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ANGULAR CORRELATION IN ALLOWED β-TRANSITIONS

M. E. Rose

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PHYSICS DIVISION

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ANGULAR CORRELATION IN ALLOWED β-TRANSITIONS

M. E. Rose

1. INTRODUCTION

It is well known that, at this time, the importance of the angular correlation between recoil nucleus and β-particle lies in the fact that experiments of this type can be used to determine the composition of the Fermi interaction. From the linearity (within present experimental precision) of the allowed Kurie plots it can be said that the Fermi interaction is predominantly S or predominantly V. The choice between these two possibilities can be made once data on the angular correlation are available.

The correlation for the case that no γ-ray follows the β-decay has been discussed by Reynolds et al.[1] While the numerical results presented there refer to the neutron decay (that is, the endpoint energy corresponds to this case) the formal part should be generally applicable. However, aside from a number of misprints which have now been corrected[1], there seems to be an error of principle in RBB which will be brought out explicitly in the following. This concerns the role of the finite mass of the recoil nucleus and the singularity in the (differential) angular correlation.

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1. Reynolds, Biedenharn and Beard, ORNL report No. 1444. See also Erratum sheet. This report is referred to as RBB. See also, M. E. Rose, Phys. Rev. 91, 197 (1953).

The purpose of the present report is (1) to give a somewhat more complete discussion of the calculations (2) to correct the aforementioned misunderstanding and (3) to present analytical results whereby, for high energy spectra, the labor of preparing theoretical curves should be very much minimized. As will be seen, the theoretical curves are obtained after two integrations which, in general, have to be done numerically. The first is over β-energy and the second is over the finite solid angles of the detectors. The latter integration is important in the most interesting region, $\theta \approx \pi$, where θ is the angle between recoil and β-particle. The reason for this is that the correlation function before integration over solid angles is infinite at $\theta = \pi$ and the experimental curve will therefore differ considerably from this. Obviously, the singularity is integrable. In order to avoid two numerical integrations, the energy integration is carried out analytically. In order to do this, the Coulomb field effects are neglected but, especially for high energy spectra, this will introduce an error negligible compared to the experimental one. Moreover, it should be recalled that it is usually necessary to distinguish between two essentially non-overlapping classes of curves, one for S with possible T admixture and the other for V with possible T admixture (see Eq. (2) below). It should also be mentioned, at this point, that the analytical energy integration is possible only by expressing the results in non-closed form. However, the infinite series involved converge very rapidly for high energy spectra. In making this statement it is recognized

that it will be desirable for experimental reasons to accept only those coincidences for which the β-energy $W \geqslant W_1$, a lower cut-off. The windows (if present) automatically accomplish this. For the purpose of using these results in a simple way W_1 should be chosen to be somewhat higher than is usual; perhaps $W_1 > 2$ corresponding, in the units used here[2] to kinetic energies above 500 kev.

It should also be mentioned that there are several interesting spectra with large endpoints. The following list is illustrative:

TABLE I

	E_o (Mev)	W_o
A^{35}	4.4	9.6
Cl^{34}	4.45	9.7
Cl^{33}	4.1	9.0
Ne^{23}	4.2	9.2
O^{19}	4.5	9.8
P^{34}	5.1	11.0

In all these cases, with the possible exception of P^{34}, the β-transition goes to the ground state of the daughter nucleus and there is no γ following the β.

---

2. We use m = c = 1. E nergies and momenta are, as usual, in mc^2 and mc respectively. W is the kinetic plus rest energy.

When there is a γ-ray the correlation can also be calculated.[3] No further consideration of such cases is necessary here.

2. THE CORRELATION FUNCTION

The number of coincidences with the angle θ between recoil and β-particle is directly obtained from the electron-neutrino correlation. Omitting the Coulomb factor the latter is (per dW and per $d\mu = d\cos\vartheta$)

$$P(\vartheta, W) = \text{const. } pWq^2\left(1 + \frac{ap}{W}\mu + \frac{b}{W}\right) \tag{1}$$

where $\mu = \cos\vartheta$ and ϑ is the angle between β-particle and neutrino. As usual $q = W_0 - W$ with infinite recoil mass (see Appendix) and $p = \sqrt{W^2 - 1}$. The constant b represents possible Fierz interference (between S and V since admixture of A with T is known to be small from He^6 angular correlation measurements[4]). Actually, from the energy spectra $|b| \lesssim 0.1$ and in this range the effect on the correlation function is too small to be detected. We henceforth take b = 0 although the inclusion of the b-term presents no problem.

