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DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS

Frank W. Stallard

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DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS

Frank W. Stallard

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TABLE OF CONTENTS

	PAGE
INTRODUCTION	1
CHAPTER	
I. DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS AT k POINTS	7
Section 1. Notation and Definitions	7
Section 2. A Fundamental Theorem	10
Section 3. The General Solution.	12
Section 4. Nonhomogeneous Systems and Green's Matrix .	14
Section 5. A Formula for Non-Singular Solutions. . . .	19
Section 6. An Example.	22
II. THE HOMOGENEOUS INTERFACE PROBLEM DEPENDING UPON A PARAMETER	27
Section 1. Self-Adjoint and Definitely Self-Adjoint Systems	27
Section 2. The Second Order System	39
Section 3. An Example.	45
Section 4. Definitely Self-Adjoint Cases for the Example	53
III. DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS AT INFINITE SUBSETS OF $[a, b]$	59
SUMMARY.	67
BIBLIOGRAPHY	69

INTRODUCTION

This paper is concerned with linear ordinary differential systems of the nth order. It is well known [11] [1] that such systems may be replaced by an equivalent system of n first order linear differential equations and that systems of the latter type may be studied by investigating a matrix differential equation [1] of the form

$$Y' + PY = Q \tag{1}$$

where P is a square matrix with n rows and Y', Y, and Q have n rows but only one column. Therefore the paper uses the notation of matrix algebra and calculus.

The paper assumes familiarity with such operations as addition, multiplication, differentiation and integration as applied to matrices. Capital letters are used to denote matrices; small letters are used to denote scalar quantities. If $A = (a_{ij})$ is a matrix, then a_{ij} is the element in its ith row and jth column. $A^T = (a_{ji})$ is the transpose matrix of A and is obtained by interchanging rows and columns of A. $E = (e_{ij})$, where $e_{ii} = 1$ $(e_{ij} = 0$ if $i \neq j)$ is the identity matrix. The zero matrix (with all its elements equal to zero) is denoted by the ordinary zero symbol. If A is a square matrix with n rows, then $|A|$

---

[1] Numerals in square brackets refer to works listed in the bibliography at the end of the paper.

denotes the nth ordered determinant of A . If $|A| \neq 0$, then the unique inverse matrix A^I exist and satisfies $A\,A^I = A^I\,A = E$.

A matrix is said to be continuous, integrable, differentialable, etc. when all its elements have the indicated properties. If $Y(x) = (y_{ij})$ is a matrix, then

$$Y'(x) = \left(\frac{d\,y_{ij}}{dx}\right)$$

and

$$\int_a^b Y(x)dx = \left(\int_a^b y_{ij}\,dx\right) .$$

The symbols $Y(x_\alpha^+)$ and $Y(x_\alpha^-)$ denote, respectively, the matrices $\left(\lim_{\substack{x \to x_\alpha \\ x > x_\alpha}} y_{ij}(x)\right)$ and $\left(\lim_{\substack{x \to x_\alpha \\ x < x_\alpha}} y_{ij}(x)\right)$. Since most of the matrices used in this paper are square with n rows, hereafter, if the term matrix is used without any reference to the number of rows or number of columns, then the matrix is square with n rows.

Boundary value problems consisting of equation (1) and a set of boundary conditions involving the values of the solution Y at one or more points have been considered extensively (for references to this work see Reid [22] and Whyburn [34]). In these investigations, there are several implicit definitions of the term solution. In general, a solution $Y(x)$ is required to be a continuous matrix having a certain

number of continuous derivatives and to satisfy equation (1) everywhere on some fundamental interval $[a, b]$. This sense of the term solution is used with the Riemann theory of integration and, therefore, may be called an R-solution.

Investigations of equation (1) and boundary conditions at k-points of the fundamental interval led some authors (see for example Reid [21] and Mansfield [15]) to consider as solutions matrices which have all of the properties of the R-solution at each point of the interval except possibly at the k points involved in the boundary conditions. Thus these solutions and their derivatives might have discontinuities of the first kind[2] at each of the k points.

Another understanding of the term solution (see for example Whyburn [34] and [35]) requires only that a solution $Y(x)$ be an absolutely continuous[3] matrix and satisfy the differential equation (1) <u>almost everywhere</u> on the interval $[a, b]$. This sense of the term solution is used with the Lebesgue theory of integration and, therefore may be referred to as a <u>L-solution</u>, or an absolutely continuous solution.

---

[2] A matrix $Y(x)$ is said to have a <u>discontinuity of the first kind</u>, or an <u>ordinary discontinuity</u>, at the point x_α if the limits $Y(x_\alpha^+)$ and $Y(x_\alpha^-)$ exist and $Y(x_\alpha^+) \neq Y(x_\alpha^-)$.

[3] A function $f(x)$ is said to be <u>absolutely continuous</u> on $[a, b]$ if $f(x)$ is defined on $[a, b]$ and has the property that for each positive number ϵ, there exist a positive number δ such that if $x_i \leqslant x \leqslant x_i'$ $(i = 1,2,\ldots)$ is any finite or infinite collection of non overlapping subintervals of $[a, b]$ such that $\sum |x_i - x_i'| < \delta$ then $\sum |f(x_i) - f(x_i')| < \epsilon$.

Obviously, the last two meanings of the term solution are generalizations of the first. In this paper the absolute continuity condition of the L-solution is relaxed in that the solutions may have discontinuities of the first kind and any point of a closed subset Γ of measure zero on $[a, b]$.

Definition 1. The statement that the matrix $Y(x)$ is a Γ-solution on $[a, b]$ of equation (1) means

(a) Γ is a closed subset of measure zero of $[a, b]$,

(b) $Y(x)$ is absolutely continuous on each component[4] of $\left([a, b] - \Gamma\right) = c\,\Gamma$,

(c) the limits $Y(x_\alpha^+)$ and $Y(x_\alpha^-)$ exist for each x on $[a, b]$, and

(d) $Y(x)$ satisfies equation (1) almost everywhere on $[a, b]$.

The points of Γ shall be called interfaces. If it is assumed that for each point x_α in Γ there is given a non-singular matrix B_α and that the Γ-solution $Y(x)$ must satisfy the relation

$$(2) \qquad Y(x_\alpha^+) = B_\alpha\, Y(x_\alpha^-) , \qquad x_\alpha \text{ in } \Gamma ,$$

for each point x_α in Γ then conditions (2) are called interface conditions. A problem composed of equation (1) and conditions (2) given at a suitable set Γ will be called an interface problem.

---

[4] By a component of a point set Ω one understands a maximal connected subset of Ω .

In Chapter I, Γ is assumed to be finite and it is shown in Section 3 that for the interface problem (1) and (2) there exist an associated matrix equation of the same type as equation (1),

$$W' + PW = Q \quad . \tag{3}$$

These systems are associated in the sense that there is a non-singular transformation $D(x)$ such that if $W(x)$ is an L-solution of the associated equation then $Y(x) = D(x)\,W(x)$ is a Γ-solution of (1) satisfying conditions (2) and conversely if $Y(x)$ is a Γ-solution of (1) satisfying conditions (2) then there is an L-solution $W(x)$ of (3) such that $Y(x) = D(x)\,W(x)$.

This association of the two systems is exploited to establish the existence and uniqueness of Γ-solutions and the form of the general Γ-solution. System (1) and (2) is studied (see Section 4, Chapter I) in conjunction with boundary conditions of the form

$$A\,Y(a) + B\,Y(b) = C \quad . \tag{4}$$

The adjoint system is introduced and the Green's matrix for system (1), (2) and (4) is found and its properties discussed.

