. The slowing-down density, $q(E,\underline{r})$, is defined as the number of neutrons slowing down past energy E per cubic centimeter at the position point $\underline{r}$. If q can be determined, that is, if $q(E_{th},\underline{r})$ in particular can be determined, the source term for the thermal-neutron balance equation is then known.

A differential equation can be written for $q(E,\underline{r})$, but it involves $\phi(E,\underline{r})$ the flux per unit energy at $\underline{r}$. (This will be shown in the next section.) An auxiliary relationship between $q(E,\underline{r})$ and $\phi(E,\underline{r})$ is therefore needed.

First, an integral equation in $\phi(E,\underline{r})$ is established by writing that the number of neutrons being degraded into the energy interval dE per cubic centimeter at $\underline{r}$ by scattering at higher energies (the possible energies are E to E/α, as can be seen from Eq. 2.6) is equal to the number being lost from dE (by degradation through a scattering collision or by spatial leakage), that is,

$$\phi(E,\underline{r})\,\Sigma_s\,dE - \frac{\lambda_{tr}}{3}\nabla^2\,\phi(E,\underline{r})\,dE = \int_E^{E/\alpha}\frac{\phi(E',\underline{r})\,\Sigma_s\,(E')\,dE'\,dE}{E'-\alpha E'}\,. \tag{2.11}$$

It is to be recalled that it was assumed that there was no absorption above thermal energies. The net leakage term on the left is small compared with the scattering term, and therefore Eq. 2.11 can be written, approximately, as

$$\phi(E,\underline{r})\,\Sigma_s = \int_E^{E/\alpha}\frac{\phi(E,\underline{r})\,\Sigma_s\,(E')\,dE'}{E'-\alpha E'}\,, \tag{2.12}$$

after dividing out the dE from both sides.

This integral equation is satisfied by

$$\phi(E,\underline{r})\,\Sigma_s = \frac{C}{E}\,, \tag{2.13}$$

as can be verified by substitution, where C is an arbitrary constant.

On the other hand, an expression for the slowing-down density can be written as

$$q(E,\underline{r}) = \int_E^{E/\alpha}\phi(E',\underline{r})\,\Sigma_s\,(E')\,\frac{E-\alpha E'}{E'-\alpha E'}\,dE'\,, \tag{2.14}$$

since $q(E,\underline{r})$ is made up of neutrons making collisions at $\underline{r}$ at energies above E that are degraded below E. The integration limits of Eq. 2.14 are E to E/α because it is seen, from the formulas for maximum energy loss, that only neutrons of energy E to E/α can be degraded below E. Since neutrons making collisions in dE' are evenly distributed from E' to $\alpha E'$ (see Fig. 2.7), it follows that

$$\frac{E-\alpha E'}{E'-\alpha E'}$$

is the fraction landing below E; this fraction is the integrand of Eq. 2.14.

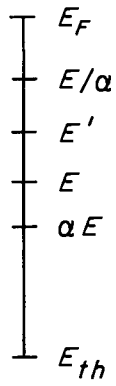


Fig. 2.7

By substituting Eq. 2.13 in Eq. 2.14 it is found that

$$q(E,\underline{r}) = C\xi \ . \qquad (2.15)$$

Then by eliminating C from Eqs. 2.13 and 2.15, the desired relationship follows:

$$\frac{q(E,\underline{r})}{\xi E \Sigma_s} = \phi(E,\underline{r}) \ . \qquad (2.16)$$

Differential Equation for $q(E,\underline{r})$. The differential equation for $q(E,\underline{r})$ is obtained by writing that the slowing-down density at $E + dE$ must equal the slowing-down density at E plus the net loss of neutrons from the cubic centimeter at $\underline{r}$ due to diffusion, that is,

$$q(E+dE,\underline{r}) = q(E,\underline{r}) - \frac{\lambda_{tr}}{3} \nabla^2 \phi(E,\underline{r}) \, dE \ . \qquad (2.17)$$

When the left side is expanded in two terms of a Taylor expansion about E, Eq. 2.17 reduces to

$$-\frac{\lambda_{tr}}{3} \nabla^2 \phi(E,\underline{r}) = \frac{\partial q(E,\underline{r})}{\partial E} \ . \qquad (2.18)$$

By substituting Eq. 2.16, Eq. 2.18 becomes

$$-\frac{\lambda_{tr}}{3} \frac{1}{\xi E \Sigma_s} \nabla^2 q(E,\underline{r}) = \frac{\partial q(E,\underline{r})}{\partial E} \ . \qquad (2.19)$$

The fundamental constant, called the "age", is defined as

$$\tau(E) \equiv \int_E^{E_F} \frac{\lambda_s \lambda_{tr}}{3\xi} \frac{dE}{E} \ , \qquad (2.20)$$

where E_F is the fission energy (assumed to be monoenergetic, as an approximation). Definition 2.20 will be discussed further in later sections. Upon differentiating Eq. 2.20, it is seen that

$$\frac{d\tau}{dE} = -\frac{\lambda_s \lambda_{tr}}{3\xi E} \ ,$$

and by using this in Eq. 2.19, the latter becomes

$$\frac{d\tau}{dE} \nabla^2 q(E,\underline{r}) = \frac{\partial q(E,\underline{r})}{\partial E} \ , \qquad (2.21)$$

or

$$\nabla^2 q(\tau,\underline{r}) - \frac{\partial q(\tau,\underline{r})}{\partial \tau} = 0 \ . \qquad (2.22)$$

Equation 2.22 is called the "age equation."

Solution for q_{th}. To solve Eq. 2.22, an attempt is made to separate the variables by writing

$$q(\tau,\underline{r}) = R(\underline{r}) \, T(\tau) \ , \qquad (2.23)$$

and substituting this into Eq. 2.22, which then becomes

$$\frac{\nabla^2 R(\underline{r})}{R(\underline{r})} - \frac{\frac{dT(\tau)}{d\tau}}{T(\tau)} = 0 \ . \qquad (2.24)$$

Solution for Energy-Dependent Component. Since the first term involves only $\underline{r}$ and the second term involves only τ, both terms must be equal to a constant, say β^2. Thus

$$\frac{dT(\tau)}{d\tau} + \beta^2 T(\tau) = 0 \ , \qquad (2.25)$$

the solution of which is

$$T(\tau) = A \, e^{-\beta^2 \tau} \ . \qquad (2.26)$$

From Eqs. 2.23 and 2.26 it follows that

$$q(\tau,\underline{r}) = R(\underline{r}) \, A \, e^{-\beta^2 \tau} \ . \qquad (2.27)$$

Evaluation of Spatial Component in Terms of ϕ_{th}. From definition 2.20 it is clear that the age of virgin neutrons is zero, and, from Eq. 2.27, the slowing-down density at fission energy (the assumption here of monoenergetic fission neutrons causes no great error in thermal reactors) is

$$q(0,\underline{r}) = R(\underline{r}) \, A \ . \qquad (2.28)$$

On the other hand, the slowing down density at fission energy is just equal to the number of neutrons being produced by fission, which is

$$\phi_{th}(\underline{r}) \, \Sigma_a \, k_{th} \ ,$$

where $\phi_{th}(\underline{r})$ is the thermal flux at $\underline{r}$ and k_{th} is the average number of

fission neutrons produced per absorption of a thermal neutron, that is, $2.5\ \Sigma_f/\Sigma_a$, for U^{235}. Hence

$$q(0,\underline{r}) = R(\underline{r})\ A = \phi_{th}(\underline{r})\ \Sigma_a\ k_{th}\ . \tag{2.29}$$

Equation 2.27 now becomes

$$q(\tau,\underline{r}) = \phi_{th}(\underline{r})\ \Sigma_a\ k_{th}\ e^{-\beta^2\tau}\ , \tag{2.30}$$

and the source term for the thermal group is

$$q(\tau_{th},\underline{r}) = \phi_{th}(\underline{r})\ \Sigma_a\ k_{th}\ e^{-\beta^2\tau_{th}}. \tag{2.31}$$

Balance Equation for Thermal Neutrons. The balance equation for the absorption, leakage, and production of thermal neutrons can be written in the manner described in chapter 1;

$$-\frac{\lambda_{tr}}{3}\nabla^2\ \phi_{th}(\underline{r}) + \phi_{th}(\underline{r})\ \Sigma_a - \phi_{th}(\underline{r})\ \Sigma_a\ k_{th}\ e^{-\beta^2\tau_{th}} = 0\ , \tag{2.32}$$

which can be written as

$$\nabla^2\ \phi_{th}(\underline{r}) + B^2\ \phi_{th}(\underline{r}) = 0\ , \tag{2.33}$$

where

$$B^2 = \frac{1}{L^2}\left(k_{th}\ e^{-\beta^2\tau_{th}} - 1\right) \tag{2.34}$$

in which

$$L^2 = \frac{\lambda_{tr}}{3\Sigma_a}\ . \tag{2.35}$$

The constant L^2 is called the thermal diffusion area. More will be said about this fundamental constant later.

The constants B^2 and β^2 of Eq. 2.34 are actually the same constant, as can be seen from the following. From Eq. 2.29

$$R(\underline{r}) = \frac{\Sigma_a\ k_{th}}{A}\phi_{th}(\underline{r})\ , \tag{2.36}$$

but from Eq. 2.24 it follows that

$$\frac{\nabla^2\ R(\underline{r})}{R(\underline{r})} = -\beta^2\ . \tag{2.37}$$

By substituting Eq. 2.36 in Eq. 2.37, the following expression is obtained:

$$\frac{\nabla^2\ \phi_{th}(\underline{r})}{\phi_{th}(\underline{r})} = -\beta^2\ .$$

Hence from this equation and Eq. 2.33, it follows that

$$\beta^2 = B^2\ .$$

Therefore Eq. 2.34 becomes

$$B^2 = \frac{1}{L^2}\left(k_{th}\ e^{-B^2\tau_{th}} - 1\right)$$

or

$$\frac{k_{th}}{1 + L^2\ B^2}\ e^{-B^2\tau_{th}} = 1\ . \tag{2.38}$$

This equation is the thermal reactor criticality equation.

Flux Distribution. Since Eq. 2.33 for ϕ_{th} is a differential equation of the same form as was obtained for ϕ in the one-group bare-reactor treatment, it follows that the solutions for ϕ_{th} are the same in form as those for ϕ. Thus, for slabs

$$\phi_{th}(x) = A\ \cos Bx\ ;$$

for spheres

$$\phi_{th}(r) = \frac{A\ \sin Br}{r}\ ;$$

for cylinders

$$\phi_{th}(r,z) = A\ J_0(\mu r)\ \cos \gamma z\ ,$$

where $\mu^2 + \gamma^2 = B^2$. The constant A in all cases depends on the power level.

Since $q(\underline{r},\tau)$ and $\phi_{th}(\underline{r})$ are spatially proportional, as can be seen from Eq. 2.30, it follows that $q(\underline{r},\tau)$ has the same spatial distribution as ϕ_{th}. Therefore for slabs,

$$q(x,\tau) = T(\tau)\ \cos Bx\ ;$$

for spheres,

$$q(r,\tau) = T(\tau)\ \frac{\sin Br}{r}\ ;$$

for cylinders,

$$q(r,z,\tau) = T(\tau)\ J_0(\mu r)\ \cos \gamma z\ ,$$

where $T(\tau)$ is a function of the age and the power level.

From Eq. 2.16 it then follows that the flux per unit energy above thermal, $\phi(E,\underline{r})$, for all E, likewise has a spatial distribution, such as given

above for the thermal flux; that is, for slabs,

$$\phi(E,x) = V(E) \cos Bx \ ;$$

for spheres,

$$\phi(E,r) = V(E) \frac{\sin Br}{r} \ ;$$

for cylinders,

$$\phi(E,r,z) = V(E) \ J_0(\mu r) \cos \gamma z \ ,$$

where $V(E)$ is a function of the energy of the flux in question and is, also, a function of the power level at which the reactor is operating. In each case, B is given by Eq. 2.38.

Critical Size. The boundary conditions are the same as for the bare-reactor one-group model. By the arguments of chapter 1, the first root of the solution of Eq. 2.33 (for each geometry) determines the critical size. Thus, for slabs,

$$x_0 = \frac{\pi}{2B} - \delta \ , \qquad (2.39)$$

where x_0 is critical half thickness, δ is extrapolation distance, that is, $0.71 \ \lambda_{tr}$;

for spheres,

$$R = \frac{\pi}{B} - \delta \ , \qquad (2.40)$$

where R is the critical radius;

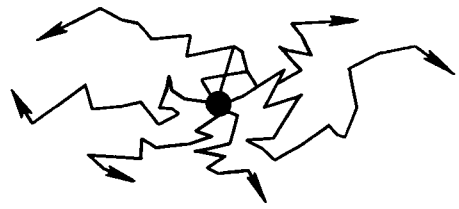
for cylinders,

$$B^2 = \left(\frac{2.405}{R + \delta}\right)^2 + \left[\frac{\pi}{2\left(\frac{H}{2} + \delta\right)}\right]^2 \ , \qquad (2.41)$$

where R is the critical radius and H is the critical height. In each case, B is given by Eq. 2.38.

Physical Interpretation of L^2. The quantity $\lambda_a \ \lambda_{tr}/3$ that was denoted by L^2 in the thermal criticality equation has a simple physical interpretation in terms of the average of the square of the crow-flight distances that thermal neutrons diffuse before capture. To find this average, the distribution of the thermal capture density (or the flux, which is proportional to it) due to a point source is needed. The neutron-balance equation affords the required differential equation for the flux.

Consider a point source in an infinite, three-dimensional space with neutrons diffusing away from it. The



neutron-balance equation for points away from the source is that for a nonproducing medium. The spherical coordinate system is chosen because the source is easily located at $r = 0$. The equation is, then

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - \frac{1}{L^2}\phi = 0 \ . \qquad (2.42)$$

This is the same equation as was previously obtained for the reflector of a spherical reactor. The general solution was

$$\phi(r) = A_1 \frac{\sinh \frac{r}{L}}{r} + C_1 \frac{\cosh \frac{r}{L}}{r} \ . \qquad (2.43)$$

This can be written in a more convenient form for the present case as

$$\phi(r) = A \frac{e^{-r/L}}{r} + C \frac{e^{r/L}}{r} \ . \qquad (2.44)$$

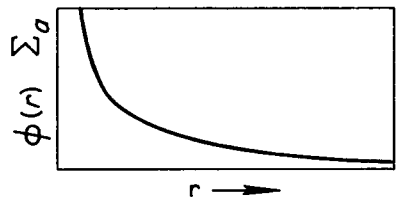
It is clear that $C = 0$, for, otherwise, the flux would increase without bound as r increases. Thus Eq. 2.44 becomes

$$\phi(r) = A \frac{e^{-r/L}}{r} \ . \qquad (2.45)$$

The number of neutron absorptions per cubic centimeter per second at r is then given by

$$\phi(r) \ \Sigma_a = A \ \Sigma_a \frac{e^{-r/L}}{r} \ , \qquad (2.46)$$

which has the appearance



However, the quantity of more interest is the relative number of *captures per unit r at different radii,* that is, Eq. 2.46 weighted by $4\pi r^2$:

$$f(r) = 4\pi r^2 \, \phi(r) \, \Sigma_a = 4\pi A r^2 \, \Sigma_a \, \frac{e^{-r/L}}{r} \, . \tag{2.47}$$

The physical interpretation of L^2 is obtained by finding $\overline{r^2}$, the average of the square of the crow-flight distance a neutron diffuses before capture:

$$\overline{r^2} = \frac{\int_0^\infty r^2 \, f(r) \, dr}{\int_0^\infty f(r) \, dr}$$

$$= \frac{\int_0^\infty r^2 \, 4\pi r^2 A \, \frac{e^{-r/L}}{r} \, \Sigma_a \, dr}{\int_0^\infty 4\pi r^2 A \, \frac{e^{-r/L}}{r} \, \Sigma_a \, dr} = 6L^2 \tag{2.48}$$

and thus

$$L^2 = \frac{\overline{r^2}}{6} \, , \tag{2.49}$$

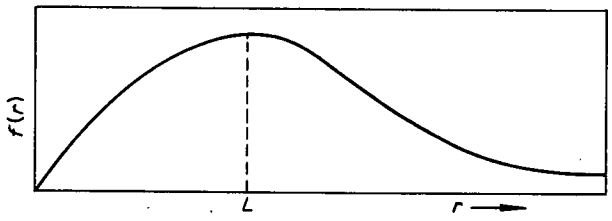
or L^2 is one-sixth the average of the square of the crow-flight distances neutrons travel from the point they become thermal to the point at which they are captured.

Students often ask why not simply determine $\overline{r}$ from

$$\overline{r} = \frac{\int_0^\infty r \, f(r) \, dr}{\int_0^\infty f(r) \, dr} = 2L \, .$$

This does indeed provide further understanding of L, but since the quantity appears in the criticality formula in the square form, L^2, the interpretation Eq. 2.49 is the one usually given.

It is well to bear in mind the appearance of the distribution of Eq. 2.47.



The *most probable* crow-flight diffusion distance before capture is $r = L$.

Physical Interpretation of τ_{th}. The physical interpretation of the age from fission energy to thermal energy, τ_{th}, is quite similar to that for L^2. It is found that τ_{th} is one-sixth the average of the square of the crow-flight distances neutrons diffuse from the points of birth to the points at which they become thermal neutrons. (The quantity τ is in units of area and, perhaps unfortunately, it is not denoted by a squared symbol such as that for the thermal diffusion "area," L^2. Some early authors did use the notation L_s^2, but the symbol τ_{th} seems to have been more widely adopted.)

It is required to find the distribution of neutrons becoming thermal from a point source of neutrons at fission energy. The differential equation for q as a function of τ, previously derived, is

$$\nabla^2 \, q(\underline{r},\tau) = \frac{\partial q(r,\tau)}{\partial \tau} \tag{2.50}$$

or, in spherical coordinates,

$$\frac{\partial^2 q(r,\tau)}{\partial r^2} + \frac{2}{r} \frac{\partial q(r,\tau)}{\partial r} = \frac{\partial q(r,\tau)}{\partial \tau} \, . \tag{2.51}$$

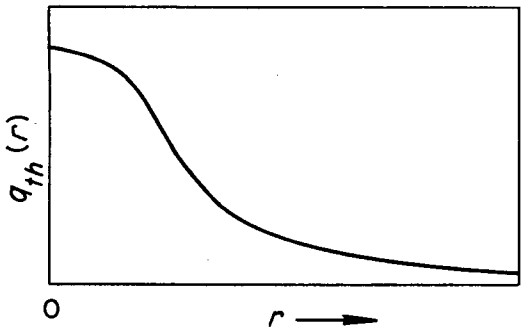
The solution to this, which can be verified by substitution, is

$$q(r,\tau) = A \, \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}} \, . \tag{2.52}$$

For neutrons of age $\tau = \tau_{th}$, Eq. 2.52 becomes

$$q_{th}(r) = A \, \frac{e^{-r^2/4\tau_{th}}}{(4\pi\tau_{th})^{3/2}} \, , \tag{2.53}$$

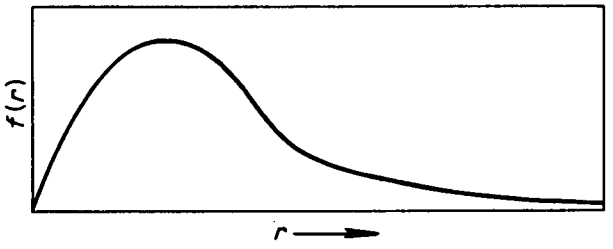
which has the appearance



The distribution with respect to r (the fraction of neutrons that became thermal in the shell between r and $r + dr$) is really Eq. 2.53 weighted by $4\pi r^2$, that is,

$$f(r) = 4\pi r^2 \, q_{th}(r) = 4\pi r^2 A \frac{e^{-r^2/4\tau_{th}}}{(4\pi\tau_{th})^{3/2}} , \tag{2.54}$$

which has the appearance



The average of the square of the crow-flight distance that neutrons travel from the source to the point at which they become thermal neutrons is

$$\overline{r^2} = \frac{\int_0^\infty r^2 \, f(r) \, dr}{\int_0^\infty f(r) \, dr}$$

$$= \frac{\int_0^\infty r^2 4\pi r^2 A \cdot \frac{e^{-r^2/4\tau_{th}}}{(4\pi\tau_{th})^{3/2}} \, dr}{\int_0^\infty 4\pi r^2 A \frac{e^{-r^2/4\tau_{th}}}{(4\pi\tau_{th})^{3/2}} \, dr} = 6\tau_{th} . \tag{2.55}$$

Thus

$$\tau_{th} = \frac{\overline{r^2}}{6} ,$$

which is the physical interpretation of τ_{th}.

Physical Interpretation of Components of Thermal-Reactor Criticality Equations. It is necessary to return to Eq. 2.38 for a moment and to note the physical interpretation of the component parts of the equation. The equation is

$$\frac{k_{th}}{1 + L^2B^2} e^{-B^2\tau_{th}} = 1 . \tag{2.38}$$

First, it will be recalled that k_{th} is the average number of neutrons produced per absorption of a neutron in any core material. The component

$$e^{-B^2\tau_{th}}$$

can be interpreted as the probability that a neutron will not be lost by leakage while slowing down from fission energy to thermal energy. This can be seen most easily from Eq. 2.31, which states that the slowing-down density at thermal energy is equal to the number of neutrons (with fission energy) born per second multiplied by the factor

$$e^{-B^2\tau_{th}} ,$$

that is,

$$q(\tau_{th}, \underline{r}) = \phi_{th}(\underline{r}) \, \Sigma_a \, k_{th} \, e^{-B^2\tau_{th}} . \tag{2.31}$$

It follows that

$$e^{-B^2\tau_{th}}$$

is the "leakage escape probability."

The factor $1/(1 + L^2B^2)$ can be interpreted as the probability that a neutron will not be lost by leakage after it has become a thermal neutron. Since a thermal neutron has only two possible fates - either it is lost through leakage or is absorbed - the factor $1/(1 + L^2B^2)$ can also be

interpreted as the probability that a thermal neutron will be absorbed. This interpretation is clear if the substitution that

$$\nabla^2\phi_{th} + B^2\phi_{th} = 0$$

or

$$B^2 = -\frac{\nabla^2\phi_{th}}{\phi_{th}}$$

is made, that is,

$$\frac{1}{1 + L^2B^2} = \frac{1}{1 + L^2\left(-\dfrac{\nabla^2\phi_{th}}{\phi_{th}}\right)} = \frac{1}{1 + \dfrac{\lambda_a\lambda_{tr}}{3}\left(-\dfrac{\nabla^2\phi_{th}}{\phi_{th}}\right)}$$

$$= \frac{\phi_{th}\Sigma_a}{\phi_{th}\Sigma_a + \left(-\dfrac{\lambda_{tr}}{3}\nabla^2\phi_{th}\right)} = \frac{\text{absorptions/cm}^3\cdot\text{sec}}{(\text{absorptions/cm}^3\cdot\text{sec}) + (\text{leakage/cm}^3\cdot\text{sec})} .$$

It follows that $1/(1+L^2B^2)$ represents the fraction of thermal neutrons being absorbed, or the probability that a thermal neutron will be absorbed, or the probability that a thermal neutron will not be lost by leakage.

When these facts are put together, the physical meaning of the criticality equation (Eq. 2.38) becomes clear. For each thermal absorption, k_{th} neutrons start out at fission energy. Of these neutrons,

$$k_{th}\, e^{-B^2\tau_{th}}$$

survive fast leakage to become thermal neutrons. Of the thermal neutrons

$$\frac{k_{th}\, e^{-B^2\tau_{th}}}{1 + L^2B^2}$$

survive total leakage and are absorbed. If this quantity is exactly one neutron, the neutron population remains constant – the reactor is critical.

The quantity

$$\frac{k_{th}\, e^{-B^2\tau_{th}}}{1 + L^2B^2}$$

is called the effective reproduction constant and is denoted by k_{eff}:

$$k_{eff} \equiv \frac{k_{th}\, e^{-B^2\tau_{th}}}{1 + L^2B^2} .$$

It is clear that if $k_{eff} > 1$, the reactor is supercritical; if $k_{eff} < 1$, it is subcritical; and if $k_{eff} = 1$, it is critical.

A further quantity in common use is the excess reproduction constant denoted by k_{ex}. It has the definition

$$k_{ex} = k_{eff} - 1 = \frac{k_{th}\, e^{-B^2\tau_{th}}}{1 + L^2B^2} - 1 .$$

Therefore there is a positive, zero, or negative k_{ex}, respectively, in the supercritical, critical, or subcritical reactor.

The term "reactivity" is used in connection with the above two equations. Several authors have defined the term differently in the literature, and considerable confusion has been associated with the word. It is often used to mean k_{eff}. However, it now appears that the most commonly accepted definition is

$$\text{Reactivity} \equiv \rho = \frac{k_{ex}}{k_{eff}} .$$

This is the definition that will be used in chapter 6, "Kinetics," where reactivity plays a prominent role.

It is worthwhile to note that the thermal Fermi age formalism degenerates to the one-group formalism if

$$e^{-B^2\tau_{th}} = 1 ,$$

that is, if there is no leakage of neutrons while slowing down. Thus for the bare reactor in the one-group model,

$$k_{eff} = \frac{k}{1 + L^2B^2}$$

$$k_{ex} = k_{eff} - 1 = \frac{k}{1 + L^2B^2} .$$

It will be recalled that in the one-group model, the B^2 was introduced by the definition

$$B^2 = \frac{3(k - 1)\,\Sigma_a}{\lambda_{tr}}$$

or

$$B^2 = \frac{k - 1}{L^2} ,$$

from which the bare-reactor one-group criticality equation follows:

$$\frac{k}{1 + L^2B^2} = 1 .$$

REACTORS WITH ABOVE-THERMAL NEUTRON CAPTURE BUT NO APPRECIABLE ABOVE-THERMAL FISSION

Relationship Between $\phi(E,\underline{r})$ and $q(E,\underline{r})$ in an Absorbing Medium. Earlier in this chapter, a relationship (Eq. 2.16) between the flux per unit energy and the slowing-down density, in a medium in which neutrons slow down without appreciable absorption was derived. This relationship was

$$\frac{q(E,\underline{r})}{\xi E\Sigma_s} = \phi(E,\underline{r}) . \qquad (2.16)$$

When appreciable absorption is present, it has been shown that more accuracy is obtained if Eq. 2.16 is replaced by

$$\frac{q(E,\underline{r})}{\xi E\Sigma_T} = \phi(E,\underline{r}) , \qquad (2.56)$$

where Σ_T is the total macroscopic cross section. Equation 2.56 is still only approximate, but it is an improvement over Eq. 2.16.

No attempt will be made here to really justify Eq. 2.56, but it can be made plausible by going back to the derivation of Eq. 2.16 and seeing what changes need to be made for the absorbing medium and finally showing that it holds for one particular case. The integral equation for the flux (Eq. 2.12) would become

$$\phi(E,\underline{r})\,\Sigma_T = \int_E^{E/\alpha} \frac{\phi(E',\underline{r})\,\Sigma_s(E')\,dE'}{E' - \alpha E'} , \qquad (2.57)$$

and the expression for the slowing-down density (Eq. 2.14) remains

$$q(E,\underline{r}) = \int_E^{E/\alpha} \phi(E',\underline{r})\,\Sigma_s(E')\,\frac{E - \alpha E'}{E' - \alpha E'}\,dE' . \qquad (2.14)$$

The procedure is, clearly, to solve for the flux in Eq. 2.57 and then evaluate the slowing-down density from the knowledge of $\phi(E,\underline{r})$ obtained from Eq. 2.14. However, Eq. 2.57 is not, in general, easy to solve. On the other hand, it is easily shown that Eq. 2.56 follows from Eqs. 2.57 and 2.14 for a hydrogenous medium with constant cross sections. This will be shown, and the demonstration will be taken as a plausibility argument that Eq. 2.56 is approximately correct for slowing down in other moderators, such as deuterium, beryllium, and carbon.

For hydrogen, Eq. 2.57 is

$$\phi(E,\underline{r})\ \Sigma_T(E) = \int_E^\infty \frac{\phi(E',\underline{r})\ \Sigma_s(E')\ dE'}{E'}\ , \tag{2.58}$$

and Eq. 2.14 is

$$q(E,\underline{r}) = E\int_E^\infty \frac{\phi(E,\underline{r})\ \Sigma_s(E,\underline{r})\ dE'}{E'}\ . \tag{2.59}$$

By differentiating Eq. 2.58, with the restriction that the cross sections are constant, the equation becomes

$$\Sigma_T\ \phi'(E,\underline{r}) = -\frac{\phi(E,\underline{r})\ \Sigma_s}{E}\ ,$$

where

$$\phi'(E,\underline{r}) \equiv \frac{\partial\phi(E,\underline{r})}{\partial E}\ .$$

Then

$$\frac{\phi'(E,\underline{r})}{\phi(E,\underline{r})} = -\frac{\Sigma_s}{\Sigma_T}\frac{1}{E}$$

$$\phi(E,\underline{r}) = CE^{-\Sigma_s/\Sigma_T}\ . \tag{2.60}$$

When Eq. 2.60 is substituted in Eq. 2.59,

$$q(E,\underline{r}) = E\int_E^\infty C\Sigma_s E'^{-(\Sigma_s/\Sigma_T)-1}\ dE'$$

$$= EC\Sigma_s E^{-\Sigma_s/\Sigma_T}\ \frac{\Sigma_T}{\Sigma_s}$$

$$= EC\Sigma_T E^{-\Sigma_s/\Sigma_T}\ . \tag{2.61}$$

When C is eliminated from Eqs. 2.60 and 2.61, it follows that

$$\phi(E,\underline{r}) = \frac{q}{E\Sigma_T}$$

or

$$\phi(E,\underline{r}) = \frac{q}{E\xi\Sigma_T}\ , \tag{2.56}$$

since $\xi = 1$ for hydrogen. Equation 2.56 is taken as plausible when the moderator is not hydrogen.

Differential Equation for $q(E,\underline{r})$. When absorption is present at above-thermal energies, the balance equation comparable to Eq. 2.17 is

$$q(E + dE,\underline{r}) = q(E,\underline{r}) + \phi(E,\underline{r})\ \Sigma_a dE - \frac{\lambda_{tr}}{3}\ \nabla^2\phi(E,\underline{r})\ dE\ , \tag{2.62}$$

where Σ_a and λ_{tr} are, of course, functions of energy. When the left side is expanded in two terms of a Taylor's expansion about E, this equation reduces to

$$-\frac{\lambda_{tr}}{3}\ \nabla^2\phi(E,\underline{r}) + \phi(E,\underline{r})\ \Sigma_a = \frac{\partial q(E,\underline{r})}{\partial E}\ . \tag{2.63}$$

When Eq. 2.56 is substituted in Eq. 2.63, the equation becomes

$$-\frac{\lambda_{tr}\nabla^2 q(E,\underline{r})}{3E\xi\Sigma_T} + \frac{q(E,\underline{r})\ \Sigma_a}{\xi E\Sigma_T} = \frac{\partial q(E,\underline{r})}{\partial E}\ . \tag{2.64}$$

Solution for q_{th}. An attempt is made to separate variables by writing

$$q(E,\underline{r}) = R(\underline{r})\ T(E)\ . \tag{2.65}$$

By substituting this into Eq. 2.64 and dividing by $q(E,\underline{r})$, the result is

$$\frac{\nabla^2 R(\underline{r})}{R(\underline{r})} = \frac{3\Sigma_a}{\lambda_{tr}} - \frac{3E\xi\Sigma_T}{\lambda_{tr}}\ \frac{1}{T(E)}\ \frac{dT(E)}{dE}\ . \tag{2.66}$$

The separation of variables was successful, and, for reasons now familiar, both sides must equal a constant, say $-\beta^2$. The two ordinary differential equations that result are:

$$\nabla^2 R(\underline{r}) + \beta^2 R(\underline{r}) = 0 \tag{2.67}$$

and

$$\frac{dT(E)}{dE} + \left(-\frac{\Sigma_a}{\xi E\Sigma_T} - \beta^2\frac{\lambda_{tr}\lambda_T}{3E\xi}\right)T(E) = 0\ . \tag{2.68}$$

Upon solving this equation, the result is

$$T(E) = A \exp\left(-\int_E^{E_F} \frac{\Sigma_a}{\xi E \Sigma_T}\, dE\right) \exp\left(-\beta^2 \int_E^{E_F} \frac{\lambda_{tr}\lambda_T}{3E\xi}\, dE\right). \qquad (2.69)$$

Now by defining the age in an absorbing medium to be,

$$T(E) \equiv \int_E^{E_F} \frac{\lambda_{tr}\lambda_T}{3E\xi}\, dE\ , \qquad (2.70)$$

and defining the "resonance escape probability" $P(E)$ to be

$$P(E) \equiv \exp\left(-\int_E^{E_F} \frac{\Sigma_a}{E\xi\Sigma_T}\, dE\right), \qquad (2.71)$$

Eq. 2.69 becomes

$$T(E) = A\ P(E)\ e^{-\tau\beta^2}\ . \qquad (2.72)$$

Definition 2.71 is an important one. The symbol P and the name "resonance escape probability" appear quite frequently in the literature.

Upon substituting Eq. 2.72 into Eq. 2.65, the latter becomes

$$q(E,\underline{r}) = R(\underline{r})\ A\ P(E)\ e^{-\tau\beta^2}\ , \qquad (2.73)$$

or, since $\tau = 0$ when $E = E_F$,

$$q(E_F,\underline{r}) = R(\underline{r})\ A\ . \qquad (2.74)$$

However,

$$q(E_F,\underline{r}) = \phi_{th}\ \Sigma_{a_{th}}\ k_{th}\ , \qquad (2.75)$$

and therefore

$$R(\underline{r})\ A = \phi_{th}\ \Sigma_{a_{th}}\ k_{th}\ . \qquad (2.76)$$

Hence,

$$q(E,\underline{r}) = \phi_{th}\ \Sigma_{a_{th}}\ k_{th}\ P(E)\ e^{-\tau\beta^2}\ . \qquad (2.77)$$

In particular, the source term for the thermal group is

$$q_{th}(\underline{r}) = \phi_{th}(\underline{r})\ \Sigma_{a_{th}}\ k_{th}\ P_{th}\ e^{-\tau_{th}\beta^2}\ . \qquad (2.78)$$

Criticality Equation. The procedure in getting the thermal Fermi age equation when above-thermal absorption is present, by beginning with Eq. 2.78 as the thermal-group source term, parallels the development of Eqs. 2.31 to 2.38 for the case of no above-thermal absorption. If the steps are repeated, it is found that the thermal criticality equation for resonance absorption is

$$\frac{k_{th}\ P_{th}\ e^{-\tau_{th}B^2}}{1 + L^2B^2} = 1 \qquad (2.79)$$

where k_{th} is the number of fission neutrons produced per thermal neutron absorbed ($2.5\ \Sigma_f/\Sigma_a$, for U^{235}), P_{th} is the probability of escaping resonance absorption while slowing down, τ_{th} is the age-to-thermal, L^2 is the thermal diffusion "area," and B^2 is the "buckling," which is related to the critical size by formulas 2.39, 2.40, and 2.41.

Equation 2.79 is most applicable to the reactor in which the uranium is of low enrichment – a significant amount of U^{238} present. A further improvement can be made for this type of mixture by introducing the quantity ϵ to account for the fact that U^{238} has a significant cross section for fission for neutrons of energy greater than 1 Mev. The ϵ has the following definition: total number of fission neutrons due to the absorption of neutrons of all energies divided by the number resulting from the absorption of thermal neutrons only. With this improvement, Eq. 2.79 becomes

$$\frac{k_{th}\ P_{th}\ \epsilon\ e^{-\tau_{th}B^2}}{1 + L^2B^2} = 1\ . \qquad (2.80)$$

Several reports exist in the literature that give methods for the

determination of ϵ;[1] the problem will not be taken up here. The value of ϵ never differs greatly from unity; it is usually around 1.03 in natural uranium reactors.

Another case in which thermal reactor theory can be used with reasonable accuracy is that in which the reactor is almost, but not quite, thermal and there is no appreciable U^{238} or other resonance absorption; there may be considerable $1/v$ absorption. In this case, the following form of the criticality equation can be used:

$$\frac{k_{th} P_{th} e^{-\tau_{th} B^2}}{1 + L^2 B^2} + k_{th}(1 - P_{th}) e^{-\tau_{th} B^2} = 1 , \quad (2.81)$$

where the absorption in the fissionable atoms and other $1/v$ absorbers makes P_{th} significantly different from unity. The assumption involved in the second term of Eq. 2.81 is that the absorption of and fissioning by epithermal neutrons is due to neutrons that are very close to thermal, and therefore the full age-to-thermal as well as the thermal k is used in this term.

REACTORS WITH ABOVE-THERMAL NEUTRON ABSORPTION AND FISSION

The method of determining the critical size (or critical mass) of an intermediate-energy reactor will be taken up in this section. The plan of the derivation is as before, but the balance equations become more complicated and the solution for the critical size is much more time-consuming.

Heretofore the simplifying assumption has been made that fission neutrons are all born at the same energy. In large thermal reactors, leakage of neutrons is rather small, and neutrons that cause fission attain a long age, so that the monoenergetic fission assumption at some average energy gives good results and the results are not too sensitive to the choice of fission energy. On the other hand, in smaller reactors in which part of the fissioning is due to above-thermal neutrons, the leakage of neutrons from the reactors is large and the proper average energy for an assumption of a monoenergetic source varies with each reactor. It is therefore necessary to take into account the fission spectrum of neutrons, that is, the energy distribution of neutrons emerging from fission. The distribution of fission neutrons is given best on a logarithmic scale. The fission spectrum in histogram form is given in Table 1, in which the variable called lethargy, u, defined as

$$u = \ln \frac{10^7}{E} ,$$

where E is energy in electron volts, is introduced.

Balance Equations. The neutron balance equations for above-thermal neutrons of energy between E and $E + dE$ and for thermal energy take the form

$$q(E + dE, \underline{r}) - q(E, \underline{r}) = \frac{-\lambda_{tr}}{3} \nabla^2 \phi(E, \underline{r})\, dE + \Sigma_a \phi(E, \underline{r})\, dE - \nu f(E)\, dE \left[\Sigma_{f_{th}} \phi_{th}(\underline{r}) + \int_{E_{th}}^{\infty} \Sigma_f \phi(E', \underline{r})\, dE' \right] , \quad (2.82)$$

$$q(E_{th}, \underline{r}) = -\frac{\lambda_{tr_{th}}}{3} \nabla^2 \phi_{th}(\underline{r}) + \Sigma_{a_{th}} \phi_{th}(\underline{r}) , \quad (2.83)$$

[1] H. Castle, H. Ibser, G. Sacher, and A. M. Weinberg, *The Effect of Fast Fission on k*, CP-644 (May 4, 1943).

TABLE 1. FISSION SPECTRUM IN HISTOGRAM FORM

Δu	ΔE (ev)	FRACTION OF FISSION NEUTRONS
0 to 0.5	10^7 to 6.07×10^6	0.0210
0.5 to 1.0	6.07×10^6 to 3.68×10^6	0.1040
1.0 to 1.5	3.68×10^6 to 2.23×10^6	0.2070
1.5 to 2.0	2.23×10^6 to 1.35×10^6	0.2310
2.0 to 2.5	1.35×10^6 to 8.21×10^5	0.1830
2.5 to 3.0	8.21×10^5 to 4.98×10^5	0.1170
3.0 to 3.5	4.98×10^5 to 3.02×10^5	0.0670
3.5 to 4.0	3.02×10^5 to 1.83×10^5	0.0350
4.0 to 7.0	1.83×10^5 to 9.12×10^3	0.0350
	Total	1.0000

where $f(E)$ is the fraction of fission neutrons born in unit energy range around E. Equation 2.82 can be simplified by expanding the first term on the left in two terms of a Taylor's series about E and then dividing through the equation by dE. The result is

$$\frac{\partial q(E,\underline{r})}{\partial E} = -\frac{\lambda_{tr}}{3} \nabla^2 \phi(E,\underline{r}) + \Sigma_a \phi(E,\underline{r}) - \nu f(E) \left[\Sigma_{f_{th}} \phi_{th}(\underline{r}) + \int_{E_{th}}^{\infty} \Sigma_f \phi(E',\underline{r}) \, dE' \right] ,$$

and by using Eq. 2.56, this becomes

$$\frac{\partial q(E,\underline{r})}{\partial E} = -\frac{\lambda_{tr}}{3} \frac{1}{\xi E \Sigma_T} \nabla^2 q(E,\underline{r}) + \frac{\Sigma_a}{\xi E \Sigma_T} q(E,\underline{r}) - \nu f(E) \left[\Sigma_{f_{th}} \phi_{th}(\underline{r}) + \int_{E_{th}}^{\infty} \frac{\Sigma_f}{\xi E \Sigma_T} q(E',\underline{r}) \, dE' \right] ; \qquad (2.84)$$

as before,

$$q(E_{th},\underline{r}) = \frac{\lambda_{tr_{th}}}{3} \nabla^2 \phi_{th}(\underline{r}) + \Sigma_{a_{th}} \phi_{th}(\underline{r}) . \qquad (2.83)$$

Separation of Variables. The geometry will be restricted to that of a slab, and the sphere and cylinder will be an exercise for the student. For slab geometry,

$$\nabla^2 \phi(E,\underline{r}) = \frac{d^2 \phi(E,x)}{dx^2} ,$$

and a change of variable is introduced, that is,

$$u = \ln \frac{10^7}{E} ;$$

u has been called "lethargy," presumably because from the definition it is clear that when a neutron has little energy it has a lot of lethargy. The energy 10^7 ev is chosen here as an arbitrarily convenient number above the fission spectrum. In changing the variable from E to u, the equations become

$$-\frac{\lambda_{tr}}{3\xi\Sigma_T} \frac{\partial^2 q(u,x)}{\partial x^2} + \frac{\Sigma_a \; q(u,x)}{\xi\Sigma_T} + \frac{\partial q(u,x)}{\partial u}$$

$$= \nu \; f(u) \left[\Sigma_{f_{th}} \; \phi_{th}(x) + \int_0^{u_{th}} \frac{\Sigma_f \; q(u',x)}{\xi\Sigma_T} \, du' \right] \qquad (2.85)$$

and

$$-\frac{\lambda_{tr_{th}}}{3} \frac{d^2 \phi_{th}(x)}{dx^2} + \Sigma_{a_{th}} \; \phi_{th}(x)$$

$$= q(u_{th}, x) \; . \qquad (2.86)$$

An attempt is made to obtain a solution by letting

$$q(u,x) = q(u) \cos Bx \; , \qquad (2.87)$$

$$\phi_{th}(x) = \phi_{th} \cos Bx \; , \qquad (2.88)$$

and substituting into Eqs. 2.85 and 2.86 to see whether a relation in B results; if such a relation does exist, it will determine a value or values of B such that solutions of the form of Eqs. 2.87 and 2.88 do, in fact, exist. The substitution of Eqs. 2.87 and 2.88 into Eqs. 2.85 and 2.86 gives, after dividing through by cos Bx,

$$\frac{\lambda_{tr}}{3} \frac{B^2 q(u)}{\xi\Sigma_T} + \frac{\Sigma_a q(u)}{\xi\Sigma_T} + \frac{dq(u)}{du} = \nu \, f(u) \left[\Sigma_{f_{th}} \; \phi_{th} + \int_0^{u_{th}} \frac{\Sigma_f q(u')}{\xi\Sigma_T} \, du' \right] , \qquad (2.89)$$

and

$$\frac{\lambda_{tr_{th}}}{3} B^2 \phi_{th} + \Sigma_{a_{th}} \; \phi_{th} = q(u_{th}) \; . \qquad (2.90)$$

A unique value of B^2 exists that satisfies Eqs. 2.89 and 2.90, and a method of finding it will be given below. Hence a solution of the form of Eqs. 2.87 and 2.88 does, in fact, exist.

To find the value of B^2 that satisfies Eqs. 2.89 and 2.90, the following procedure (used by the ANP Division, ORNL) may be used. The quantity in brackets in Eq. 2.89 is proportional to the fissions per second (per cubic centimeter), which depends on the power level at which the reactor is operating. Since the critical size does not depend on the power level, the quantity in brackets may be set equal to any convenient definite number, say unity. This amounts to fixing the power level at a definite value. It is then required that

$$\Sigma_{f_{th}} \; \phi_{th} + \int_0^{u_{th}} \frac{\Sigma_f \; q(u')}{\xi\Sigma_T} \, du' = 1 \; , \qquad (2.91)$$

and the equations become

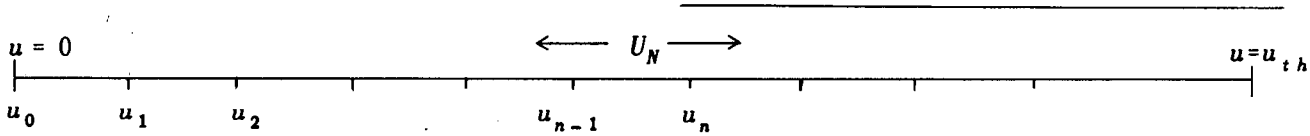
$$\frac{\lambda_{tr}}{3}\,\frac{B^2 q(u)}{\xi\Sigma_T} + \frac{\Sigma_a\, q(u)}{\xi\Sigma_T} + \frac{dq(u)}{du} = \nu\, f(u) \tag{2.92}$$

and

$$\frac{\lambda_{tr_{th}}}{3}\, B^2 \phi_{th} + \Sigma_{a_{th}} \phi_{th} = q(u_{th}) \ . \tag{2.93}$$

A trial value of B^2, say B_1^2, is now selected, and the $q(u)$ (a function of lethargy) and ϕ_{th} are determined numerically from Eqs. 2.92 and 2.93. If the $q(u)$ and ϕ_{th} corresponding to B_1^2 satisfy Eq. 2.91, the trial value of $B^2 = B_1^2$ is the correct value. If they do not, further guesses at the value of B^2 are made until the correct value has been determined to within an accuracy appropriate to the data.

The numerical procedure for finding the $q(u)$ and ϕ_{th}, from Eqs. 2.92 and 2.93, that correspond to any B^2 is the following. The lethargy range, $u = 0$ to $u = u_{th}$, is broken up into a finite number of intervals. The ANP Division of the Oak Ridge National Laboratory has found that about 25 to 30 subdivisions, depending on the thermal-neutron temperature, are appropriate.



When Eq. 2.78 is integrated over the general interval u_{n-1} to u_n, the result is

$$\int_{u_{n-1}}^{u_n} \left(\frac{\lambda_{tr} B^2}{3\xi\Sigma_T} + \frac{\Sigma_a}{\xi\Sigma_T}\right) q(u)\, du + \int_{u_{n-1}}^{u_n} \frac{dq(u)}{du}\, du = \int_{u_{n-1}}^{u_n} \nu\, f(u)\, du \ . \tag{2.94}$$

The approximation is made that

$$\int_{u_{n-1}}^{u_n} A(u) \cdot B(u)\, du = \bar{A} \cdot \bar{B}\, U_N \ , \tag{2.95}$$

where

$U_N \equiv u_n - u_{n-1}$,

$A(u) \equiv$ any function of the cross sections,

$B(u) \equiv$ any function of the cross sections,

$\bar{A} = \frac{1}{U_N}\int_{u_{n-1}}^{u_n} A(u)\, du$, in which the integral may be evaluated by Simpson's rule,

$\bar{B} = \frac{1}{U_N}\int_{u_{n-1}}^{u_n} B(u)\, du$, in which the integral may be evaluated by Simpson's rule.

With the use of Eq. 2.95, Eq. 2.94 becomes

$$\left(\overline{\frac{\lambda_{tr}B^2}{3\xi\Sigma_T} + \frac{\Sigma_a}{\xi\Sigma_T}}\right)\bar{q}_N U_N + q_n - q_{n-1} = \nu z_N \ , \tag{2.96}$$

where

$$z_N = \int_{u_{n-1}}^{u_n} f(u)\, du \ .$$

The approximation is now made that

$$\bar{q}_N = \frac{q_n - q_{n-1}}{2} \ , \tag{2.97}$$

and the definition is made that

$$C_N \equiv \left(\overline{\frac{\lambda_{tr}B^2}{3\xi\Sigma_T} + \frac{\Sigma_a}{\xi\Sigma_T}}\right)_N \frac{U_N}{2} . \tag{2.98}$$

When Eqs. 2.97 and 2.98 are used, Eq. 2.96 becomes

$$C_N(q_{n-1} + q_n) = q_{n-1} - q_n + \nu z_N$$

or

$$q_n = \frac{q_{n-1}(1 - C_N) + \nu z_n}{1 + C_N} . \qquad (2.99)$$

Equation 2.99 provides a recursion relation for obtaining $q(u)$ at $u = u_n$ in terms of the value of $q(u)$ at $u = u_{n-1}$, for which one starts with the fact that $q_0 = 0$ ($u_0 = 10^7$ ev is above the fission spectrum, by choice).

After successive solutions down to q_{th}, the ϕ_{th} is obtained from Eq. 2.93, according to the following:

$$\phi_{th} = \frac{q_{th}}{\Sigma_{a_{th}} + \frac{\lambda_{tr_{th}} B^2}{3}} . \qquad (2.100)$$

In summary, by starting with the condition that there was one fission per second per cubic centimeter, the solutions for $q(u)$ and ϕ_{th} were obtained. If the trial value of B^2 (related to the size) was the correct one for criticality, then the number of fissions per second per cubic centimeter that would result from these values of $q(u)$ and ϕ_{th} should equal unity. The procedure is then to determine the left side of Eq. 2.91:

$$k_{eff} \equiv \Sigma_{f_{th}}\phi_{th} + \sum_N \left[\left(\frac{\Sigma_f}{\xi\Sigma_T} \right)_N \left(\frac{q_{n-1} + q_n}{2} \right) U_N \right] . \qquad (2.101)$$

If $k_{eff} = 1$, the trial value of B^2 was correct. If $k_{eff} \neq 1$, a new value of B_1^2, say B_2^2, is chosen and the $k_{eff}(B_2^2)$ is determined. Usually about three guesses for B^2 and a plot of the results, that is, $k_{eff}(B^2)$ vs. B^2, is sufficient to determine the critical buckling (the B^2 such that $k_{eff}(B^2) = 1$).

A sample calculation form that is used by the Physics group (ANP Division) of the Oak Ridge National Laboratory for finding the k_{eff} corresponding to a given trial for B^2 is shown in the appendix. The form also provides for the calculation of the lethargy distribution of neutrons causing fission (F_n) and the lethargy distribution of the flux. From the F_n's, the median lethargy (or energy) for fissioning can be found; this is a useful index of the neutron spectrum of the reactor.

Equations 2.89 and 2.90 were based on the assumption of slab geometry; however, exactly the same equations are obtained for the other soluble geometries.

General Solutions. In the case of the slab geometry, assumptions as to the form of the flux or slowing-down density such as

$$q(u,x) = q(u) \sin Bx , \qquad (2.102)$$

$$\phi_{th}(x) = \phi_{th} \sin Bx , \qquad (2.103)$$

would have worked just as well. The general solution is therefore

$$q(u,x) = q(u) \left[A \cos Bx + C \sin Bx\right] , \qquad (2.104)$$

$$\phi_{th}(x) = \phi_{th} \left[A \cos Bx + C \sin Bx\right] , \qquad (2.105)$$

where ϕ_{th}, as used here, is a pure constant.

From Eq. 2.56,

$$\phi(u,x) = \frac{q(u)}{\xi\Sigma_T} \left[A \cos Bx + C \sin Bx\right] . \qquad (2.106)$$

In the case of bare, spherical geometry, the general solutions are

$$q(u,r) = q(u) \left[A \frac{\sin Br}{r} + C \frac{\cos Br}{r} \right] , \qquad (2.107)$$

$$\phi(u,r) = \frac{q(u)}{\xi\Sigma_T}\left[A\frac{\sin Br}{r} + C\frac{\cos Br}{r}\right], \tag{2.108}$$

$$\phi_{th}(r) = \phi_{th}\left[A\frac{\sin Br}{r} + C\frac{\cos Br}{r}\right]. \tag{2.109}$$

In the case of bare cylinders, the general solutions are

$$q(u,r,z) = q(u)\left[\{A\,J_0(\mu r) + C\,Y_0(\mu r)\}\{D\cos\epsilon z + E\sin\epsilon z\}\right], \tag{2.110}$$

$$\phi(u,r,z) = \frac{q(u)}{\xi\Sigma_T}\left[\{A\,J_0(\mu r) + C\,Y_0(\mu r)\}\{D\cos\epsilon z + E\sin\epsilon z\}\right], \tag{2.111}$$

$$\phi_{th}(r,z) = \phi_{th}\left[\{A\,J_0(\mu r) + C\,Y_0(\mu r)\}\{D\cos\epsilon z + E\sin\epsilon z\}\right], \tag{2.112}$$

where

$$B^2 = \mu^2 + \epsilon^2 .$$

Boundary Conditions. The boundary conditions of the problem are:

1. The flux must be finite everywhere in the reactor.
2. The flux must be symmetrical around points, lines, or planes of symmetry in the geometrical set-up.
3. The flux must be nonnegative at all points in the reactor.
4. At external surfaces, the flux at all energies extrapolates to zero at the same point. The extrapolation distance $\delta = 0.71\,\lambda_{tr}$ is actually different for each energy interval because λ_{tr} is a function of energy, but an average value of λ_{tr} is selected and δ is taken as a constant for flux at all energies, including thermal flux.

Application of Boundary Conditions and Determination of the Critical Size. Boundary conditions 1 and 2 reduce the general solutions immediately to

$$q(u,x) = A\,q(u)\cos Bx, \tag{2.113}$$

$$\phi(u,x) = A\frac{q(u)}{\xi\Sigma_T}\cos Bx, \tag{2.114}$$

$$\phi_{th}(x) = A\,\phi_{th}\cos Bx . \tag{2.115}$$

For slabs,

$$q(u,r) = A\,q(u)\frac{\sin Br}{r}, \tag{2.116}$$

$$\phi(u,r) = A\frac{q(u)}{\xi\Sigma_T}\frac{\sin Br}{r}, \tag{2.117}$$

$$\phi_{th}(x) = A\,\phi_{th}\frac{\sin Br}{r}. \tag{2.118}$$

For spheres and cylinders,

$$q(u,r,z) = A\,q(u)\,J_0(\mu r)\cos\epsilon z, \tag{2.119}$$

$$\phi(u,r,z) = A\frac{q(u)}{\xi\Sigma_T}J_0(\mu r)\cos\epsilon z, \tag{2.120}$$

$$\phi_{th}(r,z) = A\,\phi_{th}\,J_0(\mu r)\cos\epsilon z, \tag{2.121}$$

where

$$B^2 = \mu^2 + \epsilon^2 . \tag{2.122}$$

In all these equations, the B has the value that makes $k_{eff} = 1$, according to the procedure given above; the $q(u)$ and ϕ_{th} are from the normalization Eq. 2.91, and A depends on the power level at which the reactor is operating. In fact, A is equal to the fissions per second per cubic centimeter at the center of the reactor, since $q(u)$ was normalized for one fission per second per cubic centimeter and the distribution functions,

$$\cos Bx, \frac{\sin Br}{r}, \text{ and } J_0(\mu r)\cos z,$$

are all equal to unity at the center.

The critical size follows from boundary conditions 3 and 4. In the

case of slabs, boundary condition 4 states that

$$\cos B(x_0 + \delta) = 0 \ , \qquad (2.123)$$

where x_0 is the critical half thickness and δ is the extrapolation distance. From boundary condition 3, the non-negative condition

$$B(x_0 + \delta) = \frac{\pi}{2} , \qquad (2.124)$$

which is the first root, is obtained. Therefore

$$x_0 = \frac{\pi}{2B} - \delta \ . \qquad (2.125)$$

For *spheres* it follows, similarly, that

$$R = \frac{\pi}{B} - \delta \ , \qquad (2.126)$$

where R is the critical radius. For *cylinders* the relation between the critical dimensions is given by

$$B^2 = \left(\frac{2.405}{R + \delta}\right)^2 + \frac{\pi^2}{\left[2\left(\frac{H}{2} + \delta\right)\right]^2} \ . \qquad (2.127)$$

In the above procedure it was assumed that the reactor composition was given and the critical size was to be determined. The converse problem also exists in practice, perhaps even more frequently – the size is given, and the ratio of the fissionable material to moderator and structure is to be determined. The B^2 is then immediately fixed by Eqs. 2.125, 2.126, or 2.127, and various trials are made for the uranium mass, which affects the macroscopic cross sections, until $k_{eff} = 1$ in Eq. 2.101.