The constant a is given by

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3. M. E. Rose, Phys. Rev. 90, 1123 (1953). See also M. E. Rose, ORNL-1593.

4. S. L. Ruby and B. M. Rustad, Phys. Rev. 89, 880 (1953).

$$a = - \frac{C_S^2 \left|\int 1\right|^2 - \frac{1}{3} C_T^2 \left|\int \vec{\sigma}\right|^2}{C_S^2 \left|\int 1\right|^2 + C_T^2 \left|\int \vec{\sigma}\right|^2} \tag{2a}$$

for interactions and

$$a = \frac{C_V^2 \left|\int 1\right|^2 + \frac{1}{3} C_T^2 \left|\int \vec{\sigma}\right|^2}{C_V^2 \left|\int 1\right|^2 + C_T^2 \left|\int \vec{\sigma}\right|^2} \tag{2b}$$

for V interactions. Hence, $-1 < a < 1/3$ implies S and $1/3 < a < 1$ implies V. One expects $C_S^2 \sim C_T^2$ and $\int 1$ and $\int \vec{\sigma}$ can be roughly estimated from LS or jj coupling models. However, such estimates are not actually necessary unless a turns out to be near 1/3, which is not probable à priori.

The measured spectrum, aside from solid angle corrections is

$$Q(\theta) = \int_{W_1}^{W_2} dW\, P(W)\, J(\theta, W) \tag{3}$$

where $J(\theta, W)$ is the Jacobian and W_2 is a possible upper cut-off, see Eq. (10a) below. From momentum conservation alone

$$J(\theta, W) = \frac{1}{\sqrt{1 - \lambda^2 \sin^2\theta}} \left[-\epsilon \lambda \cos\theta + \sqrt{1 - \lambda^2 \sin^2\theta} \right]^2 \tag{4}$$

and in (1) we use

$$\mu = - \left[\lambda \sin^2\theta + \epsilon \cos\theta \sqrt{1 - \lambda^2 \sin^2\theta} \right] \tag{5}$$

where

$$\lambda = p/q \tag{6}$$

and

$$\epsilon = 1 \quad \text{for} \quad \lambda < 1$$
$$\epsilon = \pm 1 \quad \text{for} \quad \lambda \geqslant 1 \tag{7}$$

Contributions from both signs of ϵ must be taken for $\lambda \geqslant 1$. The value of W corresponding to $\lambda = 1$ is $\overline{W} = \frac{1}{2}(W_o + 1/W_o)$.

It will be noted that the Jacobian is singular if W is such that $\lambda = \csc\theta > 1$. This corresponds to the fact that for $\lambda > 1$, the resultant $\vec{p} + \vec{q}$ lies along $\vec{p}$ for $\vartheta = 0$ and $\vartheta = \pi$. Hence θ, which is equal to π at these two points has a minimum and for all ϑ, $\theta \geqslant \pi/2$. This also gives rise to the double sign (Eq. 7). The energy W for which $\lambda = \csc\theta$ is

$$W_m = \sec^2\theta \left[W_o - \sin\theta \sqrt{W_o^2 - \cos^2\theta}\right] \tag{8}$$

for $\theta \geqslant \pi/2$. This is the maximum energy detectable as a coincidence for the given θ. Hence unless $W_2 < W_m$ for all θ, which means $W_2 < \overline{W}$ since $W_m(\pi/2) = \overline{W}$, the singularity must appear. As will become apparent, for all θ except $\theta = \pi$, the singularity is integrable in energy but at $\theta = \pi$, the angular distribution has a weak (logarithmic) singularity. Contrary to the statement in RBB, this is not due to the assumed infinite mass of the recoil nucleus, as the appendix shows. That this is to be expected is evident from the fact that only momentum and not energy conservation is invoked.

For $\theta < \pi/2$ we note that

$$W_m = \overline{W} \qquad (\theta < \pi/2) \tag{9}$$

Since there seems to be no advantage to be gained by setting an upper cut-off we can assume that $W_2 = W_m$ if $W_1 \leqslant \overline{W}$ and if $W_1 > \overline{W}$ we can take $W_2 = W_m$ for all θ down to θ_1 where $W_m(\theta_1) = W_1$. In the latter case the correlation function $Q(\theta) = 0$ for all $\theta < \theta_1$. For W_o of the order given in the foregoing table,

$$W_m \approx W_o \left[\frac{1}{1 + \sin\theta} + \frac{\sin\theta}{2W_o^2} \right] \tag{8a}$$

with great accuracy and hence

$$\sin\theta_1 \approx \frac{W_o}{W_1} - 1 + \frac{W_o - W_1}{2W_1^3}\,; \quad W_1 > \overline{W}. \tag{8b}$$