In Chapter II, the homogeneous system depending on a parameter

$$Y' + PY = \lambda R\,Y \tag{5}$$

$$(6) \qquad Y(x_\alpha^+) = B_\alpha Y(x_\alpha^-) \qquad x_\alpha \text{ in } \Gamma$$

$$(7) \qquad A\,Y(a) + B\,Y(b) = 0$$

is studied. The definitions of self-adjoint and definitely self-adjoint as presented by Bliss [2] and [3] are extended to systems with interface conditions (see Section 1, Chapter II).

Some examples (Section 6, Chapter I and Section 3, Chapter II) are considered which are stated as chemical diffusion problems but which also have analogues in heat problems (see for example [5], [6], [7], [12] or [27]), in biological diffusion (see for example [18], [19], [20] or [36]) in reactor physics (see [8]) and even in acoustical vibrations (see [17] for example). The first example is a nonhomogeneous problem of the second order which in reactor language would be a one-group, n-region reactor problem with contact resistance type interfaces and nonhomogeneous boundary conditions.

The second example is a homogeneous problem depending on a parameter which in reactor language is a two-group n-region reactor problem with contact resistance type interfaces. Conditions (see Section 4, Chapter II) under which this problem is self-adjoint and definitely self-adjoint are also found.

In Chapter III, certain extensions of the results of Chapter I and II are considered where Γ is an infinite set.

CHAPTER I

DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS AT k POINTS

Section 1. Notation and Definitions

The system of first order linear differential equations

$$y_i' + \sum_{j=1}^{n} p_{ij}(x)\, y_j = q_i \qquad i = 1, \ldots, n$$

may be written in matrix form as

$$1.1 \qquad L(Y) = Y' + P(x)\, Y = Q(x)$$

where $P(x) = (p_{ij})$ is a square matrix of n rows and $L(Y)$, Y, Y' and $Q(x)$ are matrices with n rows but only one column. Single linear equations of order n and systems of higher order linear equations may be replaced by matrix equations similar to 1.1. To illustrate how this replacement may be effected consider the single linear differential equation

$$1.2 \qquad y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y = q \quad .$$

Let $y_1 = y$, $y_i = y_{i-1}'$, $2 \leqslant i \leqslant n$. Then a matrix equation equivalent to 1.2 is

$$1.3 \qquad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}' + \begin{pmatrix} 0 & -1 & 0 & \cdots & \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q \end{pmatrix} .$$

As a second illustration, consider a system of two second order differential equations

$$1.4 \qquad \begin{aligned} z_1'' + a_{11} z_1 + a_{12} z_2 &= q_1 \\ z_2'' + a_{21} z_1 + a_{22} z_2 &= q_2 \quad . \end{aligned}$$

Let $y_1 = z_1$, $y_2 = z_1'$, $y_3 = z_2$ and $y_4 = z_2'$ then a matrix equation equivalent to 1.4 is

$$1.5 \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' + \begin{pmatrix} 0 & -1 & 0 & 0 \\ a_{11} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & -1 \\ a_{21} & 0 & a_{22} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ q_1 \\ 0 \\ q_2 \end{pmatrix} .$$

Clearly k second order equations could be replaced by a matrix equation (each matrix having $2k$ rows) in a similar manner.

One may regard $L(Y)$, Y , Y' and Q of 1.1 as square matrices with all columns of Q alike. In this case the columns of Y each

satisfy the same system of equations and may or may not be alike. In the present paper, unless otherwise specified, the matrices in equation

$$1.6 \qquad L(Y) = Y' + P(x)\,Y = Q(x)$$

are understood to be square matrices with n rows. It is clear that the above mentioned special case occurs when the columns of $Q(x)$ are alike.

Let $Y(x_i^+)$ and $Y(x_i^-)$ denote the right and left limits respectively of the matrix $Y(x)$ at $x = x_i$. Let Γ denote a set of k points, x_i, on the real number interval $[a, b]$ such that $a < x_1 < x_2 < \dots < x_k < b$ and let $c\Gamma$ denote $([a, b] - \Gamma)$. Consider a differential system composed of equation 1.6 and a set of conditions, hereafter referred to as <u>interface conditions</u>, which are to be satisfied at the points of Γ of the form

$$1.7 \qquad I_i(Y) = Y(x_i^+) - B_i\,Y(x_i^-) = 0 \qquad i = 1, \dots, k$$

where each B_i is a non-singular matrix of constants.

<u>Definition 2</u>. The statement that the $n \times n$ matrix $U(x)$, a Γ-solution of 1.6, is a <u>non-singular Γ-solution</u> means that the determinant $|U(x)|$ is not zero on $c\Gamma$ and that the determinants $|U(x_i^+)|$ and $|U(x_i^-)|$ are not zero for any x_i in Γ.

Definition 3. The statement that a Γ-solution $Y(x)$ has the character of $D(x)$ means $Y(x)$ can be expressed as $Y(x) = D(x)\,\hat{Y}(x)$ where $\hat{Y}(x)$ is absolutely continuous on $[a, b]$ and $D(x)$ is a non-singular Γ-solution of $D' + OD = 0$, i.e., $D' = 0$.

Section 2. A Fundamental Theorem

Theorem 1. A necessary and sufficient condition that the system 1.6 have a Γ-solution with the character of $D(x)$ is that the system

$$\mathcal{L}(W) = W' + D^{I}(x)\,P\,D(x)\,W = D^{I}(x)\,Q(x) \tag{2.1}$$

have an absolutely continuous solution W, where the matrices P and Q are the same as those in equation 1.6.

Proof. Suppose that system 2.1 has an absolutely continuous solution W. Then the matrix equation 2.1 holds almost everywhere on $[a, b]$. But D is defined except at a finite set of points and is non-singular. Therefore

$$DW' + PDW = Q(x)$$

holds almost everywhere on $[a, b]$. Since $D' = 0$ by definition, the relation

$$(DW)' + P(DW) = Q(x)$$

holds almost everywhere on $[a, b]$. Clearly then $Y = DW$ satisfies the definition of a Γ-solution and has the character of D .

Suppose system 1.6 has a Γ-solution with the character of D . Then by definition, Y can be expressed as $Y = D\hat{Y}$ where $\hat{Y}$ is absolutely continuous on $[a, b]$. Since Y , by assumption, is a Γ-solution, the relation

$$(D\hat{Y})' + P(D\hat{Y}) = Q(x)$$

holds almost everywhere on $[a, b]$. Since $D' = 0$ almost everywhere on $[a, b]$

$$D(\hat{Y})' + PD(\hat{Y}) = Q(x)$$

holds and since $|D| \neq 0$

$$(\hat{Y})' + (D^{I} PD)\hat{Y} = D^{I} Q$$

holds almost everywhere on $[a, b]$. Clearly then $\hat{Y}$ satisfies the definition of an absolutely continuous solution of 2.1.

Note that a Γ-solution of 1.6 with the character of D is zero almost everywhere on $[a, b]$ if and only if $\hat{Y}$ is identically zero on $[a, b]$ hence the mapping of solutions of 2.1 onto Γ-solutions of 1.6 is one to one. As a result, the <u>existence and uniqueness of Γ-solution</u>

of 1.6 with the character of D are established by the existence and uniqueness of the solutions of system 2.1.

Section 3. The General Solution

Consider now the interface problem 1.6 and 1.7 where $P(x)$ and $Q(x)$ of 1.6 are Lebesgue integrable on $[a, b]$. Let $D_1 = E$ and $D_{i+1} = B_i D_i$, $i \leqslant k$, and then let

$$D(x) = D_{[i]} \qquad i = 1, \ldots, k+1$$

where $[i]$ denotes the smallest i such that $x < x_i$; x_i in Γ, $i \leqslant k$; and $x_{k+1} = b$.