EVALUATING THE CONSTANTS

Averaging Constants in the Thermal Group. Up to this point, thermal energy has been considered as if all thermal neutrons were at one energy. Actually, however, the thermal neutrons (neutrons that have slowed down to the point where they feel the energy of the moderator nuclei and are just as likely to gain energy from a collision as to lose energy) are spread in energy in a distribution that is given approximately by the Maxwell-Boltzmann function. If the Maxwell-Boltzmann function for velocity distribution is denoted by $M(v)$, one can write

$$M(v) \propto v^2 \ e^{-mv^2/2kT} \ , \qquad (2.128)$$

where

m = mass of the neutron,
k = the Boltzmann constant, 8.6170×10^{-5} ev/°K,
T = the temperature of the medium in degrees Kelvin.

In terms of energy, the equivalent statement is

$$M(E) \propto \sqrt{E} \ e^{-E/kT} \ . \qquad (2.129)$$

The energy given by $E = kT$ corresponds to the peak of the velocity distribution, or the energy corresponding to the most probable neutron velocity.

Actually, because of the $1/v$ behavior of most absorption cross sections, the thermal neutrons with lower energy tend to be absorbed to a greater extent than those with higher energy, and the distribution is shifted slightly in the direction of higher energies than that given by the Maxwell-Boltzmann distribution. Thus the statement is sometimes made that the "neutron temperature" is somewhat higher than the thermal temperature of the medium. However, because of the extreme difficulty in obtaining the correct distribution for any given reactor, the Maxwell-Boltzmann distribution is usually used as an approximation in obtaining average thermal cross sections.

The two thermal cross sections needed are $\Sigma_{a_{th}}$ and $\Sigma_{tr_{th}}$. In regard to $\Sigma_{a_{th}}$, the appropriate average is obviously the one that, when multiplied by ϕ_{th} (the total thermal flux) and by the atomic concentration, gives the true number of absorptions per second per cubic centimeter. Therefore one can write

$$\phi_{th}\,\overline{\Sigma}_{a_{th}} = \int_0^\infty \phi_{th}(E)\,\Sigma_a(E)\,dE \qquad (2.130)$$

or

$$\overline{\Sigma}_{a_{th}} = \frac{\int_0^\infty \phi_{th}(E)\,\Sigma_a(E)\,dE}{\int_0^\infty \phi_{th}(E)\,dE}. \qquad (2.131)$$

Since

$$\phi_{th}(E) = n(E)\,v \propto E\,e^{-E/kT},$$

assuming $n(E) \propto M(E)$, one can write

$$\overline{\Sigma}_{a_{th}} = \frac{\int_0^\infty \Sigma_a(E)\,E\,e^{-E/kT}\,dE}{\int_0^\infty E\,e^{-E/kT}\,dE}, \qquad (2.132)$$

which is the appropriate average absorption cross section for the thermal groups. It is an important fact that if $\Sigma_a(E) \propto 1/v$ (which is the case for many elements), then $\overline{\Sigma}_{a_{th}}$ is the cross section at the kT energy (0.025 ev for room-temperature reactors) divided by $2/\sqrt{\pi} = 1.128$. This can be verified by carrying out the above formula for $\overline{\Sigma}_{a_{th}}$, with the assumption that $\Sigma_a(E) = \Sigma_a(kT)\sqrt{kT/E}$. Usually, thermal cross-section data are quoted at the kT energy.

On the other hand, in respect to $\overline{\Sigma}_{tr_{th}}$, it is the thermal leakage that is to be represented accurately. Thus it is written that

$$\frac{\lambda_{tr}}{3}\nabla^2\phi_{th} = \int_0^\infty \frac{\lambda_{tr}(E)}{3}\nabla^2\phi_{th}(E)\,dE, \qquad (2.133)$$

but

$$\nabla^2\phi_{th} = -B^2\phi_{th},$$

therefore

$$\frac{\lambda_{tr}}{3}B^2\phi_{th} = \int_0^\infty \frac{\lambda_{tr}(E)}{3}B^2\,\phi_{th}(E)\,dE.$$

Then, by assuming

$$\phi_{th}(E) \propto E\,e^{-E/kT},$$

as before, it follows that

$$\frac{1}{\overline{\Sigma}_{tr_{th}}} = \frac{\int_0^\infty \frac{1}{\Sigma_{tr}(E)}E\,e^{-E/kT}\,dE}{\int_0^\infty E\,e^{-E/kT}\,dE}; \qquad (2.134)$$

thus for $\overline{\Sigma}_{tr_{th}}$ it is the reciprocal $1/\Sigma_{tr}(E)$ that is averaged over the flux.

Usually, the $\Sigma_{tr}(E)$ is constant enough so that one can choose a $\overline{\Sigma}_{tr_{th}}$ by inspection from the cross-section curve and not bother to carry out the above averaging process (which may be time-consuming).

Effect of Heterogeneity on the Constants. Up to this point, the composition of the reactor has been considered to be a homogeneous mixture of atoms of fissionable material, moderator, and, possibly, diluents. Actually, almost all reactors are designed in a cellular or lattice-like form. The fuel elements may be in plate form with the uranium alloyed with possibly aluminum or zirconium, and then clad (picture-frame construction like that of the MTR and STR); they may be in the form of small-diameter (~0.1 in.) cylinders of enriched UO_2 (for example) in a mix (the intermediate-energy submarine reactor design); or, in the case of natural uranium reactors, they may be large cylinders (~2 in.) in a can (the ORNL graphite reactor). The moderator may exist also in definite separated regions, and coolant flows between the moderator and the fuel elements.

Clearly, the critical mass and size will be affected to a greater or lesser degree by the "lumping." In some enriched-fuel reactor designs,

the thickness of the different regions is small enough to make the lumping effect negligible. An assumption of complete homogeneity causes little error. At the other extreme, in a natural uranium reactor the effect may make the difference between predicting that a mixture can be made critical or that it cannot be made critical (as may be seen in Problem 5 for graphite and uranium).

The heterogeneity effect is incorporated in the criticality formulas by adjusting the constants. In the thermal reactor equations, the constants in question are L^2, τ_{th}, k_{th}, and P_{th}. The τ_{th} can usually be assumed to be unaffected by the heterogeneity; however, the L^2 is altered somewhat. A good discussion of L^2 in the heterogeneous reactor is given by Plass.[2] A special problem in heterogeneity with respect to leakage is given by the reactor in which air passages penetrate, without steps, completely through the reactor. Neutrons tend to "stream" from the holes and cause an additional effect over and above that resulting from the decrease in over-all density. Methods have been worked out for taking into account the "streaming effect." A good reference is the work of Behrens,[3]

The effect of heterogeneity on P_{th} can be calculated, but the procedure is complicated and will not be taken up here. A good reference is the work of Dancoff and Ginsberg.[4] Qualitatively, the increase in P_{th} that results from the lumping of natural uranium is due to two effects: (1) some of the neutrons are completely slowed down below the capture resonance (that is, below about 10 ev) while in the moderator region and thus enter the fuel slug with a more favorable chance of causing fission as opposed to being captured; (2) the resonances of U^{238} apparently exist in rather sharp peaks and a neutron entering a fuel slug between peaks has an excellent chance of passing right on through without being captured, since it is not moderated in energy to any extent by the heavy nuclei of the fuel region. Upon a few such fortuitous entries, with periods of moderation between, the neutron will have been degraded below the resonance energies.

The effect of heterogeneity on the k_{th} can usually be calculated in a straightforward manner.[5] The k_{th}, as defined here, can be written

$$k_{th} = \eta f ,$$

where

$$\eta = \nu \frac{\Sigma_f}{\Sigma_a \text{ (fuel)}} ,$$

ν = neutrons per fission (= 2.5 for U^{235}),

Σ_f = macroscopic fission cross section,

Σ_a(fuel) = macroscopic absorption cross section for the fuel material (for uranium this includes U^{235} and U^{238}),

f = the thermal utilization $= \dfrac{\Sigma_a \text{ (fuel)}}{\Sigma_a \text{(total)}}$,

Σ_a(total) = macroscopic absorption cross section for all core material.

The core of the heterogeneous reactor can usually be partitioned into identical cells. For example, if the structure consists of fuel plates separated by a layer of water (coolant and moderator), the cell would consist of the slab having one face at the

(2) G. N. Plass, *The Diffusion Length and the Utilization of Thermal Neutrons in a Heterogeneous Pile*, CP-992 (Oct. 6, 1943).

(3) D. J. Behrens, *The Effect of Holes in a Reacting Material on the Passage of Neutrons, with Special Reference to the Critical Dimensions of a Reactor*, AERE-T/R 103 (May 24, 1949).

(4) S. M. Dancoff and M. Ginsburg, *Resonance Absorption in Lumps and Mixtures*, CP-1589 (Apr. 17, 1944).

(5) E. P. Wigner and G. N. Plass, *On the Utilization of Thermal Neutrons*, CP-103.

mid-plane of the fuel plate and the other face at the mid-plane of the adjacent water layer. If the structure is fuel-slug cylinders separated with moderator, the cell would consist of the square parallelepiped assignable to a particular fuel slug (Fig. 2.8). This parallepiped can be approximated by a cylinder for calculational purposes.

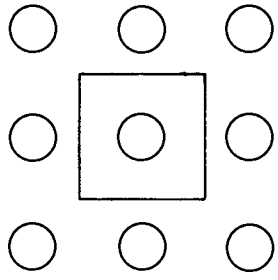


Fig. 2.8

In any event, the flux distribution within a cell will be irregular. The flux will be depressed in the center of the fuel and perhaps bulged up in the moderator. The depression of the flux in the fuel element because of the attenuation of the entering neutron current by absorption in the outer nuclei is called self-shielding. The self-shielding effect is pronounced in any localized material of large absorption cross section. The irregular behavior of the flux in a cell will not affect η, but it will change the ratio of absorptions in fuel to absorptions in other material and hence will affect the thermal utilization factor f.

As an example, the procedure for getting f in the slab-like cell will be indicated. Suppose the structure is as shown in Fig. 2.9; the fuel plates are of thickness t and the moderator layers are of thickness s. The assumption is made that no slowing down takes place in the fuel plates and that $q_{th}(x)$ is a constant at all points in the moderator. Both assumptions usually fit the facts reasonably well. The balance equations

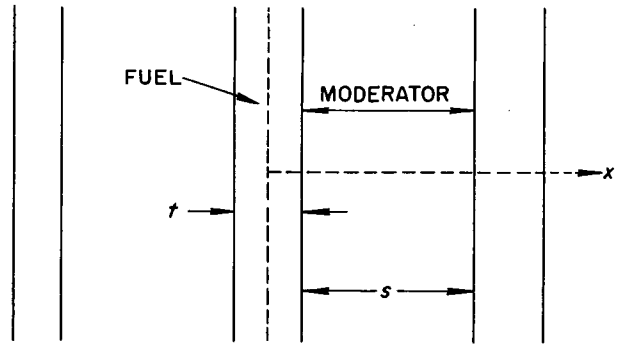


Fig. 2.9

in the fuel and moderator are therefore:

$$\frac{d^2\phi_{th_F}}{dx^2} - \frac{1}{L_F^2}\phi_{th_F} = 0 , \qquad (2.135)$$

$$\frac{d^2\phi_{th_M}}{dx^2} - \frac{1}{L_M^2}\phi_{th_M} + \frac{3q_{th}}{\lambda_{tr}} = 0 , \qquad (2.136)$$

where F refers to the fuel region and M refers to the moderator region. In obtaining Eqs. 2.135 and 2.136, the assumption has been made that the diffusion theory is applicable. Usually the absorption in the fuel plates is strong enough to cause the simple theory to be somewhat in error. However, as was stated in the section on "Derivation of Fick's Law," chap. 1, it is so complicated and time consuming to apply more accurate methods that the simple diffusion theory is usually used even in such cases as this in which it is only approximately correct.

The general solutions of Eqs. 2.135 and 2.136 are

$$\phi_{th_F}(x) = A_C \sinh \frac{1}{L_F} x + C_C \cosh \frac{1}{L_F} x , \qquad (2.137)$$

$$\phi_{th_M}(x) = A_M \sinh \frac{1}{L_M} x + C_M \cosh \frac{1}{L_M} x - \frac{3q_{th}}{\lambda_{tr_M}} . \qquad (2.138)$$

The boundary conditions are

1. symmetry

2. $\left(\frac{d\phi_F}{dx}\right)_{x=0} = 0$

3. $\left(\frac{d\phi_M}{dx}\right)_{x=\frac{t+s}{2}} = 0$

4. $\phi_F\left(\frac{t}{2}\right) = \phi_M\left(\frac{t}{2}\right)$

5. $\lambda_{tr_F}\left(\frac{d\phi_F}{dx}\right)_{x=t/2} = \lambda_{tr_M}\left(\frac{d\phi_M}{dx}\right)_{x=t/2}$

If these boundary conditions are applied to the general solutions, equations result from which A_C, C_C, A_M, and C_M can be determined as functions of L_F^2, L_M^2, q_{th}, and λ_{tr_M} (the constants).

The thermal utilization is then given by

$$f = \frac{\int_0^{t/2} \phi_{th_F}(x)\, \Sigma_{a_F}\, dx}{\int_0^{t/2} \phi_{th_F}(x)\, \Sigma_{a_F} dx + \int_{t/2}^{(t+s)/2} \phi_{th_M}(x)\, \Sigma_{a_M} dx} . \qquad (2.139)$$

In general, as the heterogeneity gets more pronounced, the P_{th} increases and f decreases. The problem of natural uranium reactors is thus to determine a lumping arrangement such that the product fP_{th} is a maximum.

In intermediate-energy enriched reactors a certain amount of self-shielding exists even for above-thermal flux. It is not convenient to compute a flux distribution throughout a cell for flux of all energies; so a method has been worked out for finding the ratio of effective to actual absorption and effective to actual fission microscopic cross section for the fissionable material by assuming neutrons to enter the fuel element in a cosine direction distribution and finding the fraction actually absorbed as compared with the number that would be absorbed if there were no self-shielding. The cross sections for the moderator, etc. above thermal can usually be taken at their actual value. The Σ_a- and Σ_f-effective cross sections of the fuel obviously depend on the shape and size of the fuel elements. The reference that gives the formulas for getting these effective cross sections from the true cross sections is the work of Bartels.[6]

REFERENCES FOR CROSS-SECTION DATA

Now that methods have been introduced that require the knowledge of cross sections at various energies, it is perhaps well to give a list of references where cross-section data may be obtained. The following is not intended to be a complete list, but is merely a suggestion of some sources that have proved to be valuable and indicative of the type of sources that give cross-section data. The various physics journals, for example, *Reviews of Modern Physics*, as well as the progress reports of the national laboratories, give continuing articles on current cross-section measurements.

Some of the older sources are:

1. C. D. Coryell *et al.*, *The Science and Engineering of Nuclear Power*, Vol. I, ed. by C. D. Goodman, Addison-Wesley Press, Cambridge, Mass., 1947.

2. N. M. Smith, *KAPL Cross Section Curves for Xe-135, U-235 and U-238*, Y-F10-51 (Apr. 17, 1951).

(6) W. J. C. Bartels, *Self-Absorption of Monoenergetic Neutrons*, KAPL-336 (May 1, 1950).

3. G. Haines and K. Way, *Graphs Showing Neutron Cross Sections as Functions of A, Z or N,* ORNL-144 (Oct. 4, 1948).

4. K. Way and G. Haines, *Thermal Neutron Cross Sections for Elements and Isotopes H-Bi,* AECD-2138.

5. K. Way and G. Haines, *Tables of Neutron Cross Sections,* MonP-405 (Oct. 31, 1947).

6. G. Haines and K. Way, *Tables of Neutron Cross Sections for Elements Po-Cm,* ORNL-86 (Sept. 9, 1948).

7. K. Way, L. Fano, M. R. Scott, and K. Thew, *Nuclear Data,* NBS Circular No. 494 (January 1950).

8. S. Bernstein *et al., Phys. Quar. Prog. Rep. Dec., Jan., Feb., 1948-49,* ORNL-325, p. 6-68.

9. H. R. Kroeger, "Thermal Neutron Cross Sections and Related Data," *Nucleonics* **5**, No. 4, 51-54 (October 1949).

10. H. H. Goldsmith, H. W. Ibser, and B. T. Feld, *Rev. Mod. Phys.* **19**, No. 4, 259 (October 1947).

11. R. K. Adair, *Rev. Mod. Phys.* **22**, No. 3, 249 (July 1950).

12. S. H. Turkel, *Neutron Cross Sections,* NEPA-851 (Dec. 5, 1948).

Two recent (1952) sources are:

13. D. J. Hughes *et al., Neutron Cross Sections, A Compilation of the AEC Neutron Cross Section Advisory Group* (unclassified), AECU-2040 (May 15, 1952).

14. D. J. Hughes *et al., Neutron Cross Sections, A Compilation of the AEC Neutron Cross Section Advisory Group* (classified), BNL-170 (May 15, 1952).

CRITICAL MASS AND SIZE RESULTS

Figures 2.10 through 2.20 give the results of some criticality calculations made by the reactivity group of the NEPA Division of Fairchild Engine and Airplane Corp. The results were compiled by Mooneyham.[7] No structural material or fission-product poisons are present in these reactors. The temperature of all components is 70°F. The assemblies considered are homogeneous, gas-cooled, enriched, cylindrical reactors containing hydrogen, beryllium oxide, beryllium carbide, graphite, or iron. No reflection is present and the method of calculation was the Fermi age model of the thermal- or intermediate-energy type.

The free-flow ratio, R_{ff}, refers to the fraction of air (or void).

PROBLEMS

1. The Materials Testing Reactor (water moderated) has the following core constants: τ_{th} = 64 cm^2; L^2 = 3.64 cm^2; λ_{tr} = 0.8 cm; k_{th} = 1.606 (unpoisoned). What are the critical sizes of the equivalent bare-spherical, cubical, and square-cylinder reactors (that is, the unreflected reactor having the same composition as the core of the reflected reactor)? Note: Strictly speaking, the two-group theory gives more accurate results than does the Fermi age theory for water-moderated reactors.

2. In the depleted and poisoned MTR, k_{th} = 1.37. If τ_{th} and L^2 remain unchanged, what must be the new critical dimensions of the soluble geometrical shapes?

3. If a spherical mixture is pierced by uniform cooling holes to the extent that R_{ff} (free-flow ratio) is the ratio of volume of void to the total volume of material plus void, find by what factor the critical extrapolated radius must increase for criticality to be maintained. By what factor is the critical mass increased?

4. Find the *most probable* crow-flight distance for neutrons to travel from birth to the point at which they enter the thermal group.

5. Can a homogeneous mixture of graphite and natural uranium (ratio of atoms of U^{235} to atoms of U^{238} of 1:140) be made critical? What ratio of N_C to N_U gives the largest

(7) A. O. Mooneyham, *Comparative Critical Conditions in Simple Nuclear Reactors,* NEPA No. 1100-EAR-R13 (Aug. 1, 1949).

k_{eff}? Use the following data in obtaining an answer:

Density of graphite, ρ_C = 167 g/cm^3, ϵ = 1.0,

σ_a (carbon) = 0.045 ($\sqrt{\pi}/2$) barns for $E = E_{th}$ (the factor $\sqrt{\pi}/2$ is discussed at the end of chap. 2),

σ_{tr} (carbon) = 4.5 barns for $E = E_{th}$,

σ_s (carbon) = 4.8 barns for $E = E_{th}$,

σ_a (carbon) = 0 barns for $E > E_{th}$,

σ_a (U^{238}) = 0 barns for $E > 450$ ev and for $0.025 < E < 14$ ev,
= 22 barns for $14 < E < 450$ ev,
= 2.58 ($\sqrt{\pi}/2$) barns for $E = E_{th}$,

σ_a (U^{235}) = 640 ($\sqrt{\pi}/2$) barns for $E = E_{th}$,
= 0 barns for $E > E_{th}$,

σ_f (U^{235}) = 540 ($\sqrt{\pi}/2$) barns for $E = E_{th}$.

6. Start with Eqs. 2.82 and 2.83 and show that Eqs. 2.89 and 2.90 are the same for spheres and cylinders as for slabs.

7. By using the data given in Table 2 and the Fermi-age, bare-reactor, calculation form given in the appendix, calculate the k_{eff} of the bare reactor with the following volume composition:

UF_4-NaF (80 lb U^{235})	7.83%
Beryllium oxide	57.15%
Stainless steel	11.38%
Sodium	23.64%

The size of the reactor is such that

$$B^2 = 0.00240957 .$$

8. Calculate the flux for each energy interval, based on the condition of one fission per second per cubic centimeter.

9. Make a plot of the normalized fission spectrum, that is, the fraction of neutrons causing fission verus u that have lethargy in the unit lethargy range around u. Indicate the thermal neutrons causing fission by a block of unit lethargy width and proper height. Determine the median lethargy of the fission-producing neutrons and from this the median energy of fission-producing neutrons.

10. Plot the flux per unit lethargy vs. lethargy. Determine the median energy of the flux.

11. Determine the size of the square cylinder that has the buckling used in problem 10, and then determine the average scaler thermal flux per cubic centimeter and the average total flux per cubic centimeter in this square cylinder if the total power output is 200 megawatts. Use the relationship that 1 watt = 3.15×10^{10} fissions per second.

12. The spatial power distribution in the bare square cylinder is given by

$$P(r,z) = A\, J_0\left(\frac{2.405}{R+\delta}\, r\right) \cos \frac{\pi}{H+2\delta}\, z .$$

Find the ratio of the peak power per cubic centimeter to the average power per cubic centimeter in the reactor.

TABLE 2. DATA SHEET FOR PROBLEM 7

N	$\left(\overline{\frac{1}{\xi\Sigma_T}}\right)_N$	$\left(\overline{\frac{\Sigma_a}{\xi\Sigma_T}}\right)_N$	$\left(\overline{\frac{\Sigma_f}{\xi\Sigma_T}}\right)_N$	$\left(\overline{\frac{1}{3\xi\Sigma_T\Sigma_{tr}}}\right)_N$	u_{n-1} to u_n	U_N
1	65.77433	0.019966	0.015792	254.01735	0 to 0.5	0.5
2	45.04023	0.013858	0.010814	130.64921	0.5 to 1.0	0.5
3	35.22225	0.011062	0.008457	83.84716	1.0 to 1.5	0.5
4	42.11434	0.013873	0.010112	110.02552	1.5 to 2.0	0.5
5	20.22131	0.006866	0.004927	24.37731	2.0 to 2.5	0.5
6	16.69578	0.005889	0.004257	16.80520	2.5 to 3.0	0.5
7	16.16205	0.006032	0.004467	13.46409	3.0 to 3.5	0.5
8	17.63337	0.007035	0.005301	15.88139	3.5 to 4.0	0.5
9	16.13459	0.008905	0.007082	13.21423	4.0 to 7.0	3.0
10	14.25246	0.027469	0.017119	9.83269	7.0 to 10.0	3.0
11	14.16492	0.070070	0.045352	9.94368	10.0 to 11.4	1.4
12	13.83022	0.123428	0.079270	9.52384	11.4 to 12.6	1.2
13	13.49418	0.163932	0.103262	9.09876	12.6 to 13.4	0.8
14	13.29129	0.177802	0.110899	8.85152	13.4 to 13.8	0.4
15	13.20091	0.242961	0.102066	8.74185	13.8 to 14.6	0.8
16	13.3[illegible]	0.103066	0.036725	8.87807	14.6 to 15.8	1.2
17	13.41421	0.262795	0.153379	8.97189	15.8 to 16.2	0.4
18	13.41766	0.233133	0.134135	8.97380	16.2 to 16.6	0.4
19	13.39895	0.313958	0.189716	8.95565	16.6 to 17.0	0.4
20	13.35766	0.569988	0.364663	8.91850	17.0 to 17.4	0.4
21	13.33254	0.699962	0.455156	8.89643	17.4 to 17.6	0.2
22	13.34117	0.692234	0.443551	8.90693	17.6 to 17.8	0.2
23	13.50018	0.735471	0.474630	9.08708	17.8 to 18.0	0.2
24	13.57861	0.797224	0.514965	9.17057	18.0 to 18.2	0.2
25	13.55810	0.870304	0.559678	9.14951	18.2 to 18.4	0.2
26	13.52962	0.965446	0.623049	9.12371	18.4 to 18.6	0.2
92*	13.51717	1.024354	0.661604	9.11577		1.0**

*The thermal group $u_{th} = 18.6$ ($E_{th} \approx 0.084$ ev in this reactor, $T = 1286^{\circ}$F).

**The thermal group is fictitiously assigned a lethargy width of unity for simplicity of mechanics in the calculation form.

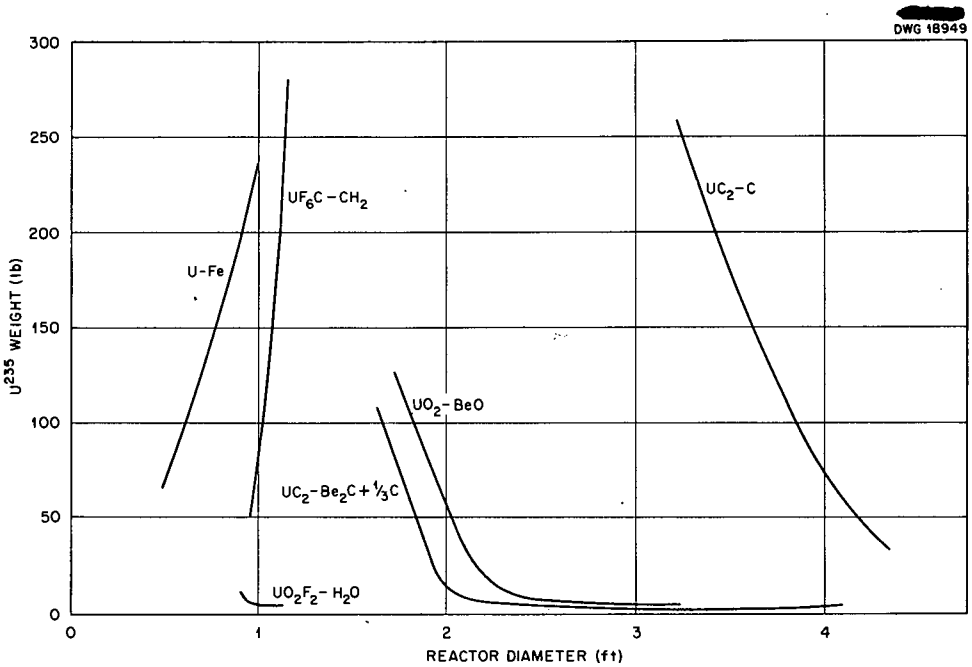


Fig. 2.10. Weight of U^{235} vs. Reactor Diameter; No Reflector; Free-Flow Ratio = 0.

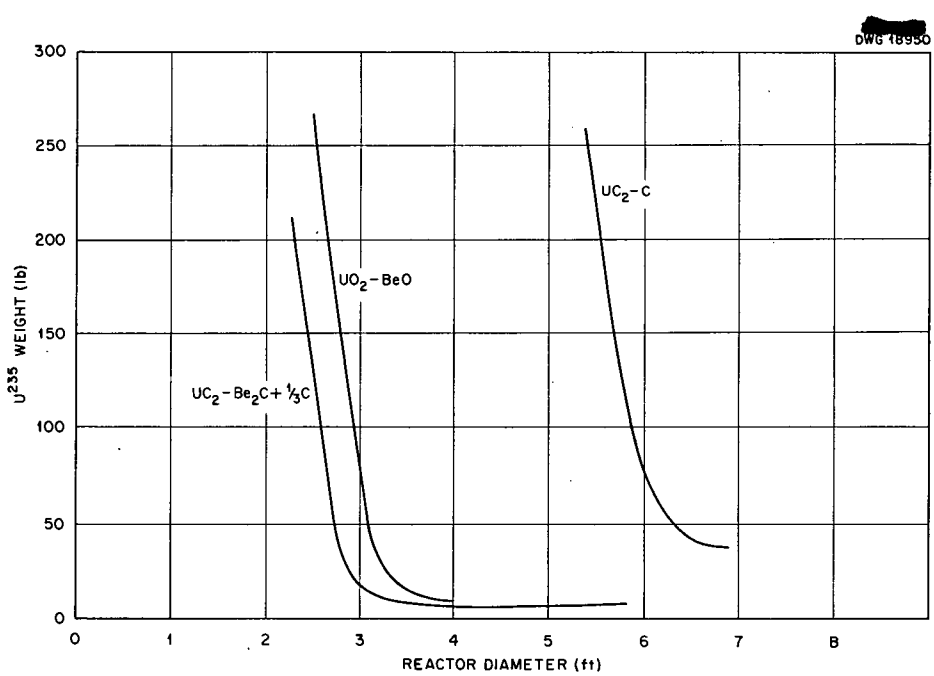


Fig. 2.11. Weight of U^{235} vs. Reactor Diameter; No Reflector; Free-Flow Ratio = 0.3.

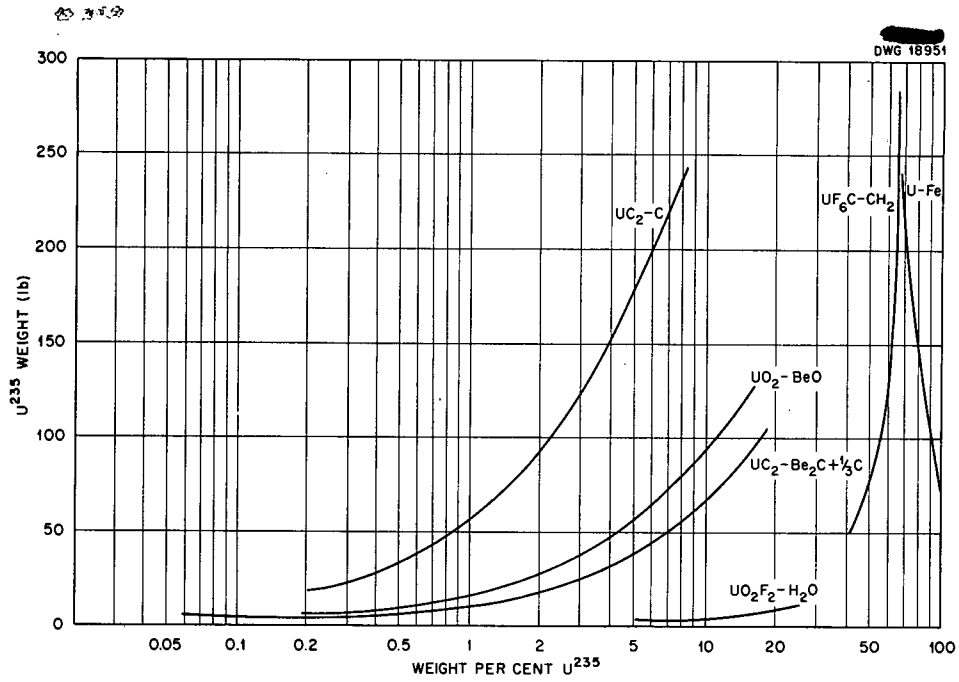


Fig. 2.12. Weight of U^{235} vs. Weight Per Cent U^{235}; No Reflector; Free-Flow Ratio = 0.

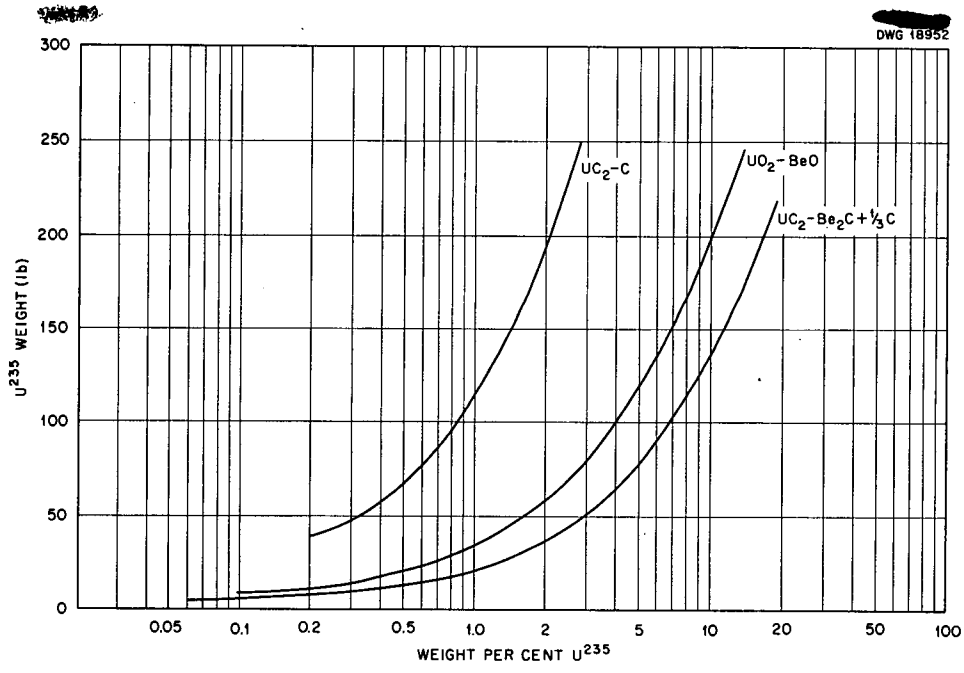


Fig. 2.13. Weight of U^{235} vs. Weight Per Cent U^{235}; No Reflector; Free-Flow Ratio = 0.3.

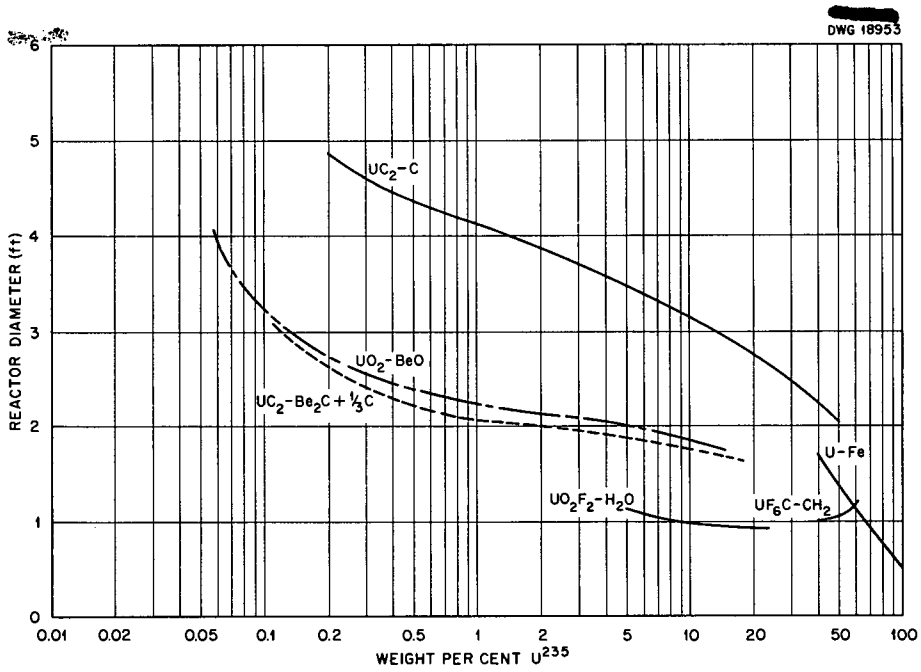


Fig. 2.14. Reactor Diameter vs. Weight Per Cent U^{235}; No Reflector; Free-Flow Ratio = 0.

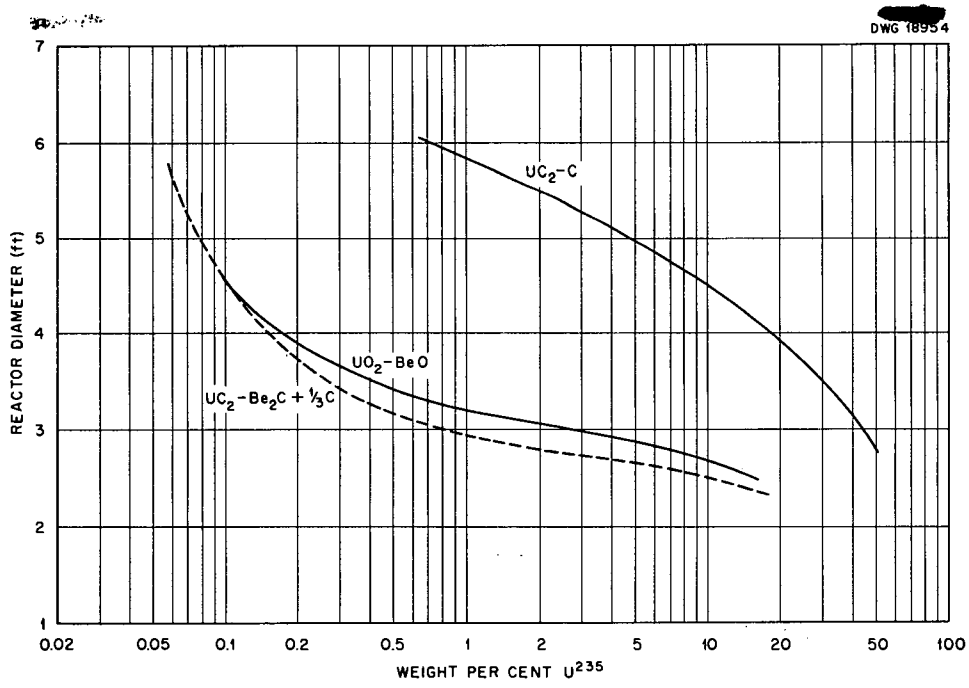


Fig. 2.15. Reactor Diameter vs. Weight Per Cent U^{235}; No Reflector; Free-Flow Ratio = 0.3.

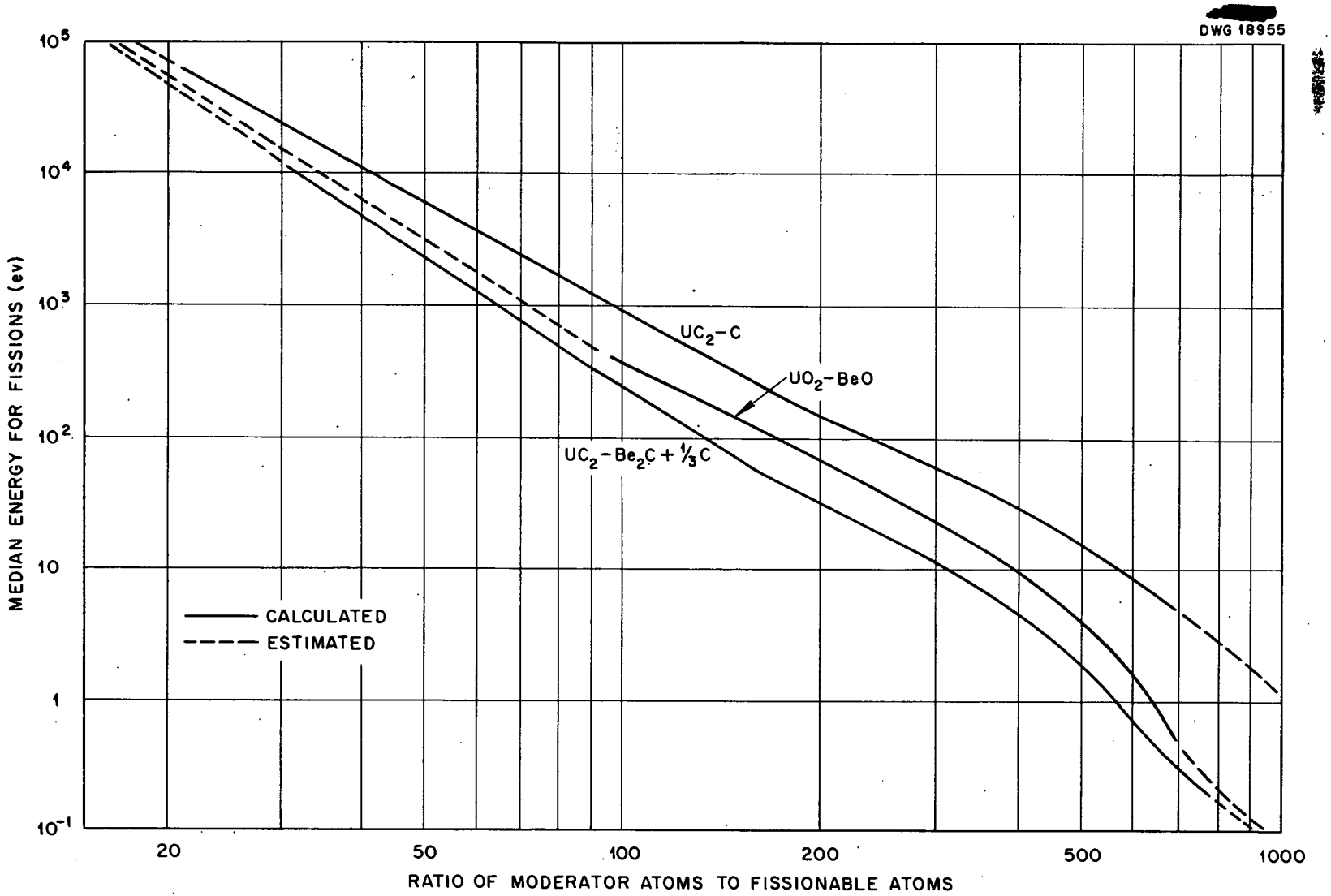


Fig. 2.16. Median Energy for Fission vs. Atomic Ratio.

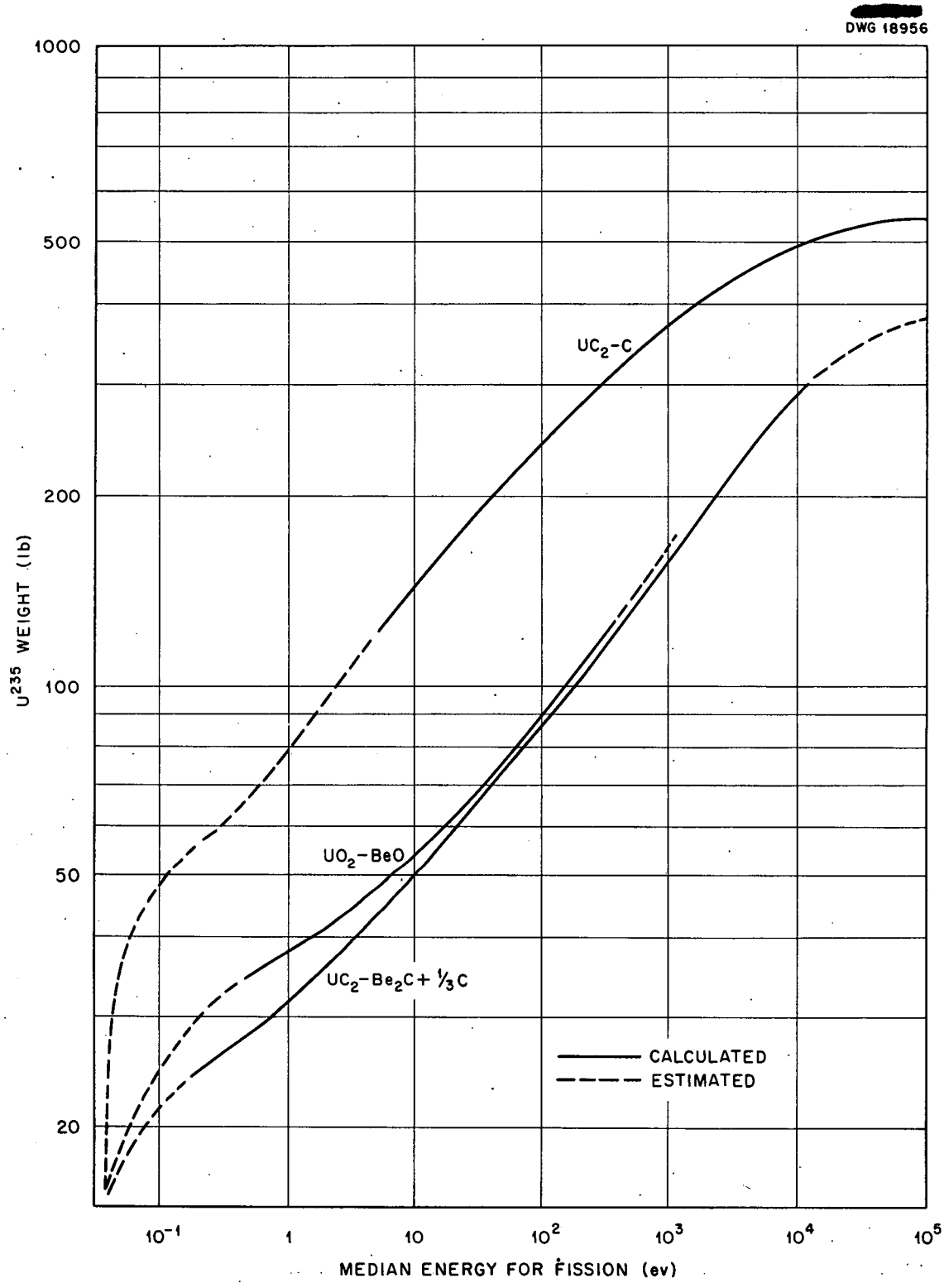


Fig. 2.17. Weight of U^{235} vs. Median Energy for Fission; No Reflector; Free-Flow Ratio = 0.

[redacted]

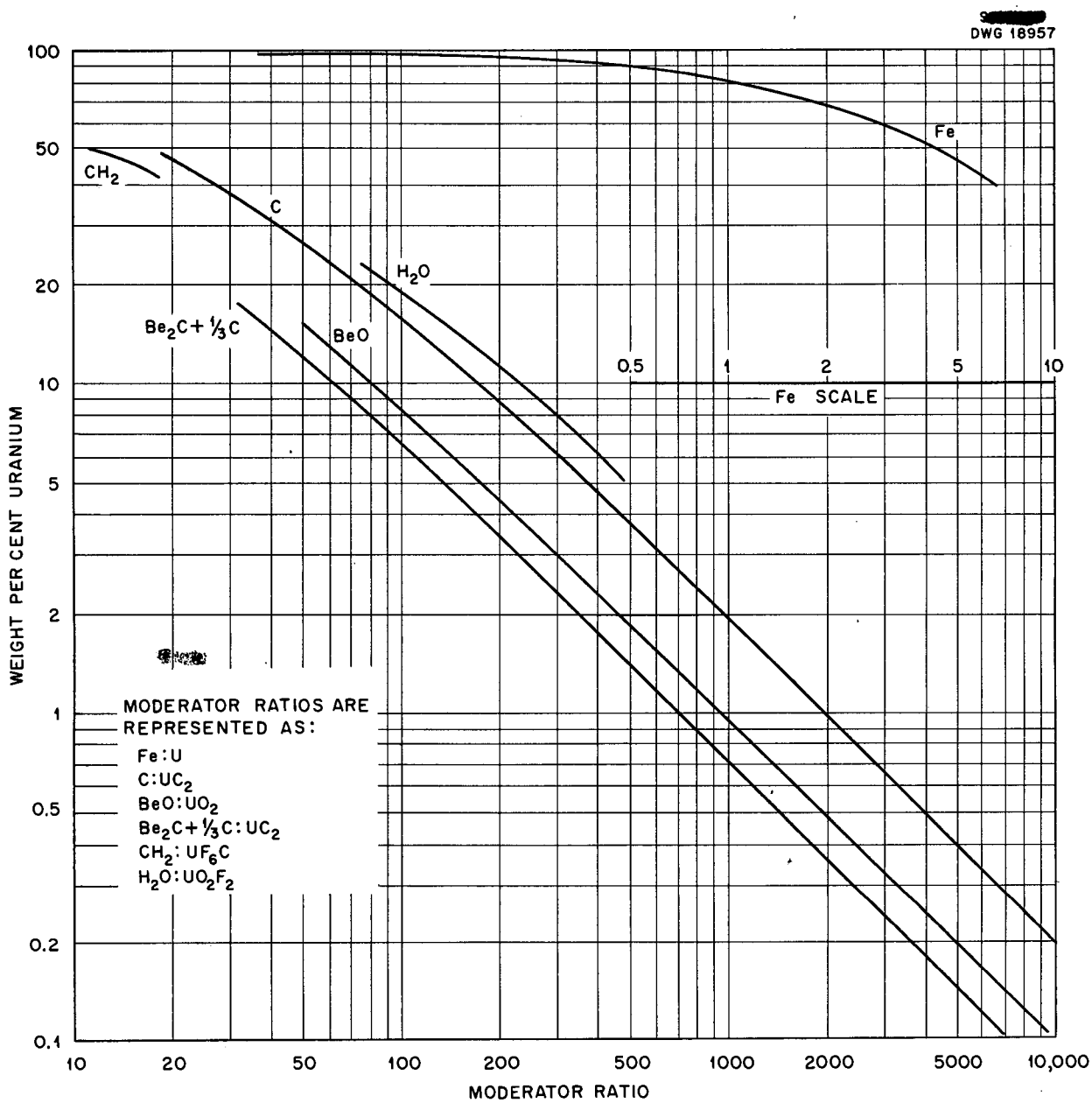


Fig. 2.18. Weight Per Cent U^{235} vs. Moderator Ratio.

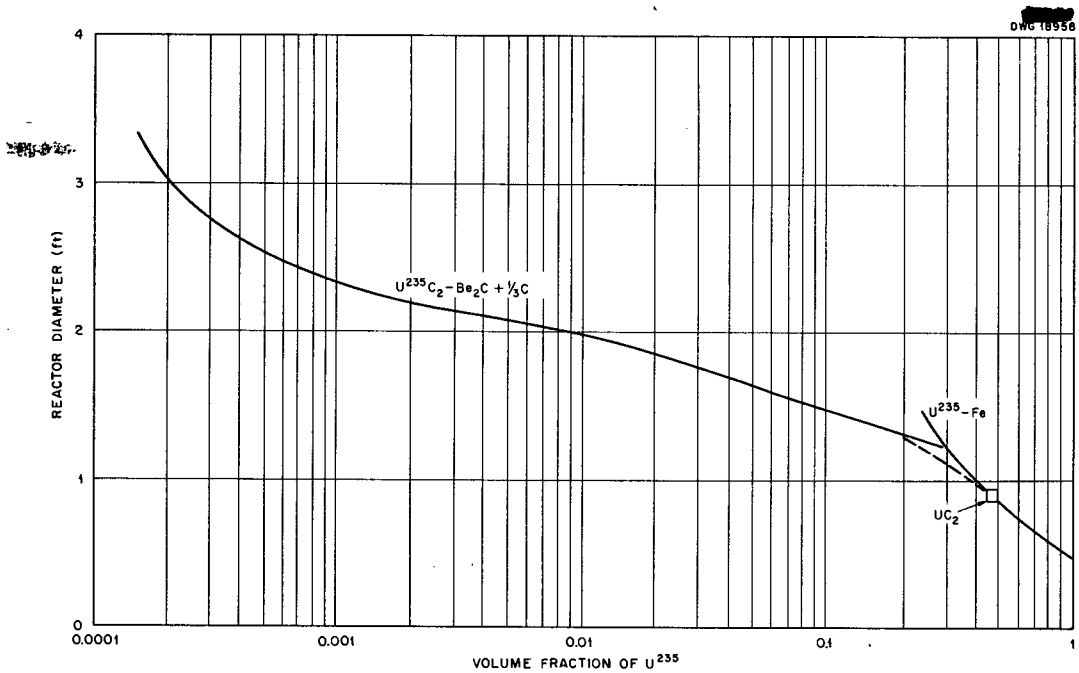


Fig. 2.19. Reactor Diameter vs. Volume Fraction of U^{235}; No Reflector; Free-Flow Ratio = 0.

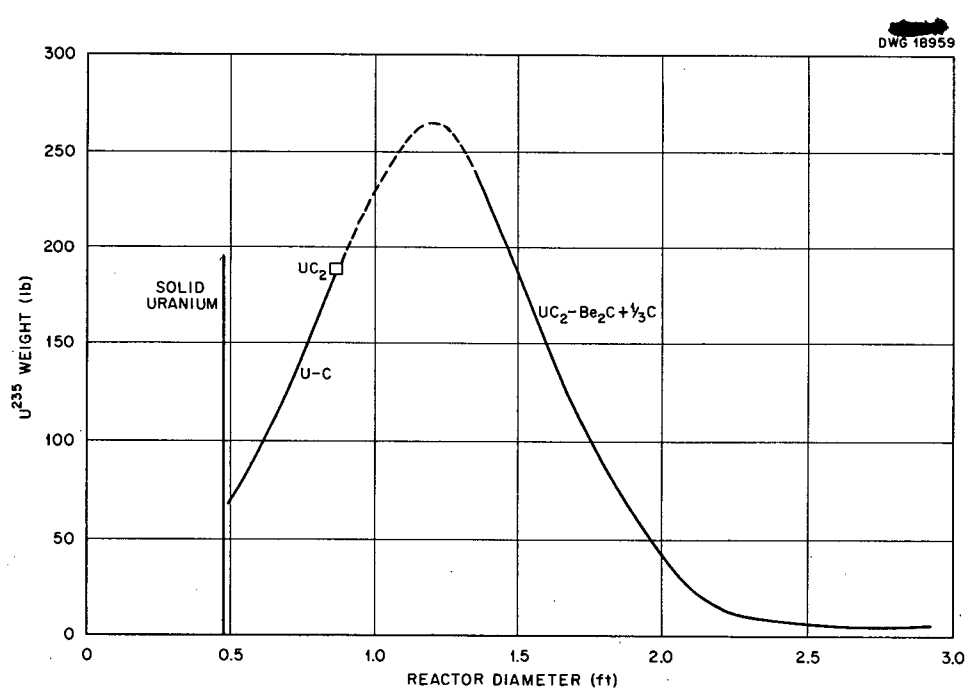


Fig. 2.20. Weight of U^{235} vs. Reactor Diameter; No Reflector; Free-Flow Ratio = 0.

Chapter 3
CRITICALITY BY TWO-GROUP THEORY

Chapter 2 presented the Fermi age treatment of criticality in which the slowing down of neutrons is taken to be continuous. The Fermi model is an excellent method of determining critical size (or mass) because of its relative accuracy and brevity. Even the intermediate-energy reactor problem can be solved in a couple of days. Unfortunately, however, the Fermi model is applicable only to reactors that are uniform in composition; that is, it cannot be applied to reflected reactors or reactors with control rods. Since almost all reactor designs have reflectors, it would seem that the Fermi model would indeed be limited. However, most reflected reactors can be represented by equivalent bare reactors for determination of the rough critical mass by simply extending the core of the given reactor a distance that is approximately equal to the reflector thickness (for reflector thicknesses of up to about 5 or 6 in.). A little experience with reflected reactor calculations makes it possible to guess fairly well the inches of core extension equivalent to a given thickness of reflector. Although the fissionable material is then treated as being throughout the entire extended bare reactor, it is, of course, only the fissionable material in the central or core region that is called the critical mass of the actual reactor. Numerous "reflector-savings" vs. reflector-thickness calculations have been reported in the literature. Figure 3.1 shows these quantities plotted from data obtained by two-group calculations.[1] It is to be noted that reflector saving is about equal to reflector thickness, up to 5 or 6 in., for reflectors that are of

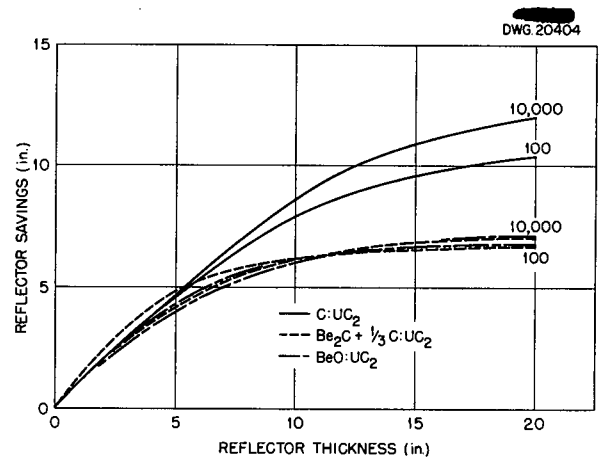


Fig. 3.1. Reflector Savings vs. Reflector Thickness for Various Moderators and Moderator Ratios; Free-Flow Ratio = 0.

about the same composition as the core. Reflectors usually have somewhat more moderator than the core (because of the decreased coolant requirement in the reflector), and thus the reflector savings are roughly equal to reflector thickness for even greater thicknesses than 5 or 6 inches.