3. EXPLICIT FORM OF INTEGRALS

We write the correlation in the form

$$Q(\theta) = Q_o(\theta) - a\, Q_1(\theta) \tag{10}$$

where

$$Q_o = \int_{W_1}^{W_2} \frac{dW\, pW\, q^2}{\sqrt{1 - \lambda^2 \sin^2\theta}} \left[\lambda^2 \cos 2\theta + 1 - 2\epsilon\lambda\cos\theta \sqrt{1 - \lambda^2 \sin^2\theta} \right] \tag{10a}$$

and

$$Q_1 = \int_{W_1}^{W_2} \frac{dW\, p^2\, q^2}{\sqrt{1 - \lambda^2 \sin^2\theta}} \left[\lambda^2 \cos 2\theta + 1 - 2\epsilon\lambda\cos\theta \sqrt{1 - \lambda^2 \sin^2\theta} \right] \times \left[\lambda \sin^2\theta + \epsilon\cos\theta \sqrt{1 - \lambda^2 \sin^2\theta} \right] \tag{10b}$$

We consider two cases:

(a) $W_2 > W_1 \geqslant \overline{W}$

(b) $W_2 > \overline{W} \geqslant W_1$

In case (a) we have $\theta > \theta_1 > \pi/2$ and

$$Q_o = 2 \int_{W_1}^{W_2} \frac{dW\, pW\, q^2}{\sqrt{1 - \lambda^2 \sin^2\theta}} \left(1 + \lambda^2 \cos 2\theta\right) \tag{11a}$$

$$Q_1 = 2 \int_{W_1}^{W_2} \frac{dW\, p^3\, q}{\sqrt{1 - \lambda^2 \sin^2\theta}} \left[1 - 3\cos^2\theta + \lambda^2 \sin^2\theta\,(4\cos^2\theta - 1)\right] \tag{11b}$$

Defining the following integrals:

$$I_o'(W_1) = \int_{W_1}^{W_2} \frac{dW\, pW\, q^2}{\sqrt{1 - \lambda^2 \sin^2\theta}}, \qquad I_1'(W_1) = \int_{W_1}^{W_2} \frac{dW\, p^3\, q}{\sqrt{1 - \lambda^2 \sin^2\theta}}$$

$$I_o(W_1) = \int_{W_1}^{W_2} \frac{dW\, p^3\, W}{\sqrt{1 - \lambda^2 \sin^2\theta}}, \qquad I_1(W_1) = \int_{W_1}^{W_2} \frac{dW\, p^5}{q\sqrt{1 - \lambda^2 \sin^2\theta}} \tag{12}$$

we have that

$$Q_o = 2\left[I_o' + I_o''\cos 2\theta\right] \tag{13a}$$

$$Q_1 = 2\left[I_1'(1 - 3\cos^2\theta) + \sin^2\theta\,(4\cos^2\theta - 1)\, I_1''\right] \tag{13b}$$

and all arguments of $I_o' \cdots I_1''$ are W_1.

For $W_2 = W_m$ it is seen that all integrals converge at all θ except I_1''.

Since

$$\sqrt{1 - \lambda^2 \sin^2\theta} = \frac{\sqrt{X}}{q} \tag{14}$$

where

$$X = a + bW + cW^2 = c(W_m - W)(W_m' - W) \tag{15}$$

and

$$a = W_o^2 + \sin^2\theta \ , \ b = -2W_o, \ c = \cos^2\theta \tag{15a}$$

$$W_m' = \sec^2\theta \left[W_o + \sin\theta \sqrt{W_o^2 - \cos^2\theta}\right] \tag{15b}$$

it follows from $W_m(\pi) = W_m'(\pi) = W_o$ that the integrand in I_1'' behaves like $1/q$ whereas at the upper limit $q \longrightarrow 0$. In all other integrals the positive powers of q insure convergence.