Clearly if $Y(x)$ is a Γ-solution of 1.6 and has the character of $D(x)$, $Y(x)$ satisfies 1.6 and 1.7, and is, therefore, a solution to this interface problem.

Since $P(x)$ and $Q(x)$ are Lebesgue integrable, it is easily seen that $D^I(x)\, P(x)\, D(x)$ and $D^I\, Q(x)$ are also Lebesgue integrable, and the existence theorem given by Whyburn [33, th. XI] gives an absolutely continuous solution, W, of

3.1 $$\mathcal{L}(W) = W' + D^I PDW = D^I Q$$

such that

3.2 $$W(c) = C$$

where $a \leqslant c \leqslant b$ and C is any constant matrix. If c is not a point of Γ then system 1.6 and 1.7 will have a Γ-solution $Y(x)$ such that

3.3 $$Y(c) = D^{I}(c)C \quad .$$

If c is a point $x_i \in \Gamma$ then system 1.6 and 1.7 will have a Γ-solution such that

$$Y(x_i^-) = D^{I}_{(i-1)} C$$

$$Y(x_i^+) = D^{I}_{i} C \quad .$$

In particular, if $Q(x)$ is identically zero on $[a, b]$ then the existence theorem gives an absolutely continuous solution $\mathcal{W}(x)$ of

3.4 $$\mathcal{L}(W) = 0$$

3.5 $$W(c) = E \quad ,$$

furthermore, by well known theorems[5] $|\mathcal{W}(x)| \neq 0$ on $[a, b]$, and every absolutely continuous solution, W, of 3.4 can be expressed as

---

[5] See Whyburn [35] for a discussion and development of the theory of absolutely continuous solutions.

3.6 $$W(x) = \mathcal{W}(x)\, C \quad .$$

It also follows from these theorems that every absolutely continuous solution, W, of 3.1 can be expressed as

3.7 $$W(x) = \mathcal{W}(x)\, C + W_p(x)$$

where W_p is a particular solution of 3.1 . It follows from Theorem 1 that every Γ-solution $Y(x)$ of 1.6 and 1.7 can be expressed as

3.8 $$Y(x) = D(x)\ (\mathcal{W}(x)\, C + W_p) \quad .$$

Then 3.8 is the general Γ-solution of 1.6, 1.7.

Section 4. Nonhomogeneous Systems and Green's Matrix

Consider, now, a boundary value problem composed of

4.1 $$L(Y) = Y' + PY = Q$$

4.2 $$I_i(Y) = Y(x_i^+) - B_i\, Y(x_i^-) = 0$$

4.3 $$H(Y) = A\, Y(a) + B\, Y(b) = C \quad .$$

Boundary conditions 4.3 are of the two point type and consist of n conditions on the values of the elements of the columns of $Y(x)$ at the end points of the interval $[a, b]$.

<u>Theorem 2</u>. A necessary and sufficient condition that the homogeneous system

4.4 $$L(Y) = Y' + PY = 0$$

4.5 $$I_i(Y) = Y(x_i^+) - B_i\, Y(x_i^-) = 0$$

4.6 $$H(Y) = A\, Y(a) + B\, Y(b) = 0$$

have a non-identically vanishing Γ-solution, is that $|H(\mathcal{Y})| = 0$ where $\mathcal{Y} = D(x)\, \mathcal{W}(x)$.

<u>Proof</u>. Since Q is identically zero W_p in 3.8 can be chosen as the zero matrix. It then follows that every Γ-solution of 4.4 and 4.5 is of the form

4.7 $$Y(x) = D(x)\, \mathcal{W}(x)\, C = \mathcal{Y}(x)\, C .$$

Substituting into 4.6 gives

4.8 $$H(\mathcal{Y}C) = A\,\mathcal{Y}(a)C + B\,\mathcal{Y}(b)C = H(\mathcal{Y})C = 0 .$$

But a necessary and sufficient condition that equation 4.8 have a solution for C which is not the zero matrix is that $|H(\mathcal{Y})| = 0$.

<u>Theorem 3</u>. If the homogeneous system 4.4, 4.5, and 4.6 is incompatible, i.e. has only the identically vanishing solution $Y \equiv 0$, then the Green's matrix $G(x, t)$ for system 4.1, 4.2, and 4.3 is

$$G(x, t) = D(x)\,\mathcal{G}(x, t)\,D^{I}(t)$$

where $\mathcal{G}(x, t)$ is the Green's matrix for the system

$$\mathcal{L}(W) = W' + D^{I}PDW = D^{I}Q \tag{4.9}$$

$$\mathcal{H}(W) = \mathcal{A}W(a) + \mathcal{B}W(b) = C \tag{4.10}$$

where $\mathcal{A} = AD(a)$ and $\mathcal{B} = BD(b)$.

<u>Proof</u>. Whyburn [35, p. 57] gives the form of the Green's matrix for system 4.9 and 4.10 as

$$\mathcal{G}(x, t) = \begin{cases} \mathcal{W}(x)\,\mathcal{H}^{I}(\mathcal{W})\,\mathcal{A}\,\mathcal{W}(a)\,\mathcal{W}^{I}(t) & \text{for } t < x \\ -\mathcal{W}(x)\,\mathcal{H}^{I}(\mathcal{W})\,\mathcal{B}\,\mathcal{W}(a)\,\mathcal{W}^{I}(t) & \text{for } t > x \end{cases} \tag{4.11}$$

and a solution of system 4.9 and 4.10 is

4.12 $$W(x) = \mathcal{W}(x)\mathcal{H}^{I}(\mathcal{W})\,C + \int_a^b \mathcal{G}(x,t)D^{I}(t)\,Q(t)dt \quad .$$

By Theorem 1 and substitutions $\mathcal{H}(W) = H(DW)$ and $\mathcal{Y}(x) = D(x)\,\mathcal{W}(x)$,

$$Y(x) = D(x)\,W(x) = \mathcal{Y}(x)\,H^{I}(\mathcal{Y})C + D(x)\int_a^b \mathcal{G}(x,t)\,D^{I}(t)\,Q(t)dt$$

or

4.13 $$Y(x) = \mathcal{Y}(x)\,H^{I}(\mathcal{Y})C + \int_a^b G(x,\,t)\,Q(t)dt$$

where

4.14 $$G(x,\,t) = D(x)\,\mathcal{G}(x,\,t)\,D^{I}(t) \quad .$$

By substituting $\mathcal{Y}(x) = D(x)\,\mathcal{W}(x)$, $\mathcal{H}(\mathcal{W}) = H(\mathcal{Y})$, $\mathcal{A} = AD(a)$ and $\mathcal{B} = BD(b)$ in 4.11 one has

$$G(x,\,t) = \begin{cases} \mathcal{Y}(x)\,H^{I}(\mathcal{Y})\,A\,\mathcal{Y}(a)\,\mathcal{Y}^{I}(t) & t < x \\ -\mathcal{Y}(x)\,H^{I}(\mathcal{Y})\,B\mathcal{Y}(b)\mathcal{Y}^{I}(t) & t > x \end{cases} .$$

$G(x,\,t)$ is the Green's matrix for Γ-solutions of 4.4, 4.5 and 4.6. Some of its properties are

(i) $G(x, t)$ is continuous in (x, t) throughout the rectangle $a \leqslant x, t \leqslant b$ except for finite discontinuities at the lines $x = x_i$, $t = x_i$ and $x = t$. The discontinuity on the line $x = t$ is given by

$$G(x, x-) - G(x, x+) = E \qquad \text{if } x \neq x_i .$$

(ii) For fixed t, $t = \bar{t}$, on $c\Gamma$; $L\left[G(\overset{\infty}{c}, \bar{t})\right] = 0$ almost everywhere on $\left[a, b\right]$.