It is clear though that the method of the equivalent bare reactor cannot be used to predict the *spatial power (or flux) distribution* in the reflected reactor; also, it cannot be used to handle the critical mass calculation if the reflector is thick (>6 in.) or if the reflector is radically different in composition from the core or to calculate a reactor with control rods. The one-group model described in chap. 1 can be used for such problems. In the one-group method, however, all neutrons are considered to be at some average velocity, and therefore the method is not at all accurate or reliable because one needs to know the answer, so to speak, in order to select an appropriate average velocity for

(1) A. O. Mooneyham, *Comparative Critical Conditions in Simple Nuclear Reactors*, NEPA No. 1100-EAR-R13 (Aug. 1, 1949).

the neutrons that will give good values for the constants. Also, it is not possible to obtain the correct power distribution with the one-group model. The two-group model, which is an improvement of the one-group model, is the subject for this chapter.

In the two-group model, all neutrons with energy above a certain value are classed as fast neutrons and the associated constants are given average values. All neutrons with energy below the dividing energy are classed as slow neutrons and, again, the associated constants are given average values. The dividing energy does not have to be taken as thermal energy, but this is the usual practice because the two-group model is most suitable for thermal reactors.

The case of the reflected reactor in which there is absorption of and fission by above-thermal, as well as thermal, neutrons is the primary topic of this chapter. The bare reactor and the completely thermal reactor are special cases of the following treatment.

BALANCE EQUATIONS

The neutron-balance differential equations in the core are

$$-\frac{\lambda_{tr_{fc}}}{3}\nabla^2\phi_{fc}(\underline{r}) + \phi_{fc}(\underline{r})\Sigma_c + \phi_{fc}(\underline{r})\Sigma_{a_{fc}} - \phi_{sc}(\underline{r})\Sigma_{a_{sc}}k_s - \phi_{fc}(\underline{r})\Sigma_{a_{fc}}k_f = 0 \quad (3.1)$$

and

$$-\frac{\lambda_{tr_{sc}}}{3}\nabla^2\phi_{sc}(\underline{r}) + \phi_{sc}(\underline{r})\Sigma_{a_{sc}} - \phi_{fc}\Sigma_c = 0\ , \quad (3.2)$$

where

$\lambda_{tr_{fc}}$ = an average transport mean free path in the core for the fast group,

Σ_c = a fictitious cross section that represents the probability per centimeter of a fast core neutron dropping into the slow core group,

$\Sigma_{a_{fc}}$ = an average macroscopic absorption cross section for core neutrons in the fast group,

$\Sigma_{a_{sc}}$ = an average macroscopic absorption cross section for core neutrons in the slow group,

$\lambda_{tr_{sc}}$ = an average transport mean free path in the core for the slow group,

k_s = number of fission neutrons produced per absorption of a slow neutron in any core material $= \nu\Sigma_{F_{sc}}/\Sigma_{a_{sc}}$,

$\Sigma_{F_{sc}}$ = an average macroscopic fission cross section for core neutrons in the slow group,

k_f = number of fission neutrons per absorption of a fast neutron in any core material $= \nu\Sigma_{F_{fc}}/\Sigma_{a_{fc}}$,

$\Sigma_{F_{fc}}$ = an average macroscopic fission cross section for core neutrons in the fast group,

ϕ_{fc} = fast-group flux in the core, neutrons/sec·cm^2,

ϕ_{sc} = slow-group flux in the core, neutrons/sec·cm^2.

The balance differential equations in the reflector are

$$-\frac{\lambda_{tr_{fR}}}{3}\nabla^2\phi_{fR}(\underline{r}) + \phi_{fR}(\underline{r})\Sigma_R + \phi_{fR}(\underline{r})\Sigma_{a_{fR}} = 0 \quad (3.3)$$

and

$$-\frac{\lambda_{tr_{sR}}}{3}\nabla^2\phi_{sR}(\underline{r}) + \phi_{sR}(\underline{r})\Sigma_{a_{sR}} - \phi_{fR}\Sigma_R = 0\ , \quad (3.4)$$

where the definitions are analagous to those for the core quantities.

GENERAL SOLUTIONS

Equations 3.1, 3.2, 3.3, and 3.4 will now be written with more compact notation: for the core,

$$\nabla^2 \phi_{fc}(\underline{r}) + a_c \phi_{fc}(\underline{r}) + b_c \phi_{sc}(\underline{r}) = 0 \quad (3.5)$$

and

$$\nabla^2 \phi_{sc}(\underline{r}) + d_c \phi_{sc}(\underline{r}) + e_c \phi_{fc}(\underline{r}) = 0 , \quad (3.6)$$

and for the reflector,

$$\nabla^2 \phi_{fR}(\underline{r}) + a_R \phi_{fR}(\underline{r}) = 0 \quad (3.7)$$

and

$$\nabla^2 \phi_{sR}(\underline{r}) + d_R \phi_{sR}(\underline{r}) + e_R \phi_{fR}(\underline{r}) = 0 , \quad (3.8)$$

where

$$a_c \equiv (- \Sigma_c - \Sigma_{a_{fc}} + \Sigma_{a_{fc}} k_f) \frac{3}{\lambda_{tr_{fc}}} , \quad (3.9)$$

$$b_c \equiv \Sigma_{a_{sc}} k_s \frac{3}{\lambda_{tr_{fc}}} , \quad (3.10)$$

$$d_c \equiv - \Sigma_{a_{sc}} \frac{3}{\lambda_{tr_{sc}}} , \quad (3.11)$$

$$e_c \equiv \Sigma_c \frac{3}{\lambda_{tr_{sc}}} , \quad (3.12)$$

$$a_R \equiv (- \Sigma_R + \Sigma_{a_{fR}}) \frac{3}{\lambda_{tr_{fR}}} , \quad (3.13)$$

$$d_R \equiv - \Sigma_{a_{sR}} \frac{3}{\lambda_{tr_{fR}}} , \quad (3.14)$$

$$e_R \equiv \Sigma_R \frac{3}{\lambda_{tr_{fR}}} . \quad (3.15)$$

The general solutions of Eqs. 3.5, 3.6, 3.7, and 3.8 will now be obtained for the three geometries: slab, sphere, and cylinder.

Slab. For the slab, the core Eqs. 3.5, and 3.6 take the form

$$\frac{d^2 \phi_{fc}(x)}{dx^2} + a_c \phi_{fc}(x) + b_c \phi_{sc}(x) = 0 \quad (3.16)$$

and

$$\frac{d^2 \phi_{sc}(x)}{dx^2} + d_c \phi_{sc}(x) + e_c \phi_{fc}(x) = 0 , \quad (3.17)$$

and reflector Eqs. 3.7 and 3.8 take the form

$$\frac{d^2 \phi_{fR}(x)}{dx^2} + a_R \phi_{fR}(x) = 0 \quad (3.18)$$

and

$$\frac{d^2 \phi_{sR}(x)}{dx^2} + d_R \phi_{sR}(x) + e_R \phi_{fR}(x) = 0 . \quad (3.19)$$

For the core, solutions of the following form are tried:

$$\phi_{fc}(x) = C_c \cos B_c x \quad (3.20)$$

and

$$\phi_{sc}(x) = A_c \cos B_c x \quad (3.21)$$

to see whether a relation in B results that gives a value or values of B such that solutions of the form 3.20 and 3.21 exist. By substituting Eqs. 3.20 and 3.21 into Eqs. 3.16 and 3.17 and dividing through by $\cos B_c x$, the results are

$$- C_c B^2 + a_c C_c + b_c A_c = 0 \quad (3.22)$$

and

$$- A_c B^2 + d_c A_c + e_c C_c = 0 , \quad (3.23)$$

or

$$C_c(-B^2 + a_c) + A_c b_c = 0 \quad (3.24)$$

and

$$C_c e_c + A_c(-B^2 + d_c) = 0 . \quad (3.25)$$

For a nontrivial solution for A_c and C_c to exist requires that the determinant of the coefficients be zero:

$$\begin{vmatrix} -B^2 + a_c & b_c \\ e_c & -B^2 + d_c \end{vmatrix} = 0. \quad (3.26)$$

This is clearly a quadratic equation in B_c^2 that determines two values of B_c^2 such that solutions for Eqs. 3.16 and 3.17 of the form 3.20 and 3.21 exist. For a reproducing medium (core) the roots are real; one root is always positive and one negative. The positive root will be denoted by B_{1c}^2 and the negative root by $-B_{2c}^2$. Then

$$\phi_{fc}(x) = C_{1c} \cos B_{1c}x \quad (3.27)$$

and

$$\phi_{sc}(x) = A_{1c} \cos B_{1c}x \quad (3.28)$$

which form a pair of particular solutions for the fast and slow flux. From Eq. 3.25, it follows that

$$C_{1c} = \frac{B_{1c}^2 - d_c}{e_c} A_{1c} , \quad (3.29)$$

and hence Eqs. 3.20 and 3.21 can be written as

$$\phi_{fc}(x) = A_{1c} \frac{B_{1c}^2 - d_c}{e_c} \cos B_{1c}x \quad (3.30)$$

and

$$\phi_{sc}(x) = A_{1c} \cos B_{1c}x \quad . \quad (3.31)$$

Also,

$$\begin{aligned} \phi_{fc}(x) &= C_{2c} \cos iB_{2c}x \\ &= C_{2c} \cosh B_{2c}x \\ &= A_{2c} \left(- \frac{B_{2c}^2 - d_c}{e_c} \right) \cosh B_{2c}x \end{aligned} \quad (3.32)$$

and

$$\phi_{sc}(x) = A_{2c} \cosh B_{2c}x \quad (3.33)$$

form another pair of particular solutions.

On the other hand, if the sine had been tried instead of the cosine in Eqs. 3.20 and 3.21 the procedure in Eqs. 3.22 to 3.26 would have been just the same and the same B's would have been determined. Hence, there are two further pairs of particular solutions:

$$\phi_{fc}(x) = A_{3c} \left(\frac{B_{1c}^2 - d_c}{e_c} \right) \sin B_{1c}x \quad (3.34)$$

$$\phi_{sc}(x) = A_{3c} \sin B_{1c}x \quad (3.35)$$

and

$$\phi_{fc}(x) = A_{4c} \left(\frac{-B_{2c}^2 - d_c}{e_c} \right) \sinh B_{2c}x \quad (3.36)$$

$$\phi_{sc}(x) = A_{4c} \sinh B_{2c}x \quad . \quad (3.37)$$

The general solution is the sum of the particular solutions, and therefore the general solution of the core equations is

$$\phi_{fc}(x) = A_{1c} \left(\frac{B_{1c}^2 - d_c}{e_c} \right) \cos B_{1c}x + A_{2c} \left(\frac{-B_{2c}^2 - d_c}{e_c} \right) \cosh B_{2c}x + A_{3c} \left(\frac{B_{1c}^2 - d_c}{e_c} \right) \sin B_{1c}x + A_{4c} \left(\frac{-B_{2c}^2 - d_c}{e_c} \right) \sinh B_{2c}x \quad (3.38)$$

$$\phi_{sc}(x) = A_{1c} \cos B_{1c}x + A_{2c} \cosh B_{2c}x + A_{3c} \sin B_{1c}x + A_{4c} \sinh B_{2c}x. \quad (3.39)$$

The same procedure is used for obtaining the general solutions of the reflector equations. If solutions of the form

$$\phi_{fR} = C \cos B_R x \tag{3.40}$$

and

$$\phi_{sR} = A \cos B_R x \tag{3.41}$$

are tried in Eqs. 3.18 and 3.19, it is found that the determinant equation for the two values of B_R^2 is

$$\begin{vmatrix} -B_R^2 + A_R & 0 \\ e_R & -B^2 + d_R \end{vmatrix} = 0. \tag{3.42}$$

It develops that both values of B^2 are negative in a nonreproducing medium. The values will be denoted by $-B_{1R}^2$ and $-B_{2R}^2$. The general solutions of the reflector equations are

$$\begin{aligned}\phi_{fR}(x) = {} & A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right) \cosh B_{1R}x \\ & + A_{2R}\left(\frac{-B_{2R}^2 - d_R}{e_R}\right) \cosh B_{2R}x + A_{3R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right) \sinh B_{1R}x \\ & + A_{4R}\left(\frac{-B_{2R}^2 - d_R}{e_R}\right) \sinh B_{2R}x\end{aligned} \tag{3.43}$$

and

$$\phi_{sR}(x) = A_{1R} \cosh B_{1R}x + A_{2R} \cosh B_{2R}x + A_{3R} \sinh B_{1R}x + A_{4R} \sinh B_{2R}x\,. \tag{3.44}$$

However, from the determinant equation in B_R^2 (Eq. 3.42), it can be seen that the two solutions for B_R^2 are

$$-B_{1R}^2 = +a_R \tag{3.45}$$

and

$$-B_{2R}^2 = +d_R \tag{3.46}$$

Hence, the second and fourth terms in Eq. 3.43 become zero, and the general solutions reduce to

$$\phi_{fR}(x) = A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right) \cosh B_{1R}x + A_{3R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right) \sinh B_{1R}x \tag{3.47}$$

and

$$\phi_{sR}(x) = A_{1R} \cosh B_{1R}x + A_{2R} \cosh B_{2R}x + A_{3R} \sinh B_{1R}x + A_{4R} \sinh B_{2R}x. \tag{3.48}$$

Sphere. By a procedure analagous to that used for the slab, it is found that the general solutions of the core equations of the reflected spherical reactor are

$$\phi_{fc}(r) = A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\cos B_{1c}r}{r} + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\cosh B_{2c}r}{r}$$

$$+ A_{3c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\sin B_{1c}r}{r} + A_{4c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\sinh B_{2c}r}{r} \quad (3.49)$$

and

$$\phi_{sc}(r) = A_{1c}\frac{\cos B_{1c}r}{r} + A_{2c}\frac{\cosh B_{2c}r}{r} + A_{3c}\frac{\sin B_{1c}r}{r} + A_{4c}\frac{\sinh B_{2c}r}{r}, \quad (3.50)$$

where the two solutions of the core buckling are B_{1c}^2 and $-B_{2c}^2$ (as given by Eq. 3.26).

The general solutions of the reflector equations are

$$\phi_{fR}(r) = A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\frac{\cosh B_{1R}r}{r} + A_{3R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\frac{\sinh B_{1R}r}{r} \quad (3.51)$$

and

$$\phi_{sR}(r) = A_{1R}\frac{\cosh B_{1R}r}{r} + A_{2R}\frac{\cosh B_{2R}r}{r} + A_{3R}\frac{\sinh B_{1R}r}{r} + A_{4R}\frac{\sinh B_{2R}r}{r}, \quad (3.52)$$

where the two solutions of the reflector buckling are $-B_{1R}^2$ and $-B_{2R}^2$ (as given by Eq. 3.42).

Cylinder. The reactor with a cylindrical core and an all-around reflector is not soluble by analytical means. However, the case of a cylindrical core with a jacket reflector and no end reflectors is soluble, and this useful geometry will be treated here.

The general solution of the core equations in cylindrical coordinates is

$$\phi_{fc}(r,z) = \left(\frac{B_{1c}^2 - d_c}{e_c}\right)\left[A_{1c}\,J_0(\mu_{1c}r) + A_{3c}\,Y_0(\mu_{1c}r)\right]\left[\cos\epsilon_{1c}z + E_{1c}\sin\epsilon_{1c}z\right]$$

$$+ \left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\left[A_{2c}\,I_0(\mu_{2c}r) + A_{4c}\,K_0(\mu_{2c}r)\right]\left[\cos\epsilon_{2c}z + E_{2c}\sin\epsilon_{2c}z\right] \quad (3.53)$$

and

$$\phi_{sc}(r,z) = \left[A_{1c}\,J_0(\mu_{1c}r) + A_{3c}\,Y_0(\mu_{1c}r)\right]\left[\cos\epsilon_{1c}z + E_{1c}\sin\epsilon_{1c}z\right]$$

$$+ \left[A_{2c}\,I_0(\mu_{2c}r) + A_{4c}\,K_0(\mu_{2c}r)\right]\left[\cos\epsilon_{2c}z + E_{2c}\sin\epsilon_{2c}z\right], \quad (3.54)$$

where

$$B_{1c}^2 = \mu_{1c}^2 + \epsilon_{1c}^2 \quad , \tag{3.55}$$

$$-B_{2c}^2 = -\mu_{2c}^2 + \epsilon_{2c}^2 \quad , \tag{3.56}$$

and B_{1c}^2 and $-B_{2c}^2$ are the positive and negative root of Eq. 3.26. Actually, $\epsilon_{1c}^2 = \epsilon_{2c}^2$, as will be discussed further, because the reactor was chosen to be uniform in composition in the axial direction.

The general solution of the jacket reflector equations in cylindrical coordinates is

$$\phi_{fR}(r,z) = \left(\frac{-B_{1R}^2 - d_R}{e_R}\right) \left[A_{1R}\ I_0(\mu_{1R}r) + A_{3R}\ K_0(\mu_{1R}r)\right] \left[\cos \epsilon_{1R}z + E_{1R} \sin \epsilon_{1R}z\right] \tag{3.57}$$

and

$$\phi_{sR}(r,z) = \left[A_{1R}\ I_0(\mu_{1R}r) + A_{3R}\ K_0(\mu_{1R}r)\right] \left[\cos \epsilon_{1R}z + E_{1R} \sin \epsilon_{1R}z\right] + \left[A_{2R}\ I_0(\mu_{2R}r) + A_{4R}\ K_0(\mu_{2R}r)\right] \left[\cos \epsilon_{2R}z + E_{2R} \sin \epsilon_{2R}z\right] \quad , \tag{3.58}$$

where

$$-B_{1R}^2 = -\mu_{1R}^2 + \epsilon_{1R}^2 \quad , \tag{3.59}$$

$$-B_{2R}^2 = -\mu_{2R}^2 + \epsilon_{2R}^2 \quad , \tag{3.60}$$

and $-B_{1R}^2$ and $-B_{2R}^2$ are the two negative values of the buckling that result from the reflector determinant Eq. 3.42. Actually,

$$\epsilon_{1R}^2 = \epsilon_{2R}^2 = \epsilon_{1c}^2 = \epsilon_{2c}^2 \quad , \tag{3.61}$$

because the jacket reflector is the same length as the core. This equality will be discussed further in a later section.

BOUNDARY CONDITIONS

The boundary conditions of the problem are:

1. The fast and slow flux must be everywhere finite.
2. The fast and slow flux must be symmetric about any points, lines, or planes of symmetry in the geometrical setup.
3. The fast and slow flux must be nonnegative at all points in the reactor.
4. The fast and slow flux must extrapolate to zero at the external surfaces of the reflector.
5. The flux and the current must be continuous across the interface between core and reflector.

For slabs,

$$\phi_{fc}(x_0) = \phi_{fR}(x_0) \quad , \tag{3.62}$$

$$\phi_{sc}(x_0) = \phi_{sR}(x_0) \quad , \tag{3.63}$$

$$-\frac{\lambda_{tr_{fc}}}{3}\left[\frac{d\phi_{fc}(x)}{dx}\right]_{x=x_0} = -\frac{\lambda_{tr_{fR}}}{3}\left[\frac{d\phi_{fR}(x)}{dx}\right]_{x=x_0} , \tag{3.64}$$

$$-\frac{\lambda_{tr_{sc}}}{3}\left[\frac{d\phi_{sc}(x)}{dx}\right]_{x=x_0} = -\frac{\lambda_{tr_{sR}}}{3}\left[\frac{d\phi_{sR}(x)}{dx}\right]_{x=x_0} , \tag{3.65}$$

where x_0 is the half thickness of the core.

For spheres,

$$\phi_{fc}(R) = \phi_{fR}(R) \quad , \tag{3.66}$$

$$\phi_{sc}(R) = \phi_{sc}(R) \quad , \tag{3.67}$$

$$-\frac{\lambda_{tr_{fc}}}{3}\left[\frac{d\phi_{fc}(r)}{dr}\right]_{r=R} = -\frac{\lambda_{tr_{fR}}}{3}\left[\frac{d\phi_{fR}(r)}{dr}\right]_{r=R} , \tag{3.68}$$

$$-\frac{\lambda_{tr_{sc}}}{3}\left[\frac{d\phi_{sc}(r)}{dr}\right]_{r=R} = -\frac{\lambda_{tr_{sR}}}{3}\left[\frac{d\phi_{sR}(r)}{dr}\right]_{r=R} , \tag{3.69}$$

where R is the radius of the spherical core.

For cylinders (jacket reflector, no end reflectors),

$$\left.\begin{aligned} \phi_{fc}(R,z) &= \phi_{fR}(R,z) \quad (3.70) \\ \phi_{sc}(R,z) &= \phi_{sR}(R,z) \quad (3.71) \end{aligned}\right\} \text{for all } z \text{ over the length,}$$

$$-\frac{\lambda_{tr_{fc}}}{3}\left[\frac{\partial\phi_{fc}(r,z)}{\partial r}\right]_{r=R} = -\frac{\lambda_{tr_{fR}}}{3}\left[\frac{\partial\phi_{fR}(r,z)}{\partial r}\right]_{r=R} , \tag{3.72}$$

$$-\frac{\lambda_{tr_{sc}}}{3}\left[\frac{\partial\phi_{sc}(r,z)}{\partial r}\right]_{r=R} = -\frac{\lambda_{tr_{sR}}}{3}\left[\frac{\partial\phi_{sR}(r,z)}{\partial r}\right]_{r=R} , \tag{3.73}$$

where R is the radius of the cylindrical core.

APPLICATION OF BOUNDARY CONDITIONS TO GENERAL SOLUTIONS FOR THE CRITICAL SIZE

Slab. Upon application of boundary condition 2, the general solutions, Eqs. 3.38 and 3.39, for the fluxes in the core of the reflected slab reduce immediately to

$$\phi_{fc}(x) = A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\cos B_{1c}x + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\cosh B_{2c}x \tag{3.74}$$

and

$$\phi_{sc}(x) = A_{1c}\cos B_{1c}x + A_{2c}\cosh B_{2c}x \quad . \tag{3.75}$$

Boundary condition 4 can be satisfied by choosing A_{3R} and A_{4R} so that the flux expressions in the reflector, Eqs. 3.43 and 3.44, become

$$\phi_{fR}(x) = A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\left[\cosh B_{1R}x - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x\right] \tag{3.76}$$

and

$$\phi_{sR}(x) = A_{1R}\left[\cosh B_{1R}x - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x\right] + A_{2R}\left[\cosh B_{2R}x - \frac{\cosh B_{2R}(x_0 + T + \delta)}{\sinh B_{2R}(x_0 + T + \delta)}\sinh B_{2R}x\right] , \tag{3.77}$$

where T is the reflector thickness and δ is the extrapolation distance. Boundary condition 5 provides the following four homogeneous equations in the four unknown constants A_{1c}, A_{2c}, A_{1R}, and A_{2R}:

$$A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\cos B_{1c}x_0 + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\cosh B_{2c}x_0 - A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\left[\cosh B_{1R}x_0 - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x_0\right] = 0 , \tag{3.78}$$

$$A_{1c}\cos B_{1c}x_0 + A_{2c}\cosh B_{2c}x_0 - A_{1R}\left[\cosh B_{1R}x_0 - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x_0\right] - A_{2R}\left[\cosh B_{2R}x_0 - \frac{\cosh B_{2R}(x_0 + T + \delta)}{\sinh B_{2R}(x_0 + T + \delta)}\sinh B_{2R}x_0\right] = 0 , \tag{3.79}$$

$$A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3} B_{1c} \sin B_{1c}x_0$$

$$- A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3} B_{2c} \sinh B_{2c}x_0$$

$$+ A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3} B_{1R}\left[\sinh B_{1R}x_0\right.$$

$$\left. - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)} \cosh B_{1R}x_0\right] = 0 , \quad (3.80)$$

$$A_{1c}B_{1c}\frac{\lambda_{tr_{sc}}}{3} \sin B_{1c}x_0$$

$$- A_{2c}B_{2c}\frac{\lambda_{tr_{sc}}}{3} \sinh B_{2c}x_0 + A_{1R}\frac{\lambda_{tr_{sR}}}{3} B_{1R}\left[\sinh B_{1R}x_0\right.$$

$$\left. - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)} \cosh B_{1R}x_0\right] + A_{2R}\frac{\lambda_{tr_{sR}}}{3} B_{2R}\left[\sinh B_{2R}x_0\right.$$

$$\left. - \frac{\cosh B_{2R}(x_0 + T + \delta)}{\sinh B_{2R}(x_0 + T + \delta)} \cosh B_{2R}x_0\right] = 0 . \quad (3.81)$$

If nontrivial solutions are to exist for A_{1c}, A_{2c}, A_{1R}, and A_{2R} in these homogeneous equations, the determinant of their coefficients in Eq. 3.78 to 3.81 must be equal to zero. The determinantal equation involves x_0 and T and is the *criticality equation*. One can either specify x_0 and solve for T, by trial and error, or specify T and solve for x_0.

The determinantal criticality equation is

$$\begin{vmatrix}
\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\cos B_{1c}x_0 & \left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\cosh B_{2c}x_0 & \left(\frac{B_{1R}^2 + d_R}{e_R}\right)\left[\cosh B_{1R}x_0 - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x_0\right] & 0 \\
\cos B_{1c}x_0 & \cosh B_{2c}x_0 & \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\sinh B_{1R}x_0 - \cosh B_{1R}x_0 & \frac{\cosh B_{2R}(x_0 + T + \delta)}{\sinh B_{2R}(x_0 + T + \delta)}\sinh B_{2R}x_0 - \cosh B_{2R}x_0 \\
\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}B_{1c}\sin B_{1c}x_0 & \left(\frac{B_{2c}^2 + d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}B_{2c}\sinh B_{2c}x_0 & \left(\frac{B_{1R}^2 + d_R}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3}B_{1R}\left[\frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\cosh B_{1R}x_0 - \sinh B_{1R}x_0\right] & 0 \\
B_{1c}\frac{\lambda_{tr_{sc}}}{3}\sin B_{1c}x_0 & -B_{2c}\frac{\lambda_{tr_{sc}}}{3}\sinh B_{2c}x_0 & \frac{\lambda_{tr_{sR}}}{3}B_{1R}\left[\sinh B_{1R}x_0 - \frac{\cosh B_{1R}(x_0 + T + \delta)}{\sinh B_{1R}(x_0 + T + \delta)}\cosh B_{1R}x_0\right] & \frac{\lambda_{tr_{sR}}}{3}B_{2R}\left[\sinh B_{2R}x_0 - \frac{\cosh B_{2R}(x_0 + T + \delta)}{\sinh B_{2R}(x_0 + T + \delta)}\cosh B_{2R}x_0\right]
\end{vmatrix} = 0 . \quad (3.82)$$

In review, the procedure for getting the critical size when the composition is given is to evaluate a_c, b_c, d_c, e_c, a_R, d_R, and e_R from the macroscopic cross-section averages by using Eqs. 3.9 through 3.15, obtain B_{1c}^2 and $-B_{2c}^2$ as the roots of Eq. 3.26, obtain $-B_{1R}^2$ and $-B_{2R}^2$ as the roots of Eq. 3.42, specify the reflector thickness T, and solve the determinantal equation, Eq. 3.82, for the critical half thickness of the core, x_0, by trial and error.

The converse problem is, as before, to have the x_0 and T given and the uranium concentration as the unknown. The procedure is then, clearly, to make trial solutions with various uranium concentrations until the set of constants, a_c, b_c, d_c, e_c, is found that, together with a_R, d_R, and e_R and the resulting buckling constants, satisfies Eq. 3.82. Once criticality has been established, three of the constants A_{1c}, A_{2c}, A_{1R}, and A_{2R} can be solved for in terms of the remaining constant by using Eqs. 3.78 to 3.81. The remaining constant is the power constant. By substituting the constants thus obtained into Eqs. 3.74 through 3.77, the fast and slow fluxes in the core and reflector can be evaluated to within the one unknown constant, which depends on the power level at which the reactor is operating.

Sphere. Upon application of boundary condition 1, the general solutions Eqs. 3.49 and 3.50, for the fluxes in the core of the reflected sphere reduce immediately to

$$\phi_{fc}(r) = A_{3c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\sin B_{1c}r}{r} + A_{4c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\sinh B_{2c}r}{r} \quad (3.83)$$

and

$$\phi_{sc}(r) = A_{3c}\frac{\sin B_{1c}r}{r} + A_{4c}\frac{\sinh B_{2c}r}{r} . \quad (3.84)$$

Boundary condition 4 can be satisfied by choosing A_{3R} and A_{4R} so that the reflector flux expressions, Eqs. 3.51 and 3.52, become

$$\phi_{fR}(r) = A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\left[\frac{\cosh B_{1R}r}{r} - \frac{\cosh B_{1R}(R + T + \delta)}{\sinh B_{1R}(R + T + \delta)}\frac{\sinh B_{1R}r}{r}\right] \quad (3.85)$$

and

$$\phi_{sR}(r) = A_{1R}\left[\frac{\cosh B_{1R}r}{r} - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\,\frac{\sinh B_{1R}r}{r}\right]$$

$$+ A_{2R}\left[\frac{\cosh B_{2R}r}{r} - \frac{\cosh B_{2R}(R+T+\delta)}{\sinh B_{2R}(R+T+\delta)}\,\frac{\sinh B_{2R}r}{r}\right], \quad (3.86)$$

where T is the reflector thickness and $\delta = 0.71\ \lambda_{tr}$ is the extrapolation distance.

Boundary condition 5 provides the following four homogeneous equations in the four unknown constants A_{3c}, A_{4c}, A_{1R}, and A_{2R}.

$$A_{3c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\sin B_{1c}R}{R} + A_{4c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\sinh B_{2c}R}{R}$$

$$- A_{1R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\left[\frac{\cosh B_{1R}R}{R} - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\,\frac{\sinh B_{1R}R}{R}\right] = 0\ , \quad (3.87)$$

$$A_{3c}\,\frac{\sin B_{1c}R}{R} + A_{4c}\,\frac{\sinh B_{2c}R}{R}$$

$$- A_{1R}\left[\frac{\cosh B_{1R}R}{R} - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\,\frac{\sinh B_{1R}R}{R}\right]$$

$$- A_{2R}\left[\frac{\cosh B_{2R}R}{R} - \frac{\cosh B_{2R}(R+T+\delta)}{\sinh B_{2R}(R+R+\delta)}\,\frac{\sinh B_{2R}R}{R}\right] = 0\ , \quad (3.88)$$

$$A_{3c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\left[\frac{RB_{1c}\cos B_{1c}R - \sin B_{1c}R}{R^2}\right]$$

$$+ A_{4c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\left[\frac{RB_{2c}\cosh B_{2c}R - \sinh B_{2c}R}{R^2}\right]$$

$$- A_{1R}\left(\frac{-B_{1R}^2 - d_c}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3}\left[\frac{RB_{1R}\sinh B_{1R}R - \cosh B_{1R}R}{R^2}\right.$$

$$\left. - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\,\frac{RB_{1R}\cosh B_{1R}R - \sinh B_{1R}R}{R^2}\right] = 0\ , \quad (3.89)$$

$$A_{3c}\frac{\lambda_{tr_{sc}}}{3}\left[\frac{RB_{1c}\cos B_{1c}R - \sin B_{1c}R}{R^2}\right] + A_{4c}\frac{\lambda_{tr_{sc}}}{3}\left[\frac{RB_{2c}\cosh B_{2c}R - \sinh B_{2c}R}{R^2}\right]$$

$$- A_{1R}\frac{\lambda_{tr_{sR}}}{3}\left[\frac{RB_{1R}\sinh B_{1R}R - \cosh B_{1R}R}{R^2} - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\,\frac{RB_{1R}\cosh B_{1R}R - \sinh B_{1R}R}{R^2}\right]$$

$$- A_{2R}\frac{\lambda_{tr_{sR}}}{3}\left[\frac{RB_{2R}\sinh B_{2R}R - \cosh B_{2R}R}{R^2} - \frac{\cosh B_{2R}(R+T+\delta)}{\sinh B_{2R}(R+T+\delta)}\,\frac{RB_{2R}\cosh B_{2R}R - \sinh B_{2R}R}{R^2}\right] = 0\ , \quad (3.90)$$

As in the slab geometry, if nontrivial solutions are to exist for A_{3c}, A_{4c}, A_{1R}, and A_{2R}, the determinant of their coefficients in Eqs. 3.87 through 3.90 must be equal to zero. The determinantal equation involves R and T and is the *criticality equation*. One can either specify R and solve for T, by trial and error, or specify T and solve for R.

The determinantal criticality equation is

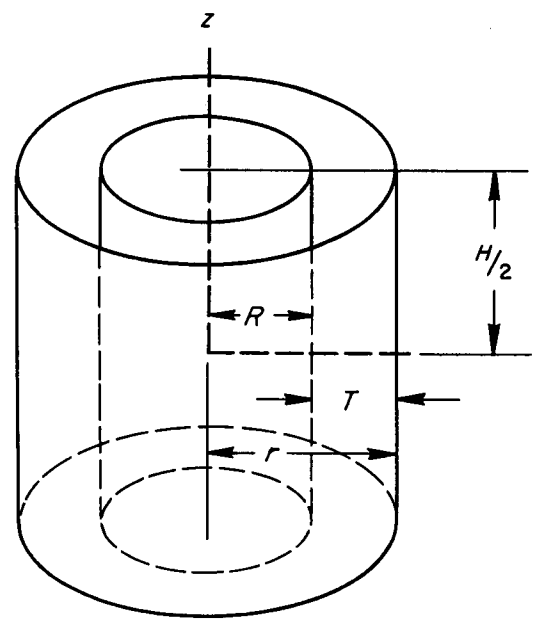
$$\begin{vmatrix}
\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\sin B_{1c}R & \left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\sinh B_{2c}R & \left(\frac{B_{1R}^2 + d_R}{e_R}\right)\left[\cosh B_{1R}R - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\sinh B_{1R}R\right] & 0 \\
\sin B_{1c}R & \sinh B_{2c}R & \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}\sinh B_{1R}R - \cosh B_{1R}R & \frac{\cosh B_{2R}(R+T+\delta)}{\sinh B_{2R}(R+T+\delta)}\sinh B_{2R}R - \cosh B_{2R}R \\
\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\left[RB_{1c}\cos B_{1c}R - \sin B_{1c}R\right] & \left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\left[RB_{2c}\cosh B_{2c}R - \sinh B_{2c}R\right] & \left(\frac{B_{1R}^2 + d_R}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3}\left[RB_{1R}\sinh B_{1R}R - \cosh B_{1R}R - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}(RB_{1R}\cosh B_{1R}R - \sinh B_{1R}R)\right] & 0 \\
\frac{\lambda_{tr_{sc}}}{3}\left[RB_{1c}\cos B_{1c}R - \sin B_{1c}R\right] & \frac{\lambda_{tr_{sc}}}{3}\left[RB_{2c}\cosh B_{2c}R - \sinh B_{2c}R\right] & \frac{\lambda_{tr_{sR}}}{3}\left[\cosh B_{1R}R - RB_{1R}\sinh B_{1R}R - \frac{\cosh B_{1R}(R+T+\delta)}{\sinh B_{1R}(R+T+\delta)}(\sinh B_{1R}R - RB_{1R}\cosh B_{1R}R)\right] & \frac{\lambda_{tr_{sR}}}{3}\left[\cosh B_{2R}R - RB_{2R}\sinh B_{2R}R - \frac{\cosh B_{2R}(R+T+\delta)}{\sinh B_{2R}(R+T+\delta)}(\sinh B_{2R}R - RB_{2R}\cosh B_{2R}R)\right]
\end{vmatrix} = 0\ . \quad (3.91)$$

As in the slab case, the procedure for getting the critical radius of the spherical core when the composition and reflector thickness are given is to evaluate the constants a_c, b_c, d_c, e_c, a_R, d_R, and e_R by using Eqs. 3.9 through 3.15, obtain B_{1c}^2 and $-B_{2c}^2$ as the roots of Eq. 3.26, obtain $-B_{1R}^2$ and $-B_{2R}^2$ as the roots of Eq. 3.42, and solve the determinantal equation, Eq. 3.91, for the critical core radius, R, by trial and error.

For the converse problem in which R and T are specified and the uranium concentration (critical mass) is unknown, it is, of course, necessary to make trial solutions with various uranium concentrations until the set of constants a_c, b_c, d_c, e_c is found that, together with a_R, d_R, and e_R and the resulting buckling constants, satisfies Eq. 3.91.

Once criticality has been established, three of the constants A_{3c}, A_{4c}, A_{1R}, and A_{2R} can be solved for in terms of the remaining constant by using Eqs. 3.87 through 3.90. The remaining constant is the power constant. By substituting the constants thus obtained into Eqs. 3.83 through 3.86 the fast and slow fluxes in the core and reflector can be evaluated to within the one unknown constant, which depends on the power level at which the reactor is operating.

Cylinder with Jacket Reflector and No End Reflector. Upon application of boundary conditions 1 and 2, the general solutions, Eqs. 3.53 and 3.54, for the fluxes in the core of the cylindrical reactor with jacket reflector and no end reflectors reduce to



From boundary condition 4, it follows that

$$\cos \epsilon_{1c}\left(\frac{H}{2} + \delta\right) = \cos \epsilon_{2c}\left(\frac{H}{2} + \delta\right) = 0 \; ; \qquad (3.94)$$

and from boundary condition 3 and the arguments of chap. 1, it follows that the first root of the cosine is set equal to the arguments of the cosines in Eq. 3.94:

$$\epsilon_{1c}\left(\frac{H}{2} + \delta\right) = \epsilon_{2c}\left(\frac{H}{2} + \delta\right) = \frac{\pi}{2}$$

and hence

$$\epsilon_{1c} = \epsilon_{2c} = \frac{\pi}{2\left(\frac{H}{2} + \delta\right)} \; . \qquad (3.95)$$

$$\phi_{fc}(r,z) = A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right) J_0(\mu_{1c}r) \cos \epsilon_{1c}z + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right) I_0(\mu_{2c}r) \cos \epsilon_{2c}z \qquad (3.92)$$

and

$$\phi_{sc}(r,z) = A_{1c}\, J_0(\mu_{1c}r) \cos \epsilon_{1c}z + A_{2c}\, I_0(\mu_{2c}r) \cos \epsilon_{2c}z \; . \qquad (3.93)$$

Equations 3.92 and 3.93 for the core fluxes then become

$$\phi_{fc}(r,z) = \left[A_{1c}\left(\frac{B_{1c}^2 - a_c}{e_c}\right) J_0(\mu_{1c}r) + A_{2c}\left(\frac{-B_{2c}^2 - a_c}{e_c}\right) I_0(\mu_{2c}r)\right] \cos\frac{\pi z}{2\left(\frac{H}{2} + \delta\right)} \tag{3.96}$$

and

$$\phi_{sc}(r,z) = \left[A_{1c}\, J_0(\mu_{1c}r) + A_{2c}\, I_0(\mu_{2c}r)\right] \cos\frac{\pi z}{2\left(\frac{H}{2} + \delta\right)} . \tag{3.97}$$

By the same argument, it follows that

$$\epsilon_{1R} = \epsilon_{2R} = \frac{\pi}{2\left(\frac{H}{2} + \delta\right)} \quad (= \epsilon_{1c} = \epsilon_{2c}). \tag{3.98}$$

Furthermore, boundary condition 4 can be satisfied by choosing A_{1R} and A_{2R} so that the reflector flux expressions, Eqs. 3.57 and 3.58, take the form

$$\phi_{fR}(r,z) = A_{3R}\left(\frac{-B_{2R}^2 - d_R}{e_R}\right)\left\{-\frac{K_0[\mu_{1R}(R + T + \delta)]}{I_0[\mu_{1R}(R + T + \delta)]} I_0(\mu_{1R}r) + K_0(\mu_{1R}r)\right\} \cos\frac{\pi z}{2\left(\frac{H}{2} + \delta\right)} \tag{3.99}$$

and

$$\phi_{sR}(r,z) = A_{3R}\left\{-\frac{K_0[\mu_{1R}(R + T + \delta)]}{I_0[\mu_{1R}(R + T + \delta)]} I_0(\mu_{1R}r) + K_0(\mu_{1R}r)\right\} \cos\frac{\pi z}{2\left(\frac{H}{2} + \delta\right)}$$

$$+ A_{4R}\left\{-\frac{K_0[\mu_{2R}(R + T + \delta)]}{I_0[\mu_{2R}(R + T + \delta)]} I_0(\mu_{2R}r) + K_0(\mu_{2R}r)\right\} \cos\frac{\pi z}{2\left(\frac{H}{2} + \delta\right)} , \tag{3.100}$$

where T is the thickness of the jacket reflector and $\delta = 0.71\ \lambda_{tr}$ is the extrapolation distance.

Boundary condition 5 provides the following four homogeneous equations in the four unknown constants, A_{1c}, A_{2c}, A_{3R}, and A_{4R} (after dividing through by $\cos \pi/2[(H/2) + \delta]$, which is common to all four flux expressions):

$$A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right) J_0(\mu_{1c}R) + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right) I_0(\mu_{2c}R)$$

$$- A_{3R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\left\{-\frac{K_0[\mu_{1R}(R + T + \delta)]}{I_0[\mu_{1R}(R + T + \delta)]} I_0(\mu_{1R}R) + K_0(\mu_{1R}R)\right\} = 0 , \tag{3.101}$$

$$A_{1c}\, J_0(\mu_{1c}R) + A_{2c}\, I_0(\mu_{2c}R) - A_{3R}\left\{-\frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_0(\mu_{1R}R) + K_0(\mu_{1R}R)\right\} - A_{4R}\left\{-\frac{K_0[\mu_{2R}(R+T+\delta)]}{I_0[\mu_{2R}(R+T+\delta)]}\, I_0(\mu_{2R}R) + K_0(\mu_{2R}R)\right\} = 0\ , \tag{3.102}$$

$$-A_{1c}\left(\frac{B_{1c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\,\mu_{1c}\, J_1(\mu_{1c}R) + A_{2c}\left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\,\mu_{2c}\, I_1(\mu_{2c}R) + A_{3R}\left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3}\,\mu_{1R}\left\{\frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_1(\mu_{1R}R) + K_1(\mu_{1R}R)\right\} = 0\ , \tag{3.103}$$

$$-A_{1c}\frac{\lambda_{tr_{sc}}}{3}\,\mu_{1c}\, J_1(\mu_{1c}R) + A_{2c}\frac{\lambda_{tr_{sc}}}{3}\,\mu_{2c}\, I_1(\mu_{2c}R) + A_{3R}\frac{\lambda_{tr_{sR}}}{3}\,\mu_{1R}\left\{\frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_1(\mu_{1R}R) + K_1(\mu_{1R}R)\right\} + A_{4R}\frac{\lambda_{tr_{sR}}}{3}\,\mu_{2R}\left\{\frac{K_0[\mu_{2R}(R+T+\delta)]}{I_0[\mu_{2R}(R+T+\delta)]}\, I_1(\mu_{2R}R) + K_1(\mu_{2R}R)\right\} = 0\ . \tag{3.104}$$

As before, if nontrivial solutions are to exist for A_{1c}, A_{2c}, A_{3R}, and A_{4R}, the determinant of their coefficients in Eqs. 3.101 through 3.104 must be equal to zero. The determinantal equation involves R and T and is the *criticality equation*. The determinantal criticality equation is, then,

$$\begin{vmatrix}
\left(\frac{B_{1c}^2 - d_c}{e_c}\right) J_0(\mu_{1c}R) & \left(\frac{-B_{2c}^2 - d_c}{e_c}\right) I_0(\mu_{2c}R) & \left(\frac{B_{1R}^2 + d_R}{e_R}\right)\left\{K_0(\mu_{1R}R) - \frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_0(\mu_{1R}R)\right\} & 0 \\
J_0(\mu_{1c}R) & I_0(\mu_{2c}R) & \frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_0(\mu_{1R}R) - K_0(\mu_{1R}R) & \frac{K_0[\mu_{2R}(R+T+\delta)]}{I_0[\mu_{2R}(R+T+\delta)]}\, I_0(\mu_{2R}R) - K_0(\mu_{2R}R) \\
\left(\frac{-B_{1c}^2 + d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\,\mu_{1c}\, J_1(\mu_{1c}R) & \left(\frac{-B_{2c}^2 - d_c}{e_c}\right)\frac{\lambda_{tr_{fc}}}{3}\,\mu_{2c}\, I_1(\mu_{2c}R) & \left(\frac{-B_{1R}^2 - d_R}{e_R}\right)\frac{\lambda_{tr_{fR}}}{3}\,\mu_{1R}\left\{\frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_1(\mu_{1R}R) + K_1(\mu_{1R}R)\right\} & 0 \\
-\frac{\lambda_{tr_{sc}}}{3}\,\mu_{1c}\, J_1(\mu_{1c}R) & \frac{\lambda_{tr_{sc}}}{3}\,\mu_{2c}\, I_1(\mu_{2c}R) & \frac{\lambda_{tr_{sR}}}{3}\,\mu_{1R}\left\{\frac{K_0[\mu_{1R}(R+T+\delta)]}{I_0[\mu_{1R}(R+T+\delta)]}\, I_1(\mu_{1R}R) + K_1(\mu_{1R}R)\right\} & \frac{\lambda_{tr_{sR}}}{3}\,\mu_{2R}\left\{\frac{K_0[\mu_{2R}(R+T+\delta)]}{I_0[\mu_{2R}(R+T+\delta)]}\, I_1(\mu_{2R}R) + K_1(\mu_{2R}R)\right\}
\end{vmatrix} = 0\ . \tag{3.105}$$

To summarize again, the procedure for getting the critical radius of the cylindrical core when the composition, length, and jacket reflector thickness are given is to evaluate the constants a_c, b_c, d_c, e_c, a_R, d_R, and e_R by using Eqs. 3.9 through 3.15, obtain B_{1c}^2 and $-B_{2c}^2$ as the roots of Eq. 3.26, obtain $-B_{1R}^2$ and $-B_{2R}^2$ as the roots of Eq. 3.42, evaluate $\epsilon_{1c} = \epsilon_{2c} = \epsilon_{1R} = \epsilon_{2R} = \pi/2[(H/2) + \delta]$, evaluate μ_{1c} and μ_{2c} by using Eqs. 3.55 and 3.56, evaluate μ_{1R} and μ_{2R} by using Eqs. 3.59 and 3.60, and solve the determinantal equation, Eq. 3.105, for the critical core radius, R, by trial and error.

It is clear that Eq. 3.105 and the other defining equations can be used to solve for any one of the following quantities when the others are specified: R, T, H, or critical uranium concentration (or critical mass).

Once criticality has been established, three of the constants A_{1c}, A_{2c}, A_{3R}, and A_{4R} can be solved for in terms of the remaining constant by using Eqs. 3.101 through 3.104. The remaining constant is the power constant. By substitution of the constants thus obtained into Eqs. 3.96 through 3.100, the fast and slow fluxes in the core and reflector can be evaluated to within the one unknown constant, which depends on the power level at which the reactor is operating.

REDUCTION TO SECOND-ORDER DETERMINANT

The criticality equation evaluated in the preceding sections for the three different geometries has, in each case, turned out to be a fourth-order determinant. The first two columns involve only core constants and the last two columns involve only reflector constants, with two zeros in the last column.

It is not difficult to show that the determinant of this fourth-order matrix can be replaced by the determinant of the product of a 2 by 4 matrix that depends only on the reflector constants and a 4 by 2 matrix that depends only on the core constants. Since the product of a 2 by 4 matrix and a 4 by 2 matrix is a 2 by 2 matrix, the result is a reduction of the original 4 by 4 determinant to a 2 by 2 determinant.

If the second and third rows are interchanged in the original 4 by 4 determinant (criticality equation) as it has been written, a determinant of the type

$$\begin{vmatrix} a_{11} & a_{12} & A_{11} & 0 \\ a_{21} & a_{22} & A_{21} & 0 \\ a_{31} & a_{32} & A_{31} & A_{32} \\ a_{41} & a_{42} & A_{41} & A_{42} \end{vmatrix} = 0 \qquad (3.106)$$

is obtained in which the a's depend on the core constants and the A's depend on the reflector constants. It is then easy to verify that the following equation, which is a second-order determinantal equation, is equivalent to Eq. 3.106:

$$\left| \begin{pmatrix} 0 & 0 & 1 & -\dfrac{A_{31}}{A_{41}} \\ -\dfrac{A_{22}}{A_{12}} & 1 & \dfrac{A_{11}}{A_{31}} & \dfrac{A_{22}}{A_{12}} - \dfrac{A_{21}}{A_{31}} & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \right| = 0 \,. \qquad (3.107)$$

Equation 3.107 is somewhat easier to evaluate for each trial of R, T, or N_U, as the case may be.

EVALUATING THE CONSTANTS

The remarks at the end of chap. 2 about "Averaging Constants in the Thermal Group" and "Effect of Heterogeneity on Constants" are applicable in the two-group problem also. However, certain problems in evaluating the constants that are peculiar to the two-group model will be discussed.

Thermal Reactors. If the reactor is a thermal reactor (most of the fissioning caused by thermal neutrons) and if no appreciable quantities of resonance absorbers are present, the two-group equations take a particularly simple form. Equations 3.1 through 3.4, for the fast and slow flux in core and reflector, immediately reduce to

$$-\frac{\lambda_{tr_{fc}}}{3}\nabla^2\phi_{fc}(\underline{r}) + \phi_{fc}(\underline{r})\,\Sigma_c - \phi_{thc}(\underline{r})\,\Sigma_{a_{thc}}k_{th} = 0 \quad (3.108)$$

and

$$-\frac{\lambda_{tr_{thc}}}{3}\nabla^2\phi_{thc}(\underline{r}) + \phi_{thc}(\underline{r})\,\Sigma_{a_{thc}} - \phi_{fc}(\underline{r})\,\Sigma_c = 0\ , \quad (3.109)$$

since $\Sigma_{a_{fc}} = 0 = k_f$ if the slow-neutron group has been selected as the true thermal neutrons only and the fast-neutron group comprises all neutrons with energy above thermal. If each of these equations, Eqs. 3.108 and 3.109, is divided through by the diffusion coefficient, the equations become

$$\nabla^2\phi_{fc}(\underline{r}) - \frac{3\Sigma_c}{\lambda_{tr_{fc}}}\phi_{fc}(\underline{r}) + \frac{\lambda_{tr_{th}}}{\lambda_{tr_f}}\frac{k_{th}}{L^2}\phi_{thc} = 0 \quad (3.110)$$

and

$$\nabla^2\phi_{thc}(\underline{r}) - \frac{1}{L^2}\phi_{thc}(\underline{r}) + \frac{\lambda_{tr_{fc}}}{\lambda_{tr_{thc}}}\frac{3\Sigma_c}{\lambda_{tr_{fc}}}\phi_{fc} = 0\ . \quad (3.111)$$

If neutron diffusion away from a point source of fast neutrons is described by a two-group model, the balance equation for the fast neutrons in a cubic centimeter away from the source is the first part of Eq. 3.110; that is,

$$\nabla^2\phi_{fc}(r) - \frac{3\Sigma_c}{\lambda_{tr_{fc}}}\phi_{fc}(r) = 0\ . \quad (3.112)$$

If

$$L_f^2 \equiv \frac{\lambda_{tr_{fc}}}{3\Sigma_c}\ , \quad (3.113)$$

Eq. 3.112 becomes

$$\nabla^2\phi_{fc}(r) - \frac{1}{L_f^2}\phi_{fc}(r) = 0\ . \quad (3.114)$$

The solution of Eq. 3.114 in the spherical coordinate system is

$$\phi_{fc}(r) = A\,\frac{e^{-r/L_f}}{r}\ . \quad (3.115)$$

If the average of the square of the crow-flight distances that the fast neutrons diffuse before dropping into the slow-neutron group is evaluated, it is found that

$$\overline{r^2} = \frac{\int_0^\infty 4\pi r^2\cdot\overline{r^2}A\Sigma_c\,\frac{e^{-r/L_f}}{r}\,dr}{\int_0^\infty 4\pi r^2 A\Sigma_c\,\frac{e^{-r/L_f}}{r}\,dr} = 6L_f^2\ . \quad (3.116)$$

However, in chap. 2, it was shown that $6\tau_{th} = \overline{r^2}$, also, in the continuous slowing-down theory. Thus L_f^2 has the same significance as τ_{th}, and $3\Sigma_c/\lambda_{tr_{fc}}$ can be replaced by $1/\tau_{thc}$ in Eqs. 3.110 and 3.111. These equations then become

$$\nabla^2 \phi_{fc}(\underline{r}) - \frac{1}{\tau_{thc}} \phi_{fc}(\underline{r}) + \frac{\lambda_{tr_{thc}}}{\lambda_{tr_{fc}}} \frac{k_{th}}{L_c^2} \phi_{thc} = 0 \quad (3.117)$$

and

$$\nabla^2 \phi_{thc}(\underline{r}) - \frac{1}{L_c^2} \phi_{thc}(\underline{r}) + \frac{1}{\tau_{thc}} \frac{\lambda_{tr_{fc}}}{\lambda_{tr_{thc}}} \phi_{fc} = 0 . \quad (3.118)$$

Hence, in the notation of Eqs. 3.5 and 3.6, it follows that

$$a_c = -\frac{1}{\tau_{thc}}, \quad (3.119)$$

$$b_c = \frac{\lambda_{tr_{thc}}}{\lambda_{tr_{fc}}} \frac{k_{th}}{L_c^2}, \quad (3.120)$$

$$d_c = -\frac{1}{L_c^2}, \quad (3.121)$$

$$e_c = \frac{1}{\tau_{thc}} \frac{\lambda_{tr_{fc}}}{\lambda_{tr_{thc}}}. \quad (3.122)$$

A good average value for $\lambda_{tr_{fc}}$ can usually be chosen by inspection, because Σ_{tr} is usually fairly constant for most elements. However, it is perhaps preferable to average $\lambda_{tr_{fc}}(E)$ over the fast flux distribution. If the reactor is large enough for the leakage to be small, $q(E)$ is essentially a constant (assuming no epithermal absorption). It follows that

$$\phi(E) = \frac{q}{\xi E \Sigma_s} \approx \frac{1}{E}$$

and

$$\lambda_{tr_{fc}} \approx \frac{\int_{E_{th}}^{E_F} \lambda_{tr_{fc}}(E) \frac{1}{E} dE}{\int_{E_{th}}^{E_F} \frac{dE}{E}} . \quad (3.123)$$

A somewhat better scheme, perhaps, is to take

$$q(E) \propto e^{-B^2 \tau_c(E)} ,$$

where B^2 is evaluated from a best guess as to the size of the equivalent bare reactor. Then

$$\phi(E) \propto \frac{e^{-B^2 \tau_c}}{\xi E \Sigma_s}$$

and

$$\lambda_{tr_{fc}} \approx \frac{\int_{E_{th}}^{E_F} \lambda_{tr_{fc}}(E) \frac{e^{-B^2 \tau_c(E)}}{\xi E \Sigma_s} dE}{\int_{E_{th}}^{E_F} \frac{e^{-B^2 \tau_c(E)}}{\xi E \Sigma_s} dE} . \quad (3.124)$$

Similarly, for the reflector

$$a_R = -\frac{1}{\tau_{thR}}, \quad (3.125)$$

$$d_R = -\frac{1}{L_R^2}, \quad (3.126)$$

$$e_R = \frac{1}{\tau_{thR}} \frac{\lambda_{tr_{fR}}}{\lambda_{tr_{thR}}}, \quad (3.127)$$

and $\lambda_{tr_{fR}}$ can be evaluated by inspection or as in Eq. 3.123.

Epithermal Reactors. In epithermal reactors the lower limit of the fast-neutron group (upper limiting energy of slow-neutron group) will, in general, be some energy above thermal, and the slow-neutron group will consist of thermal neutrons, as well as those with energies from thermal up to the dividing energy. Absorption of and fission by neutrons in the fast neutron, as well as slow neutron, core group will, in general, have to be considered. The problem of obtaining good average values for the constants in this situation is a difficult one. The difficulty arises because the neutron energy distribution is not known. Various methods of averaging two-group constants have been proposed and used, but no method will, of course, be wholly satisfactory. The difficulties of cross-section averaging can be alleviated only by using a many-group calculation.