For case (b) we give the results for $\theta > \pi/2$ and $\theta \leq \pi/2$ separately:

<u>$\theta > \pi/2$</u>

$$Q_o = 2I_o(\overline{W}) + J_o(W_1) \tag{16a}$$

$$Q_1 = 2I_1(\overline{W}) + J_1(W_1) \tag{16b}$$

where

$$I_o(\overline{W}) = I_o'(\overline{W}) + I_o''(\overline{W}) \cos 2\theta \tag{17a}$$

$$I_1(\overline{W}) = (1 - 3\cos^2\theta)\, I_1'(\overline{W}) + \sin^2\theta\,(4\cos^2\theta - 1)\, I_1''(\overline{W}) \tag{17b}$$

and, with the notation

$$y_n = \frac{1}{n} (\overline{W}^n - W_1^{\,n}) \tag{18}$$

the J_o and J_1 are given by

$$J_o (W_1) = I_o (W_1) - I_o (\overline{W}) - 2 \cos\theta \left[W_o (y_4 - y_2) - (y_5 - y_3) \right] \tag{18a}$$

$$J_1 (W_1) = I_1 (W_1) - I_1 (\overline{W}) + \cos\theta \left[W_o^{\,2} (y_3 - y_1) - 2W_o (y_4 - y_2) \right.$$

$$\left. + y_5 - y_3 \right] + \cos\theta \ (4 \cos^2\theta - 3) (y_5 - 2y_3 + y_1) \tag{18b}$$

The parts of Q_o and Q_1 exclusive of J_o and J_1 respectively are, of course, $I_o (W_1) + I_o (\overline{W})$ and $I_1 (W_1) + I_1 (\overline{W})$ respectively and again, the latter diverges at $\theta = \pi$ in a logarithmic way.

$\theta \leqslant \pi/2$

Here

$$Q_o = K_o (W_1) - 2 J_o' \cos\theta \tag{19a}$$

$$Q_1 = K_1 (W_1) + J_1' \cos\theta + (4 \cos^2\theta - 3) J_1'' \tag{19b}$$

where

$$K_o (W_1) = I_o (W_1) - I_o (\overline{W}) \tag{20a}$$

$$K_1 (W_1) = I_1 (W_1) - I_1 (\overline{W}) \tag{20b}$$

and there is no divergence. K_o is obtained from (17a) and (12) by replacing the upper limit W_2 by $\overline{W}$. Similarly, K_1 is obtained from (17b)

and (12) by the same replacement of the upper limit. For this reason, in the results given below for I_o' .... I_1'' we leave the upper limit arbitrary in some places so that either W_m or $\overline{W}$ may be substituted in the results.

4. EVALUATION OF THE INTEGRALS

Define

$$\int_{W_1}^{W_2} dW \frac{pW^m}{\sqrt{X}} = R_{m+1}(W_2, W_1) \tag{21}$$

Then, omitting the argument W_1 in I_o' etc.,

$$I_o' = W_o^3 R_2 - 3W_o^2 R_3 + 3W_o R_4 - R_5 \tag{22a}$$

$$I_o'' = W_o(R_4 - R_2) - (R_5 - R_3) \tag{22b}$$

$$I_1' = W_o^2(R_3 - R_1) - 2W_o(R_4 - R_2) + R_5 - R_3$$

$$I_1'' = R_5 - 2R_3 + R_1$$

Once the integrals R_m are evaluated, proper substitution of the limits gives all required results.

The integrals R_m cannot be given in closed form but can be given in the form of an infinite series which converges rapidly for $p_1/W_1 \approx 1$ (i.e., $W_1^2 \gg 1$). Define

$$L_m(W_2, W_1) = \int_{W_1}^{W_2} \frac{dW\, W^m}{\sqrt{X}} \tag{23}$$

and use

$$p = W \sum_{0}^{\infty} \gamma_\nu / W^{2\nu}$$

and the binomial coefficients are

$$\gamma_\nu = (-) \binom{\frac{1}{2}}{\nu} = - \frac{1.3.5 \ldots . (2\nu - 3)}{2^\nu \, \nu!} \tag{24}$$

$$\gamma_0 = 1, \; \gamma_1 = -\tfrac{1}{2}$$

then

$$R_{m+1} = \sum_{\nu=0}^{\infty} \gamma_\nu \; L_{m+1-2\nu} \tag{25}$$

Thus

$$I'_0 = \sum_{\nu=0}^{\infty} \gamma_\nu \left[W_0^3 L_{2-2\nu} - 3W_0^2 L_{3-2\nu} + 3W_0 L_{4-2\nu} - L_{5-2\nu} \right] \tag{25a}$$

$$I''_0 = \sum_{\nu=0}^{\infty} \gamma_\nu \left[W_0 (L_{4-2\nu} - L_{2-2\nu}) - (L_{5-2\nu} - L_{3-2\nu}) \right. \tag{25b}$$

$$I'_1 = \sum_{\nu=0}^{\infty} \gamma_\nu \left[W_0^2 (L_{3-2\nu} - L_{1-2\nu}) - 2W_0 (L_{4-2\nu} - L_{2-2\nu}) + L_{5-2\nu} - L_{3-2\nu} \right] \tag{25c}$$

$$I''_1 = \sum_{\nu=0}^{\infty} \gamma_\nu \left[(L_{5-2\nu} - 2L_{3-2\nu} + L_{1-2\nu}) \right. \tag{25d}$$