(iii) For fixed t, $t = \bar{t}$, on $c\Gamma$, $G(x, \bar{t})$ satisfies the interface conditions $I_i(Y) = 0$.

(iv) For fixed t, $t = \bar{t}$, on $c\Gamma$, $H\left[G(x, \bar{t})\right] = 0$.

(v) $G(x, t)$ is unique.

If $|A|$ and $|B|$ are both not zero one can then define the pair of homogeneous systems 4.4, 4.5 and 4.6 and

4.15 $$M(Z) = Z' - ZP = 0$$

4.16 $$F_i(Z) = Z(x_i^+) - Z(x_i^-)\, B_i^I = 0$$

4.17 $$J(Z) = Z(a)\, A^I + Z(b)\, B^I = 0$$

to be <u>adjoint</u>, either being the adjoint of the other.

In this case one has the further properties for $G(x, t)$

(vi) For fixed x, $x = \bar{x}$ on $c\Gamma$, $M\left[G(\bar{x}, t)\right] = 0$ almost everywhere on $\left[a, b\right]$.

(vii) For fixed x, $x = \bar{x}$, on $c\Gamma$, $G(\bar{x}, t)$ satisfies the interface conditions $F_i(Z) = 0$.

(viii) For fixed x, $x = \bar{x}$, on $c\Gamma$, $J\left[G(\bar{x}, t)\right] = 0$.

(ix) $\bar{G}(x, t) = -G(t, x)$ where $\bar{G}(x, t)$ is the Green's matrix for the adjoint system 4.15, 4.16 and 4.17.

Properties (i)-(ix) may be established by an examination of the properties of the Green's matrix $\mathcal{G}(x, t)$ for the associated homogeneous system and the matrices $D(x)$ and $D^I(x)$ and making use of 4.14.

Section 5. A Formula for Non-Singular Solutions

Consider again the homogeneous, interface problem

5.1 $$L(Y) = Y' + PY = 0$$

5.2 $$I_i(Y) = Y(x_i^+) - B_i\, Y(x_i^-) \qquad x_i \text{ in } \Gamma .$$

In Section 3, it was shown that the general Γ-solution was of the form

$$Y(x) = D(x)\, \mathcal{W}(x) C$$

C an arbitrary $n \times n$ matrix, and $\mathcal{W}(x)$ a non-singular solution of

5.3 $$\mathcal{L}(W) = W' + D^I(x)\, P(x)\, D(x)\, W = 0$$

and

5.4 $$\mathcal{W}(c) = E \quad .$$

Let $a = x_0$ and $b = x_{k+1}$. The fundamental existence theorem for absolutely continuous solution gives that on each interval $x_{i-1} \leqslant x \leqslant x_i$, $i = 1, \ldots, k+1$, there exist a non-singular solution $U_i(x)$ of 5.1, such that $U_i(x_{i-1}) = E$. Let $W_i = D_i^I U_i(x) D_i$, then on the same interval, W_i is a solution of 5.3, and $W_i(x_{i-1}) = E$. Therefore, a non-singular absolutely continuous solution on $[a, b]$ of 5.3 is

5.5 $$\underline{W}(x) = W_{[i]}(x)\, W_{i-1}(x_{i-1}) W_{i-2}(x_{i-2}) \ldots W_1(x_1)$$

where $[i]$ is the smallest i such that $x_i > x$, and $W(x_j) = W(x_j^+) = W(x_j^-)$, x_j in Γ .

Note that

$$\underline{W}(x_j^+) = W_{j+1}(x_j)\, W_j(x_j) \ldots W_1(x_1)$$

$$\underline{W}(x_j^-) = W_j(x_j)\, W_{j-1}(x_{j-1}) \ldots W_1(x_1)$$

and since $W_{j+1}(x_j) = E$, $\underline{W}(x_j^+) = \underline{W}(x_j^-)$.

Let $\prod_k^1$ denote $A_k B_k \ldots A_2 B_2 A_1 B_1$ and $\prod_1^k$ denote $A_1 B_1 A_2 B_2 \ldots A_k B_k$. Then 5.4 can be written as

$$\underline{W}(x) = D^I_{[i]} U_{[i]}(x) D_{[i]} \prod_{i-1}^{1} D^I_j U_j(x_j) D_j$$

or since $D_{j+1} D^I_j = B_j$, 5.5 can be expressed as

5.6 $$\underline{W}(x) = D^I_{[i]} U_{[i]}(x) \prod_{i-1}^{1} B_j U_j(x_j)$$

and

$$\underline{W}(a) = E \quad .$$

Clearly 5.6 satisfies the definition of $\mathcal{W}(x)$ in the formula (3.8) for the general Γ-solution. Therefore every Γ-solution of 5.1, and 5.2 is of the form

$$Y(x) = D(x) \underline{W}(x) C$$

and

5.7 $$\mathcal{Y}(x) = D(x) \underline{W}(x) = U_{[i]}(x) \prod_{i-1}^{1} B_j U_j(x_j)$$

is a non-singular Γ-solution of 5.1 and 5.2, also $\mathcal{Y}(a) = E$. Clearly the Γ-solutions of 5.1 and 5.2 then have the form

5.8 $$Y(x) = \mathcal{Y}(x)C$$

where C is a constant matrix. In the example which follows this form of the general Γ-solution will be employed and C most often will be a 2×2 matrix whose columns are all alike, which can be written as a column vector, the ith component being the common number which appears in the ith row of each column of C.

Section 6. An Example

Consider a n region diffusion problem for a spherically symmetric geometry, in particular, a central sphere and $n - 2$ concentric shells in an infinite medium. The diffusion equation for steady state solutions is

$$6.1 \qquad D(r) \nabla^2 \varphi + K(r) \varphi = 0 \quad ,$$

where $D(r)$ and $K(r)$ are step functions. In the ith region, $D(r) = D_i$, the diffusion coefficient of the material in the ith region, and $K(r) \varphi = K_i \varphi$, the rate of production of the diffusing substance in the ith region ($K_i < 0$ indicates the diffusing substance is being consumed). Assume (i) all $D_i > 0$ and $K_{n+1} = 0$; (ii) (1) $\varphi(r)$ is bounded, (2) $\lim_{r \to \infty} \varphi(r) = C_0$; (iii) at the juncture of two regions, $r = r_i$, (1) $h_i \left[\varphi(r_i^+) - \varphi(r_i^-)\right] = D_i \nabla \varphi(r_i^-)$, (2) $D_{i+1} \nabla \varphi(r_i^+) = D_i \nabla \varphi(r_i^-)$, ($h_i$ is the permeability of the membrane at the ith interface).

$\nabla^2 \varphi$ for a sphere is $\varphi''(r) + \frac{2}{r} \varphi'(r)$ using this and that $D(r) > 0$ one has that 6.1 can be written as

6.2 $$\varphi'' + \frac{2}{r}\varphi' + \frac{K(r)}{D(r)}\varphi = 0 \quad .$$

In order to employ the results of Sections 1-5, equation 6.2 is replaced by

6.3 $$L(Y) \equiv Y' + PY \equiv Y' + \begin{pmatrix} 0 & -1 \\ \frac{K(r)}{D(r)} & \frac{2}{r} \end{pmatrix} Y' = 0$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv \begin{pmatrix} \varphi(r) \\ \varphi'(r) \end{pmatrix}$.