Probably the best method of averaging the *core* constants in the two-group model as applied to epithermal reactors is the following. The simplifying assumption is made that the spectrum of the neutrons in the equivalent bare reactor is the same as that in the core of the reflected reactor, and the bare, intermediate-reactor, criticality-calculation form discussed in chap. 2 is worked out. Then since the reactor in question is not quite thermal, a decision is made to let the slow-neutron group include some of the lower energy lethargy intervals, for example, 24, 25, 26, 27, 28, 29, 30, 31 (see the calculation form in the Appendix), as well as the true thermal group (interval 31 is just above thermal). The fast-neutron group will then be the neutrons in intervals 1 through 23. Then, for example, $\lambda_{tr_{fc}}$ in the two-group equation is chosen to have the value that makes the fast-neutron group leakage equal to that for the first 23 intervals in the Fermi model; that is,

$$\lambda_{tr_{fc}} = \frac{\sum_{N=1}^{23} (\lambda_{tr})_N B^2 \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}{B^2 \sum_{N=1}^{23} \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}. \tag{3.128}$$

The other physical quantities are then evaluated:

$$\Sigma_c = \frac{q_{23}}{\sum_{N=1}^{23} \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}, \tag{3.129}$$

$$\Sigma_{a_{fc}} = \frac{\sum_{N=1}^{23} (\Sigma_a)_N \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}{\sum_{N=1}^{23} \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}, \tag{3.130}$$

$$\Sigma_{a_{sc}} = \frac{\phi_{th}\Sigma_{a_{th}} + \sum_{N=23}^{31} (\Sigma_a)_N \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}{\phi_{th} + \sum_{N=23}^{31} \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}, \tag{3.131}$$

$$\lambda_{tr_{sc}} = \frac{\phi_{th}\lambda_{tr_{th}} B^2 + \sum_{N=23}^{31} (\lambda_{tr})_N B^2 \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}{\phi_{th} B^2 + \sum_{N=23}^{31} B^2 \frac{\bar{q}_N}{\xi(\Sigma_T)_N} U_N}, \tag{3.132}$$

$$k_s = \frac{\phi_{th}\Sigma_{f_{th}}\nu + \sum_{N=23}^{31} (\Sigma_f)_N \nu \frac{\overline{q}_N}{\xi(\Sigma_T)_N} U_N}{\phi_{th}\Sigma_{a_{th}} + \sum_{N=23}^{31} (\Sigma_a)_N \frac{\overline{q}_N}{\xi(\Sigma_T)_N} U_N}, \tag{3.133}$$

$$k_f = \frac{\sum_{N=1}^{23} (\Sigma_f)_N \nu \frac{\overline{q}_N}{\xi(\Sigma_T)_N} U_N}{\sum_{N=1}^{23} (\Sigma_a)_N \frac{\overline{q}_N}{\xi(\Sigma_T)_N} U_N}. \tag{3.134}$$

Even less is known *a priori* about the neutron spectrum in the reflector. However, since there is no fissionable material or nonfissionable isotope of the fissionable material in the reflector, the reflector presents, in general, a small absorption cross section to neutrons that are slowing down. It therefore seems reasonable to consider the slowing-down density as being a constant, insofar as finding average quantities is concerned. This means that $\phi(u)$ is approximately constant and therefore

$$\lambda_{tr_{fR}} = \frac{1}{u_l}\int_0^{u_l} \lambda_{tr}(u)\, du, \tag{3.135}$$

where u_l is the lower limit (in units of lethargy) of the fast-neutron group; and

$$\Sigma_{a_{fR}} = \frac{1}{u_l}\int_0^{u_l} \Sigma_a(u)\, du \tag{3.136}$$

and

$$\Sigma_R = \frac{1}{\frac{\lambda_s}{\xi} \ln \frac{E_F}{E_l}}, \tag{3.137}$$

where

$$\lambda_s \equiv \frac{1}{u_l}\int_0^{u_l} \lambda_s(u)\, du, \tag{3.138}$$

$E_F \equiv$ an average fission energy,
$E_l \equiv$ lower limit (in units of energy) of the fast-neutron group,
$\xi \equiv$ average logarithmic energy loss per collision.

The denominator of the expression for Σ_R is the number of centimeters a neutron travels, on the average, in slowing down into the slow-neutron group. The inverse is then the probability per centimeter of a fast neutron dropping into the slow-neutron group.

There is no problem in regard to the slow-neutron group constants in the reflector if E_l is equal to E_{th} (the actual energy corresponding to the temperature of the true thermal neutrons), that is, if the slow-neutron group has been chosen to be the actual thermal-neutron group. If E_l has been chosen as greater than E_{th}, it is difficult to ascertain a weighting factor for the thermal cross section relative to the average cross section between E_{th} and E_l. It can be written that

$$\lambda_{tr_{sR}} = A\lambda_{tr_{thR}} + B\frac{1}{(u_{th} - u_l)}\int_{u_l}^{u_{th}} \lambda_{tr}(u)\, du \tag{3.139}$$

and

$$\Sigma_{a_{sR}} = A\Sigma_{a_{thR}} + B\frac{1}{(u_{th} - u_l)}\int_{u_l}^{u_{th}} \lambda_{tr}(u)\, du, \tag{3.140}$$

where

$A + B = 1$,
$A \equiv$ fraction of slow-neutron group flux that is thermal,
$B \equiv$ fraction of slow-neutron group flux that is not thermal.

An educated guess is made of the value of A and B. If the reflector is very

thin, less than 3 in., or nonmoderating, A and B can be taken as the same as A and B for the equivalent bare reactor. If the moderating reflector is very thick, greater than 9 in., the value of A will be close to unity and the value of B will be close to zero. If the reflector is neither thick nor thin, an intermediate guess for A is required.

RESULTS OF CRITICALITY STUDIES

This section presents some graphs of critical mass, flux, and power distribution to provide a brief, quantitative picture of the numerical values of these quantities in various kinds of reflected reactors.

Figure 3.2[2] shows the critical mass of "cold, clean," beryllium oxide moderated reactors for various beryllium oxide reflector thicknesses. These are not practical reactors – they contain no fission-product poisons or structural material and they are at room temperature. When these conditions are present, the uranium investment is much higher, as will be discussed in the next chapter. It can be seen from Fig. 3.2 that for a given core diameter, the reflector is very effective in reducing the critical mass. These results were obtained by two-group calculations.

Figures 3.3[3] and 3.4[3] show experimental results of critical mass determinations of "cold, clean," water-moderated reactors with water reflectors. It can be seen from Fig. 3.3 that water-moderated reactors can be very small – of all elements, hydrogen has the largest slowing-down power for neutrons. In the current, G-E, airplane-reactor design, advantage is taken of the much smaller fraction of the core volume that must be devoted to moderator when water, rather than beryllium oxide, for example, is used. More surface is

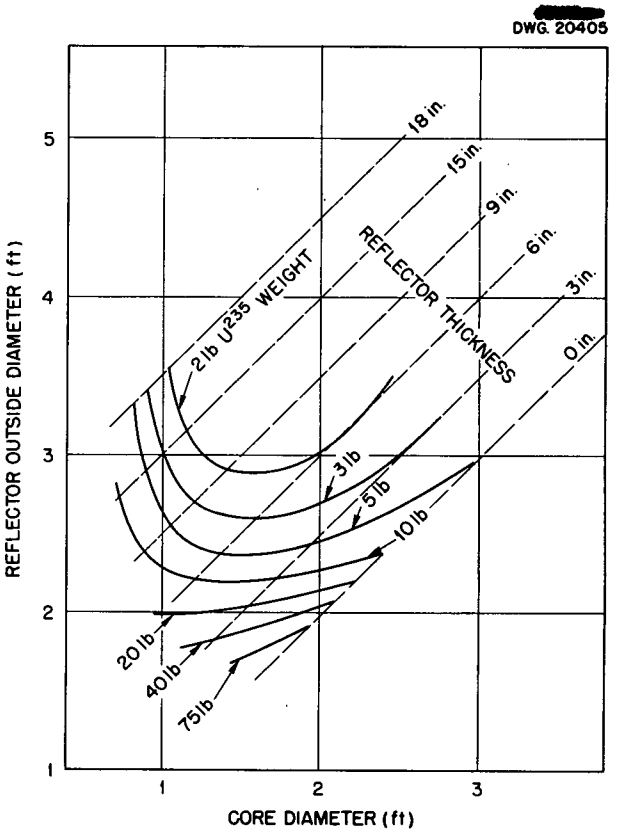


Fig. 3.2. Critical $U^{235}O_2$-BeO Reactors; Free-Flow Ratio = 0.

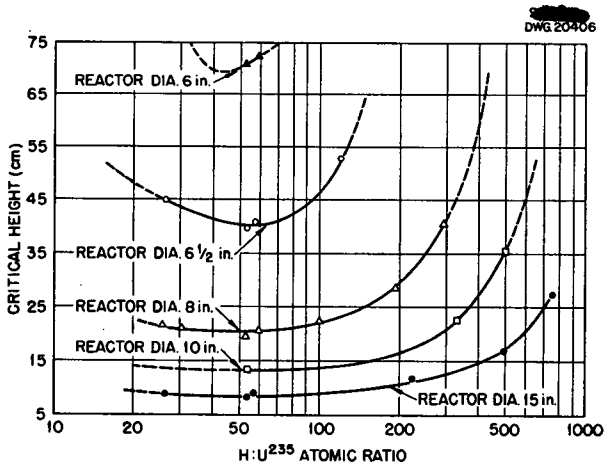


Fig. 3.3. Critical Height vs. Moderator Ratio; Aluminum Tank; Water Reflector.

(2) L. L. Gartner, *Criticality for Uranium Dioxide Beryllium Oxide Reactors*, NEPA No. 949-EAR-R6 (Mar. 14, 1949).

(3) C. K. Beck, A. D. Callihan, J. W. Morfitt, and R. L. Murray, *Critical Mass Studies Part III*, K-343 (Apr. 19, 1949).

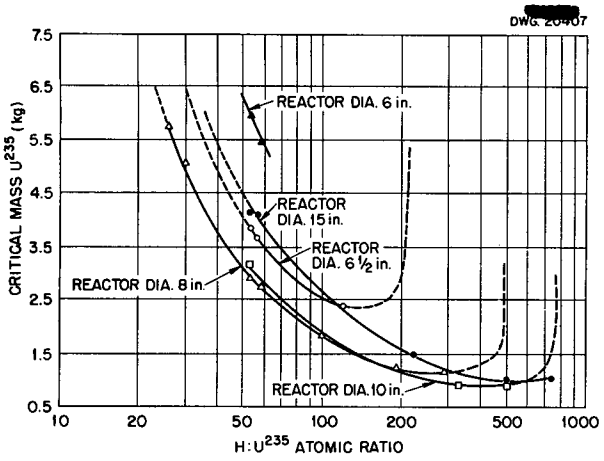


Fig. 3.4. Critical Mass of U^{235} vs. Moderator Ratio; Aluminum Tank; Water Reflector.

available for heat transfer to the air (the GE-ANP design is air-cooled).

Figure 3.5[4] shows what the critical mass of a fast, unmoderated reactor would be. In the reactor treated here, 40% of the core volume is bismuth coolant. The fast, unmoderated design is practical for a power reactor from a heat transfer standpoint, but it can be seen that the uranium investment is so large that this type of design is unfeasible

[4] D. S. Selengut, *Uranium Weight for All Metal Reactors*, NEPA-IC-50-2-25 (Feb. 10, 1950).

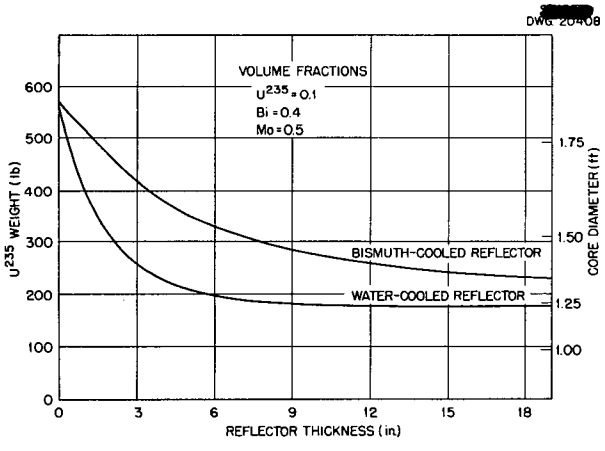


Fig. 3.5. Bismuth-Cooled, Molybdenum, Fast Reactors with Square-Cylinder Geometry.

at the present time. The calculations presented in Fig. 3.5 were made by transport theory.

Figure 3.6[5] shows the calculated critical mass of some "operable" Be_2C + 1/3 C moderated, bismuth-cooled, power reactors. For these reactors, which would operate at a temperature of 1600°F, the composition used for the calculations included considerable stainless steel tubing and fission-product poisons. The calculations were therefore realistic estimates of the U^{235} necessary for this type of design. It can be seen that 2.5 to 4 ft is the best size range for this design.

[redacted]

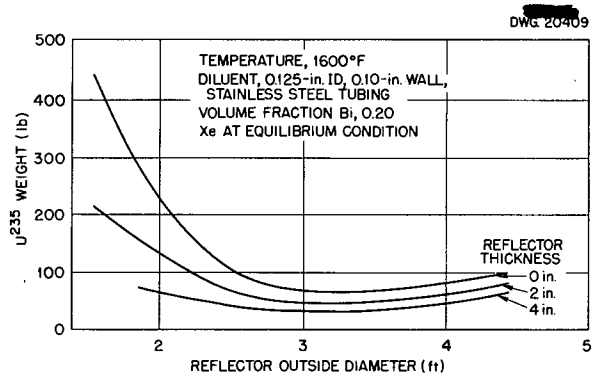


Fig. 3.6. Requirements for Critical Bismuth-Cooled UC_2 Reactors Moderated with Be_2C + 1/3 C.

Figures 3.7[2] and 3.8[2] show typical thermal- and fast-flux spatial distributions in beryllium oxide moderated reactors with beryllium oxide reflectors, Fig. 3.7 shows a thermal reactor, and Fig. 3.8 shows a reactor in which the neutrons that cause fission have median energy that is somewhat above thermal energy. It is customary to find a rise in the thermal-flux distribution in the core near the reflector when the reflector is made of moderating elements.

[5] T. R. Mitchell, *Criticality Maps for Bismuth Cooled Reactors*, NEPA-IC-49-12-36 (Dec. 16, 1949).

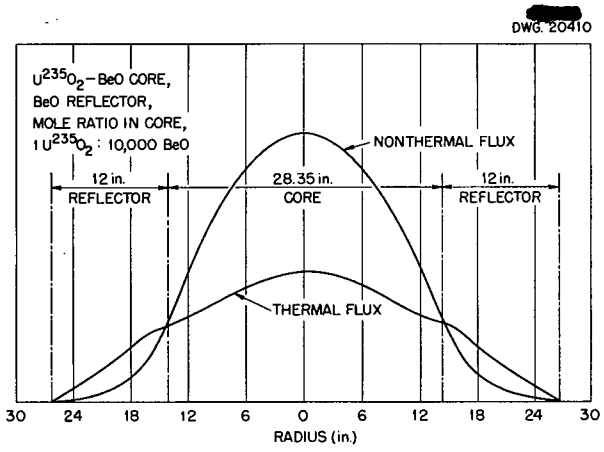


Fig. 3.7. Radial Neutron-Flux Distribution in a Spherical Reactor.

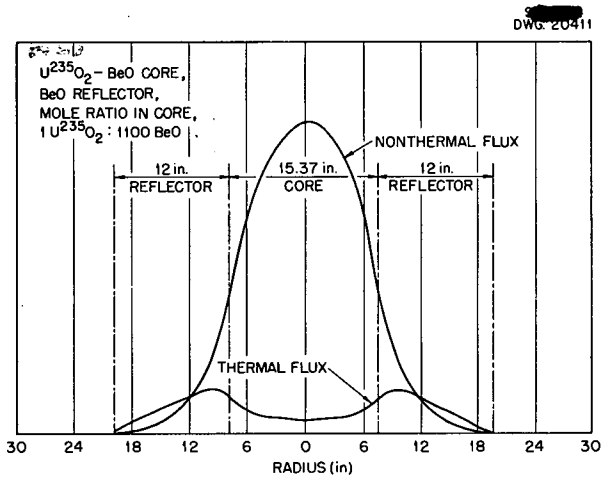


Fig. 3.8. Radial Neutron-Flux Distribution in a Spherical Reactor.

Figure 3.9[6] shows the north-south flux distributions in the MTR for a thin, slab type of loading of the fuel elements. The core is H_2O and Al. The first reflector is beryllium and the second is graphite.

It is to be noted that in thermal reactors in which nearly all the fissioning is due to thermal neutrons, the thermal flux distribution *in the core* also gives the power distribution. In nonthermal reactors calculated by group methods, the power distribution must be obtained by summing the products of the group fluxes and the respective average fission cross sections for the groups.

Figures 3.10 through 3.13[7] give the thermal- and fast-flux spatial distributions for an intermediate, beryllium oxide moderated reactor design. These calculations were made by using a 27-group model on IBM computing machines. Figure 3.10 gives the spectrum of the neutrons escaping from the core. It is interesting to note the effect of the "window" in the beryllium cross section at high energy. Figure 3.11 gives the spectrum of the neutrons causing fission at various spatial positions in the core. The thermalizing effect of the reflector is apparent. Figure 3.12 is Fig. 3.11 integrated over the core, and Fig. 3.13 is the spatial power distribution. The rise in the power density near the reflector is quite pronounced in intermediate reactors with moderating reflectors.

Figure 3.14[8] shows the spatial distribution of flux at different energy levels (on lethargy levels at which lethargy $u = \ln 10^7/E$) in an intermediate, beryllium oxide moderated reactor.

PROBLEMS

1. The MTR has the following constants:

Core (Al + U + H_2O)

$$k_{th} = 1.34 ,$$

$$L^2 = 3.64 \text{ cm}^2 ,$$

$$\frac{\lambda_{tr_{th}}}{3} = 0.263 \text{ cm} ,$$

(6) J. H. Buck and C. F. Leyse, *Materials Testing Reactor Project Handbook*, ORNL-963 (May 7, 1951).

(7) J. W. Webster and O. A. Schulze, *Some Results of Criticality Calculations on BeO and Be Moderated Reactors*, ANP-66 (Oct. 15, 1951).

(8) C. B. Mills, *Statics of the ANP Reactor–A Preliminary Report*, Y-F10-81 (Jan. 8, 1952).

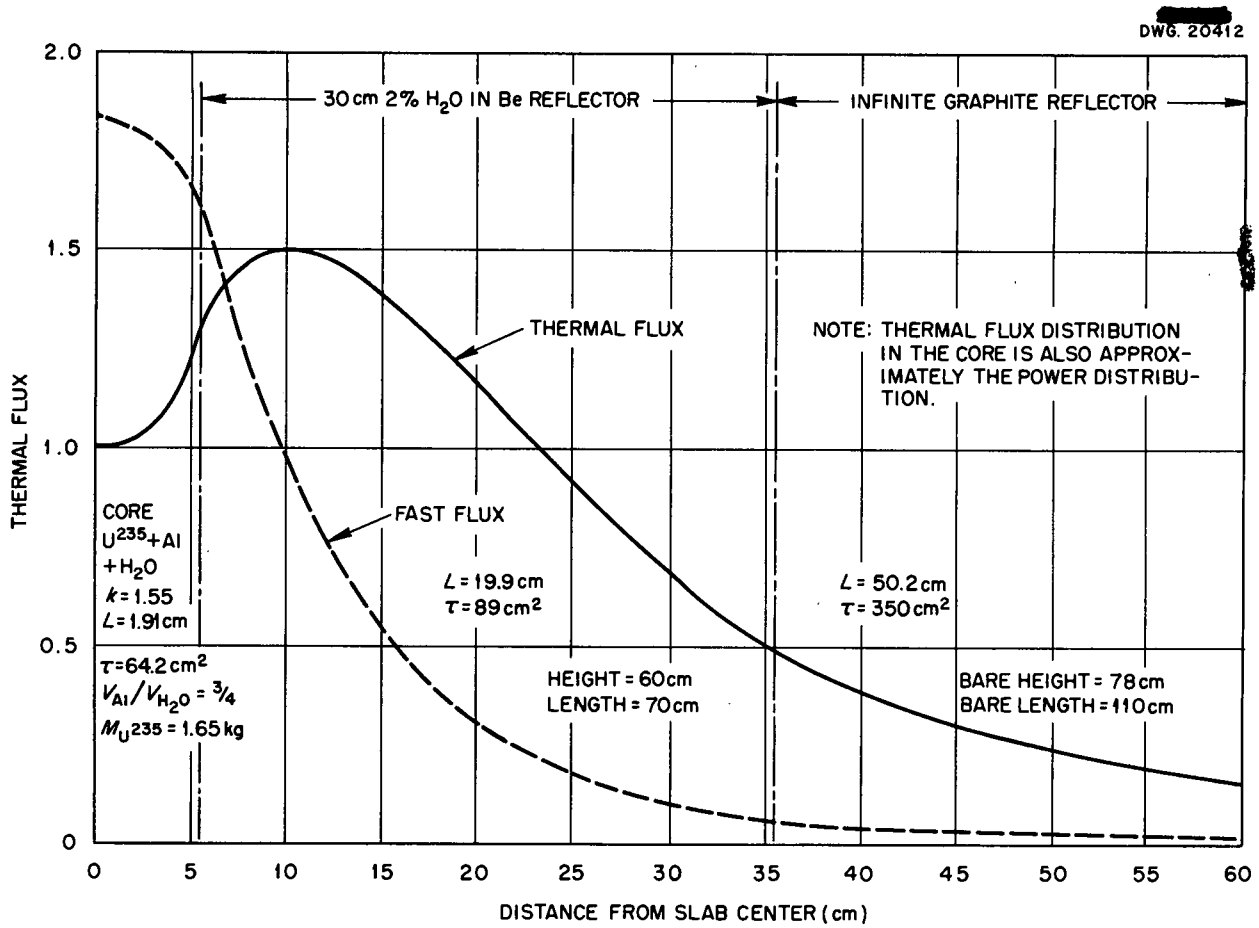


Fig. 3.9. MTR Lateral Flux Distribution Calculated as Thin Slab (11 by 60 by 70 cm).

$$\tau_{th} = 64 \text{ cm}^2 ,$$

$$\frac{\lambda_{tr_f}}{3} = 1.243 \text{ cm} .$$

Reflector (Be)

Thickness $\approx$ 46 cm ,

$$L^2 = 396 \text{ cm}^2 ,$$

$$\frac{\lambda_{tr_{th}}}{3} = 0.64 \text{ cm} ,$$

$$\tau = 89 \text{ cm}^2 ,$$

$$\frac{\lambda_{tr_f}}{3} = 0.60 \text{ cm} .$$

Assuming no absorption of or fissioning by above-thermal neutrons (a good assumption in the MTR), find the critical size of the sphere, the cylinder with jacket reflector only (take the height of core and reflector as 78 cm), and the parallelepiped with reflectors on the north and south faces only (take the length of core and reflector in the east-west direction as 113 cm and in the vertical direction as 78 cm).

2. Find the spatial distribution of the fast and thermal fluxes in the different geometries of problem 1, and plot the fluxes.

[redacted] DWG. 20526

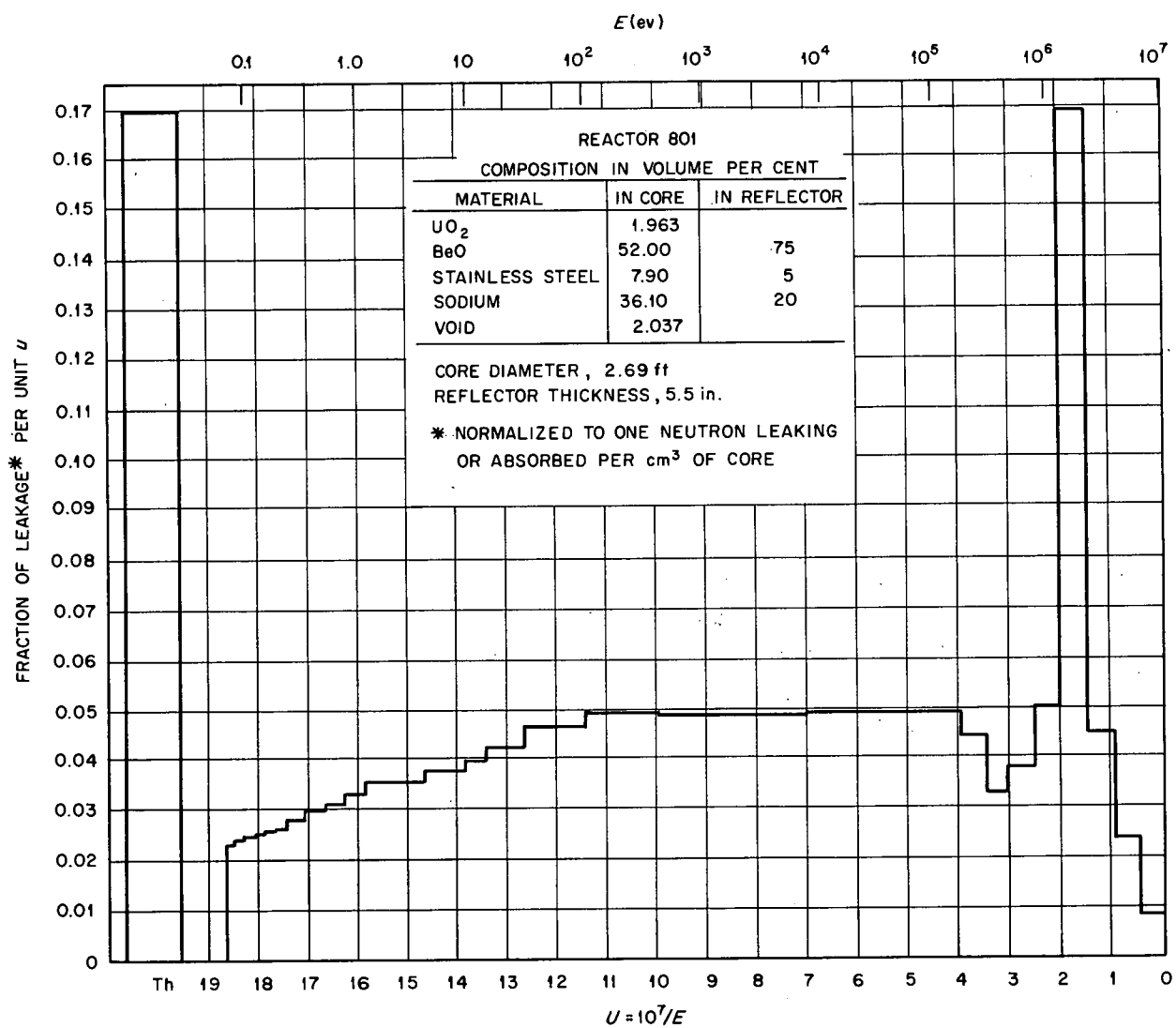


Fig. 3.10. Leakage Spectrum.

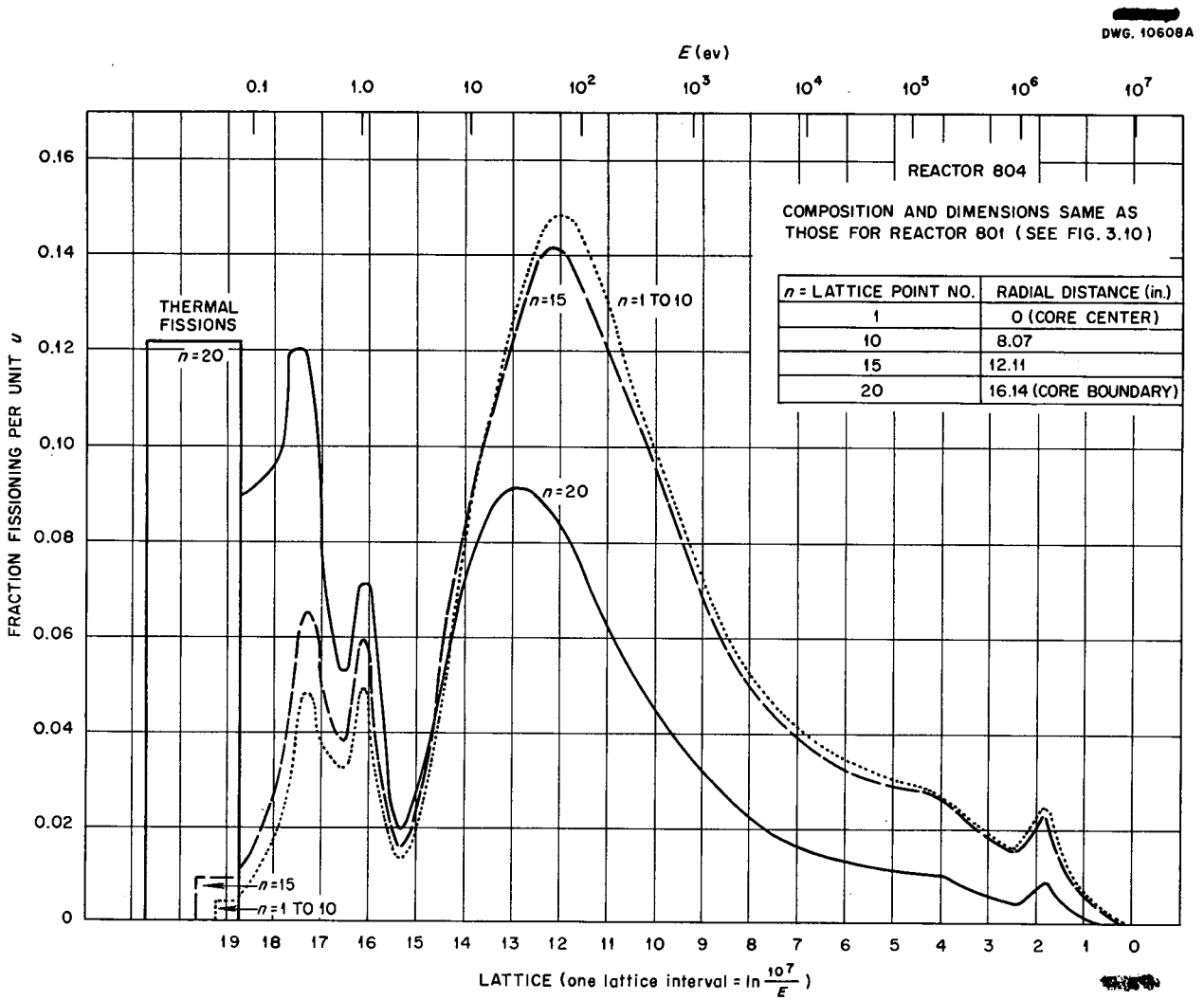


Fig. 3.11. Normalized Fissioning Spectrum.

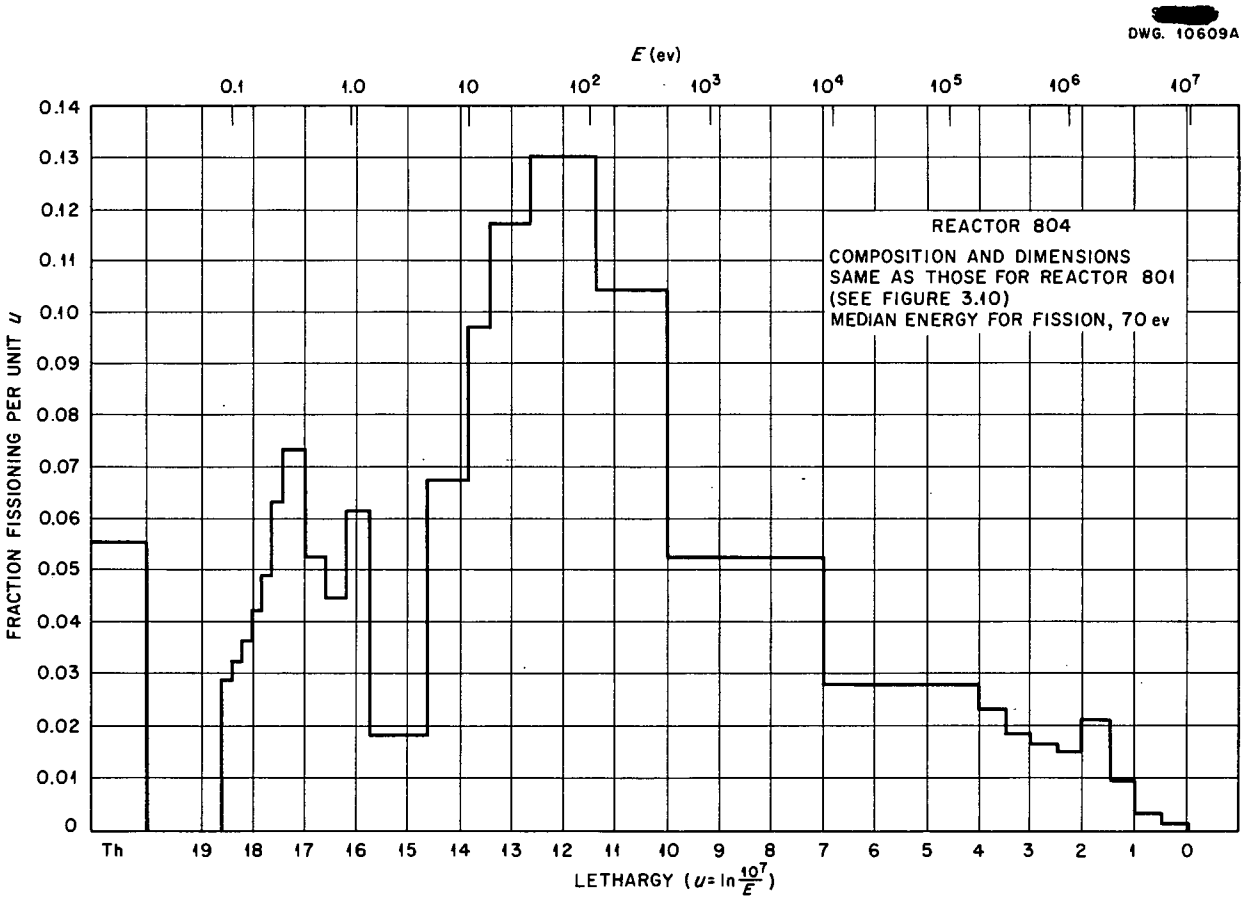


Fig. 3.12. Normalized Fissioning Spectrum Integrated over Core.

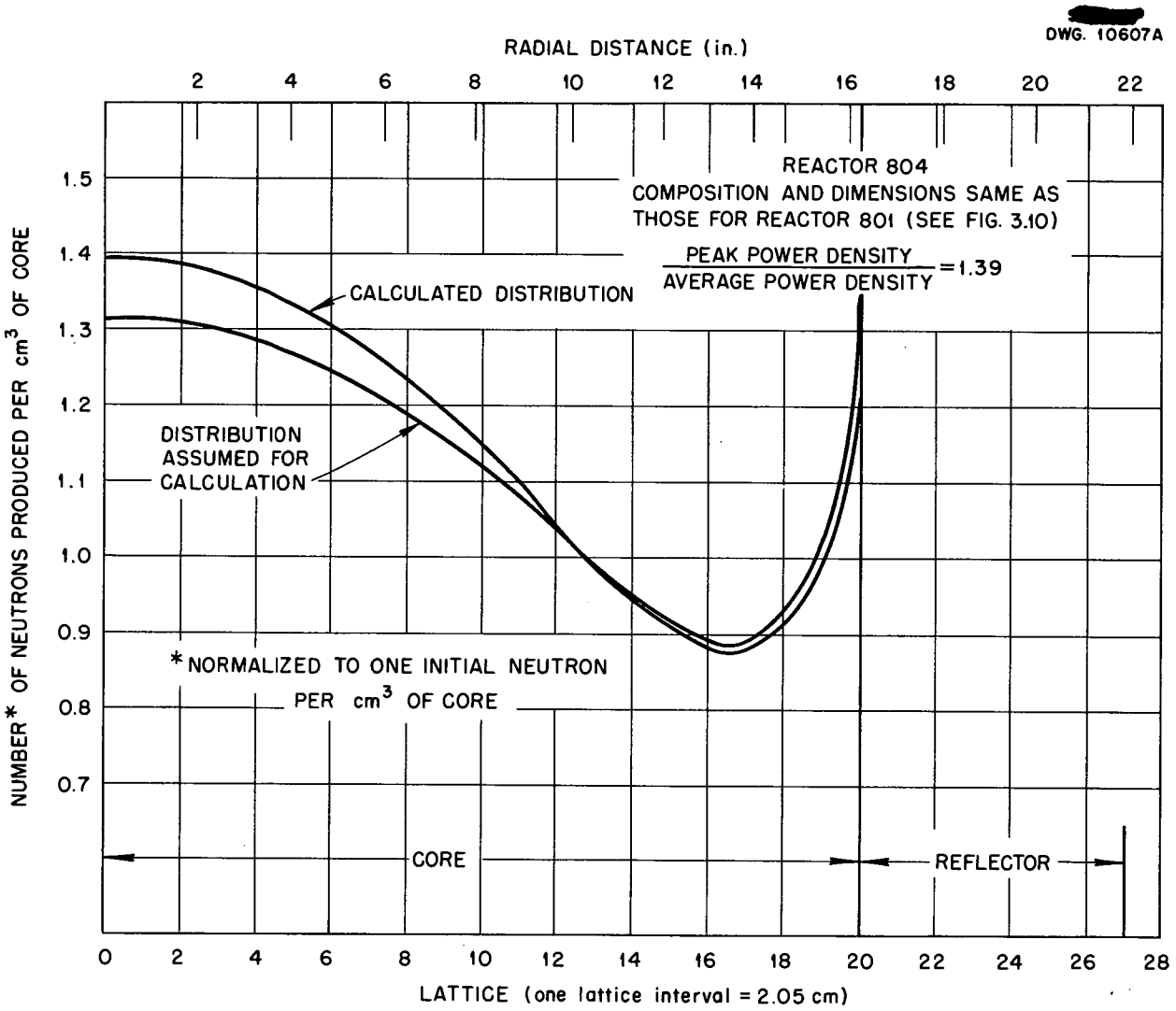


Fig. 3.13. Spatial Power Distribution.

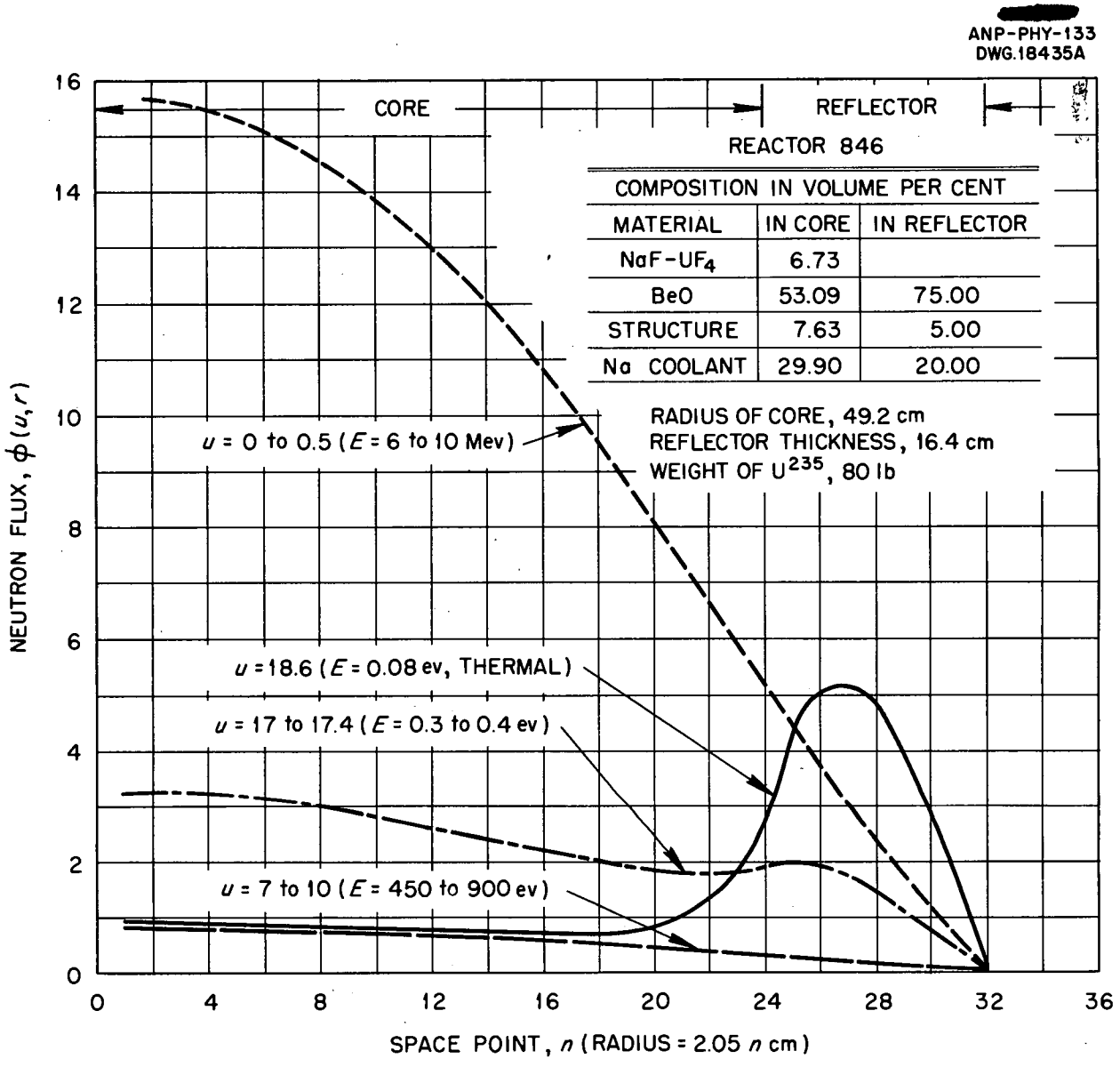


Fig. 3.14. Neutron Flux vs. Radius for Several Lethargy Values. U^{235} mass of 80 pounds.

Chapter 4
REACTIVITY EFFECTS

When a reactor is operated, it heats up thermally, the fissionable material is consumed, and fission products, some of which (such as xenon and samarium) have high capture cross sections, accumulate. The last two effects, and in most reactors the first effect, act to decrease the probability that a neutron will produce a fission because they increase the likelihood of nonproductive capture or leakage of the neutrons. The reactor must, therefore, contain enough fissionable material to ensure criticality when it is in the hot, poisoned, depleted condition. It follows, then, that at startup, when the reactor is cold, clean, and undepleted, a degree of virtual or potential supercriticality will exist that must be offset by control rods. Such control rods, called "shim rods," are fully inserted at startup and are gradually withdrawn (if they are absorber rods) as the temperature approaches the operating level, the fission-product poisons build up, and the fuel is depleted. This type of control is distinct from the "fine" control necessary to make the reactor momentarily supercritical or subcritical when power changes are required.

No new methods are needed for calculating the increased critical mass required to offset the effects of temperature, poisons, and depletion. It is necessary only to evaluate the constants in the light of (1) the expanded reactor and the higher kT energy of the thermal neutrons and (2) the increase of the Σ_a because of the fission products. Enough extra uranium must be used to offset the burnup of fissionable material that will take place at design power during the operating time prescribed by the fuel operating cycle.

This chapter presents a discussion of temperature, uranium depletion, and fission-product poisons and their virtual reactivity effects. (The word "virtual" is used to indicate that the effect never really occurs in the reactor but is always offset by a repositioning of the shim rods.) Chapter 5 presents a discussion of the calculation of the number and size of control rods needed to offset the virtual supercriticality present at startup.

It is clear that since the effect of the shim control rods is to counterbalance the effects of temperature, depletion, and poisons, all four effects should be measured in the same terms. Much confusion has existed in reactor physics about the units of measure for these effects. *However, as far as shim control calculations are concerned, the only important requirement is that the four effects be expressed by a common measure.* It has been found that in calculating the strength of a control rod a convenient procedure is to find the fictitious value of ν (physically equal to 2.5 for U^{235}, the number of neutrons per fission), say, the ν_{ca} that would be necessary to make the reactor critical with the rod inserted. The reactor may have had a degree of virtual supercriticality before rod insertion required a fictitious ν_{cb} for criticality. The measure of the effectiveness of the rod is chosen to be

$$\frac{\nu_{cb} - \nu_{ca}}{\nu_{ca}} \times 100\% . \qquad (4.1)$$

If the control rod has been selected of just the right size so that the reactor is made critical when the control rod is inserted, the effectiveness is given by

$$\frac{\nu_{cb} - 2.5}{2.5} \times 100\% ,$$

where 2.5 is the physical value of ν.

It would be possible to take simply $(\nu_{cb} - 2.5)$ or even just ν_{cb} as the measure in this case, but in chap. 6, "Kinetics," it will be found that a convenient measure of reactivity is k_{ex}/k_{eff}, and it can be easily shown that

$$\frac{k_{ex}}{k_{eff}} = \frac{2.5 - \nu_c}{2.5},$$

which is the negative value of Eq. 3.1. Since calculations of control rod effectiveness have to be made for the *regulating rods* also and the unit of measure should conform to that used in the kinetic formulas, the choice was made to express the effectiveness of the *shim rods* as well as the effects of temperature, poisons, and depletion in the same terms. *The fact remains, however, that for the "shim effects" described in chaps. 4 and 5, it does not matter how the effect is expressed so long as there is consistency.*

It is first necessary to calculate the critical uranium concentration, N_U (final), in the hot, poisoned, depleted reactor. In this calculation the additional macroscopic absorption cross section due to the xenon is found by the methods to be given in this chapter. The value of ν in this calculation will be its physical value, 2.5.

The effect of the depletion is given by

$$\frac{2.5 - \nu_d}{2.5} \times 100\% , \qquad (4.2)$$

where ν_d is the fictitious value of ν necessary to make the hot, poisoned reactor critical with extra uranium added to offset depletion. The uranium concentration for this calculation is then

$$N_U \text{ (final)} + \Delta N_U \text{ (depletion)} .$$

It follows that ν_d is less than 2.5, and Eq. 4.2 measures a virtual *supercriticality*.

The effect of fission-product poisons is given by

$$\frac{\nu_d - \nu_{dp}}{2.5} \times 100\% , \qquad (4.3)$$

where ν_{dp} is the fictitious value of ν necessary to make the hot, clean, undepleted reactor critical. The uranium concentration in this calculation is also

$$N_U \text{ (final)} + \Delta N_U \text{ (depletion)} ,$$

but the macroscopic absorption cross sections are for the unpoisoned situation.

The effect of temperature is given by

$$\frac{\nu_{dp} - \nu_{dpt}}{2.5} \times 100\% , \qquad (4.4)$$

where ν_{dpt} is the fictitious value of ν necessary to make the cold, clean, undepleted reactor critical. The N_U in this calculation is still

$$N_U \text{ (final)} + \Delta N_U \text{ (depletion)} ,$$

but the macroscopic cross sections are for the cold, clean situation.

Expressions 4.2, 4.3, and 4.4 are positive and measure a virtual increase in reactivity since we are working backward with respect to the life history of the reactor operation.

Enough control rods are then provided to give control strength, as measured by Eq. 4.1, to offset the effects of temperature, poisons, and depletion, as given by the sum of Eqs. 4.2, 4.3, and 4.4, that is,

$$\frac{2.5 - \nu_{dpt}}{2.5} \times 100\% .$$

DEPLETION

The amount of fissionable material consumed at design power during the

operating time prescribed by the fuel operating cycle can be easily found from the relation

$$1 \text{ watt} \approx 3.15 \times 10^{10} \text{ fissions per second.}$$

(This value varies slightly with each reactor because the energy of the gammas emitted upon neutron capture depends upon the reactor composition.) When this amount of fuel is added to the concentration, N_U (final), of the hot, poisoned, depleted reactor in obtaining the ν_d of Eq. 4.2, the macroscopic cross sections Σ_f and Σ_a will be affected, primarily.

FISSION-PRODUCT POISONS

The fission products that present the largest capture cross sections to neutrons in a thermal reactor are xenon and samarium, with xenon presenting the largest. Xenon arises through the decay chain

$$_{52}Te^{135} \xrightarrow{2m} {}_{53}I^{135} \xrightarrow{6.6h} {}_{54}Xe^{135} \xrightarrow{9.2h} {}_{55}Cs^{135} .$$

The half life of tellurium is so short that the iodine can be considered as being formed directly from fission. About 5% of the xenon comes directly from fission and the rest from iodine decay. The calculation of the concentration of xenon before and after shutdown will be discussed in detail in the following. The samarium arises through the decay chain

$$_{60}Nd^{149} \xrightarrow{1.7h} {}_{61}Pm^{149} \xrightarrow{47h} {}_{62}Sm^{149} \text{ (stable)} .$$

The mathematical formulation for the samarium buildup is quite similar to that for xenon; so it will not be given here. The other fission products can be taken into account, roughly, by assuming that they add 50 barns[1] (thermal) per fission to the poisons in the reactor.

The Xe^{135} isotope, which is the isotope of xenon with the high capture cross section, comes principally from decay of I^{135}, as indicated above. An equation for the rate of change of iodine can be written:

$$\frac{dN_I}{dt} = Fy_I - N_I\lambda_I , \qquad (4.5)$$

where

N_I ≡ atoms of I^{135} per unit volume,
y_I ≡ yield of I^{135} per fission,
F ≡ fission rate of the reactor in fissions per unit volume per unit time,
λ_I ≡ decay constant of I^{135} ≡ probability of disintegration per unit time.

The parent of the iodine, tellurium, which is formed directly from fission, has a half life so short that the iodine can be considered as coming directly from fission. Upon solving the differential equation, Eq. 4.5, and using the boundary condition that no iodine is present at startup ($t = 0$), the following equation results:

$$N_I = \frac{Fy_I}{\lambda_I}(1 - e^{-\lambda_I t}) . \qquad (4.6)$$

For the rate of change of xenon, the differential equation is

$$\frac{dN_{Xe}}{dt} = N_I\lambda_I + Fy_{FXe} - N_{Xe}\lambda_{Xe} - \int_E \phi(E)\, N_{Xe}\, \sigma_{Xe}(E)\, dE , \qquad (4.7)$$

where

N_{Xe} ≡ atoms of Xe^{135} per unit volume,

(1) J. H. Buck and C. F. Leyse, *Materials Testing Reactor Project Handbook*, ORNL-963 (May 7, 1951).

y_{FXe} ≡ yield of xenon directly from fission,
λ_{Xe} ≡ decay constant of Xe^{135},
$\phi(E)$ ≡ reactor flux per unit energy,
$\sigma_{Xe}(E)$ ≡ xenon capture cross section as a function of energy.

The first two terms of Eq. 4.7 account for the accrual of xenon per unit time, the third term is the decay per unit time, and the last term is the burnout due to neutron capture per unit time. Equation 4.7 can be rewritten as

$$\frac{dN_{Xe}}{dt} = N_I\lambda_I + Fy_{FXe} - N_{Xe}\lambda_{DXe} , \quad (4.8)$$

where

$$\lambda_{DXe} = \lambda_{Xe} + \int_E \phi(E)\ \sigma_{Xe}(E)\ dE . \quad (4.9)$$

Thus λ_{DXe} is the "disappearance" constant. If Eq. 4.6 is substituted in Eq. 4.8 and the resulting differential equation is solved under the boundary condition that $N_{Xe} = 0$ at startup ($t = 0$), the following formula is obtained:

$$N_{Xe} = \frac{Fy_{Xe}}{\lambda_{DXe}} - \frac{Fy_I}{\lambda_{DXe} - \lambda_I} e^{-\lambda_I t} + \left(\frac{Fy_I}{\lambda_{DXe} - \lambda_I} - \frac{Fy_{Xe}}{\lambda_{DXe}}\right) e^{-\lambda_{DXe} t} , \quad (4.10)$$

where $y_{Xe} = y_I + y_{FXe}$ = total yield of xenon per fission. If t is large, Eq. 4.10 reduces to

$$N^0_{Xe} = \frac{Fy_{Xe}}{\lambda_{DXe}} = \frac{Fy_{Xe}}{\lambda_{Xe} + \int_E \phi(E)\ \sigma_{Xe}(E)\ dE} \quad (4.11)$$

or

$$Fy_{Xe} = N^0_{Xe}\left[\lambda_{Xe} + \int_E \phi(E)\ \sigma_{Xe}(E)\ dE\right] , \quad (4.12)$$

which says that for equilibrium conditions the rate of production of xenon equals the rate of disappearance of xenon.

It is interesting to note the form of this equation for a high-flux thermal reactor.[2] For a thermal reactor, Eq. 4.12 becomes

$$\phi N_U \sigma_f y_{Xe} = N^0_{Xe}(\lambda_{Xe} + \phi\sigma_{Xe}) , \quad (4.13)$$

where
N_U ≡ atoms of U^{235} per unit of volume,
σ_f ≡ fission cross section,
ϕ ≡ the thermal flux,

and all quantities are at thermal energy. Equation 4.13 can be rewritten as

$$\frac{N^0_{Xe}\sigma_{Xe}}{N_U\sigma_f} = y_{Xe}\ \frac{\phi\sigma_{Xe}}{(\lambda_{Xe} + \phi\sigma_{Xe})} . \quad (4.14)$$

If the flux, ϕ, is very large, $\phi\sigma_{Xe}$ is much greater than λ_{Xe}, and the quantity in parentheses becomes approximately unity. Equation 4.14 then reduces to

$$\frac{N^0_{Xe}\sigma_{Xe}}{N_U\sigma_f} \approx y_{Xe} \quad (4.15)$$

or

$$N^0_{Xe} \propto N_U . \quad (4.16)$$

Thus in a thermal reactor, the density of xenon saturates with respect to increasing flux and depends only on the uranium concentration when the flux is large. It follows that the xenon concentration is uniform over the core of such a reactor. However, this is not true in an intermediate-energy reactor or in a thermal reactor with low flux in which the burnout is not necessarily large compared with the decay per unit time.

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[2] G. Young, *Critical Mass Needed to Over-Ride Xe*, MonP-457 (Dec. 29, 1947).

If the xenon reaches its equilibrium density for the fission rate F and the power is suddenly reduced to fF, the differential equation for the iodine behavior is

$$\frac{dN_I}{dt} = fFy_I - N_I\lambda_I \ . \tag{4.17}$$

When Eq. 4.17 is solved under the boundary condition that $N_I = N_I^0$, the equilibrium iodine concentration at $t = 0$, the result is

$$N_I = \frac{1}{\lambda_I}\left[fFy_I(1 - e^{-\lambda_I t}) + \lambda_I N_I^0 \ e^{-\lambda_I t}\right] \ , \tag{4.18}$$

where

$N_I^0 \equiv$ the equilibrium iodine density,

$t \equiv$ time measured from the moment of instantaneously reduced power.

The differential equation for the xenon behavior is

$$\frac{dN_{Xe}}{dt} = N_I\lambda_I + fFy_{FXe} - N_{Xe}\left(\lambda_{Xe} + f\int_E \phi\sigma_{Xe} \ dE\right) \ , \tag{4.19}$$

where ϕ is the flux before the power is reduced. When Eq. 4.19 is solved under the boundary condition that $N_{Xe} = N_{Xe}^0$ at $t = 0$, the result is

$$N_{Xe} = \frac{fFy_{Xe}}{\lambda_{DXe}} + \frac{\lambda_I N_I^0 - fFy_I \ e^{-\lambda_I t}}{\lambda_{DXe} - \lambda_I} + \left(N_{Xe}^0 - \frac{fFy_{Xe}}{\lambda_{DXe}} - \frac{\lambda_I N_I^0 - fFy_I}{\lambda_{DXe} - \lambda_I}\right) e^{-\lambda_{DXe} t} \ , \tag{4.20}$$

where

$$\lambda_{DXe} \equiv \lambda_{Xe} + f\int_E \phi(E) \ \sigma_{Xe}(E) \ dE \ .$$

If the reactor is completely shut down, Eq. 4.20 becomes

$$N_{Xe} = \frac{\lambda_I N_I^0}{\lambda_{Xe} - \lambda_I} e^{-\lambda_I t} + \left(N_{Xe}^0 - \frac{\lambda_I N_I^0}{\lambda_{Xe} - \lambda_I}\right) e^{-\lambda_{Xe} t}$$

$$= N_{Xe}^0 \ e^{-\lambda_{Xe} t} + \frac{\lambda_I N_I^0}{\lambda_{Xe} - \lambda_I}(e^{-\lambda_I t} - e^{-\lambda_{Xe} t}) \ . \tag{4.21}$$

The first term gives what is left of the equilibrium xenon at time t and the second term gives the xenon that has come from decay of the equilibrium iodine.