For $L_m(W_2, W_1)$ we have the following recurrence formula:

$$L_m = A_m - \frac{2m - 1}{2m} \frac{b}{c} L_{m - 1} - \frac{m - 1}{m} \frac{a}{c} L_{m - 2} \tag{26}$$

with

$$A_m = \left. \frac{W^{m - 1} \sqrt{X}}{mc} \right]_{W_1}^{W_2} \tag{26a}$$

and

$$L_o = \frac{1}{\sqrt{c}} \log \left(\sqrt{X} + W \sqrt{c} + \frac{b}{2 \sqrt{c}} \right) \Bigg]_{W_1}^{W_2} \tag{26b}$$

In the form (26) m is restricted to positive values. However, if (26) is written in the form

$$L_{m - 2} = B_m - \frac{2m - 1}{2m - 2} \frac{b}{a} L_{m - 1} - \frac{m}{m - 1} \frac{c}{a} L_m \tag{27}$$

where

$$B_m = \left. \frac{W^{m - 1} \sqrt{X}}{(m - 1)a} \right]_{W_1}^{W_2} \tag{27a}$$

and (27) is used in conjunction with

$$L_{-1} = - \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{X} + \sqrt{a}}{W} + \frac{b}{2 \sqrt{a}} \right) \Bigg]_{W_1}^{W_2} \tag{27b}$$

then by giving m all non-positive values (m = 0, -1, -2 ..) one obtains all L_m for $m < 0$.

If one evaluates the right-hand side of (26b) at $W_2 = W_m$ one obtains

$$\frac{1}{|\cos\theta|} \log\left[- \sin\theta \sec\theta \sqrt{W_o^2 - \cos^2\theta} \right]$$

whereas for $W_2 = \overline{W}$ one obtains

$$\frac{1}{|\cos\theta|} \log\left[- \frac{W_o \sin^2\theta}{|\cos\theta|} \right]$$

wherein one uses

$$\sqrt{X(\overline{W})} = |\cos\theta| \quad \frac{W_o^2 - 1}{2W_o}$$

It may be noted that for arbitrary W_1 the argument of the log in L_o vanishes like $\sin^2\theta$ so that L_o is finite in all cases except when one limit is W_m, see Eq. (44d).

It is seen, then, that the singularity appears at $\theta = \pi$ in L_o and only in L_o. However, when I_o', I_o'' and I_1' are evaluated at $\theta = \pi$ the singular term L_o has a vanishing coefficient. This coefficient vanishes like $\sin^2\theta$. Therefore, in evaluating the above for $\theta = \pi - \delta$ ($\delta \ll 1$), it may be disadvantageous to use the recurrence formulae to calculate a very small quantity as the difference of several large quantities. That these quantities will be large may be recognized from the fact that the leading terms ($\nu = 0$ in (25)) all contain a dominant term proportional to W_o^5. Hence, we proceed to isolate the ''singular'' term L_o. That is, we write

$$L_m = Z_m L_o + D_m \tag{28}$$

D_m fulfills the recurrence formulae (26) and (27) with

$$D_o = 0 \tag{28a}$$

and D_{-1} will be given below. The Z_m fulfills the homogeneous recurrence formulae obtained from (26) and (27) by setting A_m and $B_m = 0$. Also

$$Z_o = 1 \tag{28b}$$

and Z_{-1} will be given below (see Eq. (33b)). We consider the problem of solving the recurrence formulae for Z_m. This will be done in two parts: first an approximate solution valid near $\theta = \pi$ will be obtained. Then the omitted terms will be added to obtain an exact solution for specific m values.