Boundary conditions ii(1) and (2) are replaced by

6.4

(1) $Y(0)$ is bounded

(2) $$\lim_{r \to \infty} Y(r) = \begin{pmatrix} C_0 \\ 0 \end{pmatrix} ,$$

and the interface conditions (iii)(1) and (2) are replaced by

6.5 $$I_i(Y) \equiv Y(r_i^+) - \begin{pmatrix} 1 & -\frac{D_i}{h_i} \\ 0 & \frac{D_i}{D_{i+1}} \end{pmatrix} Y(r_i^-) \equiv Y(r_i^+) - B_i\, Y(r_i^-) = 0 \; .$$

It was shown in Section 5 under the assumptions that $Q(r) \equiv 0$ and that $P(r)$ was Lebesgue integrable on $[a, b]$ there exist a non-singular Γ-solution of 6.3 and 6.5 of the form

$$6.6 \qquad U(r) = U_{[i]}(r) \prod_{i-1}^{1} B_j \, U_j(r_j)$$

where $U_j(r)$ is a non-singular solution of 6.3 on the jth interval $r_{j-1} < r < r_j$ such that $U_j(r_{j-1}) = E$. It was also shown that every Γ-solution, $Y(x)$, of 6.3 and 6.5 was of the form

$$Y(r) = U(r)\, C \quad ;$$

C a constant matrix. For vector solutions C has all its columns alike and hence no confusion will arise if one understands by $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ the matrix $\begin{pmatrix} C_1 & C_1 \\ C_2 & C_2 \end{pmatrix}$. In this example, however, $P(r)$ is unbounded at the origin and is not Lebesgue integrable on any interval which has zero as a limit point. This difficulty is circumvented by modifying 6.6 in choosing $U_1(r)$ differently. Choose

$$6.7 \qquad U_1(r) = \begin{cases} \begin{pmatrix} \dfrac{\sin \alpha_1 r}{\alpha_1 r} & \dfrac{\cos \alpha_1 r}{r} \\ \left(\dfrac{\sin \alpha_1 r}{\alpha_1 r}\right)' & \left(\dfrac{\cos \alpha_1 r}{r}\right)' \end{pmatrix} & \text{if } \alpha_1 \neq 0 \\ \begin{pmatrix} 1 & \dfrac{1}{r} \\ 0 & -\dfrac{1}{r^2} \end{pmatrix} & \text{if } \alpha_1 = 0 \end{cases} ,$$

where $\alpha_1^2 = \dfrac{K_1}{D_1}$. Now to satisfy the boundary conditions 6.4 (1) gives that C must have the form

$$C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} ,$$

where $C_1 = \varphi(0)$. To satisfy condition 6.3 (2) one has that

$$\lim_{r \to \infty} U_n(r) \left(\prod_{n-1}^{1} B_j \, U_j(r_j) \right) \begin{pmatrix} C_1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_0 \\ 0 \end{pmatrix} ,$$

where C_0 is the concentration in the outside, or infinite, region. Since $U_n(r)$ is the only thing that is changing as r increases, one has

$$\begin{pmatrix} 1 & -r_{n-1} \\ 0 & 0 \end{pmatrix} \left(\prod_{n-1}^{1} B_j \, U_j(r_j) \right) \begin{pmatrix} C_1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_0 \\ 0 \end{pmatrix}.$$

Letting $T = \prod_{n-1}^{1} B_j \, U_j(r_j)$, one clearly has

$$(1, -r_{n-1}) \, (T) \begin{pmatrix} C_1 \\ 0 \end{pmatrix} = C_0$$

which always has a solution if $(1, -r_{n-1})T$ is not orthogonal to $\begin{pmatrix} C_1 \\ 0 \end{pmatrix}$.

Here, one should note that if $j \neq 1$, $U_j(r, \alpha_j)$ for this example is

$$6.8 \quad U_j(r,\alpha_j) = \begin{pmatrix} \dfrac{r_{j-1}\alpha_j \cos \alpha_j(r-r_{j-1}) + \sin \alpha_j(r-r_{j-1})}{\alpha_j\, r} & \dfrac{r_{j-1}}{\alpha_j} \quad \dfrac{\sin \alpha_j(r-r_{j-1})}{r} \\ \left(\dfrac{r-r_{j-1}}{r}\right) \dfrac{\cos \alpha_j(r-r_{j-1})}{r} - \left(r_{j-1}\,\alpha_j + \dfrac{1}{\alpha_j r}\right) \sin \alpha_j(r-r_{j-1}) & r_{j-1}\left[\dfrac{\cos \alpha_j(r-r_{j-1})}{r} - \dfrac{\sin \alpha_j(r-r_{j-1})}{\alpha_j\, r}\right] \end{pmatrix}$$

As a function of α_j alone $U(r, \alpha_j)$ is continuous at $\alpha_j = 0$. Further, each element of $U(r, \alpha_j)$ is an even function of α_j which is real valued if α_j^2 is real. If $\alpha_j = 0$ then $U_j(r, \alpha_j) = U_j(r, 0)$ and

$$6.9 \qquad U_j(r, 0) = \begin{pmatrix} 1 & \left(\dfrac{r_{j-1}^2}{r} - r_{j-1}\right) \\ 0 & \dfrac{r_{j-1}^2}{r^2} \end{pmatrix} .$$

Thus the original problem is reduced to grinding out the $n-1$ matrices $U_j(r_j)$, $j = 1, 2, \ldots, n-1$ and the $n-1$ matrices B_j and multiplying them together in the order specified in (6.6). Then, C_1 is determined so that the inner product

$$6.10 \qquad (1, -r_n)\, T \begin{pmatrix} C_1 \\ 0 \end{pmatrix} = C_0 ,$$

where $C_1 = \varphi(0)$ and $C_0 = \varphi(\infty)$.

CHAPTER II

THE HOMOGENEOUS INTERFACE PROBLEM DEPENDING UPON A PARAMETER

Section 1. Self-Adjoint and Definitely Self-Adjoint Systems

Assume that Γ is a finite set of points x_i such that on interval $[a, b]$, $a < x_1 < x_2 \ldots < x_k < b$, and consider the interface problem

1.1 $$L(Y) = Y' + PY = \lambda RY ,$$

1.2 $$I_i(Y) = Y(x_i^+) - B_i\, Y(x_i^-) = 0 \qquad x_i \text{ in } \Gamma$$

1.3 $$H(Y) = A\, Y(a) + B\, Y(b) = 0 ,$$

where λ is a complex parameter.

Definition 4. The values of λ for which the system 1.1, 1.2 and 1.3 has a Γ-solution, which is not identically zero, are said to be characteristic values of λ for system 1.1, 1.2 and 1.3.

Let $D(x)$ be defined as in Chapter I, Section 3 and let $\mathcal{P} = D^I(x)\, P\, D(x)$, $\mathcal{R} = D^I(x)\, R\, D(x)$, $\mathcal{A} = A\, D(a)$ and $\mathcal{B} = B\, D(b)$.

Theorem 4. The characteristic values of system 1.1, 1.2 and 1.3 are the same as the characteristic values for absolutely continuous

solutions of the system

1.4 $$\mathcal{L}(W) = W' + \mathcal{P} W = \lambda \mathcal{R} W$$

1.5 $$\mathcal{H}(W) = \mathcal{A} W(a) + \mathcal{B} W(b) \quad .$$

Proof. For each value of λ, Theorem 1 gives that a necessary and sufficient condition that 1.1, 1.2 and 1.3 have a Γ-solution is that system 1.4 and 1.5 have an absolutely continuous solution.

Systems of the form 1.4 and 1.5 have been investigated by a number of writers (see Reid [23]) and many interesting results are readily available for the interface problem 1.1, 1.2 and 1.3 by way of the transformation $D(x)$. One of the most interesting of these results is the idea of definitely self-adjoint systems as presented by Bliss [2] and [3].