From Eq. 4.21, the dN_{Xe}/dt can be found, and the equation

$$\frac{dN_{Xe}}{dt} = 0$$

can then be solved for t. The result is the time after shutdown at which the maximum concentration will occur. The result is

$$t_{max} = \frac{1}{\lambda_I - \lambda_{Xe}} \ln \frac{\lambda_I}{\lambda_{Xe} + \dfrac{N_{Xe}^0}{N_I^0}\dfrac{\lambda_{Xe}}{\lambda_I}(\lambda_I - \lambda_{Xe})} \ . \tag{4.22}$$

The average cross section of xenon for thermal neutrons is found by weighting the $N_{Xe}\,\sigma_{Xe}(E)$ over the thermal flux distribution (usually assumed to be Maxwellian), that is,

$$\overline{N_{Xe}\,\sigma_{Xe}}_{th} = \frac{\int_0^\infty N_{Xe}\,\sigma_{Xe}(E)\,E\,e^{-E/kt}\,dE}{\int_0^\infty E\,e^{-E/kt}\,dE},$$

where the cross section of xenon is given by the Breit-Wigner formula:

$$\sigma_{Xe}(E) = \frac{\sigma_0 \frac{\Gamma^2}{4}}{(E - E_0)^2 + \frac{\Gamma^2}{4}} \sqrt{\frac{E_0}{E}},$$

where

$\Gamma = 0.107$ ev,
$E_0 = 0.0863$,
$\sigma_0 = 3.4 \times 10^6$ barns.

The other constants of xenon are

$y_I = 0.0686$,
$y_{FXe} = 0.003$,
$\lambda_{Xe} = 2.103 \times 10^{-5}$ per second,
$\lambda_I = 2.900 \times 10^{-5}$ per second.

The effect of the xenon (and, similarly, the other fission-product poisons) is found by modifying the Σ_a of the reactor to take into account the macroscopic xenon cross section, $N_{Xe}\sigma_{Xe}$.

A reactor may be designed with enough uranium in it to override the xenon poisoning at its maximum transient concentration after shutdown (for example, the STR), or it may be designed to only override the xenon at a somewhat lower concentration, depending on the operational requirements of the reactor (for example, the MTR). In the latter case, if the reactor is not restarted within a short time after shutdown, it is necessary to wait for a period of time to allow part of the xenon to decay to cesium before the reactor can be restarted.

TEMPERATURE

Increasing the temperature of a reactor has three effects: (1) it raises the energy of the moderator atoms and hence raises the average energy of the thermal neutrons, with the consequence that the effective cross sections for thermal neutrons are changed (sometimes called the thermal-base effect); (2) the reactor expands and the atomic density of the constituents is reduced; and (3) the reactor size is increased.

Since the temperature is changed, the first effect results in a new Maxwell Boltzmann distribution $M(E)$, which is given by

$$M(E) \propto \sqrt{E}\,e^{-E/kT}.$$

The effective thermal cross sections, as given by

$$\overline{\sigma_a}_{th} = \frac{\int_0^\infty \sigma_a(E)\,E\,e^{-E/kT}\,dE}{\int_0^\infty E\,e^{-E/kT}\,dE}$$

and

$$\frac{1}{\overline{\sigma}_{tr}}_{th} = \frac{\int_0^\infty \frac{1}{\sigma_{tr}(E)}\,E\,e^{-E/kT}\,dE}{\int_0^\infty E\,e^{-E/kT}\,dE},$$

will thus be changed.

The second effect will result in decreased atomic concentrations. The third effect will result in a changed B^2 in the case of Fermi age formulas and, simply, changed dimensions in the case of the group determinants.

RESULTS OF CALCULATIONS ON REACTIVITY EFFECTS

Figures 4.1, 4.2, and 4.3 show the results[3] of calculations made of the virtual reactivity effect of fission-product poisons and depletion in the

(3) J. H. Buck and C. F. Leyse, *Materials Testing Reactor Project Handbook*, ORNL-963 (May 7, 1951).

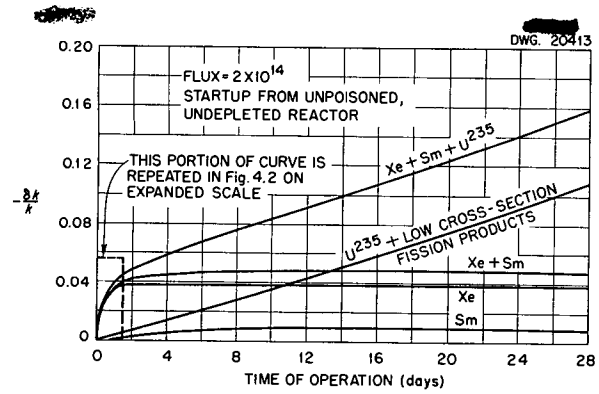

Fig. 4.1. Reactivity Loss in the MTR Owing to Xe^{135}, Sm^{149}, U^{235} Depletion, and Build-up of Low Cross-Section Fission Products as a Function of Time of Operation.

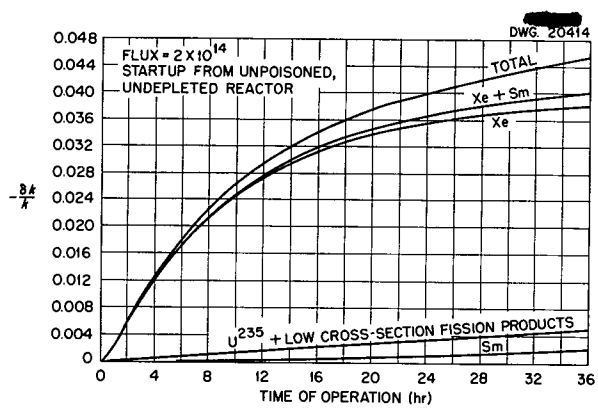

Fig. 4.2. Reactivity Loss in the MTR Owing to Xe^{135}, Sm^{135}, U^{235} Depletion, and Build-up of Low Cross-Section Fission Products as a Function of Time of Operation up to 36 Hours.

MTR. The quantitative data given are typical of thermal reactors. The effect of equilibrium xenon, for example, is usually about 4% in $\delta k/k$ ($= \Delta\nu/\nu$). It is interesting to note the magnitude of the effect of the after-shutdown xenon growth in this high-flux, thermal assembly. The effect is so large that the MTR was designed with only enough fuel to enable the reactor to be restarted within about 30 min after shutdown. If not started within 30 min, the

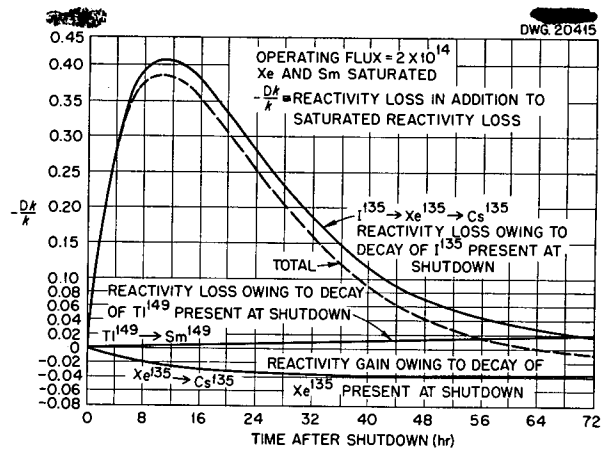

Fig. 4.3. Extra Reactivity Change in MTR After Shutdown.

reactor must lie idle for about two days, as can be seen from Fig. 4.3.

Figure 4.4 shows the results of some calculations[4] of the virtual reactivity effects for various airplane reactor proposals that have various neutron spectra. In each case the operating temperature is 1286°F, the power is 200 megawatts, the moderator is beryllium oxide, the coolant is sodium, and the tubing is stainless steel.

Figure 4.5 shows the after-shutdown xenon growth for various fractional power reductions from 200 megawatts in a slightly epithermal reactor.

PROBLEMS

1. Use the thermal, Fermi age criticality formula to find the effect of temperature, depletion, and xenon poisoning at equilibrium and maximum transient density in a bare, graphite reactor, 5 ft in diameter, consisting only of carbon and U^{235}. The temperature at startup is 70°F. The constants have the values:

$$\tau_{th} = 337.7 \text{ cm}^2,$$
$$\sigma_{a_{th}}(C) = 0.001 \text{ barn } (0.025 \text{ ev}),$$

(4) B. T. Macauley and J. W. Webster, *Results of Some Bare Pile Calculations of Critical Mass and Reactivity Effects*, Y-F10-22 (Dec. 1, 1950).

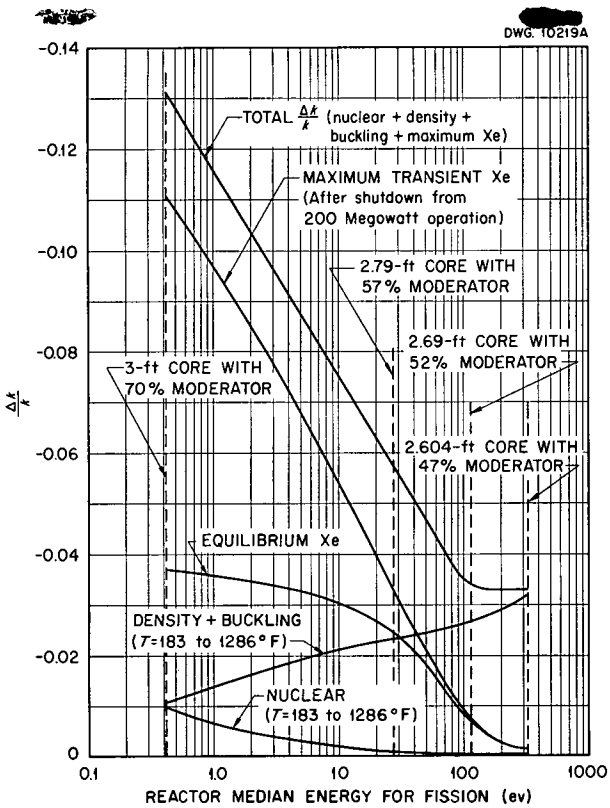

Fig. 4.4. Reactivity Effects as a Function of Reactor Spectrum.

$$\sigma_{s_{th}}(C) = 4 \text{ barns},$$
$$\sigma_{a_{th}}(U) = 640 \text{ barns}(0.025 \text{ ev}),$$
$$\sigma_{s_{th}}(U) = 10 \text{ barns},$$
$$\sigma_{f_{th}}(U) = 550 \text{ barns } (0.025 \text{ ev}),$$
$$\rho_{c70°F} = 1.67 \text{ g/cm}^3$$

Assume throughout the problem that τ_{th} is unaffected. The power level is 600 megawatts and the reactor operates for 24 hours. The reactor is a bare, square cylinder. The operating temperature is 2200°F.

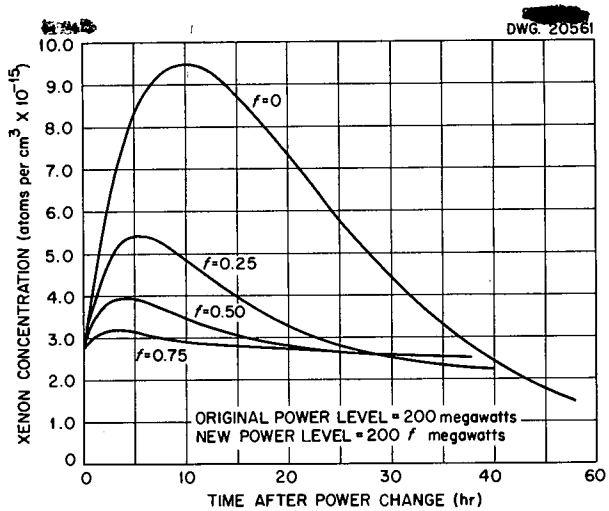

Fig. 4.5. Behavior of Xenon Concentration After Reduction in Reactor Power Level.

In calculating the thermal base effect assume that the absorption and fission cross sections are proportional to $1/v$ and that the scattering cross sections are constant. For the expansion, assume that the coefficient of linear expansion is

$$\alpha = 7.86 \times 10^{-6} \text{ per } °C .$$

In connection with the xenon, take the constants as having the values:

$$\sigma_a(Xe) = 3 \times 10^6 \text{ barns},$$
$$y_I = 0.0686,$$
$$y_{Xe} = 0.0716,$$
$$y_{FXe} = 0.003,$$
$$\lambda_{Xe} = 2.103 \times 10^{-5} \text{ per sec } (= 0.0757 \text{ per hr}),$$
$$\lambda_I = 2.900 \times 10^{-5} \text{ per sec } (= 0.1044 \text{ per hr}).$$

2. Derive formulas for the concentration of samarium before and after shutdown.

Chapter 5
CONTROL RODS

In chapter 4 it was pointed out that when a reactor operates it heats up, fission-product poisons accumulate, and the fissionable material is gradually consumed. All these things tend to increase the probability of nonproductive neutron capture or neutron leakage. It is thus necessary to build into the reactor enough extra uranium to ensure that the chain reaction is still self-sustaining when these effects are at their worst. However, with all this extra uranium in the core, the reactor will be supercritical – to the greatest degree at startup – when the effects are absent, unless shim control rods are provided. If shim control is obtained by absorber rods, these rods would normally be fully inserted at startup and then gradually withdrawn as the reactor heats up, the poisons accumulate, and the fuel burns out. The shim control rods are thus normally used to keep the reactor critical. The regulating rods are used to introduce a small amount of k_{ex} for power changes.

Several means of obtaining shim control are available: absorbers, fuel removal, and moderator or reflector removal. In reactors which are thermal or near-thermal, however, the most control for a given volume and mass movement is usually obtained by absorber control rods. The discussion here will be restricted to the absorber type of control rod. Calculations of the effect of fuel, moderator, or reflector removal are usually best done by perturbation theory.[1,2,3,4]

In chapter 4, the effect of temperature, fission-product poisons, and depletion was measured by the expression

$$\frac{2.5 - \nu_{dpt}}{2.5}, \qquad (5.1)$$

where ν_{dpt} is the fictitious value of ν necessary to make the reactor critical in the cold, clean, undepleted condition. In this calculation, the uranium concentration is taken as

$$N_U = N_U \text{ (final)} + \Delta N_U \text{ (depletion)} ,$$

where N_U (final) is the necessary concentration in the hot, poisoned, depleted reactor and ΔN_U (depletion) is the uranium per cubic centimeter that must be added to take care of depletion.

Expression 5.1 therefore measures the degree of virtual supercriticality that will exist in the reactor at startup. The problem is then to find the necessary quantity, size, and pattern of shim (absorber) control rods that will give a virtual decrease in reactivity that is equal in magnitude and opposite in sign to the virtual supercriticality given by expression 5.1. The reactor to be used for the calculation is the cold, clean, undepleted (startup) assembly with

$$N_U = N_U \text{ (final)} + N_U \text{ (depletion)}$$

and $\nu = \nu_{dpt}$ for criticality. The effectiveness of any trial pattern of control rods is therefore

$$\frac{\nu_{dpt} - \nu_c}{\nu_c}, \qquad (5.2)$$

where ν_c is the fictitious value of ν necessary for criticality after the rods are inserted. When ν_c equals

(1) M. J. Nielson, *Bare Pile Adjoint Solution*, Y-F10-18 (Oct. 27, 1950).

(2) M. J. Nielson, *The Adjoint Equations and Perturbation Theory for a Reflected Reactor*, Y-F10-31 (Jan. 18, 1951).

(3) H. Brooks, *A First Approach to Computing Control Effectiveness for Fast Piles*, A-4269 (Dec. 2, 1946).

(4) S. Glasstone and M. C. Edlund, *The Elements of Nuclear Reactor Theory* (2d ed., unpublished at this date), Van Nostrand, New York.

2.5, a correct array of control rods has been found.

The calculation of the effectiveness of the *regulating rods*, which is necessary for prescribing their size and location, can be done by the same methods. With respect to the reactor condition for this calculation, it is perhaps best to use the hot, poisoned, depleted assembly with $N_U = N_U$ (final) and hence $\nu = 2.5$. The effectiveness of the regulating rod would then be given by

$$\frac{2.5 - \nu_c}{2.5}, \qquad (5.3)$$

where ν_c is the fictitious value of ν necessary to keep the hot, poisoned, depleted reactor critical with the regulating rod inserted. The quantity 5.3 is equal to the actual (as opposed to virtual) reactivity, k_{ex}/k_{eff}, that the regulating rod can create in the reactor, as can be seen from the thermal, Fermi age formula,

$$k_{eff} = \nu \frac{\frac{\Sigma_f}{\Sigma_a} - e^{-B^2\tau}}{1 + L^2B^2}.$$

The ν_c for criticality is clearly given by $(\nu_c/\nu)\, k_{eff} = 1$, and hence $k_{eff} = \nu/\nu_c$. Therefore

$$\frac{k_{ex}}{k_{eff}} = \frac{k_{eff} - 1}{k_{eff}} = \frac{\frac{\nu}{\nu_c} - 1}{\frac{\nu}{\nu_c}}$$

$$= \frac{\nu - \nu_c}{\nu} = \frac{2.5 - \nu_c}{2.5},$$

which is sometimes denoted by $\Delta\nu/\nu$. Still another way of writing the same thing (for a thermal reactor) is

$$\frac{k_{th} - k_{th_c}}{k_{th}} \equiv \frac{\Delta k}{k}, \qquad (5.4)$$

where k_{th} is the physical value of the thermal reproduction constant and k_{th_c} is the fictitious value of k_{th} necessary for criticality after the rod is inserted. The notation $\Delta k/k$ for definition 5.4 is rather commonly used in the literature. The expressions 5.3 and 5.4 are equivalent, and both are equal to k_{ex}/k_{eff} if an actual reactivity change as opposed to a virtual reactivity effect is being considered. Some of the results of control rod calculations reported in the literature are given simply by $\Delta k \equiv k_{th} - k_{th_c}$. In such a case, it is necessary to know the k_{th} of the reactor to interpret the results in terms of expression 5.4 or the equivalent expression 5.3.

It is well to be on guard with respect to the notation $\Delta k/k$. For example, if the virtual effect on reactivity of a uniform poison is being expressed, the effect is sometimes measured by the actual change in the k_{th} divided by the k_{th} or, since

$$\frac{\Delta k_{th}}{k_{th}} = \frac{\nu \frac{\Sigma_f}{\Sigma_{a_1}} - \nu \frac{\Sigma_f}{\Sigma_{a_2}}}{\nu \frac{\Sigma_f}{\Sigma_{a_2}}} = -\frac{\Delta\Sigma_a}{\Sigma_{a_1}},$$

by the change in the macroscopic absorption cross section divided by the macroscopic absorption cross section. (In the above equation, if k_{th} on the left is the k_{th} without the poison, then the Σ_{a_1} on the right is the Σ_a of the clean reactor plus the $\Delta\Sigma_a$ of the added poison.) These physical changes $\Delta k_{th}/k_{th}$ and $\Delta\Sigma_a/\Sigma_a$ are only approximately equal to the reactivity change as given by expressions 5.4 or 5.3, the condition for them to be accurate being that the neutron leakage probability shall have been unaffected by the uniform poison. This condition is only satisfied for a small amount of uniform poison or for reactors in which the leakage is small. Expression 5.3

then is in the units that will be used in the kinetic formulas of chap. 6, as it should be, since the regulating rods are associated with the time-dependent behavior of the reactor.

The best mathematical method for determining the ν_c for use in expressions 5.2 or 5.3 is that devised by Nordheim and Scalettar.[5] The best and most complete description of their method is that written by the Power Pile Section of ORNL in 1947.[6] Two cases will be discussed here: (1) the single, cylindrical, axial, control rod in the cylindrical-reactor, two-group model; and (2) the pattern of seven cylindrical rods in the cylindrical-reactor, two-group model. (These seven rods are arranged symmetrically as shown in Fig. 5.1.) The discussion of the second case is similar to that presented by the Power Pile Section.[6]

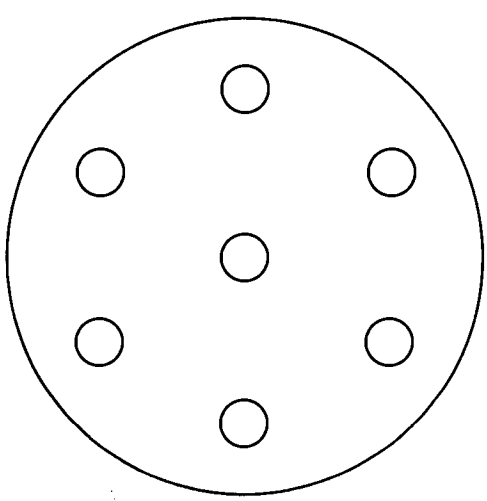

Fig. 5.1.

Considerable simplification is is achieved if the given reflected reactor is replaced by its equivalent bare reactor (the bare reactor with the same composition as that of the core of the given reflected reactor) for control rod calculations. Agreement between the calculated effect of control rods in a reflected reactor and its equivalent bare reactor is good. Thus, the first step for any pattern of rods is to reduce the given reflected reactor to its equivalent bare reactor.

SINGLE, AXIAL CONTROL ROD

The problem is to calculate the effectiveness, along the longitudinal axis of a bare, cylindrical reactor, of a cylindrical absorber rod of a given diameter. The two-group theory provides the most suitable model for the calculation. Fermi age cannot be used because the reactor is not uniform in composition. One-group theory would be too crude, and the calculations with more than two groups would be numerically too complicated.

Balance Equations. The two-group differential balance equations are

$$\nabla^2 \phi_f(r,z) + a\,\phi_f(r,z) + b\,\phi_s(r,z) = 0 \quad (5.5)$$

and

$$\nabla^2 \phi_s(r,z) + d\,\phi_s(r,z) + e\,\phi_f(r,z) = 0\,, \quad (5.6)$$

as discussed in chap. 3. The quantities a and b have

$$k_f = \nu \frac{\Sigma_{f_f}}{\Sigma_{a_f}}$$

and

$$k_s = \nu \frac{\Sigma_{f_s}}{\Sigma_{a_s}}$$

involved in their definitions, in general, and they are thus functions of ν (that is, ν_c), which is now the unknown of the problem. Equations 5.5 and 5.6 can be rewritten as

$$\nabla^2 \phi_f(r,z) + a(\nu_c)\,\phi_f(r,z) + b(\nu_c)\,\phi_s(r,z) = 0 \quad (5.7)$$

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[5] R. Scalettar and L. W. Nordheim, *Theory of Pile Control Rods*, MDDC-42 (June 17, 1946).

[6] C. R. McCullough, *Summary Report on Design and Development of High Temperature Gas-Cooled Power Pile*, Appendix, MonN-383 (Sept. 15, 1947).

and

$$\nabla^2 \phi_s(r,z) + d\,\phi_s(r,z) + e\,\phi_f(r,z) = 0 \quad (5.8)$$

to emphasize the dependency of a and b on ν_c. The d and e are pure constants.

Remarks are given in the following section on boundary conditions concerning the proper energy to select as the upper limit for the slow group (lower limit for the fast group).

General Solutions. The general solutions of Eqs. 5.7 and 5.8 for the radial components of the fluxes are

$$\phi_f(r) = A\left(\frac{B_1^2(\nu_c) - d}{e}\right) J_0[\mu_1(\nu_c)\,r] + \left(\frac{B_1^2(\nu_c) - d}{e}\right) Y_0[\mu_1(\nu_c)\,r]$$

$$+ C\left(\frac{-B_2^2(\nu_c) - d}{e}\right) I_0[\mu_2(\nu_c)\,r] + D\left(\frac{-B_2^2(\nu_c) - d}{e}\right) K_0[\mu_2(\nu_c)\,r] \quad (5.9)$$

and

$$\phi_s(r) = AJ_0[\mu_1(\nu_c)\,r] + Y_0[\mu_1(\nu_c)\,r] + CI_0[\mu_2(\nu_c)\,r] + DK_0[\mu_2(\nu_c)\,r]\,, \quad (5.10)$$

where $B_1^2(\nu_c)$ and $-B_2^2(\nu_c)$ are the roots of the determinantal equation

$$\begin{vmatrix} -B^2(\nu_c) + a(\nu_c) & b(\nu_c) \\ e & -B^2(\nu_c) + d \end{vmatrix} = 0 \quad (5.11)$$

and hence functions of ν_c. The $\mu_1(\nu_c)$ and $\mu_2(\nu_c)$ are obtained from the equations

$$\mu_1^2(\nu_c) = B_1^2(\nu_c) - \frac{\pi^2}{\left[2\left(\frac{H}{2} + \delta\right)\right]^2} \quad (5.12)$$

and

$$-\mu_2^2(\nu_c) = -B_2^2(\nu_c) - \frac{\pi^2}{\left[2\left(\frac{H}{2} + \delta\right)\right]^2}, \quad (5.13)$$

where H is the length of the equivalent bare reactor. Equations 5.9 through 5.13 follow from the derivations of chap. 3. The power constant has been so chosen to make the coefficient of the second term of ϕ_s equal to unity. In Eqs. 5.9 through 5.13, the B^2 and μ have been written $B_1^2(\nu_c)$, $\mu_1(\nu_c)$, etc. to emphasize that they are functions of the unknown of the problem, ν_c. This makes the argument $\mu_1(\nu_c)r$ of the Bessel functions $Y_0[\mu_1(\nu_c)r]$ a little confusing, unfortunately.

Boundary Conditions. The boundary conditions of the problem are:

1. $\phi_f(R + \delta) = \phi_s(R + \delta) = 0$,

where

R = the radius of the equivalent bare reactor,

δ = extrapolation distance = $0.71\,\lambda_{tr}$.

2. The fast flux is finite at the rod axis, $r = 0$; that is, the rod is transparent to the neutrons in the fast group.

3. The slow flux extrapolates to zero at a certain distance inside the rod, that is,

$$\left[\frac{1}{\phi_s}\frac{d\phi_s}{dr}\right]_{r=r_0} = \frac{1}{\delta_R}\,,$$

where

r_0 = radius of the rod,

δ_R = the extrapolation distance inside the rod.

If cadmium rods are used (or any other material that has a sharp drop in cross section at a certain energy), it is clear from boundary condition 2 that the slow group should be defined

to include all flux below the cadmium cut-off and the fast flux should include all flux above the cadmium cut-off. If the rods are boron, however, the cross section falls off in a manner inversely proportional to the neutron velocity. If the boron rod is fairly thick, it may capture a substantial portion of the epithermal neutrons that enter its surface. Probably the best procedure for this case is to choose the upper limit of the slow group to be that neutron energy for which there is still about an 85% probability that the average neutron entering the rod with this energy will be captured; that is, twice the rod wall thickness should be about 1.9 mean free paths for capture at the limiting energy.

The general form for boundary condition 3 follows from setting the return current equal to zero. The proper value of δ_R can be obtained from the results of the work of Davison and Kushneriuk[7] in which values of δ_R were determined for various sizes of "black" cylinders by transport theory in such a way that the asymptotic flux of diffusion theory will be correct in the core except for the region within two or three mean free paths of the rod. The latter region is unimportant because it is small and the flux density in it is low. Figure 5.2 is a reproduction of Davison and Kushneriuk's graph. It is best to interpret the unit of length on this graph as the transport mean free path.

Application of Boundary Conditions to General Solution. In regard to boundary condition 2, it can be seen from the series expansion for K_0 and Y_0 that

$$\lim_{x\to 0} K_0(x) = -\lim_{x\to 0}\left(\gamma + \ln\frac{x}{2}\right) \qquad (5.14)$$

[7] B. Davison and S. Kushneriuk, *Linear Extrapolation Length for a Black Sphere and a Black Cylinder*, MT-214 (March 30, 1946).

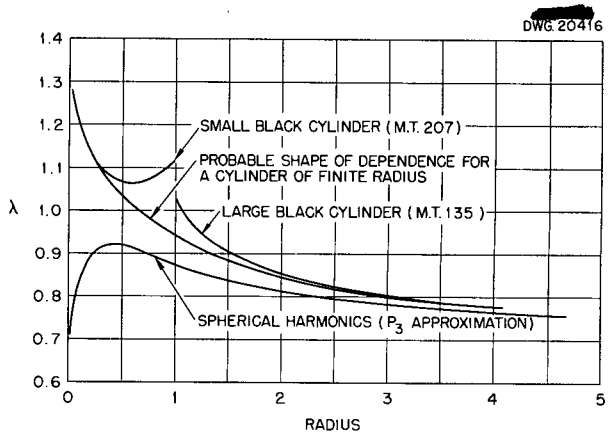


Fig. 5.2. Extrapolation Length, λ, in Terms of Radius of the Cylinder; Unit of Length = Mean Free Path of Surrounding Medium.

and

$$\lim_{x\to 0} Y_0(x) = \frac{2}{\pi}\lim_{x\to 0}\left(\gamma + \ln\frac{x}{2}\right), \qquad (5.15)$$

where γ is Euler's constant. Thus, if D is chosen with the value

$$D = \frac{2}{\pi}\frac{\left[\dfrac{B_1^2(\nu_c) - d}{e}\right]}{\left[\dfrac{-B_2^2(\nu_c) - d}{e}\right]}, \qquad (5.16)$$

the second and fourth terms in ϕ_f, as given by Eq. 5.9, will cancel as $r \to 0$ and the ϕ_f will remain finite as $r \to 0$.

Application of boundary condition 1 determines that

$$A = -\frac{Y_0\ [\mu_1(\nu_c)(R+\delta)]}{J_0\ [\mu_1(\nu_c)(R+\delta)]} \qquad (5.17)$$

and

$$C = -\frac{DK_0\ [\mu_2(\nu_c)(R+\delta)]}{I_0\ [\mu_2(\nu_c)(R+\delta)]}, \qquad (5.18)$$

where δ is the extrapolation distance at the outer reactor surface. Again it is to be noted that $[(\mu_1)(R+\delta)]$ is the argument of Y_0 in Eq. 5.17, for example, and μ_1 is shown as a function of ν_c.

Finally, application of boundary condition 3 gives

$$\frac{Y_0[\mu_1(R+\delta)]}{J_0[\mu_1(R+\delta)]}\mu_1 J_1(\mu_1 r_0) - \mu_1 Y_1(\mu_1 r_0)$$

$$-\frac{2}{\pi}\frac{B_1^2-d}{-B_2^2-d}\frac{K_0[\mu_2(R+\delta)]}{I_0[\mu_2(R+\delta)]}\mu_2 I_1(\mu_2 r_0) - \frac{2}{\pi}\frac{B_1^2-d}{-B_2^2-d}\mu_2 K_1(\mu_2 r_0)$$

$$= \frac{1}{\delta_R}\left\{-\frac{Y_0[\mu_1(R+\delta)]}{J_0[\mu_1(R+\delta)]}J_0(\mu_1 r_0) + Y_0(\mu_1 r_0)\right.$$

$$\left.-\frac{B_1^2-d}{-B_2^2-d}\frac{2}{\pi}\frac{K_0[\mu_2(R+\delta)]}{I_0[\mu_2(R+\delta)]}I_0(\mu_2 r_0) + \frac{B_1^2-d}{-B_2^2-d}\frac{2}{\pi}K_0(\mu_2 r_0)\right\}, \qquad (5.19)$$

where B_1^2, B_2^2, μ_1, and μ_2 are all functions of ν_c and r_0 is the radius of the rod.

Equation 5.19 has only one unknown, ν_c, and is thus the desired equation which determines the new value of ν, that is, ν_c, that will assure criticality of the reactor with the rod inserted.

In review, the procedure is to

1. select a trial value of ν_c,
2. determine B_1^2 and $-B_2^2$ from Eq. 5.11,
3. determine μ_1 and μ_2 from Eqs. 5.12 and 5.13,
4. see whether Eq. 5.19 is satisfied by these values and repeat the procedure until the critical value of ν_c is found,
5. use expression 5.2 (or 5.3, in the case of a regulating rod) to find the effect of the rod on criticality.

It is sometimes necessary to deduct other small effects from the value of expression 5.2 to get the net effect of the rod, for example, the effect of (1) removal of the slug of core mixture to make room for the rod, (2) the blockage, when the rod is inserted, of the streaming of neutrons down the hole that takes place when the rod is out, (3) the depression in flux around the control rod, even when the rod is out, because of the control rod thimble, coolant, etc.

PATTERN OF RODS

Several shim control rods will usually be needed in a reactor. If only a very rough estimate of the effectiveness of a pattern of four to seven identical rods is desired, the effectiveness, y, of one of the rods in the axial position can be calculated by the method described in the previous section, and the effectiveness, z, of the pattern of n rods can then be considered as being approximately $0.8\,ny$. The factor of 0.8 provides for a 20% reduction in the strength of each individual rod. This reduction occurs because the rods are not all in a position of maximum flux (the effectiveness of a rod tends to be proportional to the square of the unperturbed flux at the point of insertion) and because the rods tend to "shadow" each other; that is, any one rod tends to "feel" a flux depression caused by all the other rods. For a pattern of one rod on the axis and a ring of three to six rods equally spaced from the axis, the 20% reduction factor is usually about right if the placement of the outer ring is chosen correctly. Normally, the ring of rods should be about one third the way out on the

radius of the equivalent, bare, cylindrical reactor.

It is not too difficult to analytically calculate the combined effectiveness of a pattern, however, and the Nordheim-Scalletar procedure for this calculation is given below. A pattern of seven identical rods – one on the axis and six in a symmetrical ring off the axis – was chosen to illustrate the procedure; however, in principle, the effectiveness of any number of rods can be calculated. The more symmetrical the pattern, the simpler is the formulation.

The problem is the same as that for the axial rod. A value of ν, that is, ν_c, is sought that will assure criticality of the reactor with the pattern of rods inserted.

Differential Balance Equations. The differential balance equations, 5.7 and 5.8, are again the starting point in deriving an equation which will determine ν_c:

$$\nabla^2 \phi_f(r,\theta,z) + a(\nu_c)\, \phi_f(r,\theta,z) + b(\nu_c)\, \phi_s(r,\theta,z) = 0 \quad (5.20)$$

and

$$\nabla^2 \phi_s(r,\theta,z) + d\, \phi_s(r,\theta,z) + e\, \phi_f(r,\theta,z) = 0 \ , \quad (5.21)$$

where a and b are functions of ν_c, as indicated, and the flux is now a function of the angle θ, as well as r and z. The angle θ is measured with respect to the base line (or plane, actually) drawn through the reactor axis and one of the off-center control rod axes (the axis of rod number 2), as shown in Fig. 5.3.

General Solutions. In more general language, the procedure given in chap. 3 for solving the balance equations is to try the relations

$$\nabla^2\phi_f + B^2\phi_f = 0 \quad (5.22)$$

and

$$\nabla^2\phi_s + B^2\phi_s = 0 \quad (5.23)$$

to see whether solutions exist for B^2 such that solutions of Eqs. 5.22 and

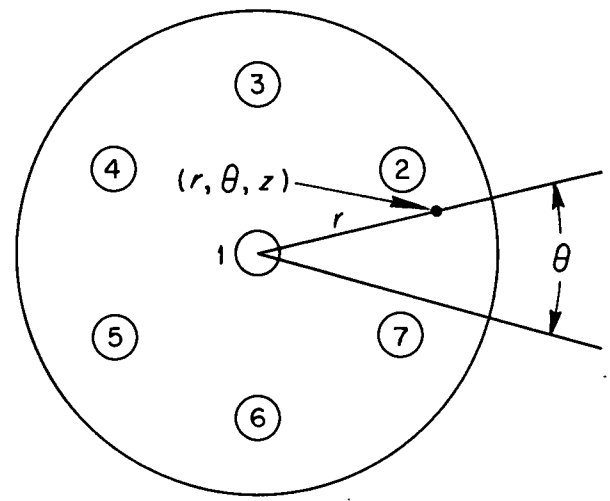


Fig. 5.3.

5.23 are solutions of Eqs. 5.20 and 5.21. Substituting Eqs. 5.22 and 5.23 into Eqs. 5.20 and 5.21 yields the familiar determinantal equation

$$\begin{vmatrix} -B^2(\nu_c) + a(\nu_c) & b(\nu_c) \\ e & -B^2(\nu_c) + d \end{vmatrix} = 0 . \quad (5.24)$$

Equation 5.24 is a quadratic in B^2 and says that two values of B^2 exist, $B_1^2(\nu_c)$ and $-B_2^2(\nu_c)$, such that solutions of Eqs. 5.22 and 5.23 are solutions of Eqs. 5.20 and 5.21. The problem reduces then to finding general solutions for Eq. 5.22 for each value of the buckling and adding them to get the general solution of Eq. 5.20 for the fast flux. Following the same procedure with Eq. 5.23 gives the general solution of Eq. 5.21 for the slow flux. Equations 5.22 and 5.23 written in complete detail are

$$\frac{\partial^2\phi_f}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_f}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_f}{\partial\theta^2} + \frac{\partial^2\phi_f}{\partial z^2} \pm B_i^2\phi_f = 0 \quad (5.25)$$

and

$$\frac{\partial^2\phi_s}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_s}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_s}{\partial\theta^2} + \frac{\partial^2\phi_s}{\partial z^2} \pm B_i^2\phi_s = 0 \ , \quad (5.26)$$

where $i = 1, 2$. Since Eqs. 5.25 and 5.26 are the same in form, the subscripts can be dropped, for the moment, and an attempt can be made to separate the variables by letting

$$\phi = R(r)\ \Theta(\theta)\ Z(z)\ . \tag{5.27}$$

Substituting Eq. 5.27 into Eq. 5.25 (or 5.26) and dividing by R, Θ, Z gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\left(\frac{Z''}{Z} + B^2\right). \tag{5.28}$$

According to the now-familiar reasoning, both sides of Eq. 5.28 must equal a constant, that is

$$\frac{Z''}{Z} + B^2 = \mu^2\ , \tag{5.29}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\mu^2\ , \tag{5.30}$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \mu^2 r^2 = -\frac{\Theta''}{\Theta}\ . \tag{5.31}$$

Again, both sides of Eq. 5.31 must equal a constant:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \mu^2 r^2 \equiv n^2 \tag{5.32}$$

and

$$\frac{\Theta''}{\Theta} = -n^2\ . \tag{5.33}$$

The three equations that determine the components of the general solutions are

$$Z'' + (B^2 - \mu^2)Z = 0\ , \tag{5.34}$$

$$\Theta'' + n^2\Theta = 0\ , \tag{5.35}$$

$$r^2R'' + rR' + (\mu^2 r^2 - n^2)R = 0\ . \tag{5.36}$$

The possible solutions are, then,

$$Z = \begin{cases} \cos\sqrt{B^2 - \mu^2}\ z\ , \\ \sin\sqrt{B^2 - \mu^2}\ z\ , \end{cases} \tag{5.37}$$

$$\Theta = \begin{cases} \cos n\theta\ , \\ \sin n\theta\ , \end{cases} \tag{5.38}$$

and

$$R = \begin{cases} J_n(\mu r)\ , \\ I_n(\mu r)\ , \end{cases} \text{ or } \begin{cases} Y_n(\mu r)\ , \\ K_n(\mu r)\ . \end{cases} \tag{5.39}$$

The most general solution to equations of the type 5.25 and 5.26 is therefore

$$\phi(r,\theta,z) = \left(F\sin\sqrt{B^2 - \mu^2}\ z + G\cos\sqrt{B^2 - \mu^2}\ z\right)\sum_{n=0}^{\infty}\Big\{\big[L_n J_n(\mu r) + M_n Y_n(\mu r)\big]\ (A_n\cos n\theta + E_n\sin n\theta)\Big\}\ , \tag{5.40}$$

where F, G, L_n, M_n, A_n, and E_n are arbitrary constants and J_n and Y_n are arbitrarily used instead of I_n and K_n. A solution of the type given by Eq. 5.40 exists for each solution of the buckling, that is, B_1^2 and $-B_2^2$. Adding the solutions gives the following expression for the most general solution of Eq. 5.21 for the slow flux:

$$\phi_s(r,\theta,z) = \left(F_1 \sin\sqrt{B_1^2 - \mu_1^2}\, z + G_1 \cos\sqrt{B_1^2 - \mu_1^2}\, z\right) \sum_{n=0}^{\infty} \left\{\left[L_{1n} J_n(\mu_1 r) + M_{1n} Y_n(\mu_1 r)\right] + (A_{1n} \cos n\theta + E_{1n} \sin n\theta)\right\}$$

$$+ \left(F_2 \sin\sqrt{-B_2^2 + \mu_2^2}\, z + G_2 \cos\sqrt{-B_2^2 + \mu_2^2}\, z\right) \sum_{n=0}^{\infty} \left\{\left[L_{2n} I_n(\mu_2 r) + M_{2n} K_n(\mu_2 r)\right] (A_{2n} \cos n\theta + E_{2n} \sin n\theta)\right\} . \tag{5.41}$$

As before, the ratio of fast to slow flux is known for each buckling; so the general solution for the fast flux can immediately be written as

$$\phi_f(r,\theta,z) = \frac{B_1^2 - d}{e} \left(F_1 \sin\sqrt{B_1^2 - \mu_1^2}\, z + G_1 \cos\sqrt{B_1^2 - \mu_1^2}\, z\right) \sum_{n=0}^{\infty} \left\{\left[L_{1n} J_n(\mu_1 r) + M_{1n} Y_n(\mu_1 r)\right] (A_{1n} \cos n\theta + E_{1n} \sin n\theta)\right\}$$

$$+ \frac{-B_2^2 - d}{e} \left(F_2 \sin\sqrt{-B_2^2 + \mu_2^2}\, z + G_2 \cos\sqrt{-B_2^2 + \mu_2^2}\, z\right) \sum_{n=0}^{\infty} \left\{\left[L_{2n} I_n(\mu_2 r) + M_{2n} K_n(\mu_2 r)\right] (A_{2n} \cos n\theta + E_{2n} \sin n\theta)\right\} . \tag{5.42}$$

Boundary Conditions. The boundary conditions of the problem are

1. $\phi_f(R + \delta) = \phi_s(R + \delta) = 0$. (5.43)

2. The fast flux is finite everywhere, including the rod positions; that is, the rods are transparent to the neutrons in the fast group.

3. The asymptotic slow flux extrapolates to zero behind each rod surface; that is,

$$\left[\frac{1}{\phi_s} \frac{\partial \phi_s}{\partial \rho}\right]_{\rho=\rho_0} \equiv \frac{1}{\partial_R} , \tag{5.44}$$

where ρ is measured from the rod center and where ρ_0 is the radius of the rods. The source for obtaining δ_R, given in the discussion of the axial rod case, applies here also.

As will be seen later, it is difficult to apply Eq. 5.44 to an off-axis rod. An approximately equivalent condition can be attained by finding for an axial rod the radius (ρ_{eff}) inside the rod at which $\phi_s \to \theta$ when Eq. 5.44 is satisfied. The condition 5.44 is then replaced by

$$\phi_s(\rho_{eff}) = 0 \tag{5.45}$$

for the off-axis rods.

Application of Boundary Conditions to General Solutions. It will be recalled that in the case of the axial rod the general solution involved the terms $Y_0(\mu_1 r)$ and $K_1(\mu_2 r)$, which go to $-\infty$ at $r = 0$, and that none of the boundary conditions forced their removal; in fact, they formed the necessary sink terms for assuring that the third boundary conditions could be met. It is of no consequence that with these functions the slow flux becomes negatively unbounded and hence fictitious inside the rod, because it is only the flux in the active region that is of physical importance.

The expressions for the flux in a multirod pattern can be put in a similar form; that is, a Y_0 and a K_0 term can be established for each rod. The *radial* components of the fast and slow fluxes for the seven-rod pattern, shown in Fig. 5.4, are given by the expressions

$$\begin{aligned}\phi_s(r,\theta) = {} & \sum_{n=0}^{\infty} J_n(\mu_1 r)\,(A_{1n}\cos n\theta + E_{1n}\sin n\theta) \\ & + Y_0(\mu_1\rho_1) + C\sum_{m=2}^{7} Y_0(\mu_1\rho_m) \\ & + \sum_{n=0}^{\infty} I_n(\mu_2 r)\,(A_{2n}\cos n\theta + E_{2n}\sin n\theta) \\ & + DK_0(\mu_2\rho_1) + C\sum_{m=2}^{7} DK_0(\mu_2\rho_m)\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}\phi_f(r,\theta) = {} & \frac{B_1^2 - d}{e}\left[\sum_{n=0}^{\infty} J_n(\mu_1 r)\,(A_{1n}\cos n\theta + E_{1n}\sin n\theta)\right. \\ & \left. + Y_0(\mu_1\rho_1) + C\sum_{m=2}^{7} Y_0(\mu_1\rho_m)\right] \\ & + \frac{-B_2^2 - d}{e}\left[\sum_{n=0}^{\infty} I_n(\mu_2 r)\,(A_{2n}\cos n\theta + E_{2n}\sin n\theta)\right. \\ & \left. + DK_0(\mu_2\rho_1) + C\sum_{m=0}^{7} DK_0(\mu_2\rho_m)\right],\end{aligned} \tag{5.47}$$

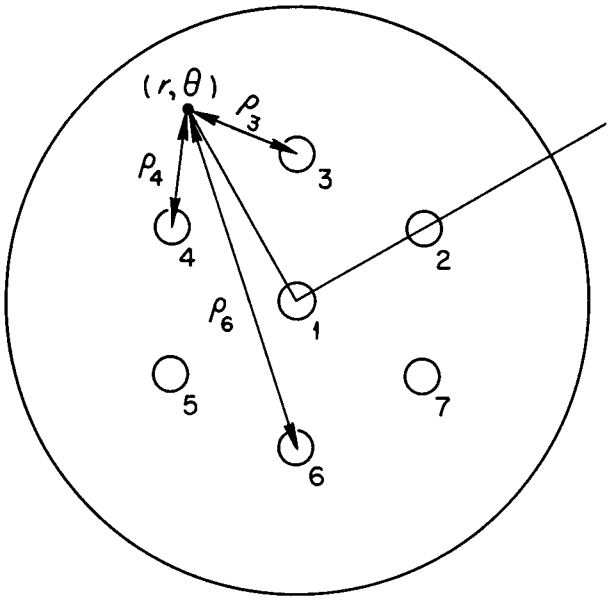


Fig. 5.4.

where B_1^2 and $-B_2^2$ are solutions of Eq. 5.24 and functions of ν_c; μ_1 and μ_2 are given by

$$\mu_1^2 = B_1^2 - \frac{\pi^2}{\left[2\left(\frac{H}{2} + \delta\right)\right]^2} \qquad (5.48)$$

and

$$-\mu_2^2 = -B_2^2 - \frac{\pi^2}{\left[2\left(\frac{H}{2} - \delta\right)\right]^2} \qquad (5.49)$$

and, hence, are also functions of ν_c; A_{1n}, E_{1n}, C, and D are arbitrary constants; ρ_m is the distance from the mth rod center to the point (r,θ) in the two-dimensional radial plane (Fig. 5.4).

If the quantities $Y_0(\mu_1\rho_m)$ and $K_0(\mu_2\rho_m)$ are replaced by their equivalent expressions, as given by the addition theorem for Bessel's functions,

$$Y_0(\mu_1\rho_m) = Y_0(\mu_1 r)\, J_0(\mu_1 r_m) + 2\sum_{n=1}^{\infty} Y_n(\mu_1 r)\, J_n(\mu_1 r_m) \cos n\theta_m \qquad (5.50)$$

and

$$K_0(\mu_2\rho_m) = K_0(\mu_2 r)\, I_0(\mu_2 r_m) + 2\sum_{n=1}^{\infty} K_n(\mu_2 r)\, I_n(\mu_2 r_m) \cos n\theta_m \,. \qquad (5.51)$$

With the substitution

$$\cos n\theta_m = \cos n\alpha_n \cos n\theta + \sin n\alpha_m \sin n\theta \,, \qquad (5.52)$$

it becomes clear that Eqs. 5.46 and 5.47 are solutions of Eqs. 5.20 and 5.21, because it is seen that Eqs. 5.46 and 5.47 can be obtained directly from the radial components of the general solutions, Eqs. 5.41 and 5.42, by assigning particular values to some of the arbitrary constants.

In Eqs. 5.50, 5.51, and 5.52 the notation is the same as that given in MonN 383;[6] that is, θ_m is the angle included between the radius vector through rod m and the radius vector through the point (r,θ), and α_m is the angle measured counterclockwise from the radius vector through rod 2 to the radius vector through rod m, as shown in Fig. 5.5.

The following development, which was worked out by the Power Pile Section at ORNL, is the same as that

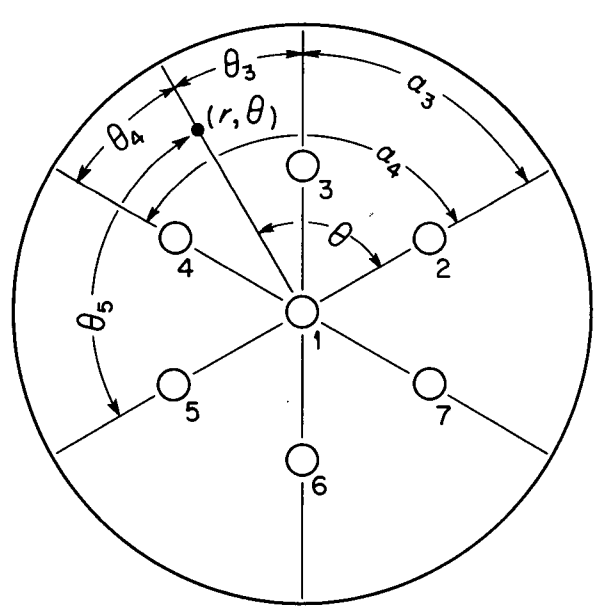


Fig. 5.5.

reported in the appendix of MonN-383,[6] except for minor changes. Upon substitution of Eqs. 5.50, 5.51, and 5.52 into Eqs. 5.46 and 5.47, the latter two equations take the form

$$\phi_s(r,\theta) = A_{10}J_0(\mu_1 r) + 6CY_0(\mu_1 r)\, J_0(\mu_1 r_m) + Y_0(\mu_1 r)$$

$$+ \sum_{n=1}^{\infty} \Bigg[J_n(\mu_1 r)\, (A_{1_n} \cos n\theta + E_{1_n} \sin n\theta)$$

$$+ 2C \sum_{m=2}^{7} Y_n(\mu_1 r)\, J_n(\mu_1 r_m)\, (\cos n\alpha_m \cos n\theta + \sin n\alpha_m \sin n\theta) \Bigg]$$

$$+ A_{20}I_0(\mu_2 r) + 6DC\, K_0(\mu_2 r)\, I_0(\mu_2 r_m) + DK_0(\mu_2 r)$$

$$+ \sum_{n=1}^{\infty} \Bigg[I_n(\mu_2 r)\, (A_{2n} \cos n\theta + E_{2n} \sin n\theta)$$

$$+ 2DC \sum_{m=2}^{7} K_n(\mu_2 r)\, I_n(\mu_2 r_m)\, (\cos n\alpha_m \cos n\theta + \sin n\alpha_m \sin n\theta) \Bigg] \quad (5.53)$$

and

$$\theta_f(r,\theta) = \frac{B_1^2 - d}{e} \Bigg\{ A_{10}J_0(\mu_1 r) + 6C\, Y_0(\mu_1 r)\, J_0(\mu_1 r_m) + Y_0(\mu_1 r)$$

$$+ \sum_{n=1}^{\infty} \Bigg[J_n(\mu_1 r)\, (A_{1n} \cos n\theta + E_{1n} \sin n\theta)$$

$$+ 2C \sum_{m=2}^{7} Y_n(\mu_1 r)\, J_n(\mu_1 r_m)\, (\cos n\alpha_m \cos n\theta + \sin n\alpha_m \sin n\theta) \Bigg] \Bigg\}$$

$$+ \frac{B_2^2 - d}{e} \Bigg\{ A_{20}I_0(\mu_2 r) + 6DC\, K_0(\mu_2 r)\, I_0(\mu_2 r_m) + DK_0(\mu_2 r)$$

$$+ \sum_{n=1}^{\infty} \Bigg[I_n(\mu_2 r)\, (A_{2n} \cos n\theta + E_{2n} \sin n\theta)$$

$$+ 2DC \sum_{m=2}^{7} K_n(\mu_2 r)\, I_n(\mu_2 r_m)\, (\cos n\alpha_m \cos n\theta + \sin n\alpha_m \sin n\theta) \Bigg] \Bigg\}. \quad (5.54)$$

The first boundary condition, that $\phi_s(R + \delta) = \phi_f(R + \delta) = 0$, is now applied and the resulting equations are

$$\begin{aligned}
\phi_s(R + \delta, \theta) = 0 = {} & A_{10} J_0[\mu_1(R + \delta)] + 6C\, Y_0[\mu_1(R + \delta)]\, J_0(\mu_1 r_m) + Y_0[\mu_1(R + \delta)] \\
& + A_{20} I_0[\mu_2(R + \delta)] + 6DC\, K_0[\mu_2(R + \delta)]\, I_0(\mu_2 r_m) + D K_0[\mu_2(R + \delta)] \\
& + \sum_{n=1}^{\infty} \Bigg(\Bigg\{ A_{1n} J_n[\mu_1(R + \delta)] \\
& + 2C \sum_{m=2}^{7} Y_n[\mu_1(R + \delta)]\, J_n(\mu_1 r_m) \cos n\alpha_m + A_{2n} I_n[\mu_2(R + \delta)] \\
& + 2DC \sum_{m=2}^{7} K_n[\mu_2(R + \delta)]\, I_n(\mu_1 r_m) \cos n\alpha_m \Bigg\} \cos n\theta \\
& + \Bigg\{ E_{1n} J_n[\mu_1(R + \delta)] + 2C \sum_{m=2}^{7} Y_n[\mu_1(R + \delta)]\, J_n(\mu_1 r_m) \sin n\alpha_m \\
& + E_{2n} I_n[\mu_2(R + \delta)] \\
& + 2DC \sum_{m=2}^{7} K_n[\mu_2(R + \delta)]\, I_n(\mu_2 r_m) \sin n\alpha_m \Bigg\} \sin n\theta \Bigg)
\end{aligned} \tag{5.55}$$

and

$$\phi_f(R+\delta,\theta) = 0 = A_{10}\frac{B_1^2-d}{e}J_0[\mu_1(R+\delta)]$$

$$+\frac{B_1^2-d}{e}\,6C\,Y_0[\mu_1(R+\delta)]\;J_0(\mu_1 r_m)$$

$$+\frac{B_1^2-d}{e}\,Y_0[\mu_1(R+\delta)] + A_{20}\frac{-B_2^2-d}{e}I_0[\mu_2(R+\delta)]$$

$$+\frac{-B_2^2-d}{e}\,6DC\,K_0[\mu_2(R+\delta)]\;I_0(\mu_2 r_m)$$

$$+\frac{-B_2^2-d}{e}\,D\,K_0[\mu_2(R+\delta)]$$

$$+\frac{B_1^2-d}{e}\sum_{n=1}^{\infty}\Bigg(\Bigg\{A_{1n}J_n[\mu_1(R+\delta)]$$

$$+\;2C\sum_{m=2}^{7}Y_n[\mu_1(R+\delta)]\;J_n(\mu_1 r_m)\;\cos n\alpha_m$$

$$+\;A_{2n}\frac{-B_2^2-d}{B_1^2-d}I_n[\mu_2(R+\delta)]$$

$$+\;2DC\frac{-B_2^2-d}{B_1^2-d}\sum_{m=2}^{7}K_n[\mu_2(R+\delta)]\;I_n(\mu_1 r_m)\;\cos n\alpha_m\Bigg\}\cos n\theta$$

$$+\Bigg\{E_{1n}J_n[\mu_1(R+\delta)] + 2C\sum_{m=2}^{7}Y_n[\mu_1(R+\delta)]\;J_n(\mu_1 r_m)\;\sin n\alpha_m$$

$$+\;E_{2m}\frac{-B_2^2-d}{B_1^2-d}I_n[\mu_2(R+\delta)]$$

$$+\;2DC\frac{-B_2^2-d}{B_1^2-d}\sum_{m=2}^{7}K_n[\mu_2(R+\delta)]\,I_n(\mu_2 r_m)\,\sin n\alpha_m\Bigg\}\sin n\theta\Bigg)\;. \tag{5.56}$$

It is seen that Eqs. 5.55 and 5.56 are of the form

$$0 = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$
$$+ b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots ;$$

that is, they are Fourier series expansions of zero. From the orthogonal properties of the Fourier series, it follows easily that the coefficients are all equal to zero: $a_i = b_i = 0$, for all i; that is

$$0 = A_{10} J_0[\mu_1(R+\delta)] + 6C\, Y_0[\mu_1(R+\delta)]\, J_0(\mu_1 r_m) + Y_0[\mu_1(R+\delta)]$$
$$+ A_{20} I_0[\mu_2(R+\delta)] + 6DC\, K_0[\mu_2(R+\delta)]\, I_0(\mu_2 r_m) + D\, K_0[\mu_2(R+\delta)] , \quad (5.57)$$

$$0 = A_{10} \frac{B_1^2 - d}{e} J_0[\mu_1(R+\delta)] + \frac{B_1^2 - d}{e} 6C\, Y_0[\mu_1(R+\delta)]\, J_0(\mu_1 r_m)$$
$$+ \frac{B_1^2 - d}{e} Y_0[\mu_1(R+\delta)] + A_{20} \frac{-B_2^2 - d}{e} I_0[\mu_2(R+\delta)]$$
$$+ \frac{-B_2^2 - d}{e} 6DC\, K_0[\mu_2(R+\delta)]\, I_0(\mu_2 r_m) + \frac{-B_2^2 - d}{e} D\, K_0[\mu_2(R+\delta)] , \quad (5.58)$$

and, for $n = 1, 2, \ldots$,

$$0 = A_{1n} J_n[\mu_1(R+\delta)] + 2C \sum_{m=2}^{7} Y_n[\mu_1(R+\delta)]\, J_n(\mu_1 r_m) \cos n\alpha_m$$
$$+ A_{2n} I_n[\mu_2(R+\delta)] + 2DC \sum_{m=2}^{7} K_n[\mu_2(R+\delta)]\, I_n(\mu_1 r_m) \cos n\alpha_m , \quad (5.59)$$

$$0 = A_{1n} J_n[\mu_1(R+\delta)] + 2C \sum_{m=2}^{7} Y_n[\mu_1(R+\delta)]\, J_n(\mu_1 r_m) \cos n\alpha_m$$
$$+ A_{2n} \frac{-B_2^2 - d}{B_1^2 - d} I_n[\mu_2(R+\delta)]$$
$$+ 2DC \frac{-B_2^2 - d}{B_1^2 - d} \sum_{m=2}^{7} K_n[\mu_2(R+\delta)]\, I_n(\mu_1 r_m) \cos n\alpha_m , \quad (5.60)$$

$$0 = E_{1n} J_n[\mu_1(R+\delta)] + 2C \sum_{m=2}^{7} Y_n[\mu_1(R+\delta)]\, J_n(\mu_1 r_m) \sin n\alpha_m$$
$$+ E_{2n} I_n[\mu_2(R+\delta)] + 2DC \sum_{m=2}^{7} K_n[\mu_2(R+\delta)]\, I_n(\mu_2 r_m) \sin n\alpha_m , \quad (5.61)$$

$$0 = E_{1n} J_n[\mu_1(R + \delta)] + 2C \sum_{m=2}^{7} Y_n[\mu_1(R + \delta)]\ J_n(\mu_1 r_m) \sin n\alpha_m$$

$$+ E_{2n} \frac{-B_2^2 - d}{B_1^2 - d} I_n[\mu_2(R + \delta)]$$

$$+ 2DC \frac{-B_2^2 - d}{B_1^2 - d} \sum_{m=2}^{7} K_n[\mu_2(R + \delta)]\ I_n(\mu_2 r_m) \sin n\alpha_m \ . \quad (5.62)$$

Equations 5.57 through 5.62 are the equations needed to evaluate A_{10}, A_{20}, A_{1n}, A_{2n} $(n > 0)$, and E_{1n} and E_{2n}. Thus,

$$A_{20} = - \frac{6DC\ K_0[\mu_2(R + \delta)]\ I_0(\mu_2 r_m) + D\ K_0[\mu_2(R + \delta)]}{I_0(\mu_2(R + \delta)]} , \quad (5.63)$$

$$A_{10} = - \frac{6C\ Y_0[\mu_1(R + \delta)]\ J_0(\mu_1 r_m) + Y_0[\mu_1(R + \delta)]}{J_0[\mu_1(R + \delta)]} , \quad (5.64)$$

$$A_{1n} = - \frac{2C \sum_{m=2}^{7} Y_n[\mu_1(R + \delta)]\ J_n(\mu_1 r_m) \cos n\alpha_m}{J_n[\mu_1(R + \delta)]} , \quad (5.65)$$

$$A_{2n} = - \frac{2DC \sum_{m=2}^{7} K_n[\mu_2(R + \delta)]\ I_n(\mu_1 r_m) \cos n\alpha_m}{I_n[\mu_2(R + \delta)]} , \quad (5.66)$$

$$E_{1n} = - \frac{2C \sum_{m=2}^{7} Y_n[\mu_1(R + \delta)]\ J_n(\mu_1 r_m) \sin n\alpha_m}{J_n[\mu_1(R + \delta)]} , \quad (5.67)$$

$$E_{2n} = - \frac{2DC \sum_{m=2}^{7} K_n[\mu_2(R + \delta)]\ I_n(\mu_2 r_m) \sin n\alpha_m}{I_n[\mu_2(R + \delta)]} . \quad (5.68)$$

Now, by applying boundary condition 2, that the fast flux must be finite at each rod, directly to Eqs. 5.46 and 5.47, it follows, by the same reasoning as that given in Eqs. 5.14, 5.15, and 5.16, that

$$D = \frac{2}{\pi}\left(\frac{B_1^2 - d}{-B_2^2 - d}\right) . \tag{5.69}$$

Thus the A's and E's are known explicitly at this point, except for the constant C.