Considering $m > 0$, we set

$$Z_m = W_o^m \, c^{\tau_m} \chi_m \tag{29}$$

and with $a \approx W_o^2$ we find

$$\chi_m = \frac{2m - 1}{m} c^{\tau_{m-1} - \tau_m - 1} \chi_{m-1} - \frac{m - 1}{m} c^{\tau_{m-2} - \tau_m - 1} \chi_{m-2} \tag{30}$$

For convenience we make $\chi_o \equiv 1$. Then, comparing (30) with the recurrence formula for the Legndre polynomials, which is

$$P_m = \frac{2m - 1}{m} x P_{m-1} - \frac{m - 1}{m} P_{m-2} \tag{31}$$

for argument x, we see that (30) and (31) are identical with identical initial conditions ($P_o \equiv 1$) if

$$\tau_m = -\frac{m}{2}$$

and

$$x = \frac{1}{\sqrt{c}}$$

Hence

$$Z_m = \frac{W_o^m}{c^{m/2}} P_m \left(\frac{1}{\sqrt{c}}\right) \qquad (\theta \approx \pi) \tag{32}$$

Hence, near $\theta = \pi$, Z_m is a polynomial in 1/c of degree m.

If the right-hand side of (27b) is evaluated at $W_2 = W_m$ we find that the argument of the log is $W_o \sqrt{W_o^2 - 1} \sin\theta - W_o^2 \sec^2\theta \sin^2\theta \longrightarrow \sin\theta$ times a positive constant as $\theta \longrightarrow \pi$.

Hence, if we write

$$L_o = \frac{1}{\sqrt{c}} \log \beta_o$$

and

$$L_{-1} = -\frac{1}{\sqrt{a}} \log \beta_{-1}$$

we have

$$\beta_{-1} = f/\beta_o$$

where f is finite at $\theta = \pi$, (see Eq. (44a, b, c, d)). Therefore

$$L_{-1} = -\frac{1}{\sqrt{a}} \log f + \sqrt{\frac{c}{a}}\, L_o \tag{33}$$

and

$$D_{-1} = -\frac{1}{\sqrt{a}} \log f \tag{33a}$$

$$Z_{-1} = \sqrt{c/a} \longrightarrow \sqrt{c}/W_o \tag{33b}$$

If in (27) with $B_m = 0$ and $a = W_o^2$ we insert

$$Z_{-m\ -1} = W_o^{-m\ -\ 1}\, c^{\sigma_m} P_m\, (1/\sqrt{c}) \tag{34}$$

we obtain

$$c^{\sigma_m} P_m = \frac{2m - 1}{m}\, c^{\sigma_{m-1}} P_{m-1} - \frac{m - 1}{m}\, c^{\sigma_{m-2}+1} P_{m-2} \tag{35}$$

Comparing (35) and (31) and taking (33b) into account we find

$$\sigma_m = \tfrac{1}{2}\,(m + 1) \tag{35a}$$

and

$$Z_{-m-1} = \left(\frac{\sqrt{c}}{W_o}\right)^{m+1} P_m(1/\sqrt{c}), \qquad (\theta \approx \pi) \tag{36}$$

with $m \geqslant 0$. With the results (32) and (36) it may be checked that at $\theta = \pi$ all integrals except I_1'' do not contain L_o and are hence finite.

Returning to the exact homogeneous recurrence formulae and noting that $a = W_o^2 + 1 - c$, we set

$$Z_m = Z_m^{(o)} + \epsilon_m \tag{37}$$

where $Z_m^{(o)}$ is given by (32) and (36) then $\epsilon_o = \epsilon_1 = 0$ and for $m > 0$

$$\epsilon_m = \frac{2m-1}{2m}\frac{2W_o}{c}\epsilon_{m-1} - \frac{m-1}{m}\left[\frac{W_o^2+1-c}{c}\epsilon_{m-2} + \frac{1-c}{c}Z_{m-2}^{(o)}\right] \tag{38}$$

Explicitly,

$$\epsilon_2 = \tfrac{1}{2}\frac{1-c}{c}Z_o^{(o)} = \tfrac{1}{2}\frac{1-c}{c} \tag{38a}$$

$$\epsilon_3 = \frac{5}{3}\frac{W_o}{c}\epsilon_2 - \frac{2}{3}\frac{1-c}{c}Z_1^{(o)} \tag{38b}$$

$$\epsilon_4 = \frac{7}{4}\frac{W_o}{c}\epsilon_3 - \frac{3}{4}\left[\frac{W_o^2+1-c}{c}\epsilon_2 + \frac{1-c}{c}Z_2^{(o)}\right] \tag{38c}$$

$$\epsilon_5 = \frac{9}{5}\frac{W_o}{c}\epsilon_4 - \frac{4}{5}\left[\frac{W_o^2 + 1 - c}{c}\epsilon_3 + \frac{1 - c}{c} Z_3^{(o)}\right] \tag{38d}$$