Definition 5. A system 1.4 and 1.5 is said to be self-adjoint in the sense of Bliss if there exist a non-singular absolutely continuous matrix $\mathcal{K}$ such that

1.6 $$\mathcal{K}' - \mathcal{P}^T \mathcal{K} - \mathcal{K} \mathcal{P} = 0$$

1.7 $$\mathcal{K} \mathcal{R} + \mathcal{R}^T \mathcal{K} = 0$$

1.8 $$\mathcal{A} \mathcal{K}^I(a) \mathcal{A}^T = \mathcal{B} \mathcal{K}^I(b) \mathcal{B}^T \quad .$$

This is not the definition given by Bliss but is equivalent since Bliss [2] showed that the conditions 1.6, 1.7 and 1.8 are necessary and sufficient conditions that system 1.4 and 1.5 be self-adjoint under the definition he gave.

Definition 6. System 1.4 and 1.5 is said to be definitely self-adjoint in the sense of Bliss if the system is self-adjoint and satisfies

(1) $\mathcal{K}\mathcal{R}$ is symmetric,

(2) $\xi^T \mathcal{K}\mathcal{R}\xi$ is positive semi-definite for real vectors ξ ,

(3) $W(x) \equiv 0$ is the only solution of $L(W) = 0$ which satisfies $\mathcal{L}(W) = 0$ and $\mathcal{R}W = 0$.

(Hereafter when the terms self-adjoint and definitely self-adjoint are used it will be in the sense of Bliss as given in the above definitions.)

Definition 7. System 1.1, 1.2 and 1.3 is said to be self-adjoint if the associated system 1.4 and 1.5 is self-adjoint.

Definition 8. System 1.1, 1.2 and 1.3 is said to be definitely self-adjoint if the associated system 1.4 and 1.5 is definitely self-adjoint.

Theorem 5. Necessary and sufficient conditions that system 1.1, 1.2 and 1.3 be self-adjoint is that there exist a non-singular Γ-solution of

$$K' - P^T K - KP = 0 \tag{1.9}$$

1.10 $$B_i^T K(x_i^+) B_i - K(x_i^-) = 0$$

satisfying the further properties

1.11 $$KR + R^T K = 0$$

1.12 $$A K^I(a) A^T = B K^I(b) B^T \quad .$$

The proof of this theorem consists of showing that if $\mathcal{K}$ exist satisfying 1.6, 1.7 and 1.8, then $K = (D^T)^I \mathcal{K} D^I$ satisfies 1.9, 1.10, 1.11 and 1.12. Furthermore if K exist satisfying 1.9, 1.10, 1.11 and 1.12 then $\mathcal{K} = D^T K D$ satisfies 1.6, 1.7 and 1.8 and $\mathcal{K}$ is absolutely continuous.

<u>Theorem 6</u>. Necessary and sufficient conditions that system 1.1, 1.2 and 1.3 be definitely self-adjoint are that K be a non-singular Γ-solution of 1.9-1.12 and K satisfies the following properties:

(1)' KR is symmetric,

(2)' $\xi^T KR\xi$ is positive semi-definite, and

(3)' $Y(x) \equiv 0$ is the only Γ-solution of $L(Y) = 0$ satisfying $I_i(Y) = 0$, $H(Y) = 0$, and $RY = 0$.

<u>Proof</u>. If $\mathcal{K}\mathcal{R}$ is symmetric, substituting $\mathcal{K} = D^T K D$ and $\mathcal{R} = D^I R D$ one has

$$D^T K D\, D^I R D = D^T R^T (D^T)^I D^T K^T D \quad ,$$

which clearly gives

$$D^T \left[KR - R^T K^T \right] D = 0 \quad ,$$

or since D is non-singular

$$KR - R^T K^T = 0 \quad .$$

Therefore, KR is symmetric.

If $\mathcal{K}\mathcal{R}$ is positive semi-definite, then it follows that $F^T \mathcal{K}\mathcal{R} F$ is positive semi-definite if F is a non-singular matrix. Letting $F = D^I$ one has that

$$(D^T)^I \mathcal{K} D^I D D^I RD D^I = KR$$

and hence KR is semi-definite.

Suppose condition 3 is satisfied, then by Theorem 1 one has that $Y(x) \equiv 0$ is the only Γ-solution of $L(Y) = 0$, $I_i(Y) = 0$ and $H(Y) = 0$. It therefore remains only to show that $RY = 0$ implies $\mathcal{R} W(x) = 0$. Multiplying by D^I gives $D^I RY = 0$, but there exist a L-solution of $\mathcal{L}(W) = 0$ such that $Y = DW$. Therefore, $\mathcal{R} W = D^I R DW = 0$.

The proof of the necessity is just as obvious.

In order to indicate the nature of the conditions 1.9-1.12 and (1)'-(3)' consider the set $\mathcal{J}$ of all matrices Y which are absolutely continuous on each component of $c\Gamma$.

<u>Statement 1.</u> If K is a non-singular Γ-solution of 1.9, then for all Y_1 and Y_2 in $\mathcal{J}$ the La Grange Identity

1.13 $$L^T(Y_1)\, K\, Y_2 + Y_1^T\, K\, L(Y_2) = (Y_1^T\, K\, Y_2)'$$

holds almost everywhere on $[a, b]$.

<u>Proof.</u>

1.14 $$L^T(Y_1) = Y_1^{T'} + Y_1^T P^T$$

1.15 $$L(Y_2) = Y_2' + P\, Y_2 \quad .$$

Multiply 1.14 by KY_2 on the right and 1.15 by $Y_1^T K$ on the left and add. Then one has

$$L^T(Y_1)\, KY_2 + Y_1^T\, KL(Y_2) = Y_1^{T'}\, KY_2 + Y_1^T(P^T K + KP)Y_2 + Y_1^T KY_2' \, .$$

Since $K' = P^T K + KP$ almost everywhere on $[a, b]$ one has that

$$L^T(Y_1)KY_2 + Y_1^T\, KLY_2 = Y_1^{T'}\, KY_2 + Y_1^T K' Y_2 + Y_1^T KY_2' = (Y_1^T KY_2)' \quad .$$

Statement 2. If K satisfies 1.9 and 1.10, then for all Y_1 and Y_2 in $\mathcal{J}$ which satisfy the interface conditions 1.2, the Green's Formula

$$1.16 \qquad \mathcal{K}(s, t) = \int_s^t \left[L(Y_1)\, KY_2 + Y_1^T K\, L(Y_2) \right] dx = Y_1^T KY_2 \Big|_s^t$$

holds for all s and t such that $a \leqslant s, t \leqslant b$.