The procedure for determining C is discussed in the following. Boundary condition 3 (Eq. 5.45) says that $\phi_s(\rho_{eff}) = 0$ for each rod, where ρ_{eff} is the effective radius of the rod. By applying boundary condition 3 at rod 2 directly to Eq. 5.46, noting that $\theta = 0$ for rod 2 (by definition), and using the notation of MonN-383[6] that

r_{12} = distance from rod 1 to rod 2 = $r_2 = r_m$,

r_{32} = distance from rod 3 to rod 2 ,

r_{m2} = distance from rod m to rod 2 ,

it can be seen that

$$\begin{aligned} 0 = \sum_{n=0}^{\infty} A_{1n} J_n(\mu_1 r_2) &+ Y_0(\mu_1 r_{12}) + C\,[Y_0(\mu_1 \rho_{eff}) + Y_0(\mu_1 r_{32}) \\ &+ Y_0(\mu_1 r_{42}) + Y_0(\mu_1 r_{52}) + Y_0(\mu_1 r_{62}) + Y_0(\mu_1 r_{72})] \\ &+ \sum_{n=0}^{\infty} A_{2n} I_n(\mu_2 r_2) + D\,K_0(\mu_2 r_{12}) + DC\,[K_0(\mu_2 \rho_{eff}) + K_0(\mu_2 r_{32}) \\ &+ K_0(\mu_2 r_{42}) + K_0(\mu_2 r_{52}) + K_0(\mu_2 r_{62}) + K_0(\mu_2 r_{72})] . \end{aligned} \tag{5.70}$$

Substitution of Eqs. 5.63 through 5.69 into Eq. 5.70 gives

$$0 = -\frac{6C\, Y_0[\mu_1(R+\delta)]\, J_0(\mu_1 r_m) + Y_0[\mu_1(R+\delta)]}{J_0[\mu_1(R+\delta)]} J_0(\mu_1 r_2)$$

$$+ \sum_{n=1}^{\infty} \frac{-2C \sum_{m=2}^{7} Y_n[\mu_1(R+\delta)]\, J_n(\mu_1 r_m) \cos n\alpha_m}{J_n[\mu_1(R+\delta)]} J_n(\mu_1 r_2)$$

$$+ Y_0(\mu_1 r_{12}) + C\left[Y_0(\mu_1 \rho_{eff}) + \sum_{m=3}^{7} Y_0(\mu_1 r_{m2})\right]$$

$$- \frac{6C \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} K_0[\mu_2(R+\delta)]\, I_0(\mu_2 r_m) + \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} K_0[\mu_2(R+\delta)]}{I_0[\mu_2(R+\delta)]} I_0(\mu_2 r_2)$$

$$+ \sum_{n=1}^{\infty} \frac{-2C \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} \sum_{m=2}^{7} K_n[\mu_2(R+\delta)]\, I_n(\mu_1 r_m) \cos n\alpha_m}{I_n[\mu_2(R+\delta)]} I_n(\mu_2 r_2)$$

$$+ \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} K_0(\mu_2 r_{12}) + C \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d}\left[K_0(\mu_2 \rho_{eff}) + \sum_{m=3}^{7} K_0(\mu_2 r_{m2})\right] . \quad (5.71)$$

By noting that

$$\cos n\alpha_2 + \cos n\alpha_3 + \ldots + \cos n\alpha_7 = 0 ,$$

when $n = 1, 2, 3, \ldots$, because of the angular spacing of the six symmetrical rods in the outer ring, and solving for C, it is found that

$$C = \frac{-Y_0(\mu_1 r_{12}) + \frac{Y_0[\mu_1(R+\delta)]}{J_0[\mu_1(R+\delta)]} J_0(\mu_1 r_2) - \frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} K_0(\mu_2 r_{12}) + \frac{\frac{2}{\pi} \frac{B_1^2 - d}{-B_2^2 - d} K_0[\mu_2(R+\delta)]}{I_0[\mu_2(R+\delta)]} I_0(\mu_2 r_2)}{L} , \quad (5.72)$$

where

$$L = -\frac{6Y_0[\mu_1(R+\delta)]\ J_0(\mu_1 r_m)}{J_0[\mu_1(R+\delta)]}\ J_0(\mu_1 r_2) + Y_0(\mu_1\rho_{eff})$$

$$+ \sum_{m=3}^{7} Y_0(\mu_1 r_{m2}) - \frac{\frac{12}{\pi}\frac{B_1^2 - d}{-B_2^2 - d} K_0[\mu_2(R+\delta)]\ I_0(\mu_2 r_m)}{I_0[\mu_2(R+\delta)]}\ I_0(\mu_2 r_2)$$

$$+ \frac{2}{\pi}\frac{B_1^2 - d}{-B_2^2 - d} K_0(\mu_2\rho_{eff}) + \frac{2}{\pi}\frac{B_1^2 - d}{-B_2^2 - d}\sum_{m=3}^{7} K_0(\mu_2 r_{m2}) \quad .$$

All the arbitrary constants have now been determined. To get the final criticality equation, which gives the desired relation in the unknown, ν_c, the third boundary condition is again applied, this time, to the axial rod, rod 1. Since $\phi_s(\rho_{eff}) = 0$ for all θ, it is legitimate to simplify the resulting equation by applying the condition at $\phi = 0$. The result is

$$0 = A_{10} + Y_0(\mu_1\rho_{eff}) + 6C\ Y_0(\mu_1 r_m)$$

$$+ A_{20} + \frac{2}{\pi}\left(\frac{B_1^2 - d}{-B_2^2 - d}\right) K_0(\mu_2\rho_{eff})$$

$$+ 6C\frac{2}{\pi}\left(\frac{B_1^2 - d}{-B_2^2 - d}\right) K_0(\mu_2 r_m) \ , \qquad (5.73)$$

where A_{10} is given by Eq. 5.64, A_{20} is given by Eq. 5.63, D is given by Eq. 5.69, C is given by Eq. 5.72, B_1^2 and $-B_2^2$ are solutions of Eq. 5.24, μ_1^2 is given by Eq. 5.48, $-\mu_2^2$ is given by Eq. 5.49, and all are functions of ν_c.

To review, then, the procedure for finding the effectiveness of the symmetrical pattern of seven rods is to

1. select a trial value of ν_c,
2. determine B_1^2 and $-B_2^2$ from Eq. 5.24,
3. determine μ_1^2 and μ_2^2 from Eqs. 5.48 and 5.49,
4. determine D from Eq. 5.69,
5. determine A_{10} and A_{20} from Eqs. 5.64 and 5.63,
6. determine the C from Eq. 5.72,
7. see whether Eq. 5.73 is satisfied by these values and, if not, repeat the procedure until the required value of ν_c is found,
8. evaluate the effect of the pattern of rods from Eq. 5.2.

RESULTS OF CALCULATIONS ON CONTROL RODS

Figure 5.6[8] shows the effect of a 2-in., solid B_4C, control rod on the axis of various reactors with fast to thermal fluxes.

Figure 5.7[9] shows the effect of an axial control rod as a function of the effective radius (ρ_{eff}, Eq. 5.45) of the rod. This figure presents the results according to two assumptions: (1) the rod is black to all neutrons (captures all neutrons that enter it), and (2) the rod is black to thermal neutrons and transparent to fast neutrons. No control rod material satisfies the first assumption, of course; so this part of the curve is of theoretical interest only.

(8) T. Rubin, H. E. Stern, and F. W. Mezger, *An Axial Control Rod in Cylindrical Reactors*, NEPA 1207-EAR-R14 (Nov. 15, 1949).

(9) C. R. McCullough, *Summary Report on Design and Development of High Temperature Gas-Cooled Power Pile*, MonN-383 (Sept. 15, 1947).

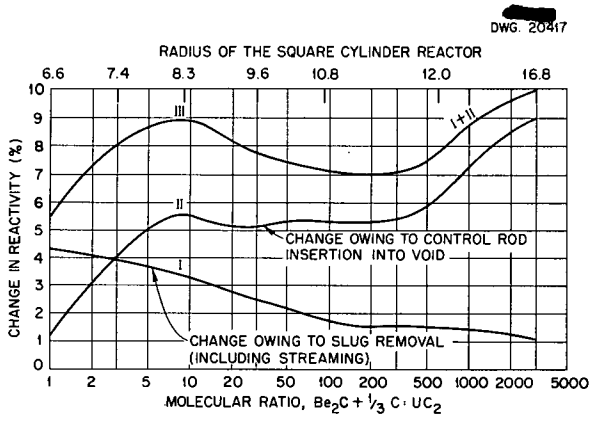


Fig. 5.6. Percentage Change in Reactivity vs. Molecular Ratio for Removal of 2-in.-dia Axial Slug and Insertion of 2-in.-dia Boron Carbide Rod into Cold, Clean, Be_2C + 1/3 C Moderated Reactors.

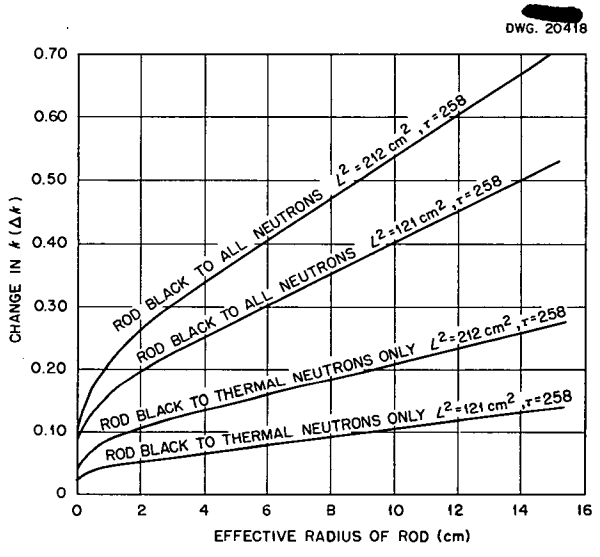


Fig. 5.7. Effectiveness of Central Control Rods.

Figure 5.8[10] shows the effect of a rod, with ρ_{eff} = 1.35 cm, as a function of distance off the reactor axis.

Figure 5.9[10] shows the effect of an array of seven control rods, one on the axis and six in a symmetrical

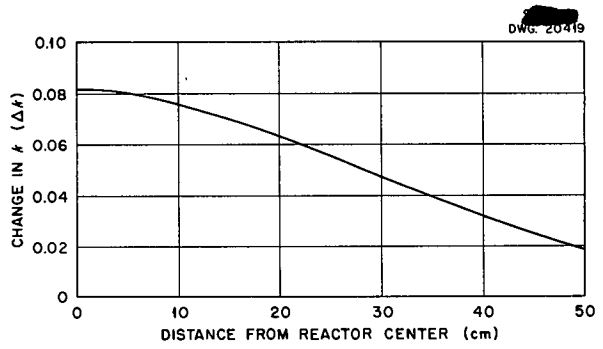


Fig. 5.8. Effect of a Single Control Rod on Reactivity. Effective radius of rod = 1.35 cm.

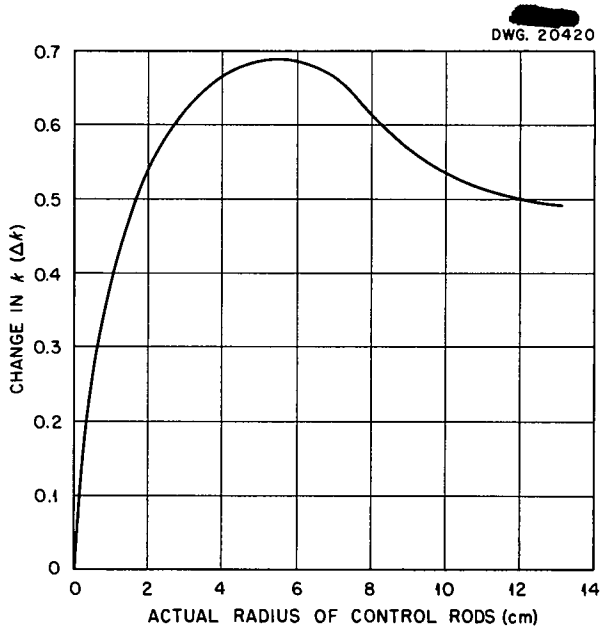


Fig. 5.9. Change in k for an Array of Seven Control Rods (Best Spacing).

ring around the axis, as a function of rod size. It is found that when the rods get to a certain size they "shadow" each other to such an extent that the total effectiveness begins to be reduced.

In Figs. 5.7, 5.8, and 5.9 the rod effect is expressed as Δk (cf., paragraph following formula 5.4). It is necessary to divide by $k_{th} \approx 1.78$ to interpret this effect in the units

(10) J. A. Wheeler, *Principles of Nuclear Power - Chapter 22: Control*, N-2292.

recommended at the beginning of this chapter.

Figure 5.10[11] shows the effect of a pattern of four 2-in. rods in a ring (no rod on axis) as a function of distance off the axis. These calculations were done by one-group theory and are overestimates. They indicate the qualitative conclusion that there is usually a "best placement" of any set of control rods.

Figure 5.11[11] shows the flux distribution in the reactor with the four-rod pattern of Fig. 5.10 for different placements of the rods. The placement where the rods are 29.1% out on the reactor radius results in the flattest power distribution and also in the maximum reactivity effectiveness.

Figure 5.12[10] shows the effect of partial insertion of an absorber control rod, expressed as a fraction of its total effectiveness.

(11) J. W. Webster, *Control Rod Effects on Reactivity and Power Distribution*, NEPA-IC-50-2-52 (Feb. 7, 1950).

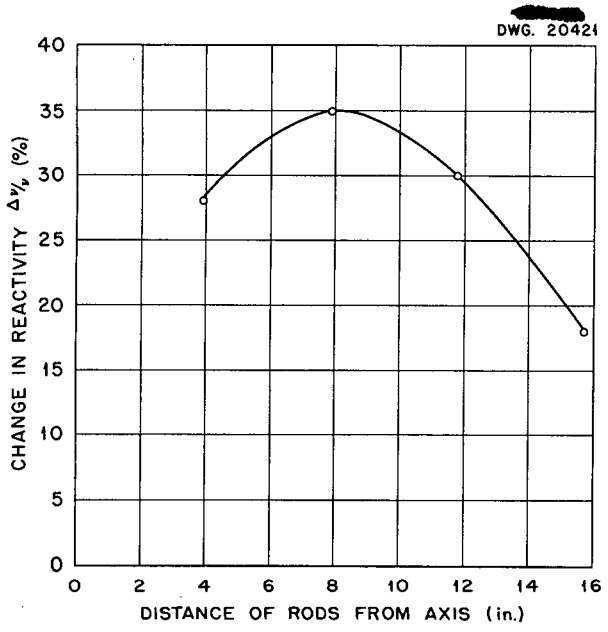
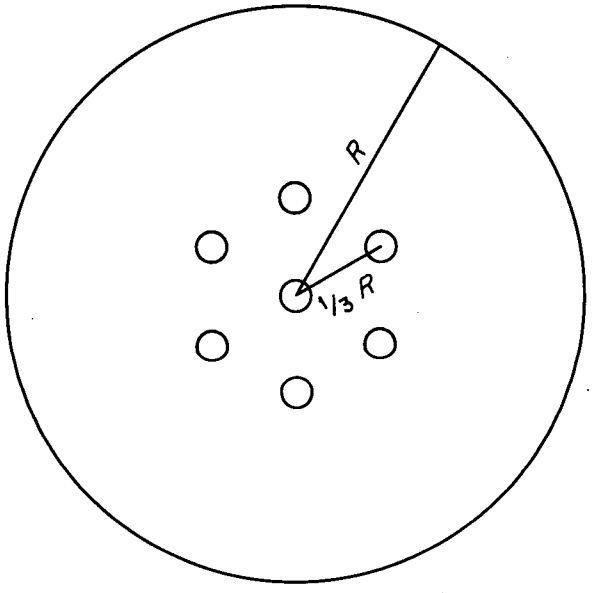


Fig. 5.10. Effectiveness of Four 2-in. Rods Placed Symmetrically at Various Distances from the Reactor Axis. Radius of square-cylinder reactor = 2.9 inches.

PROBLEM

1. Calculate the reactivity effect of a 2-in. control rod which lies on the axis and is black to thermal neutrons and transparent to fast neutrons. For this calculation, use the reactor described in problem 1 of chap. 4. Also, calculate the effect of a pattern of seven such 2-in. rods, one on the axis and six in a ring off the axis a distance equal to one third the reactor radius.



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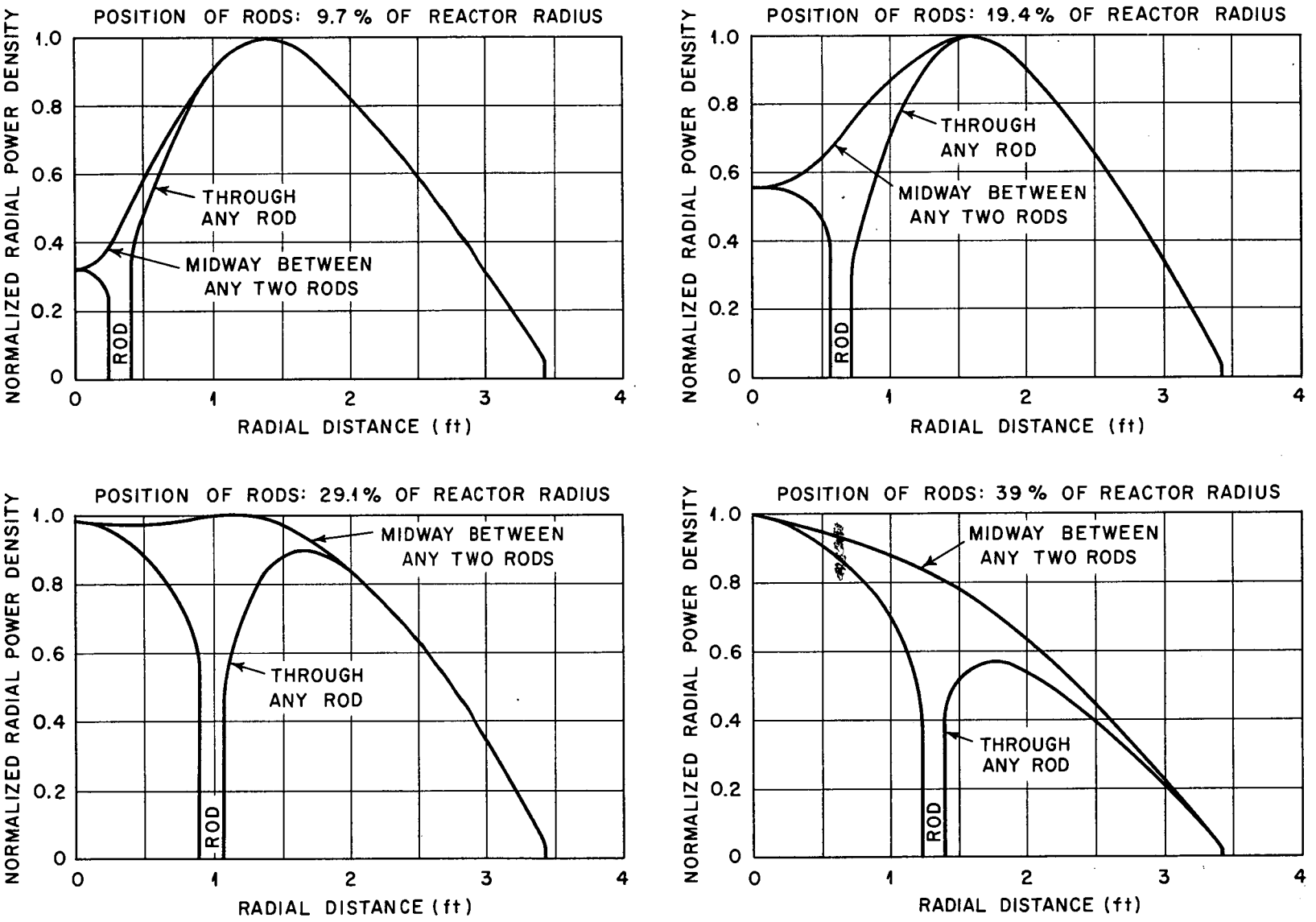

Fig. 5.11. Radial Power Distribution in a 6.8-ft Square-Cylinder Reactor with Four Symmetrically Placed 2-in. Control Rods (No Center Rod).

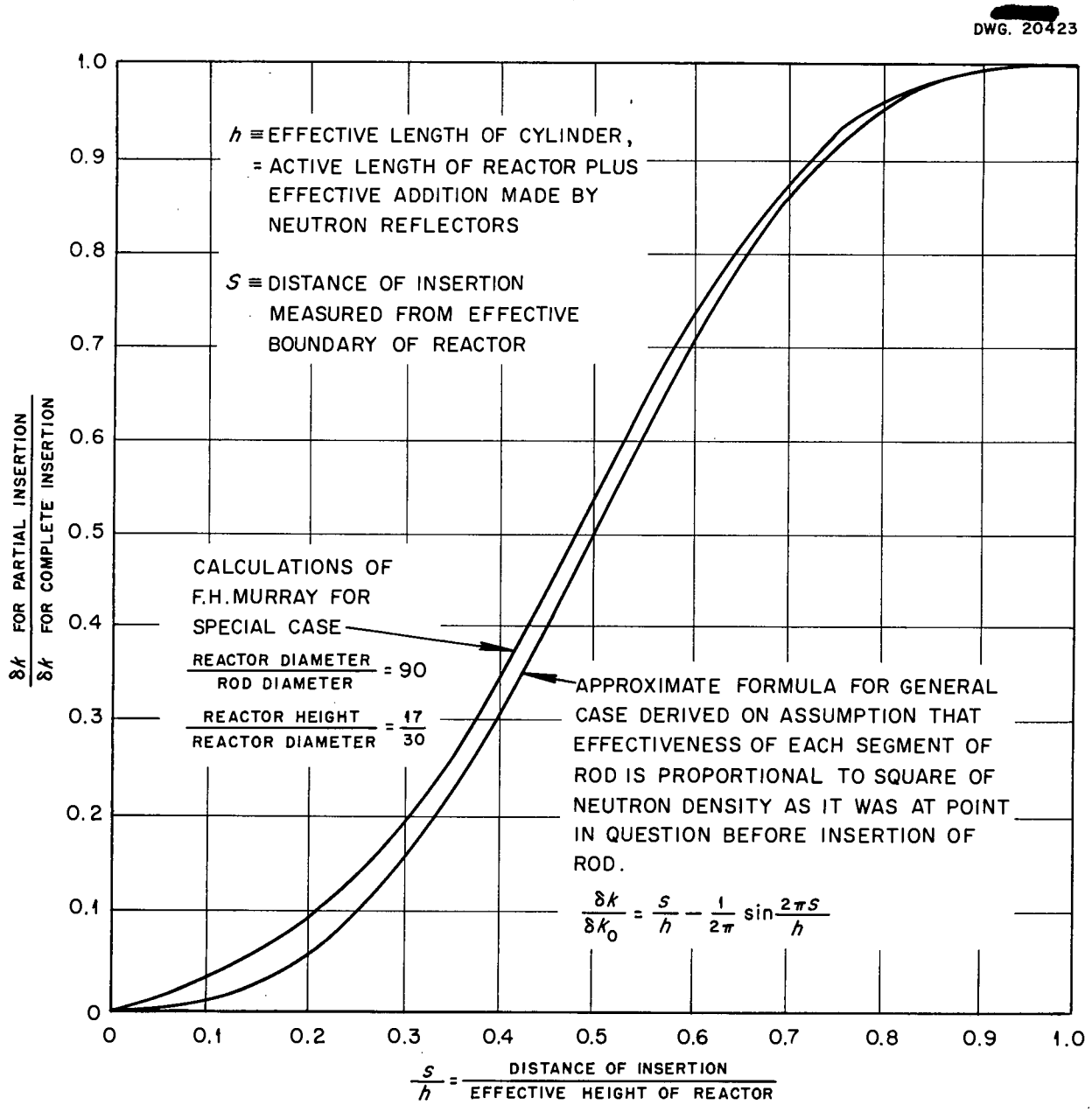

Fig. 5.12. Partial Insertion of Control Rod Parallel to Axis of Bare, Cylindrical Reactor.

Chapter 6
KINETICS

The first five chapters have dealt with subjects that are clearly important in reactor design; that is, through a joint theoretical and experimental program, the critical mass and critical size must be determined that are consistent with heat transfer and shielding constraints. The effects of temperature, fission-product poisons, and uranium depletion must also be found, and then the necessary number of shim control rods needed to offset these effects can be determined.

In respect to reactor kinetics, or the time behavior of the power, component temperatures, and fission-product-poison concentration, it is perhaps not so clear just what the theoretician should do, that is, just what concerning kinetics is important to reactor design.

Perhaps the role of the theoretician in kinetics could be summarized by saying that he helps to design a safe and easily controllable reactor. For example, some reactor designs are inherently very stable. If the power starts to change because of some inadvertent perturbation, the reactor automatically restores itself to the original power. Such reactors have a large, negative, temperature coefficient, that is, the k_{eff} decreases with increased reactor temperature; and thus when the power and hence the temperature increases, the reactor becomes subcritical and the power decreases. In a reactor with a negative temperature coefficient, if a control rod is withdrawn a short distance and left there, the power will increase for a few moments and then become steady at a new, somewhat higher, power.

It is clear, then, that a reactor proposal should be investigated theoretically to see whether the temperature coefficient is positive or negative and to determine its magnitude. The response of the power and the temperatures to various assumed perturbations should be calculated. Various questions must be answered, such as, "how fast should the control rods be designed to move?"

Probably one of the most thorough and capable investigations of the kinetics of a reactor that had been conducted at the date of this writing was that for the submarine thermal reactor (STR). The STR is a water-cooled and -moderated reactor with solid fuel plates. The water, which is heated as it passes through the core, is used to make steam which runs turbines for propulsion of the ship. As the power increases, the average water temperature increases, and the associated decrease in water density tends to decrease the k_{eff} and to send the power level down again. As a consequence, the STR is a very stable reactor; it has excellent self-regulation characteristics; and almost no accident, except that caused by sabotage, would result in damage to the reactor.

The report on the STR investigation[1] gives an excellent example of the kind of theoretical work that can be done to answer the question "is the reactor safe?" The following is quoted from the introduction of the report:

> It is the purpose of this paper to investigate accidents which will affect the reactivity of the Mark I Submarine Thermal Reactor (STR) when it is operating in its designed power range and to estimate the effect of particular accidents on the overall power plant. Events which may occur during the operation of the plant – malfunction of some component or operational

(1) A. F. Henry, *Accidents Affecting the Reactivity of the Mark I STR*, WAPD-41 (Oct. 1951).

errors – will be examined in considerable detail. Catastrophes and possible acts of sabotage will be investigated to the point of estimating the extent of damage and the time required for the catastrophe to develop.

Primary emphasis will be placed on the results of accidents rather than on the mechanism by which they may occur or the probability of their occurring. For example, the behavior of the reactor following failure of all coolant pumps will be examined. It is not intended to imply by this examination that there is any serious probability that such an accident – or any of the other accidents studied – will occur and no attempt will be made to give a quantitative estimate of this probability. Only qualitative statements of how such an accident might occur will be stated.

Table 6.1, which is a reproduction of a table from the report,[1] suggests the kind of investigation that can be made of the kinetics of a reactor design.

KINETICS EQUATIONS

The equations of reactor kinetics that are encountered in a study such as that made for the STR and an indication of how they can be solved will now be given. The equations that follow apply in particular to a reactor with a fixed moderator, a liquid coolant that flows through the reactor in one pass, and fixed fuel plates or tubes. The equations, with slight modifications, apply to the STR, the MTR, most of the airplane reactor proposals, and other reactors.

The simultaneous, first-order, differential equations will be set down for (1) the flux, (2) the concentration of delayed-neutron emitters, (3) the fuel temperature, (4) the moderator temperature, (5) the coolant temperature (as functions of position along the reactor length), (6) the iodine (xenon precursor) concentration, (7) the xenon concentration, and (8) the excess k_{eff}, which depends on items 3, 4, 5, and 7.

Flux. In chap. 1, which was on the one-group theory, it was stated that there were reasons, over and above the pedagogical ones, for presenting the one-group formulation. It does have practical uses, and the study of kinetics provides such a use. The one-group equation and the equivalent bare reactor are used here. As will be seen in the paragraphs on "Flux" in the following section on "Convenient Transformations," the average neutron lifetime is the only reactor constant needed, and therefore it is unnecessary to average the various constants entering into the one-group formulas for criticality. The lifetime should be calculated by methods more accurate than the one-group method.

It will be recalled that in the one-group, steady-state case described in chap. 1 the neutron balance was expressed as

$$-(\text{leakage}) - (\text{absorption}) + (\text{production}) = 0 .$$

It follows that in the nonsteady state the equivalent statement is

$$-(\text{leakage}) - (\text{absorption}) + (\text{production}) = (\text{rate of change of neutron density}) .$$

Flux with no Delayed Neutrons. If all fission neutrons appeared promptly after fission, the equation for the flux with no delayed neutrons could be written as

$$+\frac{\lambda_{tr}}{3}\nabla^2\phi(\underline{r},t) - \phi(\underline{r},t)\,\Sigma_a + \phi(\underline{r},t)\,\Sigma_{ak} = \frac{\partial n(\underline{r},t)}{\partial t}, \qquad (6.1)$$

where

$\underline{r}$ = the spatial position,
t = time,
n = neutrons per unit volume at $\underline{r}$ and time t.

Actually, some of the fission neutrons (about 0.76%) do not appear

TABLE 6.1. SUMMARY OF REACTOR ACCIDENTS

ACCIDENT	FIRST EFFECT	FINAL RESULTS	APPROXIMATE TIME BEFORE FINAL RESULTS COMMENCE	MEANS, OTHER THAN SCRAM, BY WHICH ACCIDENT CAN BE CONTROLLED
1. Primary coolant pump failure	Coolant temperature surge; no oscillations; reactor power falls abruptly	Coolant boils	2 minutes	Emergency cooling loop
2. Closing of steam turbine throttle	Temperatures in power plant rise (~25°F); reactor power falls abruptly	Coolant boils	10 minutes	Emergency cooling loop
3. Regulator rod introduces a step increase of 0.6% in reactivity	Reactor power increases in ~0.1 sec to 3.5 times its initial value; falls to 2.2 times its initial value in ~0.3 sec	Reactor power approaches a higher equilibrium value of 1.2 times its initial value	20 seconds	Automatic
4. Shim and xenon rods all move out at maximum speed	Rise in reactor temperatures of ~1.5°F/sec	Coolant boils	2 minutes	Return rods to core at normal insertion rate
5. Xenon rods lock in position just as the fully xenon poisoned pile goes critical	Rise in reactor temperatures of ~0.3°F/sec	Coolant boils	10 minutes	Return xenon rods to core at normal insertion rate
6. Emergency cooling loop opens during normal operation	Some of coolant about to enter the core is shunted through emergency loop	Reactor delivers power both to boilers and to emergency loop	10 seconds	Close emergency loop valves
7. At a time when only one of the two boiler loops is operating, the valves separating primary coolant from the inoperative loop are opened, the pumps in the loops are started, and the initially operating loop is closed off	Temperature of primary coolant entering core falls 60°F/sec for 5 sec; reactor power increases to 14 times initial power; no oscillations are to be expected	Power and temperatures return to initial values	4 seconds	Automatic
8. Pressurizer malfunctions so that pressure in the primary loop increases	Temperature rises 1°F for 97 psi rise in primary coolant pressure	Primary coolant system ruptures	Dependent on rate of pressure increase	Relief valves in primary line blow
9. Pressure in primary coolant system falls	Primary coolant water temperature falls slightly	Coolant boils	Dependent on rate of pressure decrease	Repressurize or shut reactor down
10. Extended boiling in core	Reactivity decreases	Equilibrium when ~50% of volume of coolant in reactor is in vapor phase	Dependent on cause of boiling	Dependent on cause of boiling
11. All control rods removed from core at scram rate (sabotage)	Reactivity increases ~3.6% per sec, reaches prompt critical in 0.2 sec, then goes as exp $(+250\ t^2)$	Pressure relief valves in primary coolant loop blow; fuel plates vaporize	0.2 seconds	None - must scram in ~0.2 seconds
12. Coolant water removed from reactor	Reactivity falls; fuel plates heat up	Fuel plates melt	1 minute	Return coolant to core
13. A block in the adjacent water channels causes a fuel plate to melt	Reactivity falls ~ -0.0006	Average coolant temperature in core falls ~2°F; channel blocking may spread	~ seconds	None - moreover once the melting has started, even scramming will not control the accident

promptly after fission, but, rather, they arise after a delay time that is dependent on the beta decay of various fission products; hence, they are referred to as the "delayed neutrons." It is worthwhile, however, to solve Eq. 6.1 to see the form of the solution, even though the results have no practical value. The right side of Eq. 6.1 can be written as

$$\frac{\partial n(\underline{r},t)}{\partial t} = \frac{1}{v}\frac{\partial n(\underline{r},t)v}{\partial t} = \frac{1}{v}\frac{\partial \phi(\underline{r},t)}{\partial t} ; \tag{6.2}$$

then, dividing Eq. 6.1 by Σ_a gives

$$L^2\nabla^2 \phi(\underline{r},t) + (k-1)\,\phi(\underline{r},t) = \frac{\lambda_a}{v}\frac{\partial \phi(\underline{r},t)}{\partial t} , \tag{6.3}$$

where, as before, $L^2 = \lambda_a\lambda_{tr}/3$. The quantity λ_a/v is the infinite reactor neutron lifetime (neutron lifetime if there were no leakage). When λ_a/v is defined as l_0, Eq. 6.3 becomes

$$L^2\nabla^2 \phi(\underline{r},t) + (k-1)\,\phi(\underline{r},t) = l_0\frac{\partial \phi(\underline{r},t)}{\partial t} . \tag{6.4}$$

An attempt is then made to separate the variables by assuming that

$$\phi(\underline{r},t) = \phi(t)\,Z(\underline{r}) . \tag{6.5}$$

Now, since the approximation of replacing the reflected reactor by its equivalent bare reactor has been made, it follows that

$$\nabla^2 Z(\underline{r}) + B^2 Z(\underline{r}) = 0 \tag{6.6}$$

(if the assembly is not too far from critical), where

$$B^2 = \left[\frac{\pi}{2(x_0+\delta)}\right]^2$$

for slabs, with x_0 being the half thickness of the equivalent, bare, slab reactor; B^2 is equal to the equivalent expressions for the other soluble geometries. Equation 6.5 simply says that it is assumed that the transient, spatial, flux distribution remains as a cosine for slabs, as $(\sin Br)/r$ for spheres, and as a zero-order Bessel's function for the radial distribution in cylinders.

Substituting Eqs. 6.5 and 6.6 into Eq. 6.4 and dividing out the $Z(\underline{r})$ gives

$$(-L^2B^2 - 1 + k)\,\phi(t) = l_0\frac{d\phi(t)}{dt} , \tag{6.7}$$

and dividing through by $(1 + L^2B^2)$ gives

$$\left(\frac{k}{1+L^2B^2} - 1\right)\phi(t) = \frac{l_0}{1+L^2B^2}\frac{d\phi(t)}{dt} . \tag{6.8}$$

The quantity $1/(1 + L^2B^2)$ is the probability that a neutron will not escape from the reactor, as was pointed out previously. Thus $k/(1 + L^2B^2)$ is the k_{eff} for the one-group model. The quantity $l_0/(1 + L^2B^2)$ is called the finite reactor lifetime, which is denoted here by $\mathfrak{L}$, that is,

$$\mathfrak{L} = \frac{l_0}{1+L^2B^2} . \tag{6.9}$$

With these substitutions, Eq. 6.8 becomes

$$(k_{eff} - 1)\,\phi(t) = \mathfrak{L}\frac{d\phi(t)}{dt} . \tag{6.10}$$

Since $k_{eff} = 1$ for the critical reactor, the quantity $(k_{eff} - 1)$ is called k-excess; that is,

$$k_{ex} = k_{eff} - 1 .$$

With this substitution, the solution of Eq. 6.10 is

$$\phi(t) = \phi(0)\,e^{\int_0^t (k_{ex}/\mathfrak{L})\,dt} , \tag{6.11}$$

since, in general, k_{ex} is a function of time. If the case is considered in which k_{ex} remains constant, that is, there are no temperature or fission product poison effects, then Eq. 6.11 reduces to

$$\phi(t) = \phi(0)\,e^{(k_{ex}/\mathfrak{L})t} . \tag{6.12}$$

As was stated in chap. 2, the most commonly accepted definition of reactivity is

$$\text{reactivity} = \frac{k_{ex}}{k_{eff}} , \qquad (6.13)$$

which is often denoted by the symbol ρ:

$$\rho = \frac{k_{ex}}{k_{eff}} . \qquad (6.14)$$

Thus, the time dependence of the one-group flux in the equivalent bare reactor, for which k_{ex} is a constant because of the false assumption that there are no delayed neutrons, is given by

$$\phi(t) = \phi(0)\, e^{(\rho k_{eff}/\mathcal{L})t} , \qquad (6.15)$$

which, when Eq. 6.5 is considered, becomes

$$\phi(\underline{r},t) = \phi(\underline{r},0)\, e^{(\rho k_{eff}/\mathcal{L})t} \qquad (6.16)$$

The quantity $\mathcal{L}/\rho k_{eff}$ in the exponent is called the period, T; that is,

$$T = \frac{\mathcal{L}}{\rho k_{eff}} . \qquad (6.17)$$

Equation 6.17 in the form

$$\rho = \frac{\mathcal{L}}{T k_{eff}} \qquad (6.18)$$

is sometimes called the inhour formula, under the false assumption of no delayed neutrons. More specifically, however, the inhour formula for the case of no delayed neutrons is

$$\rho_{ih} = \frac{\dfrac{\mathcal{L}}{T k_{eff}}}{\dfrac{\mathcal{L}}{3600\, k_{eff}}} , \qquad (6.19)$$

which gives the reactivity in units called inhours, where 1 inhour is the amount of reactivity that will cause the reactor to have a period of 1 hour.

The formulation has, perhaps, been carried a bit too far under the assumption of no delayed neutrons and a more realistic treatment will be given, but it is worthwhile to see that the reactor has a purely exponential behavior and that even with a small amount of k_{ex} the flux would assume astronomical values in a very short time if it were not for the damping effect of the delayed neutrons. For example, by taking a typical value for $\mathcal{L}$, say 10^{-4} sec, and by supposing that a conservative amount of k_{ex}, say 0.002, is injected, the flux would rise as

$$\phi(\underline{r},t) = \phi(\underline{r},0)\, e^{20T}$$

and in 1 sec would rise by a factor of 10^8 (if the reactor remained intact).

Flux with Delayed Neutrons. As stated before, some of the fission neutrons (about 0.76%) actually do not appear promptly after fission, but, rather, they arise after a delay time that is dependent on the beta decay of various fission products. These delayed neutrons have, on the basis of experimental measurements, been "peeled off" into six groups. Each group is a certain fraction, β_i, of the average number of neutrons produced per fission.

Each group has a definite, measured, mean time, l_i, for appearance after fission that is related to the half life, $\tau_{1/2\,i}$, of the fission product (called the ith-group precursor) which is the emitter of the ith group by the usual relations:

$$\frac{1}{l_i} = \frac{\ln 2}{\tau_{1/2\,i}} .$$

The probability per second for delayed neutrons of the ith group to appear is

$$\lambda_i = \frac{1}{l_i} .$$

Table 6.2 gives the experimentally measured data on the six groups of delayed neutrons.

TABLE 6.2. DELAYED NEUTRON DATA

GROUP NO.	B_i (fraction of neutrons per fission)*	$\tau_{\frac{1}{2}i}$ (half life of ith-group precursor, sec)	l_i (mean delay time of ith group, sec)	λ_i (probability per second for appearance of ith-group neutrons, sec^{-1}
1	0.00029	0.05	0.07	14
2	0.00084	0.43	0.62	1.61
3	0.0024	1.52	2.19	0.456
4	0.0021	4.51	6.50	0.151
5	0.0017	22.0	31.7	0.0315
6	0.00026	55.6	80.2	0.0124

*The total of the neutrons per fission is given by $\sum_{i=1}^{6} \beta_i = \beta = 0.000759$.

With the definitions

$$\beta = \sum_{i=1}^{6} \beta_i \qquad (6.20)$$

and $C_{i0}(\underline{r},t)$ = (concentration of ith group precursor at position $\underline{r}$ and time t), it follows that the statement that

-(leakage) - (absorption)
+ (production) = (rate of change of neutron density)

becomes

$$\frac{\lambda_{tr}}{3} \nabla^2 \phi(\underline{r},t) - \phi(\underline{r},t)\,\Sigma_a + \phi(\underline{r},t)\,\Sigma_a\, k(1-\beta) + \sum_{i=1}^{6} C_{i0}(\underline{r},t)\,\lambda_i = \frac{1}{v}\frac{\partial\phi(\underline{r},t)}{\partial t} . \qquad (6.21)$$

An attempt is made to separate the variables by assuming that

$$\phi(\underline{r},t) = \phi(t)\, Z(\underline{r})$$

and

$$C_{i0}(\underline{r},t) = C_i(t)\, Z(\underline{r}) , \qquad (6.22)$$

where

$$\nabla^2 Z(\underline{r}) + B^2 Z(\underline{r}) = 0 . \qquad (6.23)$$

It is found that the flux equation with the spatial component separated out is

$$\phi(t)\left[k_{ex}(t) - \beta\, k_{eff}(t)\right] + \frac{1}{\Sigma_a}\sum_{i=1}^{6} \lambda_i C_i(t) = \mathcal{L}\frac{d\phi(t)}{dt} , \qquad (6.24)$$

where

$$C_i(t) = \frac{C_{i0}(t)}{1 + L^2B^2} , \qquad (6.25)$$

or

$$\phi(t)\,[\rho(t) - \beta]\, k_{eff}(t) + \frac{1}{\Sigma_a}\sum_{i=1}^{6} \lambda_i C_i(t) = \mathcal{L}\frac{d\phi(t)}{dt} , \qquad (6.26)$$

where

$$\rho = \frac{k_{ex}}{k_{eff}} \quad \text{(the reactivity)} .$$

Concentration of Precursors. The six differential equations for the concentrations of the precursors of the six groups of delayed neutrons express the fact that

$$\begin{pmatrix}\text{production per sec}\\ \text{of the } i\text{th-group}\\ \text{precursor}\end{pmatrix} - \begin{pmatrix}\text{decay per sec of}\\ \text{the } i\text{th-group}\\ \text{precursor}\end{pmatrix}$$

$$= \begin{pmatrix}\text{rate of change of concen-}\\ \text{tration of the } i\text{th-group}\\ \text{precursor}\end{pmatrix}.$$

Thus

$$\phi(\underline{r},t)\ \Sigma_a k\beta_i - \lambda_i\ C_{i0}(\underline{r},t) = \frac{dC_{i0}(\underline{r},t)}{dt}\ , \qquad (6.27)$$

for $i = 1, 2, \ldots, 6$. Using Eqs. 6.22 and 6.23, separating out the spatial component, and dividing through by $1 + L^2B^2$ gives

$$\phi(t)\ \Sigma_a\ k_{eff}(t)\ \beta_i - \lambda_i\ C_i(t) = \frac{dC_i(t)}{dt}\ , \qquad (6.28)$$

where $i = 1, 2, \ldots, 6$.

Before going on to derive the equations for the temperatures and fission product poisons, which determine the time behavior of k_{eff}, it is worthwhile to consider the special case in which k_{eff} is constant (and not equal to unity). An attempt is made to obtain a solution of the seven simultaneous equations, Eqs. 6.26 and 6.28, by assuming that $\phi(t)$ and $C_i(t)$ have solutions of the form

$$\phi(t) = A\ e^{\omega t}$$

and (6.29)

$$C_i(t) = D_i\ e^{\omega t}\ , \qquad i = 1, 2, \ldots, 6\ ,$$

where A and D_i are constants, and by substituting Eq. 6.29 into Eqs. 6.26 and 6.28 to see whether solutions for ω exist which assure that solutions for $\phi(t)$ and $C_i(t)$ of the form of Eq. 6.29 exist. The substitutions yield

$$A\ e^{\omega t}\ (\rho - \beta)\ k_{eff} + \frac{1}{\Sigma_a}\sum_{i=1}^{6}\lambda_i D_i\ e^{\omega t} = \mathfrak{L}\omega A\ e^{\omega t} \qquad (6.30)$$

and

$$A\ e^{\omega t}\ \Sigma_a k_{eff}\beta_i - \lambda_i D_i\ e^{\omega t} = \omega D_i\ e^{\omega t}\ , \qquad (6.31)$$

where $i = 1, 2, \ldots, 6$. Upon dividing out the $e^{\omega t}$ and solving Eq. 6.31 for D_i, it is found that

$$D_i = A\frac{\Sigma_a k_{eff}\beta_i}{\omega + \lambda_i}\ . \qquad (6.32)$$

Substituting Eq. 6.32 into Eq. 6.30 and dividing out the $A\ e^{\omega t}$ in Eq. 6.30 gives

$$(\rho - \beta)\ k_{eff} + k_{eff}\sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\omega + \lambda_i} = \mathfrak{L}\omega\ , \qquad (6.33)$$

or

$$\rho k_{eff} + k_{eff}\sum_{i=1}^{6}\left(\frac{\lambda_i\beta_i}{\omega + \lambda_i} - \beta_i\right) = \mathfrak{L}\omega\ , \qquad (6.34)$$

or

$$\rho k_{eff} - k_{eff}\sum_{i=1}^{6}\frac{\omega\beta_i}{\omega + \lambda_i} = \mathfrak{L}\omega\ , \qquad (6.35)$$

or

$$\rho = \frac{\mathfrak{L}\omega}{k_{eff}} + \sum_{i=1}^{6}\frac{\omega\beta_i}{\omega + \lambda_i}\ . \qquad (6.36)$$

For any given reactivity, ρ, Eq. 6.36 is an equation of the form

$$\omega^7 + A_1\omega^6 + A_2\omega^5 + \ldots + A_6\omega = b\ .$$

Thus there are seven solutions for ω. If a plot is made of ρ as a function of ω, it is seen that for $\rho > 0$, one solution of ω is positive and the other six are negative. Furthermore, it is seen that for any ρ, the other six values of ω lie between consecutive values of $-\lambda_i$, one to each interval. Let ω_0 be the root which is positive when $\rho > 0$, and let ω_1 be the root lying in the interval

$$-\lambda_5 < \omega_1 < -\lambda_6 = -0.016\ ,$$

and, similarly,

$$-\lambda_4 < \omega_2 < -\lambda_5 ,$$
$$-\lambda_3 < \omega_3 < -\lambda_4 ,$$
$$-\lambda_2 < \omega_4 < -\lambda_3 ,$$
$$-\lambda_1 < \omega_5 < -\lambda_2 ,$$
$$-10^4 \approx -\frac{1}{\mathfrak{L}} < \omega_6 < -\lambda_1 .$$

In this way, it is determined that expressions of the form $D_i\ e^{\omega_i t}$ are solutions for the flux, and expressions of the form $D_i\ e^{\omega_i t}$ are solutions for the precursor concentrations, where the ω_i are those discussed above. The complete solution is, then, the sum of the particular solutions. Thus

$$\phi(t) = A_0\ e^{\omega_0 t} + A_1\ e^{\omega_1 t} + A_2\ e^{\omega_2 t} + \ldots + A_6\ e^{\omega_6 t} \quad (6.37)$$

and

$$C_i(t) = D_{i0}\ e^{\omega_0 t} + D_{i1}\ e^{\omega_1 t} + \ldots + D_{i6}\ e^{\omega_6 t} , \quad (6.38)$$

where $i = 1, 2, \ldots, 6$. The D_{ij} can be evaluated from Eq. 6.32; that is,

$$D_{ij} = A_j \frac{\Sigma_a k_{eff} \beta_i}{\omega_j + \lambda_i} ,$$

in terms of the A_j. The seven A_j can be evaluated from the initial values of the ϕ and C_i after multiplying through by the spatial component. It turns out that $A_0 > 0$ and $A_j < 0$, for $j = 1, 2, \ldots, 6$.

Even though the constants have not been evaluated precisely, enough is known to make some observations on the behavior of the flux in the special case where k_{eff} is a constant and not equal to unity. The ω_6 is the largest root in absolute value, and since it is negative, the last term in the flux expression will quickly become negligible after a time of the order of the neutron lifetime. The fifth term will become negligible next, and so on through the fourth and third terms, and, finally, after about 0.2 sec, all terms after the first will be negligible compared with the first. For this reason, $1/\omega_0$ in the first term, which it will be recalled is positive for ρ positive, is called the stable reactor period, T. An observation can be made at this point: if the flux is differentiated with respect to time and t is set equal to zero (in order to find the initial slope), it is found that the last term dominates (ω_6 is very large). Since ω_6 is close in absolute value to $1/\mathfrak{L}$, it follows that the initial behavior of the flux depends primarily on the neutron lifetime; that is, the flux behaves initially as if there were no delayed neutrons. The graphs for this chapter illustrate this point. The flux climbs very steeply for a fraction of a second, and then the delayed neutrons "catch" it and it begins to round off; finally, after about 0.2 sec, the increase steadies into that of the stable reactor period.

Going back to Eq. 6.36, it can be seen that since ω_0 is a solution of Eq. 6.36,

$$\rho = \frac{\mathfrak{L}\omega_0}{k_{eff}} + \sum_{i=1}^{6} \frac{\omega_0 \beta_i}{\omega_0 + \lambda_i} , \quad (6.39)$$

and since $T = 1/\omega_0$,

$$\rho = \frac{\mathfrak{L}}{T k_{eff}} + \sum_{i=1}^{6} \frac{\beta_i}{1 + \lambda_i T} , \quad (6.40)$$

which relates the reactivity and the stable reactor period. The equation

$$\rho_{ih} = \frac{\dfrac{\mathfrak{L}}{T k_{eff}} + \displaystyle\sum_{i=1}^{6} \dfrac{\beta_i}{1 + \lambda_i T}}{\dfrac{\mathfrak{L}}{3600\ k_{eff}} + \displaystyle\sum_{i=1}^{6} \dfrac{\beta_i}{1 + 3600\ \lambda_i}} \quad (6.41)$$

expresses the reactivity in inhour units. One inhour of reactivity is the reactivity which causes a stable reactor period of 1 hr (3600 sec).