For negative index

$$\epsilon_{m-2} = \frac{2m-1}{m-1}\frac{W_o}{a}\epsilon_{m-1} - \frac{m}{m-1}\frac{c}{a}\epsilon_m + \Lambda_{m-2} \tag{39}$$

where

$$\Lambda_{m-2} = \frac{2m-1}{m-1}\left(\frac{W_o}{a} - \frac{1}{W_o}\right) Z_{m-1}^{(o)} - \frac{mc}{m-1}\left(\frac{1}{a} - \frac{1}{W_o^2}\right) Z_m^{(o)} \tag{40}$$

The coefficients

$$\frac{W_o}{a} - \frac{1}{W_o} = -\frac{\sin^2\theta}{aW_o} \approx -\frac{\sin^2\theta}{W_o^3}$$

and

$$\frac{1}{a} - \frac{1}{W_o^2} = \frac{-\sin^2\theta}{aW_o^2} \approx -\frac{\sin^2\theta}{W_o^4}$$

are very small (for $W_o = 9$ the larger is 1.3×10^{-3} at best). Hence the Λ are all very small. Explicitly, the first few ϵ_{-m} are

$$\epsilon_{-1} = \sqrt{c}\left(\frac{1}{\sqrt{a}} - \frac{1}{W_o}\right) \tag{41a}$$

from (33b) and

$$\epsilon_{-2} = \frac{W_o}{a}\epsilon_{-1} + \Lambda_{-2} \tag{41b}$$

$$\epsilon_{-3} = \frac{3}{2}\frac{W_o}{a}\epsilon_{-2} - \frac{1}{2}\frac{c}{a}\epsilon_{-1} + \Lambda_{-3} \tag{41c}$$

and as many others as are necessary are easily obtained.

The explicit D_m are for $m > 0$

$$D_1 = A_1$$

$$D_2 = A_2 + \frac{3}{2}\frac{W_o}{a} A_1$$

$$D_3 = A_3 + \frac{5}{3}\frac{W_o}{a} D_2 - \frac{2}{3}\frac{a}{c} A_1$$

$$D_4 = A_4 + \frac{7}{4}\frac{W_o}{a} D_3 - \frac{3}{4}\frac{a}{c} D_2$$

$$D_5 = A_5 + \frac{9}{5}\frac{W_o}{a} D_4 - \frac{4}{5}\frac{a}{c} D_3 \tag{42}$$

The form (42) is most suited for numerical calculation. In this connection, we note that for $W_2 = W_m$, $W_1 = \overline{W}$

$$A_m = -\frac{1}{\sqrt{c}}\frac{W_o^2 - 1}{2W_o}\left(\frac{W_o^2 + 1}{2W_o}\right)^{m-1} \tag{43}$$

whereas for $W_2 = W_m$ and arbitrary W_1

$$A_m = -\frac{1}{c} W_1^{m-1} \sqrt{X(W_1)} \tag{43a}$$

For $W_2 = \overline{W}$, arbitrary W_1, A_m is equal to (43a) minus (43).

In all cases $\sqrt{c} > 0$.

For negative index we have the following explicit results:

$$D_{-1} = -\frac{1}{\sqrt{a}}\left[\log\left\{\frac{\sqrt{X}+\sqrt{a}}{W} + \frac{b}{2\sqrt{a}}\right\}\left\{\sqrt{X} + W\sqrt{c} + \frac{b}{2\sqrt{c}}\right\}\right]_{W_1}^{W_2} \tag{44a}$$

see (33a). For $W_2 = W_m$ and W_1 arbitrary we use the result that the argument of this log is

$$\frac{\sin^2\theta \ \sec^2\theta}{\sqrt{ac}\ W_m}\left[W_o^2 \sin\theta \sqrt{W_o^2 - \cos^2\theta} - W_o^2 + \cos^2\theta\right] \tag{44b}$$

and for $W_2 = \overline{W}$ the argument is

$$-\frac{W_o \sin^2\theta}{W_o^2 + 1}\left[W_o^2 - 1 + \frac{W_o}{\sqrt{ac}}(W_o^2 - \cos^2\theta)\right] \tag{44c}$$

For $W_2 = W_m$, $W_1 = \overline{W}$, and $\theta = \pi$ we have D_{-1} finite in contrast to L_o. For $\sin\theta \ll 1$ the argument of the log in (44a) evaluated at arbitrary W_1 is

$$\frac{(W_o - W_1)}{W_1}\left\{\frac{(1 - W_1)^2}{W_o - W_1} - (W_o + W_1)\right\}\sin^2\theta \tag{44d}$$

The quantity f_Λ $(\theta \to \pi)$ (see Eq. (33a)) is given by the ratio of (44b) or (44c) to (44d) according to whether the upper limit is W_m or $\overline{W}$.