Proof. Since Y_1 and Y_2 satisfy the interface conditions 1.2 at a point x_i of Γ, one has

$$Y_1^T(x_i^+)\, K(x_i^+)\, Y_2(x_i^+) = \left[Y_1^T(x_i^-)\, B_i^T \right] K(x_i^+) \left[B_i^T\, Y_2(x_i^-) \right]$$

$$= Y_1^T(x_i^-) \left[B_i^T\, K(x_i^+)\, B_i \right] Y_2(x_i^-) \quad .$$

But since K satisfies 1.10, one has that $K(x_i^-) = B_i^T\, K(x_i^+)\, B_i$. Hence

$$Y_1^T(x_i^+)\, K(x_i^+)\, Y_2(x_i^+) = Y_1^T(x_i^-)\, K(x_i^-)\, Y_2(x_i^-)$$

for each x_i in Γ. Hence $Y_1^T KY_2$ can be extended to a continuous matrix on $[a, b]$. But since Γ is finite and $Y_1^T KY_2$ is absolutely continuous on each component of $c\Gamma$, it follows that $Y_1^T KY_2$ is an absolutely continuous matrix. Hence by a fundamental theorem of L-integrals ([14], p. 208)

$$\int_s^t (Y_1^T KY_2)' \, dx = Y_1^T KY_2 \Big|_s^t$$

and by Statement 1, then

$$\int_s^t \left(L^T(Y_1) KY_2 + Y_1^T K L(Y_2)\right) dx = Y_1^T KY_2 \Big|_s^t \quad .$$

<u>Statement 3</u>. If A and B are non-singular and K satisfies 1.9, 1.10, and 1.12, then for all Y_1 and Y_2 in $\mathscr{J}$ satisfying 1.2 and 1.3

1.17 $$\mathscr{K}(a, b) = Y_1^T KY_2 \Big|_a^b = 0 \quad .$$

<u>Proof</u>.

1.18 $$Y_1^T(b) K(b) Y_2(b) - Y_1^T(a) K(a) Y_2(a)$$

but since Y_1 and Y_2 satisfy 1.3, one has

$$Y_1^T(a) + Y_1^T(b) B^T A^{I^T} = 0$$

and

$$Y_2(a) + A^I BY_2(b) = 0 \quad .$$

Substituting into 1.18 gives

$$Y_1^T(b)\ K(b)\ Y_2(b) \ - \ Y_1^T(b)\ B^T\ A^{I^T}\ K(a)\ A^I\ BY_2(b)$$

which gives

$$Y_1^T(b)\ \left[K(b) - B^T\ A^{I^T}\ K(a)\ A^I\ B\right]\ Y_2(b) \quad .$$

Since K satisfies

$$B\ K^I(b)\ B^T \ = \ A\ K^I(a)\ A^T$$

and A and B are non-singular

$$B^{T^I}\ K(b)\ B^I \ = \ A^{T^I}\ K(a)\ A^I$$

or

$$K(b) \ - \ B^T\ A^{I^T}\ K(a)\ A^I\ B \ = \ 0 \quad .$$

Thus

$$\mathcal{K}(a,\ b) \ = \ Y_1^T\ KY_2 \Big|_a^b \ = \ 0 \quad .$$

Statement 4. If K satisfies 1.9, 1.10, 1.11 and 1.12 and A and B are non-singular, then if Y_1 and Y_2 are Γ-solutions of 1.1, 1.2 and 1.3 for different λ, say λ_1 and λ_2 respectively, then

1.19 $$\left\{Y_1, Y_2\right\} = \int_a^b (Y_1^T KRY_2)\, dx = 0 .$$

Proof. By Statement 3, one has that

1.20 $$\int_a^b \left[L^T(Y_1)\, KY_2 + Y_1^T K\, L(Y_2)\right] dx = 0 .$$

But

$$L^T(Y_1) = \lambda_1 Y_1^T R^T$$

and

$$L(Y_2) = \lambda_2 RY_2 .$$

Hence,

$$\lambda_1 \int_a^b (Y_1^T R^T KY_2)dx + \lambda_2 \int_a^b (Y_1^T KRY_2)dx = 0 ,$$

but $KR = -R^T K$. Therefore

$$(\lambda_2 - \lambda_1) \int_a^b (Y_1^T KRY_2)dx = 0 \quad .$$

By hypothesis $\lambda_1 \neq \lambda_2$ and therefore,

$$\left\{ Y_1, Y_2 \right\} = \int_a^b (Y_1^T KRY_2)dx = 0 \quad .$$

Statement 5. If the hypothesis of Statement 4 is satisfied and K and R satisfy the conditions (1)', (2)' and (3)', then all of the characteristic values of λ for the system 1.1, 1.2 and 1.3 are real.

Proof. Since all of the conditions 1.1, 1.2, and 1.3 are conditions on the columns of the matrix Y, for any λ which is a characteristic value of this system, one can choose a corresponding characteristic Γ-solution, Y, which has all its columns alike.

Suppose $\lambda = \mu_1 + i\mu_2$, $\mu_2 \neq 0$ is a characteristic value of 1.1, 1.2 and 1.3, and let $Y = U + iV$ be a corresponding characteristic Γ-solution with its columns all alike. Then $\overline{Y} = U - iV$ is also a characteristic Γ-solution of this system corresponding to $\overline{\lambda} = \mu_1 - i\,\mu_2$. Since $\lambda \neq \overline{\lambda}$, it follows from Statement 4 that

$$\left\{ Y, \overline{Y} \right\} = \int_a^b (Y^T KR\overline{Y})dx = 0$$

$$= \int_a^b \left[U^T + iV^T \right] KR \left[U - iV \right] dx$$

$$= \int_a^b (U^T KRU)dx + \int_a^b (V^T KRV)dx + i \left[\int_a^b (V^T KRU)dx - \int_a^b (U^T KRV)dx \right] .$$

But since columns of U are all alike and the columns of V are all alike, the matrix $V^T KRU$ has all its elements alike. Hence $(V^T KRU)^T = V^T KRU$ and since KR is symmetric by (1)',

$$V^T KRU = (V^T KRU)^T = U^T (KR)^T V = U^T KRV .$$

It follows then that

$$\int_a^b (U^T KRU)dx + \int_a^b (V^T KRV)dx = 0 .$$

But KR is positive by (2)' and $RU \neq 0$ by (3)'. Hence

$$\int_a^b (U^T KRU)dx > 0 \quad \text{and} \quad \int_a^b (V^T KRV)dx > 0 .$$

This is a contradiction and the hypothesis that λ was complex is untenable. Therefore, all the characteristic values of λ for system 1.1, 1.2 and 1.3 are real.

One could easily continue in this direction and find other interesting results. The work of Bliss ([2] and [3]) and of Reid

([23] , [24] and [25]) , for the two point self-adjoint and definitely self-adjoint systems, clearly implies that corresponding results for interface problems may be obtained.

Section 2. The Second Order System

Consider the second order differential equation

$$y'' + p_1 y' + p_2 y = \lambda r y \quad , \tag{2.1}$$

where p_1 , p_2 and r are Lebesgue integrable functions on the interval $[a, b]$. Suppose that $\Gamma = \{x_i\}$ $(a < x_1 < x_2 \cdots < x_k < b)$ is a finite subset of points of $[a, b]$. Suppose also that interface conditions

$$\begin{aligned} y(x_i^+) &= b_{11}^i\, y(x_i^-) + b_{12}^i\, y'(x_i^-) \\ y'(x_i^+) &= b_{21}^i\, y(x_i^-) + b_{22}^i\, y'(x_i^-) \end{aligned} \qquad i = 1, 2, \ldots, k \tag{2.2}$$

and boundary conditions

$$\begin{aligned} \alpha_{11}\, y(a) + \alpha_{12}\, y'(a) + \beta_{11}\, y(b) + \beta_{12}\, y'(b) &= 0 \\ \alpha_{21}\, y(a) + \alpha_{22}\, y'(a) + \beta_{21}\, y(b) + \beta_{22}\, y'(b) &= 0 \end{aligned} \tag{2.3}$$

are given.