The preceding discussion was a digression that was made after obtaining Eqs. 6.26 and 6.28 to consider the special case in which k_{eff} (and ρ) is a constant. Actually, however, in every reactor, to a greater or lesser degree, k_{eff} varies with time – even after the control rod is moved to a new position and left there. The change of reactivity with temperature and with change of fission-product-poison concentration makes it necessary to couple additional first-order differential equations with Eqs. 6.26 and 6.28 (already seven in number) to account for the variation of k_{eff} and ρ. The equations are written for the temperatures of the three regions: fuel, moderator, and coolant. The reactor is assumed to be cylindrical, with the coolant flowing longitudinally. The temperatures are expressed as radial averages at a variable, longitudinal, position z.

Fuel Temperature. The kinetic equation for the fuel-plate temperatures at position z along the reactor length is

$$\frac{\partial\theta_f(z,t)}{\partial t} = \zeta_f\,\phi(z,t) - \lambda_f\,[\theta_f(z,t) - \theta_c(z,t)]\ , \qquad (6.42)$$

where

$\theta_f(z,t) \equiv$ the radial, average, fuel-plate temperature at longitudinal position z at time t,

$\theta_c(z,t) \equiv$ the radial, average, coolant temperature at longitudinal position z and time t,

$\phi(z,t) = \phi(t)\,P(z)$,

$P(z) \equiv$ the longitudinal, power distribution normalized such that

$$\int_0^L P(z)\,dz = 1\ ,$$

$L \equiv$ reactor core length,

$\zeta_f/L \equiv$ rate of increase of fuel temperature per unit flux in the absence of cooling,

$\lambda_f \equiv$ rate of loss of temperature by heat transfer from fuel plates to coolant per degree difference in temperature of fuel and coolant.

The $P(z)$ should be determined by at least a two-group calculation. The constants ζ_f and λ_f can be determined from the thermal properties of the materials and the geometrical arrangement of the core structure.

If Eq. 6.42 is multiplied by the heat capacity of the fuel, C_f, it is clear that the first term on the right is then the heat per second being supplied to the fuel, the second term is the heat being transferred to the coolant per second, and $C_f(\partial\theta_f/\partial t)$, on the left, is the net amount of heat accruing in the fuel per second.

Moderator Temperature. The kinetic equation for the moderator temperature at position z is

$$\frac{\partial\theta_m(z,t)}{\partial t} = \zeta_m\,\phi(z,t) - \lambda_m\,[\theta_m(z,t) - \theta_c(z,t)]\ , \qquad (6.43)$$

which is identical in form to Eq. 6.42 and has analogous definitions.

Coolant Temperature. The equation for the coolant temperature at position z and time t is

$$\theta_c(z,t) = \theta_c\left(\theta,\ t - \frac{z}{v}\right) + \frac{a_1}{v}\int_0^z \left[\theta_f\left(z',\ t - \frac{z-z'}{v}\right) - \theta_c\left(z',\ t - \frac{z-z'}{v}\right)\right] dz'$$
$$- \frac{a_2}{v}\int_0^z \left[\theta_m\left(z',\ t - \frac{z-z'}{v}\right) - \theta_c\left(z',\ t - \frac{z-z'}{v}\right)\right] dz'\ , \qquad (6.44)$$

where v is the coolant velocity and $\theta_c(z,t)$ is the temperature of a slug of coolant at position z and time t; the first term on the right is the temperature at which this slug entered the core at a time z/v seconds previous to time t; the second term on the right is the rise in the temperature of the slug in moving from $z = 0$ to z as a result of heat transfer from the fuel; and the last term on the right is the rise in temperature of the slug as a result of heat transfer from the moderator. The constant a_1 is the coolant temperature rise per second per unit temperature difference between fuel and coolant; the constant a_2 is the coolant temperature rise per second per unit temperature difference between moderator and coolant. The constants a_1 and a_2 can be evaluated from a heat transfer analysis.

Iodine Concentration. The kinetic equation for the spatial average of the iodine (xenon precursor) concentration is

$$\frac{d\mathcal{I}(t)}{dt} = y\,\phi(t) - \lambda_1\,\mathcal{I}(t)\ , \qquad (6.45)$$

where y is the number of iodine atoms formed per unit volume per second per unit flux and λ_1 is the decay constant for I^{135}. This equation and the following equation for xenon are discussed in chap. 4 (y has a slightly different definition here than in chap. 4).

Xenon Concentration. The kinetic equation for the spatial average of Xe^{135} concentration is

$$\frac{d\mathcal{X}(t)}{dt} = \lambda_1\,\mathcal{I}(t) - \lambda_x\,\mathcal{X}(t) - \sigma_x\,\phi(t)\,\mathcal{X}(t)\ , \qquad (6.46)$$

where λ_x is the decay constant for xenon and σ_x is the microscopic cross section of Xe^{135} averaged over the reactor flux spectrum.

The other fission-product poisons, such as samarium, obey similar equations, which are omitted here for brevity.

Reactivity. The reactivity, ρ, and the k_{eff} [$= 1/(1-\rho)$] of the reactor depend on the temperatures of coolant, moderator, and fuel plates, the concentration of fission-product poisons, and the position of the control rods. The temperatures determined above as functions of longitudinal position are averaged over the core length, and the reactivity is determined from the averages; the averages are denoted by $\theta_f(t)$, $\theta_m(t)$, and $\theta_c(t)$.

The equation for the reactivity is

$$\begin{aligned}\rho k_{eff} = k_{ex}(t) = \delta k_c(t) &+ \epsilon_f[\theta_f(t) - \theta_f(0)] \\ &+ \epsilon_m[\theta_m(t) - \theta_m(0)] \\ &+ \epsilon_c[\theta_c(t) - \theta_c(0)] \\ &+ \epsilon_x[\mathcal{X}(t) - \mathcal{X}(0)]\ , \qquad (6.47)\end{aligned}$$

where

ϵ_f = the change in k_{eff} per degree change in average fuel plate temperature evaluated at $\theta_f = \theta_f(0)$,

ϵ_c = the change in k_{eff} per degree change in average coolant temperature evaluated at $\theta_c = \theta_c(0)$,

ϵ_m = the change in k_{eff} per degree change in average moderator temperature evaluated at $\theta_m = \theta_m(0)$ and with $\mathcal{X} = \mathcal{X}(0)$,

ϵ_x = the change in k_{eff} per unit change in xenon concentration evaluated at $\mathcal{X} = \mathcal{X}(0)$ and $\theta_m = \theta_m(0)$,

$\delta k_c(t)$ = the k_{ex} introduced by the control rods.

CONVENIENT TRANSFORMATIONS

It is convenient to make certain transformations of these kinetic equations. By expressing the quantities as dimensionless variables in terms of the values at $t = 0$, many of the

constants disappear and the numbers involved are reduced to a convenient magnitude.

Flux. In the flux equation the following substitutions are made:

$$\phi(t) = \phi_0 + \phi'(t) \ , \tag{6.48}$$

$$k_{eff}(t) = k_{ex}(t) + 1 \ ,$$

$$\rho = \frac{k_{ex}}{k_{eff}} \tag{6.49}$$

$$C_i(t) = C_{i0} + C_i'(t) \ , \tag{6.50}$$

where ϕ_0 and C_{i0} are the steady-state flux and ith-group delayed-neutron-precursor density for $t = 0$, and $\phi'(t)$ and $C_i'(t)$ are the departures from the steady state values with these substitutions Eq. 6.26 becomes

$$\mathfrak{L}\,\frac{d\phi'(t)}{dt} = [\phi_0 + \phi'(t)]\left\{k_{ex}(t) - \beta\,[1 + k_{ex}(t)]\right\} + \frac{1}{\Sigma_a}\sum_i \lambda_i [C_{i0} + C_i'(t)] \ , \tag{6.51}$$

and the steady state form becomes

$$0 = \phi_0(-\beta) + \frac{1}{\Sigma_a}\sum_i \lambda_i C_{i0} \ . \tag{6.52}$$

Substracting Eq. 6.52 from Eq. 6.51 gives

$$\mathfrak{L}\,\frac{d\phi'(t)}{dt} = \phi_0\, k_{ex}(t)\,(1 - \beta) + \phi'(t)\, k_{ex}(t) - \phi'(t)\,\beta - \phi'(t)\,\beta\, k_{ex} + \frac{1}{\Sigma_a}\sum_i \lambda_i\, C_i'(t) \ . \tag{6.53}$$

Now, by transforming to the new variables

$$\Phi(t) = \frac{\phi'(t)}{\phi_0} \tag{6.54}$$

and

$$C_i(t) = \frac{C_i'(t)}{C_{i0}} \tag{6.55}$$

and by using Eq. 6.52, Eq. 6.53 becomes

$$\frac{d\Phi(t)}{dt} = \frac{1}{\zeta}\rho(t)\, k_{eff}(t)\,(1 - \beta) + \frac{1}{\zeta}\rho(t)\, k_{eff}(t)\,\Phi(t)\,(1 - \beta) - \frac{\beta}{\zeta}\Phi(t) + \sum_i \frac{\beta_i}{\zeta} C_i \ . \tag{6.56}$$

Interestingly, the only property of the reactor entering this equation is the neutron lifetime.

Concentration of Precursors. If Eqs. 6.48, 6.49, and 6.50 are substituted in the equation for the concentration of the ith-group delayed-neutron emitters. Eq. 6.28 becomes

$$\frac{dC_i'(t)}{dt} = \beta_i[1 + \rho(t)\ k_{eff}(t)]\ \Sigma_a[\phi_0 + \phi'(t)] - \lambda_i[C_{i0} + C_i'(t)] \ . \qquad (6.57)$$

By using the notations of Eqs. 6.48 and 6.50, the steady state form of Eq. 6.28 becomes

$$0 = \beta_i \Sigma_a \phi_0 - \lambda_i C_{i0} \ . \qquad (6.58)$$

Subtracting Eq. 6.58 from 6.57 gives

$$\frac{dC_i'(t)}{dt} = \beta_i \Sigma_a \phi_0\ \rho(t)\ k_{eff}(t) + \beta_i \Sigma_a\ \phi'(t) + \beta_i \Sigma_a\ \rho(t)\ k_{eff}(t) - \lambda_i\ C_i'(t) \ . \qquad (6.59)$$

By transforming to the new variables defined by Eqs. 6.54 and 6.55 and by using Eq. 6.58, Eq. 6.59 becomes

$$\frac{dC_i(t)}{dt} = \lambda_i[\rho(t)\ k_{eff}(t) + \rho(t)\ k_{eff}(t)\ \Phi(t) + \Phi(t) - C_i(t)] \ . \qquad (6.60)$$

Fuel Temperature. With the substitutions

$$\theta_f(x,t) = \theta_{f0}(z) + \theta_f'(z,t) \qquad (6.61)$$

and

$$\theta_c(z,t) = \theta_{c0}(z) + \theta_c'(z,t) \ , \qquad (6.62)$$

where $\theta_{f0}(z)$ and $\theta_{c0}(z)$ are the steady-state fuel and coolant temperatures for $t = 0$ at z, and $\theta_f'(z,t)$ and $\theta_c'(z,t)$ are the time-dependent departures, and the relationship

$$\phi(z,t) = \phi(t)\ P(z) \ , \qquad (6.63)$$

w' . $P(z)$ is the longitudinal power di.tribution normalized over the reactor length, L, so that

$$\int_0^L P(z) = 1 \ , \qquad (6.64)$$

Eq. 6.42 becomes

$$\frac{\partial\theta_f'(z,t)}{\partial t} = \zeta_f[\phi_0(z) + \phi'(t)]\ P(z) - \lambda_f[\theta_{f0}(z) + \theta_f'(z,t) - \theta_{c0}(z) - \theta_c'(z,t)] \ . \qquad (6.65)$$

By using the notations of Eqs. 6.61, 6.62, and 6.63, the steady-state form of Eq. 6.42 becomes

$$0 = \zeta_f \phi_0\ P(z) - \lambda_f[\theta_{f0}(z) - \theta_{c0}(z)] \ . \qquad (6.66)$$

Subtracting Eq. 6.66 from Eq. 6.65 gives

$$\frac{\partial\theta_f'(z,t)}{\partial t} = \zeta_f \phi'(t)\ P(z) - \lambda_f[\theta_f'(z,t) - \theta_c'(z,t)] \ . \qquad (6.67)$$

By using Eq. 6.54, Eq. 6.67 becomes

$$\frac{\partial\theta_f'(z,t)}{\partial t} = \zeta_f \phi_0\ \Phi(t)\ P(z) - \lambda_f[\theta_f'(z,t) - \theta_c'(z,t)] \ . \qquad (6.68)$$

Moderator Temperature. In a similar manner and with similar definitions, it follows that the moderator temperature equation becomes

$$\frac{\partial\theta_m'(z,t)}{\partial t} = \zeta_m \phi_0\ \Phi(t)\ P(z) - \lambda_m[\theta_m'(z,t) - \theta_c'(z,t)] \ . \qquad (6.69)$$

Coolant Temperature. By substituting Eqs. 6.61 and 6.62 and subtracting out the steady-state part, the coolant temperature equation (Eq. 6.44) becomes

$$\theta_c'(z,t) = \theta_c'\left(0, t - \frac{z}{v}\right) + \frac{a_1}{v}\int_0^z \left[\theta_f'\left(z,\ t - \frac{z-z'}{v}\right) - \theta_c'\left(z', t - \frac{z-z'}{v}\right)\right] dz'$$

$$+ \frac{a_2}{v}\int_0^z \left[\theta_m'\left(z',\ t - \frac{z-z'}{v}\right) - \theta_c'\left(z',\ t - \frac{z-z'}{v}\right)\right] dz' \ . \quad (6.70)$$

Iodine Concentration. By substituting

$$\mathcal{I}(t) = \mathcal{I}_0 + \mathcal{I}'(t) \ , \quad (6.71)$$

where $\mathcal{I}_0$ is the steady-state iodine concentration at design conditions and $\mathcal{I}'(t)$ is the time-dependent departure, subtracting out the steady-state part, and transforming to the dimensionless variable

$$I(t) = \frac{\mathcal{I}'(t)}{\mathcal{I}_0} \ , \quad (6.72)$$

the equation for the iodine concentration (Eq. 6.45) becomes

$$\frac{dI(t)}{dt} = \lambda_I[\Phi(t) - I(t)] \ . \quad (6.73)$$

Xenon Concentration. By substituting Eqs. 6.48 and 6.71 and the similar equation for xenon, that is,

$$\mathcal{X}(t) = \mathcal{X}_0 + \mathcal{X}'(t) \ , \quad (6.74)$$

the equation for the xenon concentration (Eq. 6.46) becomes

$$\frac{d\mathcal{X}'(t)}{dt} = \lambda_I[\mathcal{I}_0 + \mathcal{I}'(t)] - \lambda_x[\mathcal{X}_0 + \mathcal{X}'(t)] - \sigma_x[\phi_0 + \phi'(t)]\ [\mathcal{X}_0 + \mathcal{X}'(t)] \ . \quad (6.75)$$

In steady-state form, Eq. 6.46 is

$$0 = \lambda_I\mathcal{I}_0 - \lambda_x\mathcal{X}_0 - \sigma_x\phi_0\mathcal{X}_0 \ , \quad (6.76)$$

and subtracting Eq. 6.76 from Eq. 6.75 gives

$$\frac{d\mathcal{X}'(t)}{dt} = \lambda_I\ \mathcal{I}'(t) - \lambda_x\ \mathcal{X}'(t)$$

$$- \sigma_x\ \phi'(t)\ \mathcal{X}_0 - \sigma_x\ \phi'(t)\ \mathcal{X}'(t)$$

$$- \sigma_x\phi_0\ \mathcal{X}'(t) \ . \quad (6.77)$$

Transforming to dimensionless variables by using Eqs. 6.54 and 6.72 and

$$X(t) = \frac{\mathcal{X}'(t)}{\mathcal{X}_0} \ , \quad (6.78)$$

as well as Eq. 6.76, Eq. 6.77 becomes

$$\frac{dX(t)}{dt} = \lambda_I\frac{\mathcal{I}_0}{\mathcal{X}_0}\ [I(t) - X(t)$$

$$- \Phi(t)\ X(t) - \Phi(t)]$$

$$+ \lambda_x[\Phi(t) + \Phi(t)\ X(t)] \ . \quad (6.79)$$

Reactivity. With the use of the definitions already introduced, Eq. 6.47 for the $\rho(t)\ k_{eff}(t)$ becomes

$$\rho(t)\ k_{eff}(t) = k_{ex}(t) = \delta k_c(t)$$

$$+ \epsilon_f\ \theta_f'(t) + \epsilon_c\ \theta_c'(t) + \epsilon_m\ \theta_m'(t)$$

$$+ \epsilon_x\mathcal{X}_0\ X(t) \ . \quad (6.80)$$

REMARKS ON SOLUTION OF THE EQUATIONS

In solving for the response of the system to, for example, a sudden introduction of k_{ex} or perhaps a sudden change of inlet coolant temperatures by means of these simultaneous first-order differential equations, a rather lengthy set of equations must be used. Often it is possible to reduce the number of equations by approximations and by leaving out certain equations that affect the response only slightly.

For example, the moderator temperature and the xenon concentration sometimes change slowly and hence can be omitted if the interest is only in the behavior of, say, power for a short time after a sudden change in k_{eff}. Also, the six precursor equations can be reduced to one by the following approximation: The six groups are divided into three classes, the shortest delay group being Class A, the next three groups, Class B, and the longest two groups, Class C. Class A neutrons are considered as prompt neutrons, Class C neutrons are considered as being infinitely delayed, that is, they are never returned to the system, and Class B is given an average decay constant,

$$\lambda_B = \frac{\beta_2 + \beta_3 + \beta_4}{\beta_2 l_2 + \beta_3 l_3 + \beta_4 l_4}.$$

Even after such approximations and simplifications, however, there are usually several simultaneous differential equations, the solution of which is at best a lengthy job. Three procedures have been used with success at the various laboratories: (1) An inexpensive but sometimes slightly inaccurate method is to "linearize" the equations by neglecting products of departure quantities and then using the method of Laplace transform to solve the linearized approximate equations. This procedure has been described by Smith *et al.*[2] (2) The equations can be solved in a straightforward manner by using simple numerical techniques and a digital computer such as the IBM-CPC. This procedure has been described by Nielson and Webster.[3] (3) The equations can be solved through the use of analog simulators such as were used for the kinetic studies of the STR design.

RESULTS OF KINETIC STUDIES

Figures 6.1 through 6.9[4] are presented as illustrations of the kinetic behavior of a stable reactor. These figures show the response of flux (or power) and fuel temperature to step increases in reactivity. The data for these curves were calculated by IBM numerical methods for the ARE design of June 10, 1951. This reactor was BeO-moderated and sodium-cooled, and the fuel was in a liquid state in many small vertical tubes. The fuel tubes extended upward into a layer of boron at the top of the active lattice, and thus if the power were to increase and the fuel temperature consequently to rise, part of the fuel would expand into the boron layer and be lost to the chain reaction. The result would be a slight state of subcriticality and the power level would tend to decrease. This reactor design is therefore a very stable one.

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(2) N. M. Smith, Jr., *et al.*, *Perturbation Equations for the Kinetic Response of a Liquid-Fuel Reactor*, ANP-62.

(3) M. J. Nielson and J. W. Webster, *Solution of Kinetic Equations of Cylindrical Liquid-Fuel Reactor*, ANP-68.

(4) J. W. Webster and M. J. Nielson, *Some Results of Kinetic Studies on the ARE Design of 10 June 1951*, ORNL Y-F10-75.

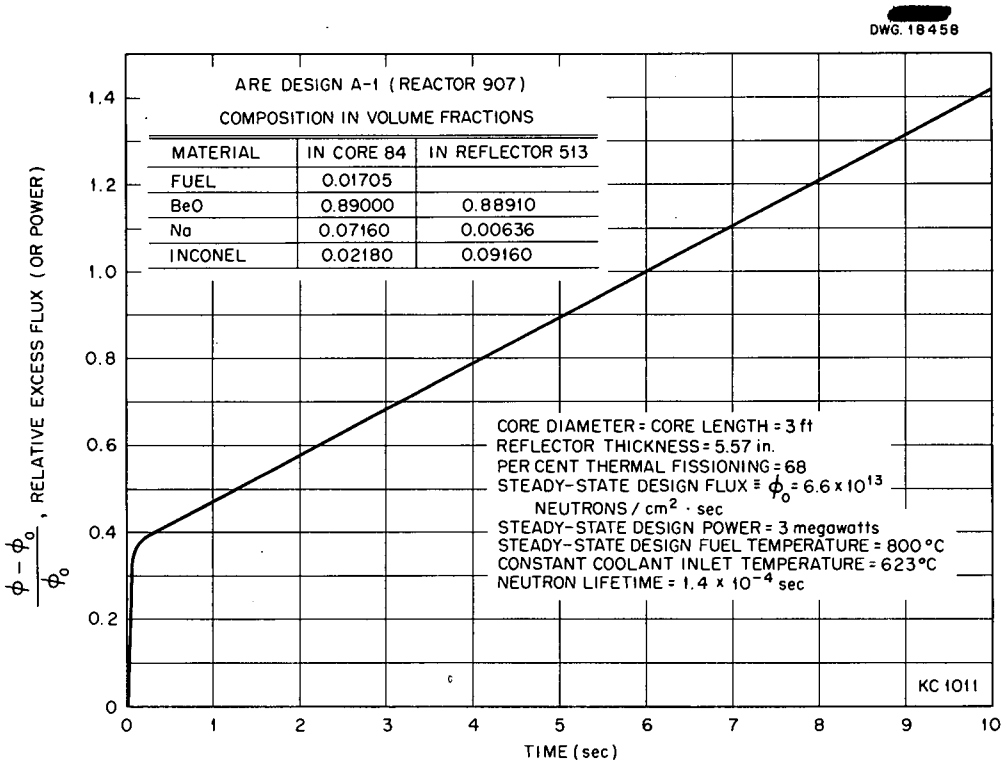


Fig. 6.1. Flux Response of ARE to Step Increase in Reactivity of 0.002. Change in reactivity because of fuel expansion not included.

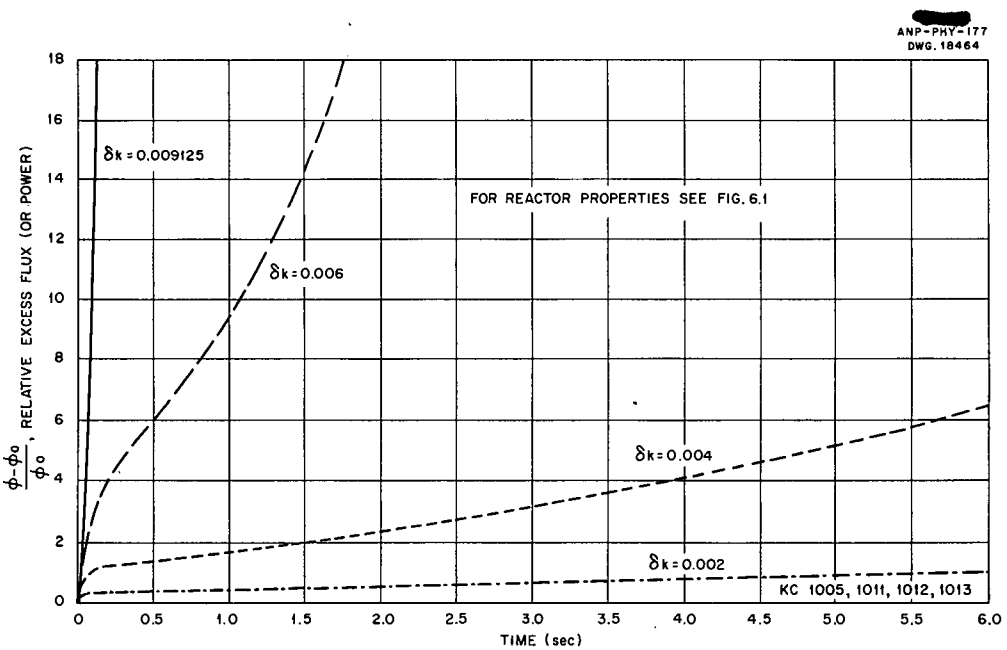


Fig. 6.2. Flux Response to a Step Increase in Reactivity for Various Reactivity Changes. Change in reactivity because of fuel expansion not included.

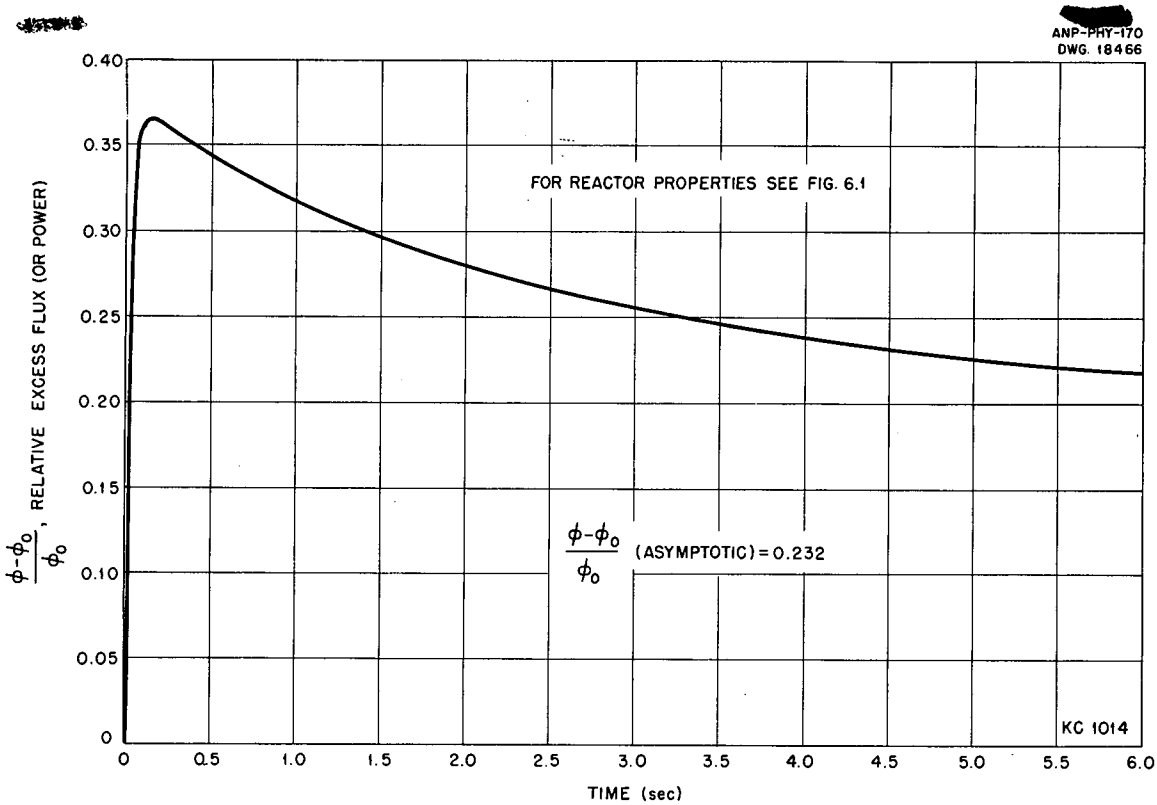


Fig. 6.3. Flux (or Power) Response to a Step Change in Reactivity of 0.002.

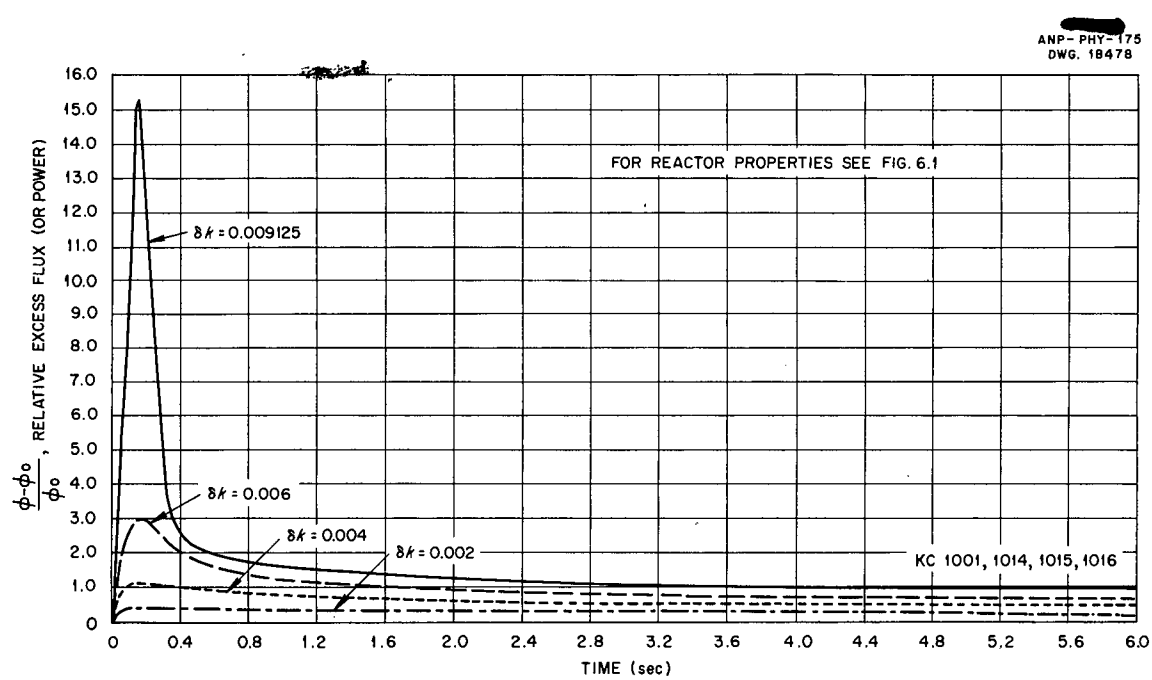


Fig. 6.4. Flux Response to a Step Increase in Reactivity for Various Reactivity Changes.

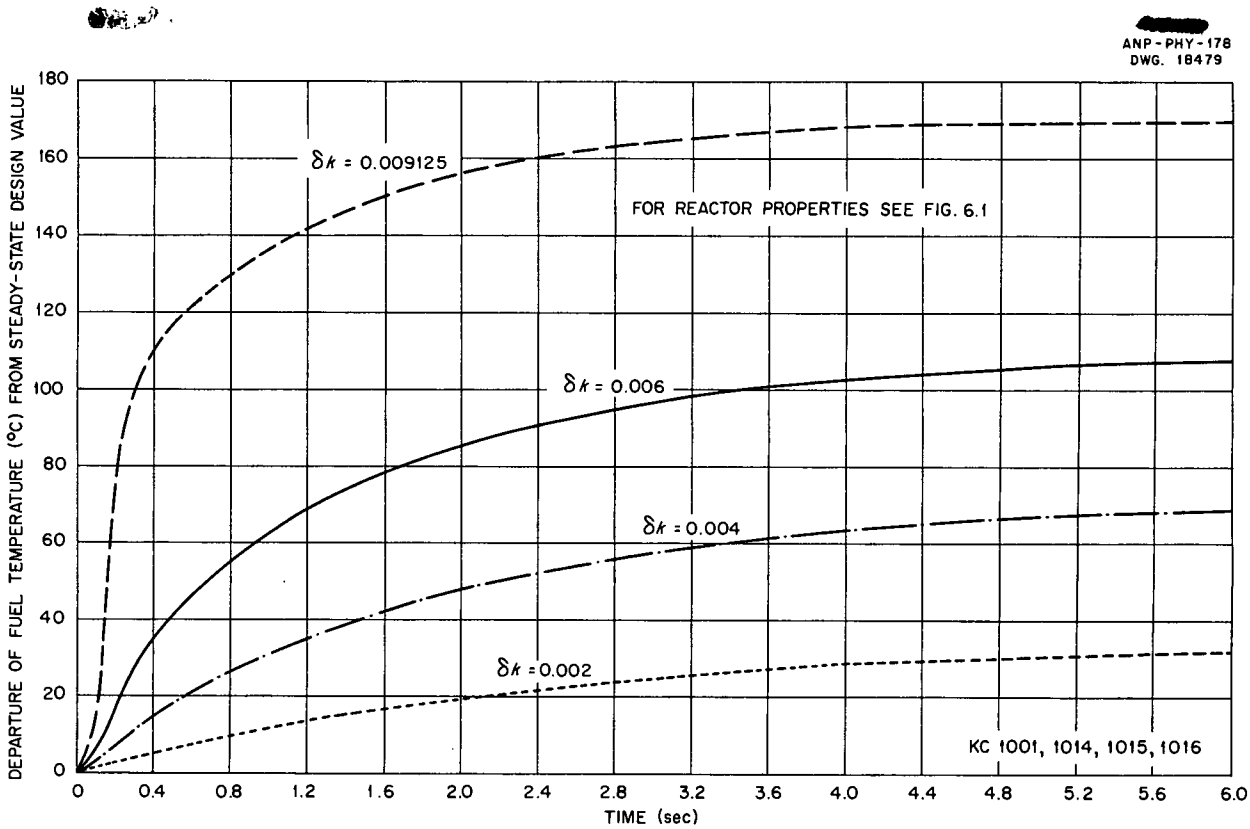


Fig. 6.5. Fuel Temperature Response to a Step Increase in Reactivity for Various Reactivity Changes.

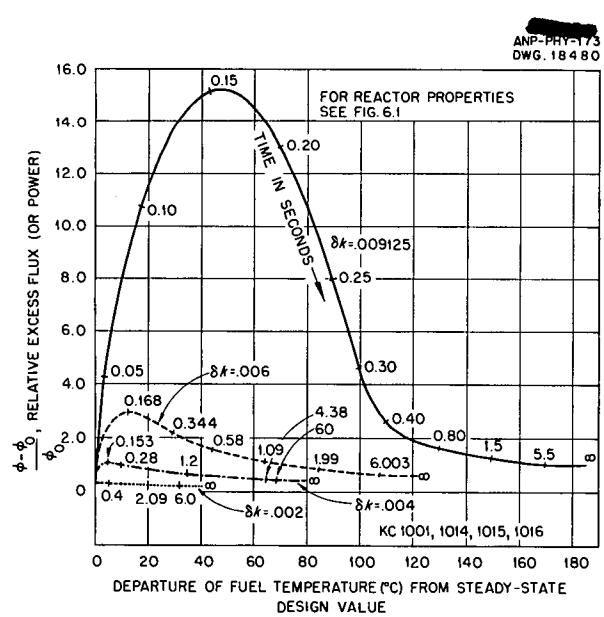


Fig. 6.6. Phase Plot of Flux vs. Fuel Temperature for Various Step Increases in Reactivity.

[redacted]

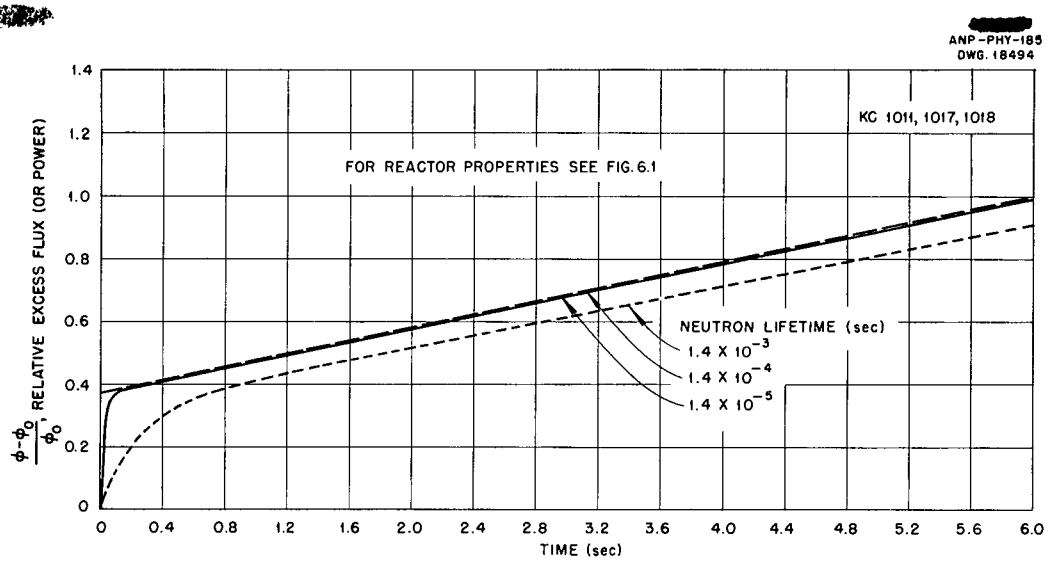


Fig. 6.7. Flux (or Power) Response to a Step Increase in Reactivity of 0.002 for Various Average Neutron Lifetimes. Reactivity change because of fuel expansion not included.

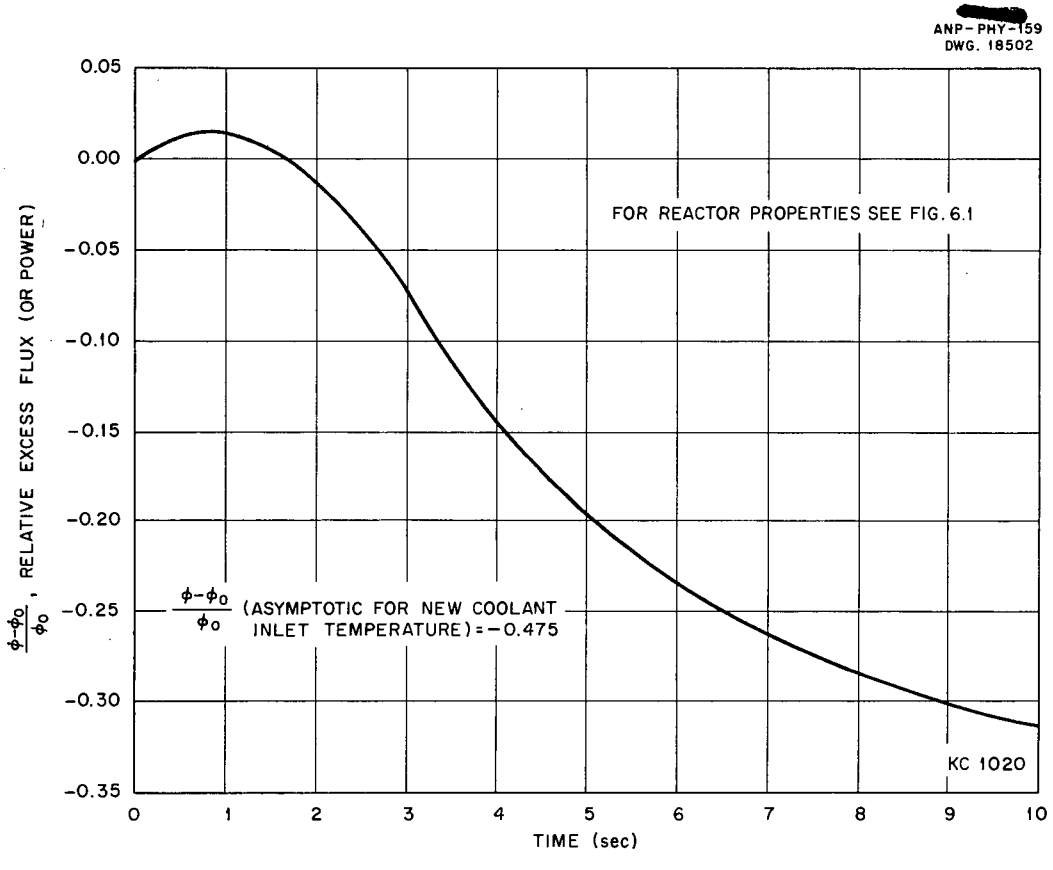


Fig. 6.8. Flux (or Power) Response to an Increase of 100°C in Coolant Inlet Temperature.

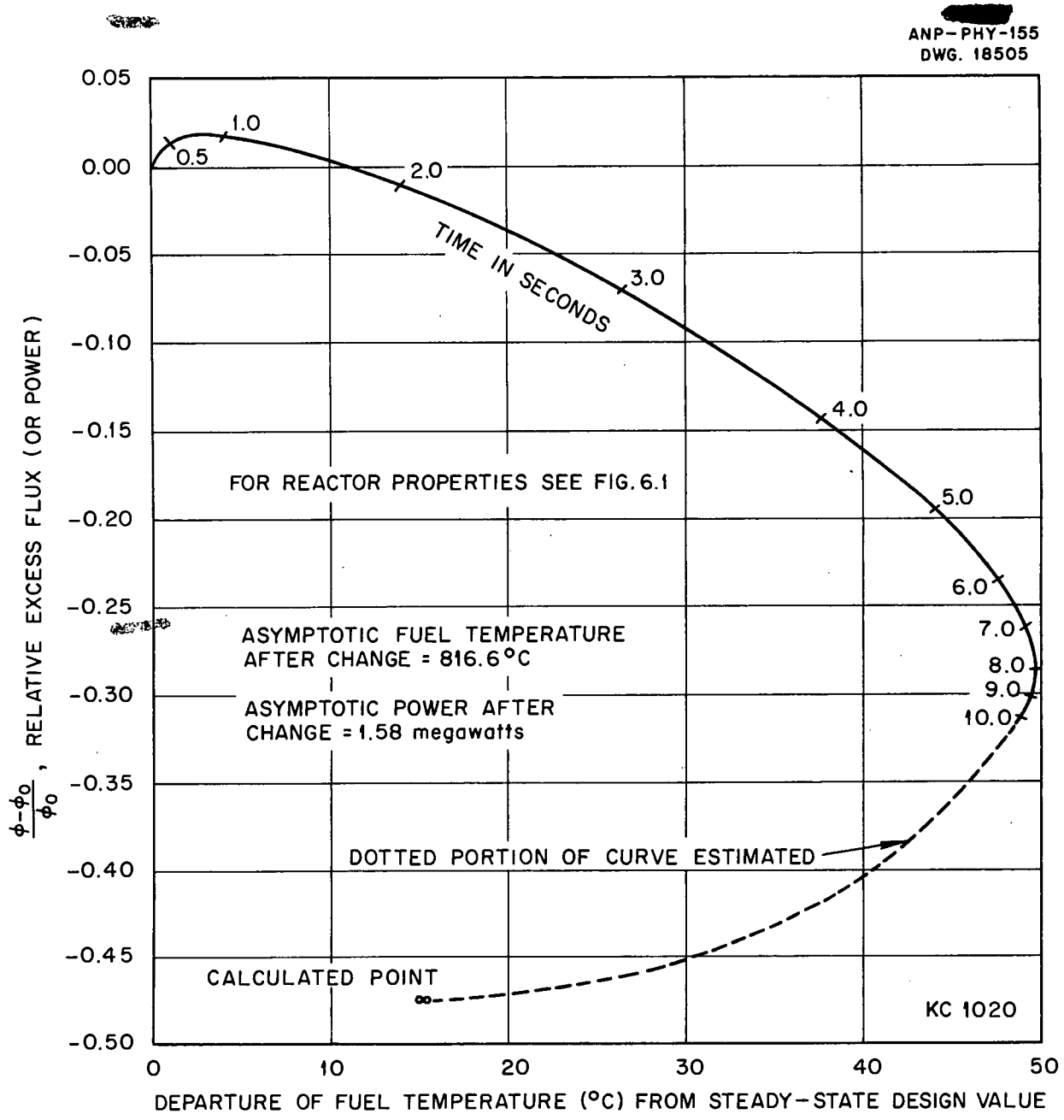


Fig. 6.9. Phase Plot of Flux (or Power) vs. Fuel Temperature After an Increase of 100°C in Coolant Inlet Temperature.

Appendix A

NUCLEAR DATA

Nuclear data are presented for (1) the thermal cross sections of the unclassified elements (Table 1),[1] (2) the fission and absorption cross sections as a function of energy for uranium (Figs. A1 through A6),[2] (3) the absorption cross section as a function of energy for xenon (Fig. A7),[2] (4) the xenon cross section averaged over a Maxwell-Boltzmann distribution for various reactor temperatures (Table 2).[3]

Another item of nuclear data frequently needed is the fission spectrum. The distribution of the energies of the fission neutrons from U^{235} is given by the semi-empirical relation[4]

$$N(E_n) = \text{const} \sinh \left(\frac{E_n}{0.487}\right)^{1/2} \times e^{-[(E_n/0.974) + (1/2)]}$$

This formula fits quite well the entire fission spectrum from 75 kev up to 15 Mev.

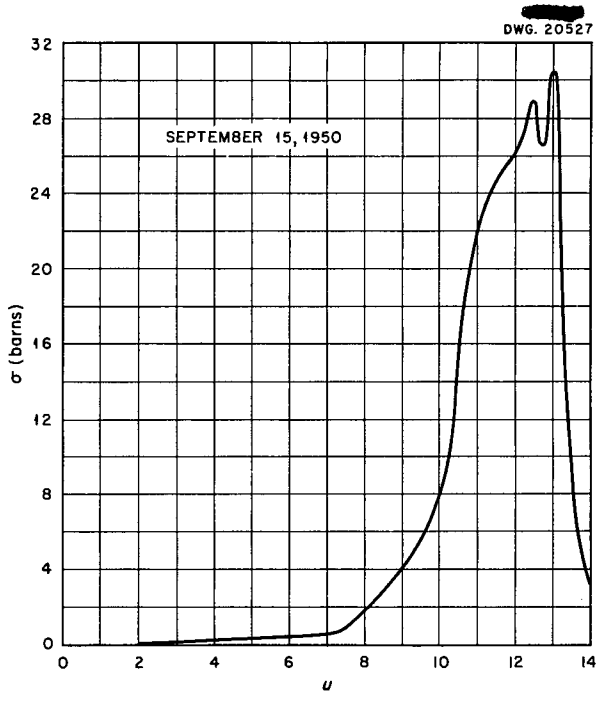


Fig. A1. Capture Cross Section of U^{238}.

(1) H. R. Kroeger, *Nucleonics*, Vol. 5, No. 4, 51-54 (1949).

(2) N. M. Smith, Jr., *KAPL Cross-Section Curves for Xe-135, U-235, and U-238*, Y-F10-51 (Apr. 17, 1951).

(3) J. W. Webster, *Xe Effect in an Epi Thermal Reactor*, Y-F10-17 (Oct. 10, 1950).

(4) T. W. Bonner, *A Study of the Spectrum of the Neutrons of Low Energy From the Fission of* U^{235}, LA-715 (Dec. 12, 1948).

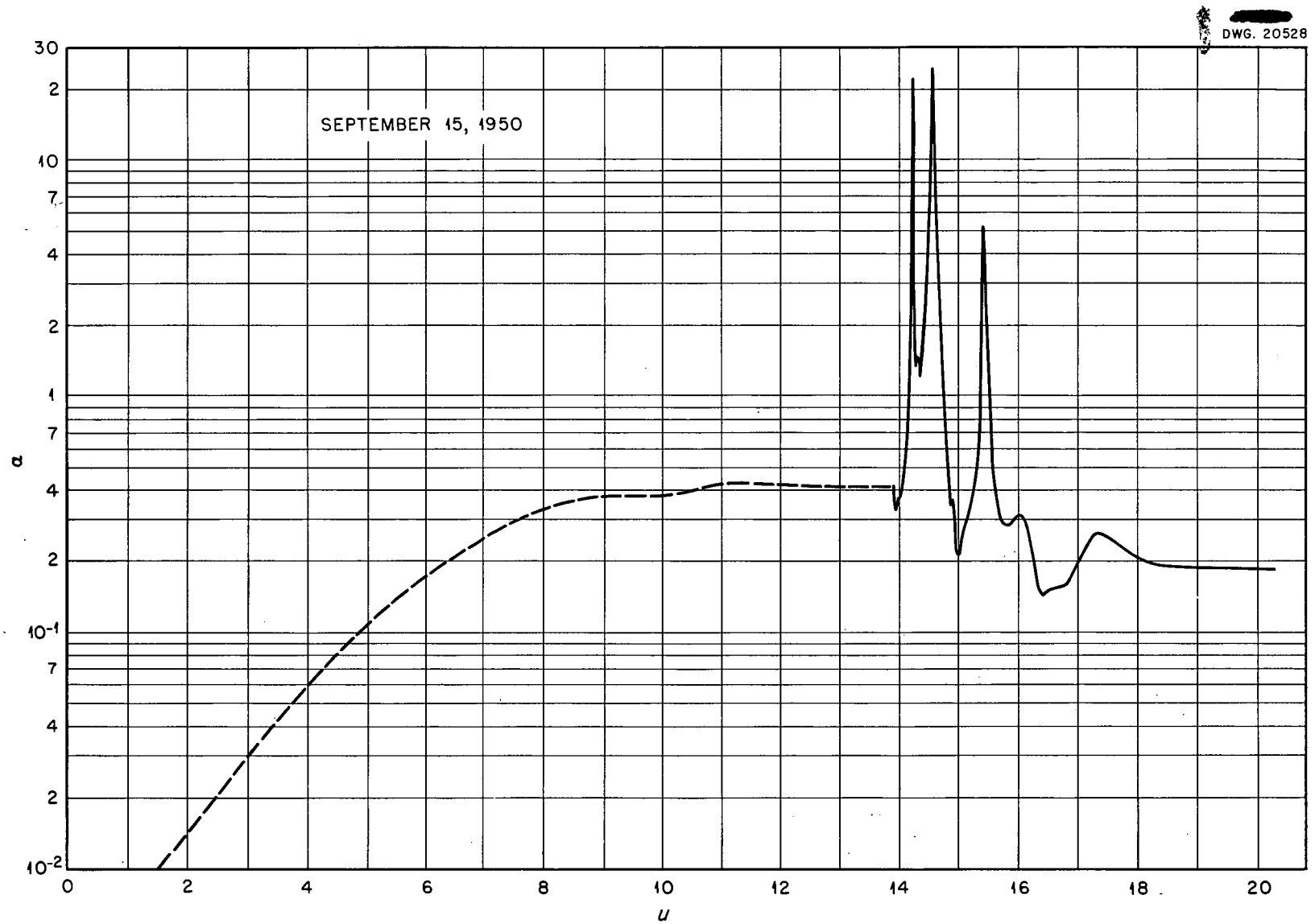


Fig. A2. Ratio of Capture-to-Fission Atomic Cross Section of 93% Enriched Uranium.

TABLE 1. THERMAL NEUTRON CROSS SECTIONS AND RELATED DATA*

Symbols:
b ≡ average cosine of the scattering angle relative to the initial direction,
σ_T ≡ total microscopic cross section for collision,
σ_S ≡ microscopic cross section for a scattering collision,
σ_{TR} ≡ microscopic cross section for a transport collision,
σ_A ≡ microscopic cross section for an absorption collision.