Hence, at $\theta = \pi$, the D_{-1} term is always finite as expected. Other values of D_{-m} are obtained directly from the recurrence formula (27):

$$D_{-2} = B_o + \frac{W_o}{a} D_{-1} \tag{45a}$$

$$D_{-3} = B_{-1} + \frac{3}{2} \frac{W_o}{a} D_{-2} - \frac{1}{2} \frac{c}{a} D_{-1} \tag{45b}$$

$$D_{-4} = B_{-2} + \frac{5}{3} \frac{W_o}{a} D_{-3} - \frac{2}{3} \frac{c}{a} D_{-2} \tag{45c}$$

Obviously, these results can be easily extended as far as is necessary. The neglect of the coulomb effects and the finite experimental precision will determine the required accuracy and the number of correction terms necessary. In this connection, as already emphasized, the calculations are made more easily if the lower energy cut-off W_1 is not too low. Clearly, since the anisotropy of the correlation arises from the higher energy electrons one doesn't lose very much if W_1 is set at a comparatively high value.

APPENDIX

Although the arguments given above may be sufficient to show that the finite mass of the recoil nucleus cannot remove the (θ - integrable) singularity in the angular correlation at $\theta = \pi$, we give an explicit demonstration of this fact in the following in order to make the point abundantly clear.

Let $\vec{P}$ be the resultant of the lepton momenta $\vec{p}$ and $\vec{q}$ and let M be the mass of the recoil nucleus, then

$$\vec{P} = \vec{p} + \vec{q} \tag{A.1}$$

and

$$W_o = W + q + P^2/2M \tag{A.2}$$

With $\xi = \cos\theta$ we find

$$\xi^2 = \frac{\left[2M(W_o - W - q) - q^2 + p^2 \right]^2}{8p^2M(W_o - W - q)} \tag{A.3}$$

The maximum energy W_m is that root of $1 - \lambda^2 \sin^2 \theta = 0$ which is not greater than W_o. This is easily found to be

$$W_m = M \left[- 1 + + \sqrt{1 + \frac{2}{M}(W_o - q) + \frac{1 + q^2}{M^2}} \right] \tag{A.4}$$

expressed as a function of q rather than θ.

One could express W_m in terms of θ by solution of the equation resulting from elimination of between (A.3) and the quadratic of which (A.4) is a root:

$$W^2 + 2MW - 2M(W_o - q) - 1 - q^2 = 0 \tag{A.5}$$

but this is not actually necessary.

We now consider

$$X = q^2 (1 - \lambda^2 \sin^2 \theta)$$

$$= \frac{\left[2M(W_o - W - q) - p^2 + q^2\right]^2}{8M(W_o - W - q)} \tag{A.6}$$

If $W = W_m - \epsilon$ ($\epsilon \geqslant 0$) we know that for $\theta \rightarrow \pi$, $X \sim \epsilon^2$ so that $\sqrt{X}$ vanishes linearly. That is, $X(W_m)$ and $(\partial X/\partial W)_{W_m} = 0$. It is now a straightforward matter to verify that this is true with finite M.

From (A.3) for fixed ξ

$$- \frac{\partial q}{\partial W} \left\{ 2M(W_o - W - q)(q + M) - q^2(q + M) + p^2(2q + M - 2\xi^2 M) \right\} \tag{A.7}$$

$$= 2M(W_o - W - q)(2W\xi^2 + M - W) + p^2(M - W - 2\xi^2 M) - q^2(M - W)$$

which is to be substituted in

$$\frac{\partial X}{\partial W} = 2q \frac{\partial q}{\partial W} - 2W \sin^2 \theta \tag{A.8}$$

At $W = W_m$

$$\frac{1}{2}\left(\frac{\partial X}{\partial W}\right)_{W_m} = \left(q\,\frac{\partial q}{\partial W}\right)_{W_m} - W_m \sin^2\theta \tag{A.9}$$

where $q(W_m) = p(W_m)\sin\theta$

Since $2M(W_o - W - q) \longrightarrow p^2 - q^2$ at $W = W_m$ we find

$$(q\,\partial q/\partial W)_{W_m} = -W_m \sin^2\theta \tag{A.10}$$

and therefore

$$\left(\frac{\partial X}{\partial W}\right)_{W_m,\pi} = 0 \tag{A.11}$$

Consequently $\sqrt{X}$ vanishes at least linearly in $W_m - W$ and the divergence of I_1'' remains.