ATOMIC NUMBER, Z	NAME	CHEMICAL ATOMIC WEIGHT, A	DENSITY, ρ (g/cc at 20°C 760 mm Hg)	NUCLEI PER cc ($\times 10^{-24}$), N	$(1 - b)$	AVG. LOG ENERGY LOSS, ξ	THERMAL (0.025 ev) NEUTRON CROSS SECTIONS, σ (barns)				THERMAL (0.025 ev) MACROSCOPIC CROSS SECTIONS, $\Sigma = \sigma N$				THERMAL (0.025 ev) MEAN FREE PATH (cm)				SLOWING-DOWN POWER $N\sigma_S\xi$	MODERATING RATIO $(\sigma_S/\sigma_A) \times \xi$	ATOMIC NUMBER, Z
							σ_T	σ_S	σ_{TR}	σ_A	Σ_T	Σ_S	Σ_{TR}	Σ_A	λ_T	λ_S	λ_{TR}	λ_A			
1	H	1.008	0.00008	0.00005	0.3329	1.0	46	46	15	0.32									3.33 (H_2O)	140	1
1	D	2.0147	0.00017	0.00005	0.6662	0.7261	5.3	5.3	3.5	0											1
2	He	4.003	0.00016	0.000025	0.8320	0.4281	1.5	1.5	1.2	0.008										80	2
3	Li	6.940	0.534	0.0463	0.9031	0.2643	66.5	1.5	1.4	65	3.1	0.07	0.07	3.0	0.33	14	15	0.33	0.018	0.006	3
4	Be	9.02	1.84	0.1229	0.9255	0.2078	6.1	6.1	5.6	0.0085	0.75	0.75	0.69	0.001	1.3	1.3	1.5	950	0.156	150	4
5	B	10.82	2.535	0.1411	0.9379	0.1756	719	3.8	3.6	715	100	0.54	0.51	100	0.01	1.9	2.0	0.01	0.094	0.001	5
6	C	12.01	1.67	0.0838	0.9440	0.1589	4.8	4.8	4.5	0.0045	0.40	0.40	0.38	0.0004	2.5	2.5	2.6	2600	0.064	170	6
7	N	14.008	0.00116	0.00005	0.9520	0.1373	11.7	10	9.5	1.7										0.81	7
8	O	16.000	0.00133	0.00005	0.9580	0.1209	4.1	4.1	3.9	0.001										500	8
9	F	19.00	0.00158	0.00005	0.9646	0.1025	4	4	4	0.01										41	9
10	Ne	20.183	0.00084	0.000025	0.9667	0.0967	2.8														10
11	Na	22.997	0.9712	0.0254	0.9708	0.0852	4.5	4	4	0.45	0.11	0.10	0.10	0.011	8.8	9.8	9.8	88	0.009	0.76	11
12	Mg	24.32	1.741	0.0431	0.9724	0.087	3.0	2.7	2.6	0.3	0.13	0.12	0.11	0.013	7.8	8.6	8.9	77	0.009	0.73	12
13	Al	26.97	2.699	0.0603	0.9751	0.0730	1.6	1.4	1.4	0.22	0.097	0.084	0.084	0.013	10	12	12	77	0.006	0.47	13
14	Si	28.06	2.42	0.0519	0.9760	0.0702	2.5	2.3	2.2	0.2	0.13	0.12	0.11	0.010	7.7	8.4	8.8	96	0.008	0.81	14
15	P	30.98	2.34	0.0455	0.9783	0.0637	4.4	4.1	4.0	0.3	0.20	0.19	0.18	0.014	5.0	5.4	5.5	73	0.012	0.87	15
16	S	32.06	2.0	0.0376	0.9790	0.0616	1.4	1	1	0.4	0.053	0.038	0.038	0.015	19	27	27	67	0.002	0.15	16
17	Cl	35.457	0.00295	0.00005	0.9810	0.0558	55	20	20	35										0.032	17
18	A	39.944	0.00166	0.000025	0.9832	0.0497	3	2	2	1										0.099	18
19	K	39.096	0.87	0.0134	0.9828	0.0507	4.0	1.5	1.5	2.5	0.054	0.020	0.020	0.034	19	50	50	30	0.001	0.030	19
20	Ca	40.08	1.54	0.0231	0.9832	0.0495	4	3.5	3.4	0.5	0.092	0.081	0.079	0.012	11	12	13	86	0.004	0.35	20
21	Sc	45.10	2.5	0.0334	0.9851	0.0441				22				0.74				1.4			21
22	Ti	47.90	4.5	0.0566	0.9860	0.0415	11	6	6	5	0.62	0.34	0.34	0.28	1.6	2.9	2.9	3.5	0.014	0.050	22
23	V	50.95	5.96	0.0705	0.9868	0.0391	12	7	7	4.5	0.85	0.49	0.49	0.32	1.2	2.0	2.0	3.2	0.019	0.061	23
24	Cr	52.01	6.92	0.0801	0.9871	0.0383	6.5	4	4	2.5	0.52	0.32	0.32	0.20	1.9	3.1	3.1	5.0	0.012	0.061	24
25	Mn	54.93	7.2	0.0789	0.9878	0.0363	15.3	2.3	2.3	13	1.2	0.18	0.18	1.0	0.83	5.5	5.5	0.95	0.007	0.006	25
26	Fe	55.85	7.85	0.0847	0.9880	0.0357	13.5	11	11	2.5	1.1	0.93	0.93	0.21	0.88	1.1	1.1	4.7	0.033	0.16	26
27	Co	58.94	8.71	0.0890	0.9886	0.0338	41	5	5	36	3.7	0.45	0.45	3.2	0.27	2.3	2.3	0.31	0.015	0.005	27
28	Ni	58.69	8.9	0.0913	0.9885	0.0340	22	17	17	4.5	2.0	1.6	1.6	0.41	0.50	0.65	0.65	2.4	0.053	0.13	28
29	Cu	63.57	8.89	0.0842	0.9894	0.0314	10.2	7.2	7.1	3	0.86	0.61	0.60	0.25	1.2	1.7	1.7	4.0	0.019	0.075	29
30	Zn	65.38	7.19	0.0662	0.9897	0.0305	4.5	3.6	3.6	0.9	0.30	0.24	0.24	0.060	3.4	4.2	4.2	17	0.007	0.12	30
31	Ga	69.72	5.9	0.0510	0.9904	0.0287	18	16	16	2.2	0.92	0.82	0.82	0.11	1.1	1.2	1.2	8.9	0.023	0.21	31
32	Ge	72.60	5.46	0.0453	0.9907	0.0275	9	6	6	2.8	0.41	0.27	0.27	0.13	2.5	3.7	3.7	7.9	0.007	0.059	32
33	As	74.91	5.73	0.0461	0.9910	0.0267	8.8	4.5	4.5	4.3	0.41	0.21	0.21	0.20	2.5	4.8	4.8	5.1	0.006	0.028	33
34	Se	78.96	4.5	0.0343	0.9915	0.0253	28	13	13	15	0.96	0.45	0.45	0.52	1.0	2.2	2.2	1.9	0.011	0.022	34
35	Br	79.916	3.12	0.0235	0.9916	0.0250	11.5	5	5	6.5	0.27	0.12	0.12	0.15	3.7	8.5	8.5	6.5	0.003	0.019	35
36	Kr	83.7	0.00348	0.000025	0.9920	0.0239	27	27	27	0.05										13	36
37	Rb	85.48	1.532	0.0108	0.9921	0.0234	12.6	12	12	0.6	0.14	0.13	0.13	0.006	7.4	7.7	7.7	150	0.003	0.47	37
38	Sr	87.63	2.6	0.0179	0.9923	0.0228	11.3	9.5	9.4	1.8	0.20	0.17	0.17	0.032	5.0	5.9	5.9	31	0.004	0.12	38
39	Y	88.92	3.8	0.0257	0.9924	0.0225	4.2	3	3	1.2	0.11	0.08	0.08	0.031	9.3	13	13	33	0.002	0.057	39
40	Zr	91.22	6.44	0.0425	0.9926	0.0220	8.4	8	8	0.4	0.36	0.34	0.34	0.017	2.8	2.9	2.9	59	0.007	0.44	40
41	Nb	92.91	8.4	0.0545	0.9928	0.0216	8.2	7	7	1.2	0.45	0.38	0.38	0.065	2.2	2.6	2.6	15	0.008	0.13	41
42	Mo	95.95	10.2	0.0640	0.9930	0.0209	9.6	7	7	2.6	0.61	0.45	0.45	0.17	1.6	2.2	2.2	6.0	0.009	0.056	42
43	Tc	99			0.9932	0.0202															43
44	Ru	101.7	12.2	0.0723	0.9934	0.0197	9	6	6	3	0.65	0.43	0.43	0.22	1.5	2.3	2.3	4.6	0.009	0.039	44
45	Rh	102.91	12.5	0.0732	0.9935	0.0195	155	6	6	149	11	0.44	0.44	11	0.088	2.3	2.3	0.092	0.009	0.001	45
46	Pd	106.7	12.16	0.0686	0.9937	0.0188	10.5	4.5	4.5	6	0.72	0.31	0.31	0.41	1.4	3.2	3.2	2.4	0.006	0.014	46
47	Ag	107.880	10.5	0.0586	0.9938	0.0186	66	6	6	60	3.9	0.35	0.35	3.5	0.26	2.8	2.8	0.28	0.007	0.002	47
48	Cd	112.41	8.65	0.0463	0.9940	0.0178	2500	6	6	2500	120	0.28	0.28	120	0.009	3.6	3.6	0.009	0.005	0	48
49	In	114.76	7.28	0.0382	0.9941	0.0175	192	2	2	190	7.3	0.08	0.08	7.3	0.14	13	13	0.14	0.001	0	49

TABLE 1 (continued)

ATOMIC NUMBER, Z	NAME	CHEMICAL ATOMIC WEIGHT, A	DENSITY, ρ (g/cc at 20°C, 760 mm Hg)	NUCLEI PER cc (× 10^{-24}), N	(1 − b)	AVG. LOG ENERGY LOSS, ξ	THERMAL (0.025 ev) NEUTRON CROSS SECTIONS, σ (barns)				THERMAL (0.025 ev) MACROSCOPIC CROSS SECTIONS, Σ = σN				THERMAL (0.025 ev) MEAN FREE PATH (cm)				SLOWING-DOWN POWER $N\sigma_S\xi$	MODERATING RATIO $(\sigma_S/\sigma_A)\times\xi$	ATOMIC NUMBER, Z
							σ_T	σ_S	σ_{TR}	σ_A	Σ_T	Σ_S	Σ_{TR}	Σ_A	λ_T	λ_S	λ_{TR}	λ_A			
50	Sn	118.7	7.3	0.0370	0.9943	0.0169	4.6	4	4	0.6	0.17	0.15	0.15	0.022	5.9	6.8	6.8	46	0.003	0.11	50
51	Sb	121.76	6.691	0.0331	0.9945	0.0165	9	4	4	5	0.30	0.13	0.13	0.17	3.4	7.6	7.6	6.0	0.002	0.013	51
52	Te	127.61	6.25	0.0295	0.9947	0.0157	9	5	5	4	0.27	0.15	0.15	0.12	3.8	6.8	6.8	8.5	0.002	0.020	52
53	I	126.92	4.94	0.0234	0.9947	0.0158	10	3	3	7	0.23	0.07	0.07	0.16	4.3	14	14	6.1	0.001	0.007	53
54	Xe	131.3	0.00546	0.000025	0.9949	0.0153	37														54
55	Cs	132.91	1.873	0.0085	0.9949	0.0151	50	24	24	26	0.43	0.20	0.20	0.22	2.4	4.9	4.9	4.5	0.003	0.014	55
56	Ba	137.36	3.78	0.0166	0.9951	0.0146	9.3	8	8	1.3	0.15	0.13	0.13	0.022	6.5	7.5	7.5	46	0.022	0.090	56
57	La	138.92	6.15	0.0267	0.9952	0.0145	28	20	20	8	0.75	0.53	0.53	0.21	1.3	1.9	1.9	4.7	0.008	0.036	57
58	Ce	140.13	6.9	0.0297	0.9952	0.0143	29	23	23	6	0.86	0.68	0.68	0.18	1.2	1.5	1.5	5.6	0.010	0.055	58
59	Pr	140.92	6.475	0.0277	0.9952	0.0142	15	5	5	10	0.42	0.14	0.14	0.28	2.4	7.2	7.2	3.6	0.002	0.007	59
60	Nd	144.27	6.96	0.0291	0.9953	0.0139	100				2.9				0.34						60
61	Pm	147			0.9954	0.0137				60											61
62	Sm	150.43	7.7	0.0308	0.9955	0.0134	8,900			8,900	270			270	0.004			0.004			62
63	Eu	152.0	5.24	0.0208	0.9956	0.0132	6,000			6,000	120			120	0.008			0.008			63
64	Gd	156.9	7.94	0.0305	0.9957	0.0128	40,000			40,000	1,200			1,200	0.001			0.001			64
65	Tb	159.2	8.33	0.0315	0.9958	0.0126				10				0.32				3.2			65
66	Dy	162.46	8.56	0.0317	0.9959	0.0124	1,200			1,200	38			38	0.026			0.026			66
67	Ho	164.94	8.76	0.0320	0.9959	0.0122	65	5	5	60	2.1	0.16	0.16	1.9	0.48	6.3	6.3	0.52	0.002	0.001	67
68	Er	167.2	9.16	0.0330	0.9960	0.0120	200				6.6				0.15						68
69	Tm	169.4	9.34	0.0332	0.9960	0.0119	120	20	20	100	4.0	0.66	0.66	3.3	0.25	1.5	1.5	0.30	0.008	0.002	69
70	Yb	173.04	7.01	0.0244	0.9961	0.0116	60				1.5				0.69						70
71	Lu	174.99	9.74	0.0335	0.9962	0.0115	170	30	30	140	5.7	1.0	1.0	4.7	0.18	1.0	1.0	0.21	0.012	0.002	71
72	Hf	178.6	11.4	0.0384	0.9962	0.0113	125	20	20	105	4.8	0.77	0.77	4.0	0.21	1.3	1.3	0.25	0.009	0.002	72
73	Ta	180.88	16.6	0.0553	0.9963	0.0111	25	5	5	20	1.4	0.28	0.28	1.1	0.73	3.6	3.6	0.90	0.003	0.003	73
74	W	183.92	19.3	0.0632	0.9963	0.0109	23	5	5	18	1.5	0.32	0.32	1.1	0.69	3.2	3.2	0.88	0.003	0.003	74
75	Re	186.31	20	0.0647	0.9964	0.0108	98	13	13	85	6.3	0.84	0.84	5.5	0.16	1.2	1.2	0.18	0.009	0.002	75
76	Os	190.2	22.48	0.0712	0.9965	0.0106	32	15	15	17	2.3	1.1	1.1	1.2	0.44	0.94	0.94	0.83	0.011	0.009	76
77	Ir	193.1	22.42	0.0699	0.9965	0.0104	464	14	14	450	32	0.98	0.98	31	0.031	1.0	1.0	0.032	0.010	0	77
78	Pt	195.23	21.37	0.0659	0.9966	0.0103	18	10	10	8	1.2	0.66	0.66	0.53	0.84	1.5	1.5	1.9	0.007	0.013	78
79	Au	197.2	19.3	0.0589	0.9966	0.0102	102	6.5	6.5	95	6.0	0.38	0.38	5.6	0.17	2.6	2.6	0.18	0.004	0.001	79
80	Hg	200.61	13.546	0.0407	0.9966	0.0100	415	15	15	400	17	0.61	0.61	16	0.059	1.6	1.6	0.061	0.006	0	80
81	Tl	204.39	11.86	0.0349	0.9967	0.0098	12	9	9	3	0.42	0.31	0.31	0.11	2.4	3.2	3.2	9.5	0.003	0.029	81
82	Pb	207.21	11.347	0.0330	0.9968	0.0097	8	8	8	0.2	0.26	0.26	0.26	0.007	3.8	3.8	3.8	150	0.003	0.39	82
83	Bi	209.00	9.8	0.0282	0.9968	0.0096	6.5	6.5	6.5	0.015	0.18	0.18	0.18	0.0004	5.5	5.5	5.5	2,400	0.002	4.2	83
84	Po	210			0.9968	0.0096															84
85	At	211			0.9968	0.0095															85
86	Rn	222	4.4	0.0119	0.9970	0.0091															86
87	Fr	223			0.9970	0.0090															87
88	Ra	226.05	5	0.0133	0.9970	0.0089															88
89	Ac	227			0.9970	0.0089															89
90	Th	232.12	11.5	0.0298	0.9971	0.0087															90
91	Pa	230.9			0.9971	0.0087															91
92	U	238.07	18.7	0.0473	0.9972	0.0084															92
93	Np	237			0.9972	0.0085															93
94	Pu	239			0.9972	0.0084															94
95	Am	241			0.9972	0.0083															95
96	Cm	242			0.9972	0.0083															96

*Cross sections for hydrogen and deuterium are for H and D atoms in combination with O (H_2O and D_2O). The density given for $_{86}Rn^{222}$ is for liquid radon.

The atomic weights and densities are from "Handbook of Chemistry and Physics" (Chemical Rubber Publishing Co., Cleveland). The macroscopic cross sections, mean free paths, slowing-down powers, and moderating powers are based on thermal-neutron cross-section data gathered from the following sources:

1. K. Way, G. Haines, Thermal neutron cross sections for elements and isotopes, CNL-33, 2/29/48.
2. E. Melkonian, L. J. Rainwater, W. W. Havens, Neutron beam spectrometer studies of oxygen, M-2554.
3. R. G. Nucholls, The total scattering cross sections of deuterium and oxygen for fast neutrons, MDDC-37, 6/17/46.
4. K. Way, G. Haines, Tables of neutron cross sections, MonP-405, 10/28/47 (supplement issued 4/20/48).
5. H. H. Goldsmith, H. W. Ibser, Neutron cross sections of the elements, MUC-HHG-7, 12/5/45.

TABLE 2. AVERAGE THERMAL-NEUTRON CROSS SECTION OF XENON FOR VARIOUS REACTOR TEMPERATURES

σ_{Xe} averaged by the formula

$$\bar{\sigma}_{Xe} = \frac{\int_0^\infty \sigma_{Xe}(E)\, E\, e^{-E/kT}\, dE}{\int_0^\infty E\, e^{-E/kT}\, dE} ,$$

where

$$\sigma_{Xe}(E) = \frac{\sigma_0 \frac{\Gamma^2}{4}}{(E - E_0)^2 + \frac{\Gamma^2}{4}} \sqrt{\frac{E_0}{E}} ,$$

where

$$\Gamma = 0.107 \text{ ev} ,$$
$$E_0 = 0.0863 ,$$
$$\sigma_0 = 3.4 \times 10^6 \text{ barns.*}$$

T (°F)	$E = kT$ (ev)	$\bar{\sigma}_{Xe}$(barns) x 10^{-6}
62.5	0.025	2.989
375.9	0.04	2.657
793.6	0.06	2.123
1211.4	0.08	1.674
1286.4	0.08359	1.607
1372.3	0.0877	1.539
1503.9	0.094	1.450
1587.4	0.098	1.378
1629.2	0.1	1.346
1673.1	0.1021	1.316
1712.8	0.104	1.290
1838.1	0.11	1.212
2047.0	0.12	1.097

*The data were taken from the work of S. Bernstein *et al.* as reported in ORNL-325.

[redacted]
DWG. 20529

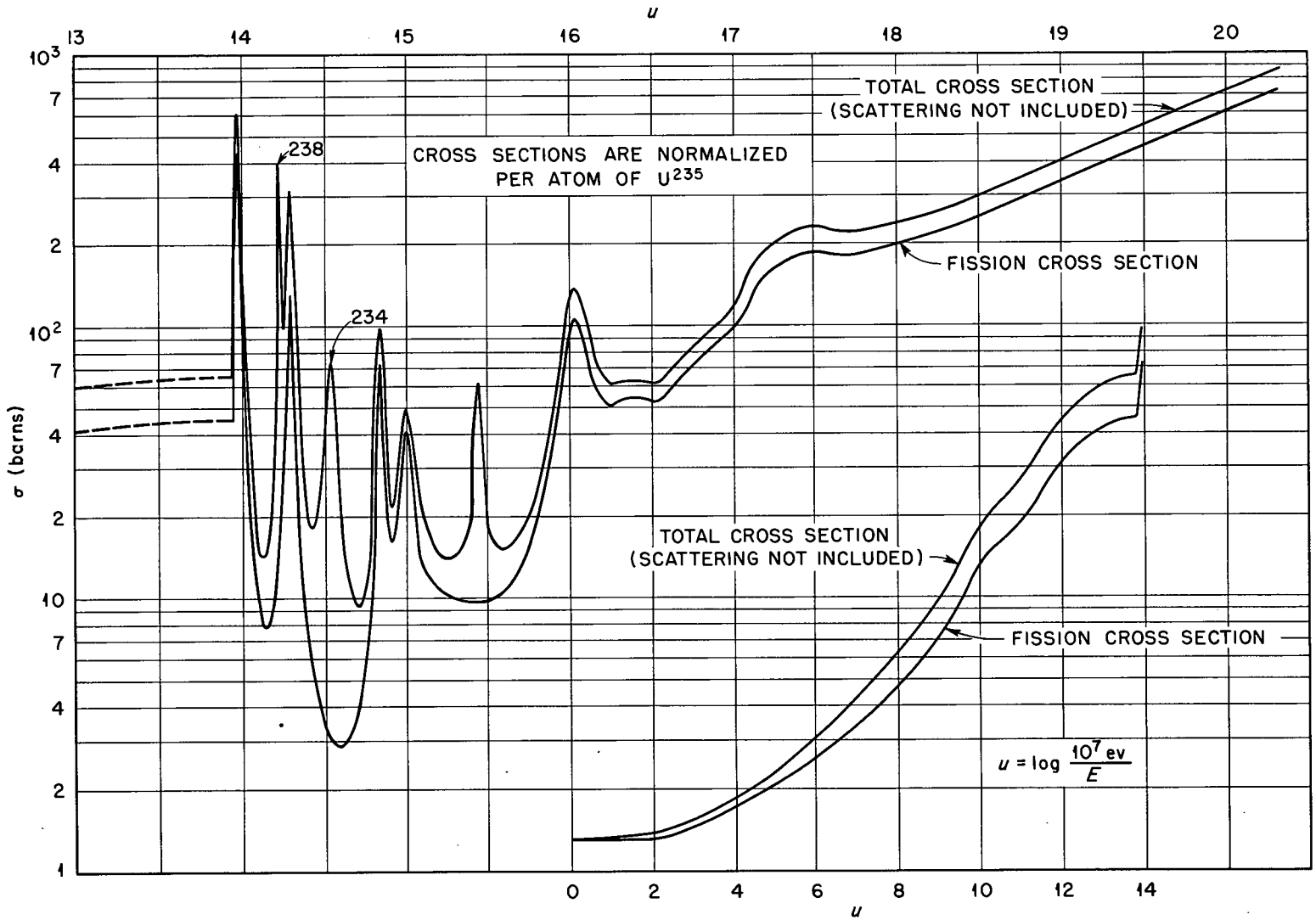


Fig. A3. Atomic Cross Section for Fission of 93% Enriched Uranium (U^{238} ~7%, U^{234} ~1.5%, U^{235} ~91.5%).

[redacted]

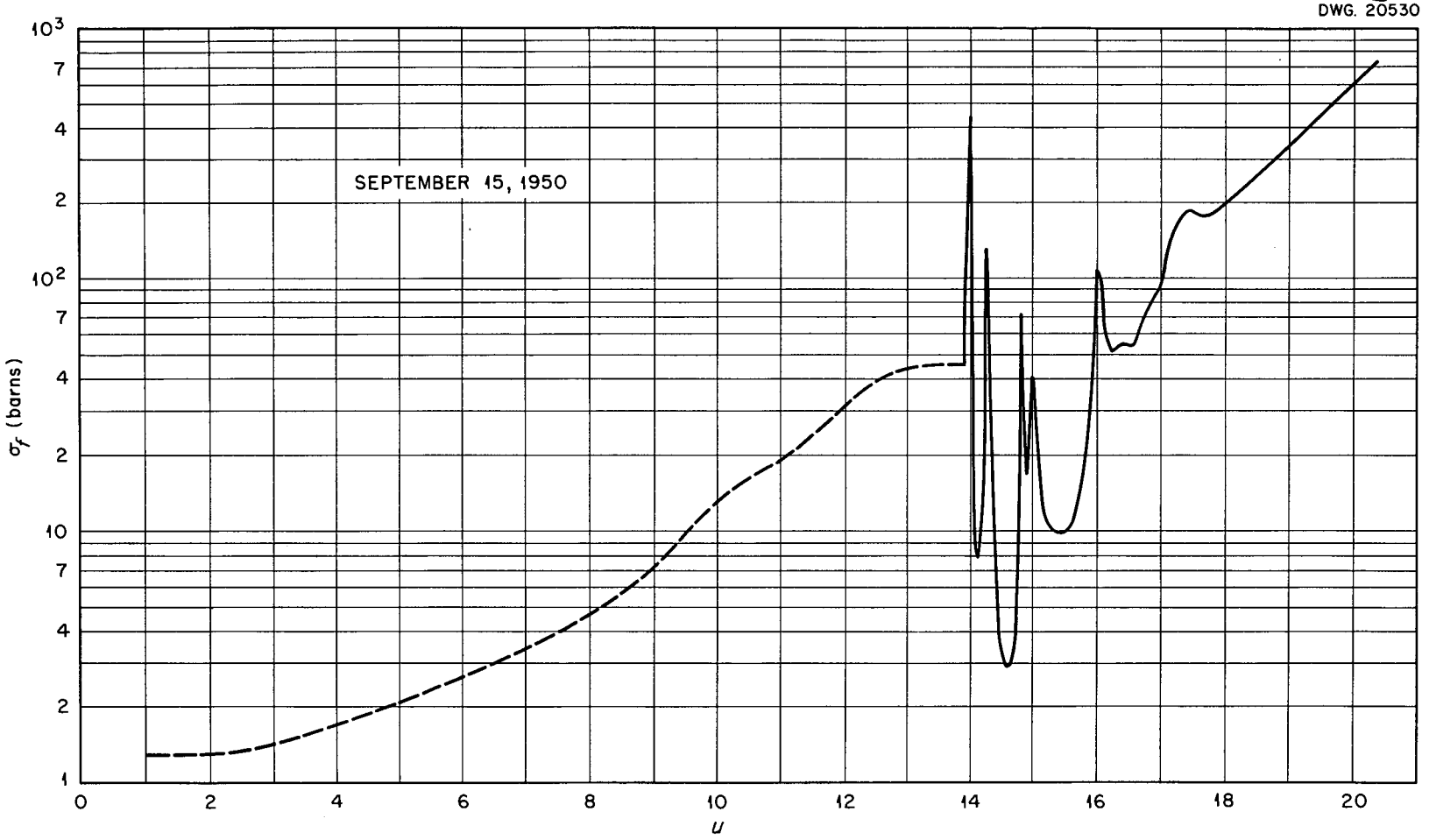
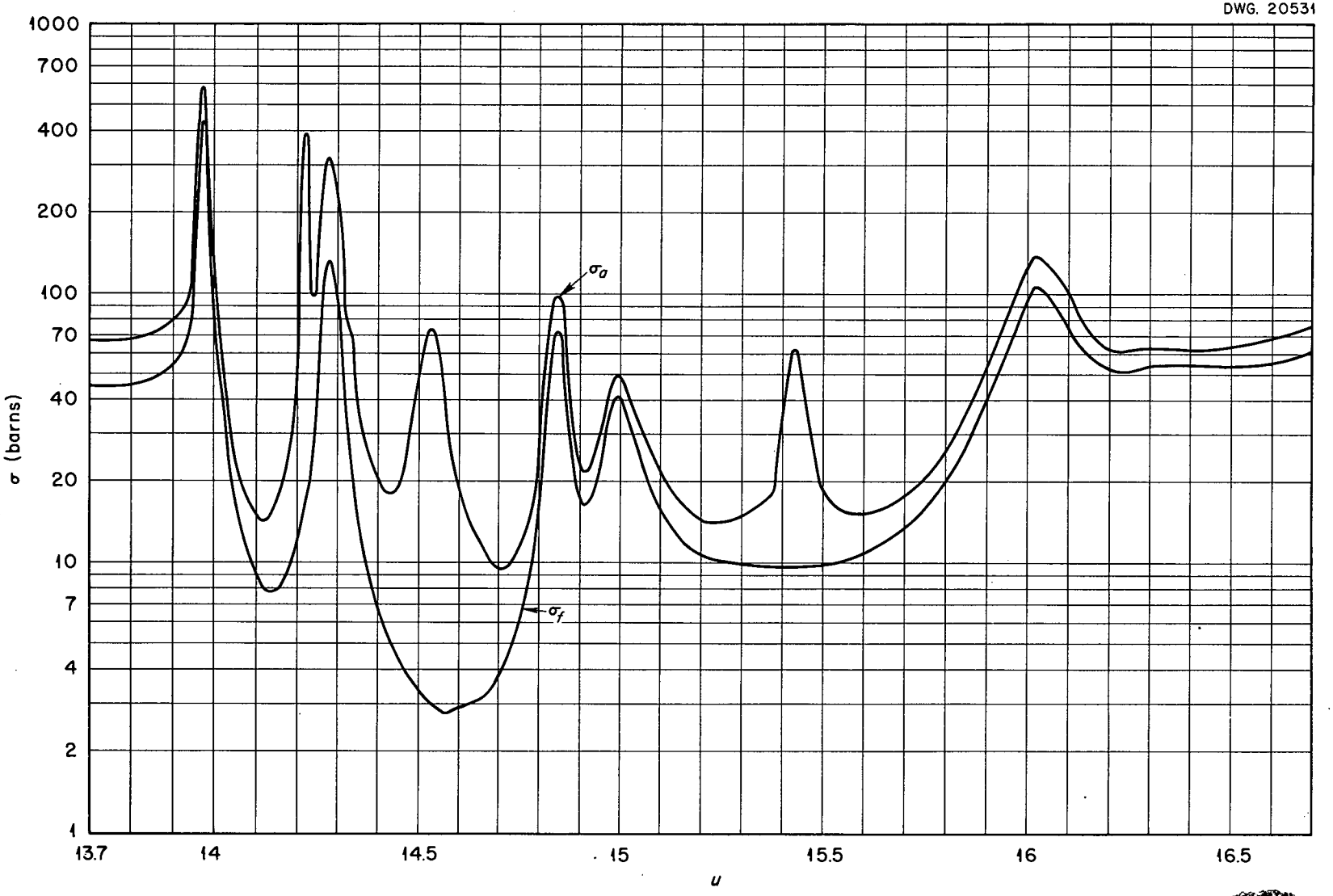


Fig. A4. Absorption and Fission Cross Section of K-25 End-Product Uranium.

[redacted]

[redacted]

DWG. 20531



[redacted]

Fig. A5. Expanded Scale Between Lethargies 13.7 and 16.7 of Absorption and Fission Cross Section of K-25 End-Product Uranium.

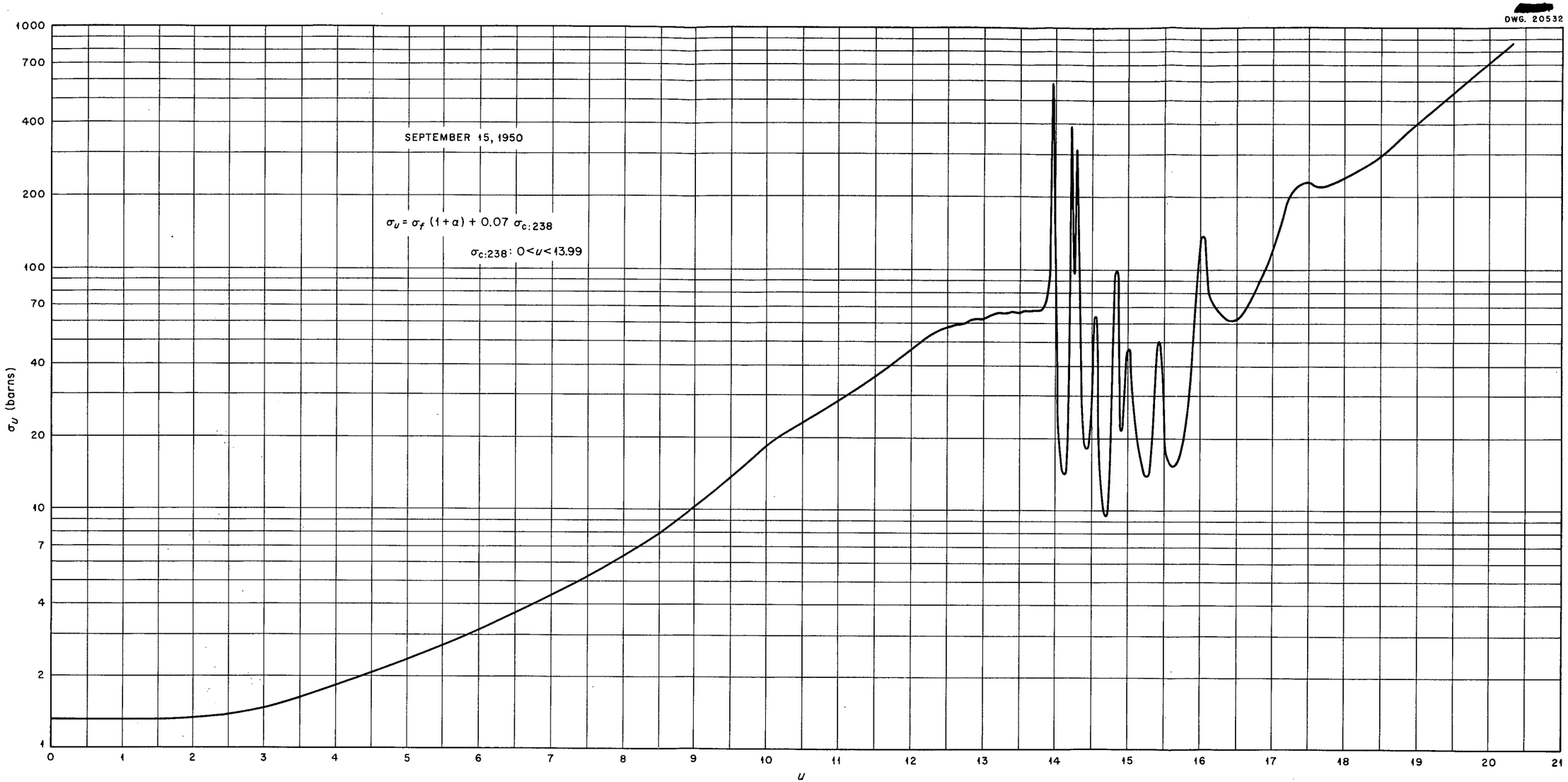


Fig. A6. Absorption Cross Section of 93% Enriched Uranium.

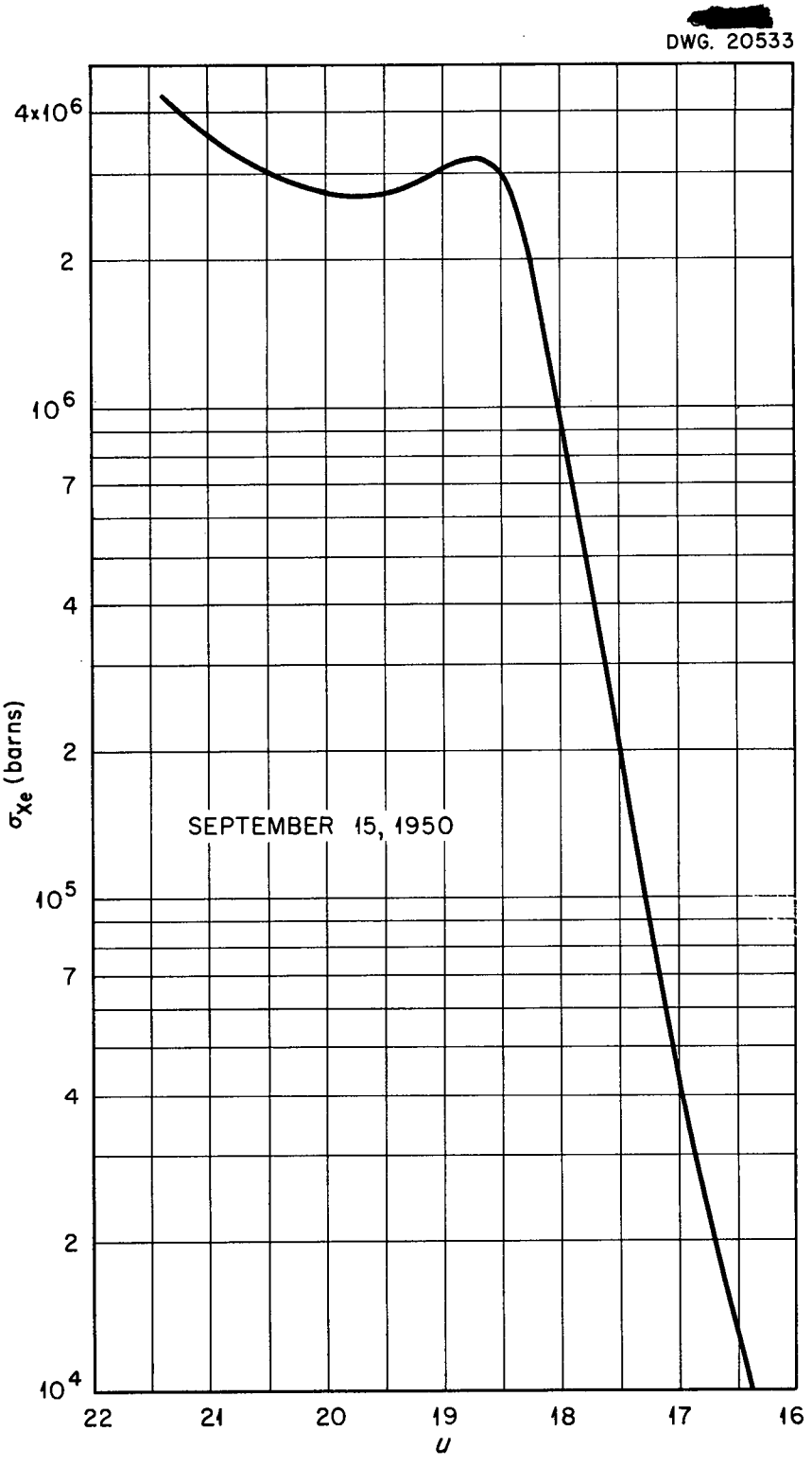


Fig. A7. Capture Cross Section of Xe^{135}.

Appendix B
CALCULATION FORMS

Calculation Form 1 is for use in applying the Fermi age theory to bare, intermediate-energy reactors. The notations and derivations are given in chap. 2. Form 2 presents procedures for solving two-group, three-region criticality problems for reactors with parallelepiped geometry and reflectors on one pair of opposite faces. Similar procedures can be devised for the other soluble geometries.

Form 1 was designed by the ANP Physics Group of the Oak Ridge National Laboratory, N. M. Smith, Jr., Chairman. Form 2 is based on the work of Garabedian and Householder, as presented in MonP-202,[1] and modeled after a similar form prepared by B. Spinrad. The symbol λ in the notation of MonP-202 is the diffusion coefficient, that is, $\lambda_{tr}/3$. The sheets are specifically designed for thermal reactors.

[1] H. L. Garabedian and A. S. Householder, *The Two Group Theory of Piles with Multiple Reflectors*, MonP-202 (Nov. 11, 1946).

[redacted]
DWG. 20652

CODE _____

FORM I

FERMI AGE FOR BARE-REACTOR CALCULATION*

GROUP NUMBER	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII
				$[\mathrm{I}+\mathrm{II}]\times\mathrm{III}$				$\frac{\mathrm{VI}\, q_{n-1}+\mathrm{VII}}{\mathrm{V}}$ SLOWING-DOWN DENSITY	$\frac{q_{n-1}+q_n}{2}$	$2\times\mathrm{I}\times\mathrm{III}\times\mathrm{IX}$	$\frac{q_{n-1}\times\mathrm{I}}{\mathrm{IV}}$	$2\times\mathrm{II}\times\mathrm{III}\times\mathrm{IX}$	$\frac{q_{n-1}\times\mathrm{II}}{\mathrm{IV}}$	BALANCE $\mathrm{VIII}+\mathrm{X}+\mathrm{XII}$	CHECKS $\mathrm{VII}+q_{n-1}$		$A_N\frac{\mathrm{XVI}}{\mathrm{II}}$	$n\neq th$ $\mathrm{IX}_N\times\mathrm{XIX}_N$
										LEAKAGE		ABSORPTIONS					FISSIONS	$N=th$ $q_{th-1}\times\left(\frac{\mathrm{XIX}}{\mathrm{I}+\mathrm{II}}\right)_{th}$
N	$B^2\left(\frac{1}{3\xi\Sigma_T\Sigma_{tr}}\right)$	$\left(\frac{\Sigma_a}{\xi\Sigma_T}\right)_N$	$U_N/2$	C_N	$1+C_N$	$1-C_N$	νZ_N	q_n	$\overline{q_N}$	E_N	E_{th}	A_N	A_{th}			$\left(\frac{\Sigma_f}{\xi\Sigma_T}\right)_n$	F_n	ϕN
0																		
1			0.25				0.0525											
2			0.25				0.2600											
3			0.25				0.5175											
4			0.25				0.5775											
5			0.25				0.4575											
6			0.25				0.2925											
7			0.25				0.1675											
8			0.25				0.0875											
9			1.50				0.0875											
10			1.50				0											
11			0.70				0											
12			0.60				0											
13			0.40				0											
14			0.20				0											
15			0.40				0											
16			0.60				0											
17			0.20				0											
18			0.20				0											
19			0.20				0											
20			0.20				0											
21			0.10				0											
22			0.10				0											
23			0.10				0											
24			0.10				0											
25			0.10				0											
Thermal (18.4)			1.00				0											
26			0.10				0											
Thermal (18.6)			1.00				0											
27			0.10				0											
28			0.10				0											
29			0.10				0											
30			0.10				0											
31			0.10				0											
Thermal (19.6)			1.00				0											

SUM (18.4)					$A+B+C+D$		SUM (18.4)		k_{eff} (18.4)
SUM (18.6)					$A+B+C+D$		SUM (18.6)		k_{eff} (18.6)
SUM (19.6)					$A+B+C+D$		SUM (19.6)		k_{eff} (19.6)
	"A	"B	"C	"D					

	FUEL	MODERATOR	METAL	COOLANT	VOID
VOLUME FRACTION					
DENSITY					
XIX		TEMPERATURE			
		CLEAN OR POISONED			
$\frac{1}{\xi\Sigma_T}$		$B^2=$			
		RADIUS _____ cm + ____ cm AUGMENTED LENGTH			

*THIS SHEET IS SPECIFICALLY DESIGNED FOR THREE POSSIBLE THERMAL BASES: $u=\ln\frac{10^7}{E}=18.4$, $u=18.6$, OR $u=19.6$. THE NOTATIONS ARE THE SAME AS THOSE GIVEN IN CHAPTER 2

FORM 2. TWO-GROUP MULTIREGION CALCULATION SHEETS FOR SLAB OR PARALLELEPIPED GEOMETRY

The calculation procedure varies slightly with the problem. The three particularly important types of problems are:

A. to calculate the uranium content to make a reactor of given dimensions, core structure, and moderator composition critical;

B. to calculate the radius to make a reactor of given core structure and composition (including uranium) critical;

C. to calculate δk for a reactor when a uniform change is made in the composition of any region.

Procedure in Case A

1. Estimate the uranium content and from this compute k and Σ_{as}.

2. Carry through the calculation of f, AB^{-1}, C, and $AB^{-1}C$ and in f leave μ_{01} tan $\mu_{01}a$ as a variable.

3. Determine the value of μ_{01} tan $\mu_{01}a$ which makes the eliminant = 0.

4. Find the value of μ_{01} and the associated k corresponding to it.

5. Repeat the process with the use of the new k and the new Σ_{as} computed from it. Only f needs to be recalculated. Repeat until the initial and the final k values are the same.

Procedure in Case B

1. k and Σ_{as} are known. Calculate AB^{-1}, C, and $AB^{-1}C$.

2. Calculate f for several values of a.

3. Plot the value of the eliminant vs. a and determine the a which makes the eliminant vanish. This procedure is also valid in Cases A and C.

Procedure in Case C

1. k and Σ_{as} are given for the unperturbed reactor. Put in the uniform perturbation and calculate the μ_{01} that makes the eliminant = 0. Calculate the k corresponding to this μ_{01}.

2. Recalculate f with the use of and new k and determine μ_{01} and k values which make the eliminant = 0.

3. Repeat until the initial and the final k values are the same.

Comments. It would seem that the procedure given for Case B would be useful for any of these cases – particularly when the initial conditions are far from criticality.

CORE*

Dimensions

H = _____ ;

L = _____ ;

Half thickness = a = _____ cm ;

$B_1^2 = (\pi/H)^2$ = _____ ;

$B_2^2 = (\pi/L)^2$ = _____ ;

$B_1^2 + B_2^2$ = _____ .

Constants

Slow Neutrons

Σ_a = _____ ;

Σ_f = _____ ;

k = _____ .

Evaluate

$k_{01}^2 = 1/L_s^2$ = _____ ;

$\lambda_{01} = D_s$ = _____ .

Fast Neutrons

$k_{02}^2 = 1/\tau$ = _____ ;

$\lambda_{02} = D_f$ = _____ .

Solve

$$(\Delta_0 - k_{01}^2)\ (\Delta_0 - k_{02}^2) = k k_{01}^2 k_{02}^2 .$$

Designate the negative and positive roots of this equation by

$\Delta_{01} = -F_{01}^2$ = – _____ ;

$\Delta_{02} = F_{02}^2$ = _____ ;

$\mu_{01}^2 = F_{01}^2 - B_1^2 - B_2^2$ = _____ ;

$\mu_{02}^2 = F_{02}^2 + B_1^2 + B_2^2$ = _____ ;

μ_{01} = _____ ;

μ_{02} = _____ .

*Notation is that of MonP-202.[1]

Compute

$$r_1 = \frac{\lambda_{01}(k_{01}^2 + F_{01}^2)}{\lambda_{02} k_{02}^2} = \underline{\qquad\qquad\qquad} ;$$

$$r_2 = \frac{k\,\lambda_{01}\, k_{01}^2}{\lambda_{02}\,(k_{02}^2 - F_{02}^2)} = - \underline{\qquad\qquad\qquad} .$$

Functions

$\mu_{02} a$ = _____

$\tanh \mu_{02} a$	$\mu_{02} \tanh \mu_{02} a$

where a = half thickness of core.

Matrix and Critical Relations

$$f = \begin{pmatrix} 1 & 1 \\ \dfrac{-\lambda_{01}\mu_{01} \tan \mu_{01} a}{(\underline{\qquad\qquad})} & \dfrac{\lambda_{01}\mu_{02} \tanh \mu_{02} a}{(\underline{\qquad\qquad})} \\ \dfrac{r_1}{(\underline{\qquad\qquad})} & \dfrac{r_2}{(-\underline{\qquad\qquad})} \\ \dfrac{-\lambda_{02} r_1 \mu_{01} \tan \mu_{01} a}{(-\underline{\qquad\qquad})} & \dfrac{\lambda_{02} r_2 \mu_{02} \tanh \mu_{02} a}{(-\underline{\qquad\qquad})} \end{pmatrix} .$$

Multiply AB^{-1} and C to obtain

$$AB^{-1}C = \begin{pmatrix} \dfrac{A_{11}}{(\underline{\qquad\qquad})} & \dfrac{A_{12}}{(\underline{\qquad\qquad})} \\ \dfrac{A_{21}}{(\underline{\qquad\qquad})} & \dfrac{A_{22}}{(\underline{\qquad\qquad})} \\ \dfrac{A_{31}}{(\underline{\qquad\qquad})} & 0 \\ \dfrac{A_{41}}{(\underline{\qquad\qquad})} & 0 \end{pmatrix} .$$

The eliminant of

$$f\begin{pmatrix} 1 \\ a_2 \end{pmatrix} = AB^{-1}C \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

must be zero. Manipulation of the eliminant yields the following two-rowed determinant, which is to equal zero.

$$\left| \begin{pmatrix} 0 & 0 & 1 & \dfrac{-A_{31}}{(\underline{\qquad})} \\ \dfrac{-A_{22}}{(\underline{\qquad})} & 1 & \dfrac{A_{11}}{A_{31}} \cdot \dfrac{A_{22}}{A_{12}} - \dfrac{A_{21}}{A_{31}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \dfrac{-\lambda_{01}\mu_{01} \tan \mu_{01} a}{(-\underline{\qquad\qquad})} & \dfrac{\lambda_{01}\mu_{02} \tanh \mu_{02} a}{(\underline{\qquad\qquad})} \\ \dfrac{r_1}{(\underline{\qquad\qquad})} & \dfrac{r_2}{(-\underline{\qquad\qquad})} \\ \dfrac{-\lambda_{02} r_1 \mu_{01} \tan \mu_{01} a}{(-\underline{\qquad\qquad})} & \dfrac{\lambda_{02} r_2 \mu_{02} \tanh \mu_{02} a}{(-\underline{\qquad\qquad})} \end{pmatrix} \right|$$

$$= \left| \begin{array}{c|c} \text{- - - - - - - -} & \text{- - - - - - - -} \\ & \\ & \end{array} \right| = \underline{\qquad\qquad\qquad} .$$

$(_____)\ \mu_{01} \tan \mu_{01} a = (_____)$;

$\mu_{01} \tan \mu_{01} a = (_____)$;

μ_{01}			
$\mu_{01} a$			
$\tan \mu_{01} a$			
$\mu_{01} \tan \mu_{01} a$			

$\mu_{01} a =$ _____ ;

$\mu_{01} =$ _____ ;

$\mu_{01}^2 =$ _____ ;

$F_{01}^2 = \mu_{01}^2 + B_1^2 + B_2^2 =$ _____ ;

$(-F_{01}^2 - k_{01}^2) =$ _____ ;

$(-F_{01}^2 - k_{02}^2) =$ _____ ;

$$\frac{(-F_{01}^2 - k_{01}^2)\ (-F_{01}^2 - k_{02}^2)}{k_{01}^2\, k_{02}^2} = k = _____ ;$$

$r_1 =$ _____ ;

$r_2 =$ _____ .

CONSISTENCY CHECK

$\cos \mu_{01} a =$ _____ ;

$\cosh \mu_{02} a =$ _____ ;

$$a_2 = \frac{r_1}{r_2} \frac{\cosh \mu_{01} a}{\cosh \mu_{02} a} \times \frac{\left[1 + \frac{A_{31}}{A_{41}} \lambda_{02} \mu_{01} \tan \mu_{01} a\right]}{\left[-1 + \frac{A_{31}}{A_{41}} \lambda_{02} \mu_{02} \tanh \mu_{02} a\right]}$$

$=$ _____ ;

$$C_1 = \frac{a_2}{A_{41}} \lambda_{02} r_2 \mu_{02} \tanh \mu_{02} a \cosh \mu_{02} a - \frac{\lambda_{02} r_1}{A_{41}} \mu_{01} \tan \mu_{01} a \cos \mu_{01} a$$

$=$ _____ ;

$$C_2 = \frac{a_2}{A_{22}} \lambda_{01} \mu_{02} \tanh \mu_{02} a \cosh \mu_{02} a - \frac{A_{21}}{A_{22}} C_1 - \frac{\lambda_{01}}{A_{22}} \mu_{01} \tan \mu_{01} a \cos \mu_{01} a$$

$=$ _____ .

The check to assure that critical conditions have been established is that the following equality exist:

$$a_2 \cosh \mu_{02} a - A_{11} C_1 - A_{12} C_2 = - \cos \mu_{01} a .$$

This does not ensure that the A_{ij} have been evaluated correctly.

INNER REFLECTOR

Dimensions

Thickness = S = _____ cm ;

$B_1^2 = (\pi/H)^2 =$ _____ ;

$B_2^2 = (\pi/L)^2 =$ _____ ;

$B_1^2 + B_2^2 =$ _____ .

Constants

Slow Neutrons

$k_{11}^2 = 1/L^2 =$ _____ ;

$\lambda_{11} = D_s =$ _____ ;

$\mu_{11}^2 = k_{11}^2 + B_1^2 + B_2^2 =$ _____ ;

$\mu_{11} =$ _____ ;

$\mu_{12}^2 = k_{12}^2 + B_1^2 + B_2^2 =$ _____ ;

$\mu_{12} =$ _____ .

Fast Neutrons

$k_{12}^2 = 1/\tau =$ _____ ;

$\lambda_{12} = D_f =$ _____ ;

Functions	sinh	cosh
$\mu_{11}S$		
$\mu_{12}S$		

$$S_1 = \frac{\lambda_{11}(k_{11}^2 - k_{12}^2)}{\lambda_{12}k_{12}^2} = -\text{ _____ .}$$

Constants

Slow Neutrons

$k_{21}^2 = 1/L^2 =$ _____ ;

$\lambda_{21} = D_s =$ _____ ;

$\mu_{21}^2 = k_{21}^2 + B_1^2 + B_2^2 =$ _____ ;

$\mu_{21} =$ _____ ;

$\mu_{22}^2 = k_{22}^2 + B_1^2 + B_2^2 =$ _____ ;

$\mu_{22} =$ _____ .

Fast Neutrons

$k_{22}^2 = 1/\tau =$ _____ ;

$\lambda_{22} = D_f =$ _____ ;

Matrix

$$AB^{-1} = \begin{pmatrix} \cosh \mu_{11}S & -\dfrac{\sinh \mu_{11}S}{\lambda_{11}\mu_{11}} & -\dfrac{\cosh \mu_{11}S - \cosh \mu_{12}S}{S_1} & \dfrac{\mu_{12} \sinh \mu_{11}S - \mu_{11} \sinh \mu_{12}S}{\lambda_{12}\mu_{11}\mu_{12}S_1} \\ (____) & (____) & (____) & (____) \\ -\lambda_{11}\mu_{11} \sinh \mu_{11}S & \cosh \mu_{11}S & \dfrac{\lambda_{11}}{S_1}(\mu_{11} \sinh \mu_{11}S - \mu_{12} \sinh \mu_{12}S) & -\dfrac{\lambda_{11}}{\lambda_{12}S_1}(\cosh \mu_{11}S - \cosh \mu_{12}S) \\ (____) & (____) & (____) & (____) \\ 0 & 0 & \cosh \mu_{12}S & -\dfrac{1}{\lambda_{12}\mu_{12}} \sinh \mu_{12}S \\ & & (____) & (____) \\ 0 & 0 & -\lambda_{12}\mu_{12} \sinh \mu_{12}S & \cosh \mu_{12}S \\ & & (____) & (____) \end{pmatrix} .$$

OUTER REFLECTOR

Dimensions

Thickness = t = _____ cm ;

$B_1^2 = (\pi/H)^2 =$ _____ ;

$B_2^2 = (\pi/L)^2 =$ _____ ;

$B_1^2 + B_2^2 =$ _____ .

Functions	sinh	cosh
$\mu_{21}t$		
$\mu_{22}t$		

$$t_1 = \frac{\lambda_{21}(k_{21}^2 - k_{22}^2)}{\lambda_{22}k_{22}^2} = -\text{ _____ .}$$

Matrix

$$C = \begin{pmatrix} \dfrac{\sinh \mu_{22} t}{(_____)} & \dfrac{\sinh \mu_{21} t}{(_____)} & 0 & 0 \\[2ex] \dfrac{-\lambda_{21}\mu_{22} \cosh \mu_{22} t}{(_____)} & \dfrac{-\lambda_{21}\mu_{21} \cosh \mu_{21} t}{(_____)} & 0 & 0 \\[2ex] \dfrac{t_1 \sinh \mu_{22} t}{(_____)} & 0 & 0 & 0 \\[2ex] \dfrac{-\lambda_{22} t_1 \mu_{22} \cosh \mu_{22} t}{(_____)} & 0 & 0 & 0 \end{pmatrix} .$$

DISTRIBUTIONS IN CORE

Coefficients

a_2 = _____ (from consistency check).

Fluxes

$\phi_{s0} = U_1 = \cos \mu_{01} X + a_2 \cosh \mu_{02} X$

$= _____$;

$\phi_{f0} = U_2 = r_1 \cos \mu_{01} X$

$+ r_2 a_2 \cosh \mu_{02} X$ = _____ .

Edge Values

$$f \begin{pmatrix} \cos \mu_{01} a \\ a_2 \cosh \mu_{02} a \end{pmatrix} = \begin{pmatrix} \phi_{s0} \\ J_{s0} \\ \phi_{f0} \\ J_{f0} \end{pmatrix} ;$$

$$\left(\begin{array}{c|c} 1 & 1 \\ \hline \cdots & \cdots \\ \hline \cdots & \cdots \end{array}\right) \begin{pmatrix} \dfrac{\cos \mu_{01} a}{(_____)} \\[2ex] \dfrac{a_2 \cosh \mu_{02} a}{(_____)} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\phi_{s0}}{(_____)} \\[2ex] \dfrac{J_{s0}}{(_____)} \\[2ex] \dfrac{\phi_{f0}}{(_____)} \\[2ex] \dfrac{J_{f0}}{(_____)} \end{pmatrix} .$$

Plots

			①	②	ϕ_{s0}	ϕ_{f0}
X	$\mu_{01}X$	$\mu_{02}X$	$\cos \mu_{01}X$	$a_2 \cosh \mu_{02}X$	① + ②	r_1 ① + r_2 ②

DISTRIBUTIONS IN INNER REFLECTOR

Coefficients

$\cosh \mu_{11}a$ = _____ ;

$\cosh \mu_{12}a$ = _____ ;

$\sinh \mu_{11}a$ = _____ ;

$\sinh \mu_{12}a$ = _____ ;

$$b_1 = \frac{\phi_{f0}\lambda_{12}\mu_{12} \cosh \mu_{12}a - J_{f0} \sinh \mu_{12}a}{\lambda_{12}\mu_{12}S_1}$$

= _____ ;

$$b_2 = \frac{J_{f0} \cosh \mu_{12}a - \phi_{f0}\lambda_{12}\mu_{12} \sinh \mu_{12}a}{\lambda_{12}\mu_{12}S_1}$$

= _____ ;

$$b_3 = \phi_{s0} \cosh \mu_{11}a - \phi_{f0} \frac{\cosh \mu_{11}a}{S_1} - J_{s0} \frac{\sinh \mu_{11}a}{\lambda_{11}\mu_{11}} + J_{f0} \frac{\sinh \mu_{11}a}{\mu_{11}\lambda_{12}S_1}$$

= _____ ;

$$b_4 = J_{s0} \frac{\cosh \mu_{11}a}{\lambda_{11}\mu_{11}} - J_{f0} \frac{\cosh \mu_{11}a}{\lambda_{12}\mu_{11}S_1} - \phi_{s0} \sinh \mu_{11}a + \phi_{f0} \frac{\sinh \mu_{11}a}{S_1}$$

= _____ .

Fluxes

$$\phi_{s1} = b_1 \cosh \mu_{12}X + b_2 \sinh \mu_{12}X + b_3 \cosh \mu_{11}X + b_4 \sinh \mu_{11}X$$

= _____ ;

$$\phi_{f1} = b_1 s_1 \cosh \mu_{12}X + b_2 s_1 \sinh \mu_{12}X$$

= _____ .

Plots

			1	2	ϕ_{f1}	3	4	5	ϕ_{s1}
X	$\mu_{12}X$	$\mu_{11}X$	$b_1 s_1 \cosh \mu_{12}X$	$b_2 s_1 \sinh \mu_{12}X$	1 + 2	$b_3 \cosh \mu_{11}X$	$b_4 \sinh \mu_{11}X$	$\frac{1}{s_1}\phi_{f1}$	3 + 4 + 5

Edge Values (necessary only if there are more than two reflectors)

$$\phi_{s1} = b_1 \cosh \mu_{12} (a + s) + b_2 \sinh \mu_{12} (a + s) + b_3 \cosh \mu_{11} (a + s) + b_4 \sinh \mu_{11} (a + s)$$

$$= _____ ;$$

$$J_{s1} = b_1\lambda_{11}\mu_{12} \sinh \mu_{12} (a + s) + b_2\lambda_{11}\mu_{12} \cosh \mu_{12} (a + s) + b_3\lambda_{11}\mu_{11} \sinh \mu_{11} (a + s) + b_4\lambda_{11}\mu_{11} \cosh \mu_{11} (a + s)$$

$$= _____ ;$$

$$\phi_{f1} = s_1 b_1 \cosh \mu_{12} (a + s) + s_1 b_2 \sinh \mu_{12} (a + s)$$

$$= _____ ;$$

$$J_{f1} = b_1\lambda_{12}s_1\mu_{12} \sinh \mu_{12} (a + s) + b_2\lambda_{12}s_1\mu_{12} \cosh \mu_{12} (a + s)$$

$$= _____ .$$

DISTRIBUTIONS IN OUTER REFLECTOR

Coefficients

C_1 = _____ (from consistency check);

C_2 = _____ (from consistency check).

Fluxes

$$\phi_{s2} = C_1 \sinh \mu_{22} (a + s + t - X) + C_2 \sinh \mu_{21} (a + s + t - X)$$

$$= _____ ;$$

$$\phi_{f2} = t_1 C_1 \sinh \mu_{22} (a + s + t - X)$$

$$= _____ .$$

Plots

	①			②	③	ϕ_{s2}	ϕ_{f2}
X	$a + s + t - X$	μ_{21} ①	μ_{22} ①	$C_1 \sinh \mu_{21}$ ①	$C_2 \sinh \mu_{21}$ ①	② + ③	t_1 ②