One may write a system equivalent to 2.1 in matrix notation as

2.4 $$Y' + PY = \lambda RY$$

where

$$P = \begin{pmatrix} 0 & -1 \\ p_2 & p_1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \text{ and } Y = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

The interface conditions 2.2 become

2.5 $$Y(x_i^+) - B_i\, Y(x_i^-) = 0 \qquad i = 1, \ldots, k$$

where

$$B_i = \begin{pmatrix} b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i \end{pmatrix}.$$

The boundary conditions 2.3 become

2.6 $$A\, Y(a) + B\, Y(b) = 0$$

where

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

The necessary and sufficient conditions of Theorem 5 must be satisfied if this system is to be self-adjoint with the matrix K. Let

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} .$$

Then condition 1.11 gives

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Hence

$$\begin{pmatrix} r\,k_{12} & 0 \\ r\,k_{22} & 0 \end{pmatrix} + \begin{pmatrix} rk_{21} & rk_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

This gives $k_{22} = 0$ and $k_{21} = -k_{12}$. Therefore, K must have the form

$$K = \begin{pmatrix} k_1 & k_2 \\ -k_2 & 0 \end{pmatrix} .$$

Condition 1.9 gives that

$$\begin{pmatrix} k_1' & k_2' \\ -k_2' & 0 \end{pmatrix} + \begin{pmatrix} 0 & p_2 \\ -1 & p_1 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \\ -k_2 & 0 \end{pmatrix} + \begin{pmatrix} k_1 & k_2 \\ -k_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ p_2 & p_1 \end{pmatrix} = 0$$

which gives

$$k_1' = 0$$

$$k_2' = -k_1 + p_1 k_2$$

$$-k_2' = -k_1 - p_1 k_2 \quad .$$

Hence $k_1 = 0$ and K must have the form

$$K(x) = k(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where k is a scalar function satisfying

2.7 $$k' - p_1 k = 0 \quad .$$

Since for any 2×2 matrix C

$$C^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C = |C| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

one has that for each i , $i = 1, 2, \ldots, k$,

$$B_i^T K(x_i^+) B_i = |B_i| \; k(x_i^+) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Thus conditions 1.10 are satisfied if k is a Γ-solution of 2.7 and interface conditions

2.8 $$|B_i| \; k(x_i^+) - k(x_i^-) = 0 .$$

Letting $D(x)$ be defined as in Section 3 of Chapter I, one has that

2.9 $$k(x) = \frac{c_1 \, e^{\int_a^x p_1 \, dt}}{|D(x)|} ,$$

where c_1 is an arbitrary constant, satisfies 2.7 and 2.8 and is indeed a Γ-solution.

Therefore if

$$K(x) = k(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

where k is given by 2.9, then the necessary and sufficient conditions of Theorem 5 are satisfied except possibly 1.12 and no further restrictions have been placed on the functions p_1 , p_2 , and r , and only the restriction that $|B_i| \neq 0$, $i = 1, \ldots, k$, on the interface conditions.

Clearly this result is analogous to the well known theorem (Ince [11], p. 215) that every linear second order differential equation can be put in a self-adjoint form.

Condition 1.12 gives that if in the boundary conditions 2.6 A and B satisfy

$$|A|\, k(b) = |B|\, k(a) \quad ,$$

which may be written

$$2.10 \qquad |A|\, e^{\int_a^b p_1\, dt} = |B|\, |D(b)| \quad ,$$

then system 2.4, 2.5 and 2.6 is self-adjoint.

The further conditions under which system 2.4, 2.5 and 2.6 is definitely self-adjoint are given in Theorem 6 as (1)', (2)' and (3)'.

If K is given by 2.9, then,

$$KR = k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} = k \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} kr & 0 \\ 0 & 0 \end{pmatrix}$$

which is symmetric. Hence condition (1)' is satisfied. Condition (2)' requires that the quadratic form $\xi^T KR\xi$ be positive semi-definite.

$$(\xi_1, \xi_2) \begin{pmatrix} kr & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_1^2\, kr \quad .$$

Since $\xi_1^2 \geqslant 0$, one has that k and r must not have different signs on $[a, b]$. Examination of 2.9 reveals that since

$$e^{\int_a^x p_1 dt} \geqslant 0 \quad \text{for} \quad a \leqslant x \leqslant b$$

and c_1 is arbitrary, this condition becomes the product $r|D|$ must not change sign on the interval $[a, b]$. Since D is constant on each component of $c\Gamma$, this means that r cannot change sign at any point of $c\Gamma$.

Condition (3)' is satisfied since $RY = 0$, if and only if, Y has the form $\begin{pmatrix} 0 & 0 \\ y_{21} & y_{22} \end{pmatrix}$. But $y_{21} = y'_{11}$ and $y_{22} = y'_{12}$. Hence, $y_{21} = y_{22} = 0$.

Thus the conditions that system 2.4, 2.5 and 2.6 be definitely self-adjoint are that

(a) Each B_i be non-singular,

(b) A and B satisfy

$$|A| e^{\int_a^b p_1 dt} = |B| |D(b)| ,$$

and

(c) $r \cdot |D|$ always has the same sign on $[a, b]$.

Section 3. An Example

Consider a n-region 2-substance diffusion problem for a spherically symmetric geometry, in particular, a central sphere and

n-1 concentric spherical shells. In certain regions, S_1 and S_2 are being produced and consumed while in other regions, S_1 is being consumed only and S_2 is being produced and consumed. The diffusion equations for the steady state solutions are

$$D_1(r) \nabla^2 \varphi_1 - k_1(r) \varphi_1 + \lambda \gamma(x) k_2(r) \varphi_2 = 0$$

3.1

$$D_2(r) \nabla^2 \varphi_2 + k_1(r) \varphi_1 - k_2(r) \varphi_2 = 0$$

where φ_i is the flux of S_i $(i = 1, 2)$; $D_i(r) = D_{ij}$ and $k_i(r) = k_{ij}$ where D_{ij} and k_{ij} are the known positive physical constants, the subscript i denotes the diffusing substance and the subscript j denotes the region; γ is a non negative step function being zero in those regions where S_1 is not produced, and λ is a parameter.

The assumed symmetry indicates that the flux φ_i is a function of r and t , but for steady state solutions φ_i is a function of r alone. Hence the equations 3.1 can be written

$$\frac{d^2 \varphi_1}{dr^2} + \frac{2}{r} \frac{d\varphi_1}{dr} - \frac{k_1}{D_1} \varphi_1 + \lambda\gamma \frac{k_2}{D_1} \varphi_2 = 0$$

3.2

$$\frac{d^2 \varphi_2}{dr^2} + \frac{2}{r} \frac{d\varphi_2}{dr} + \frac{k_1}{D_2} \varphi_1 - \frac{k_2}{D_2} \varphi_2 = 0 \quad .$$

Let r_s be the radius of the composite sphere then the boundary conditions are

(i) (1) $\varphi_i(r, \lambda)$ is bounded $0 \leqslant r \leqslant r_s$, $i = 1, 2$

(2) $\varphi_1(r_s, \lambda) = \varphi_2(r_s, \lambda) = 0$.

(ii) At the juncture of two regions $r = r_j$,

(1) $\varphi_1(r_j^+) = \varphi_1(r_j^-) - \frac{D_{1j}}{h_{1j}} \varphi_1'(r_j^-)$

(2) $D_{1j+1} \varphi_1'(r_j^+) = D_{1j} \varphi_1'(r_j^-)$

(3) $\varphi_2(r_j^+) = \varphi_2(r_j^-) - \frac{D_{2j}}{h_{2j}} \varphi_2'(r_j^-)$

(4) $D_{2j+1} \varphi_2'(r_j^+) = D_{2j} \varphi_2'(r_j^-)$,

where h_{ij} is the permeability of the <u>jth</u> membrane to the <u>ith</u> substance, $i = 1, 2$ and $j = 1, 2, \ldots, n-1$.

System 3.2 is replaced by the system

$$3.3 \qquad L(Y) = Y' + P(x,\lambda) = Y' + \begin{pmatrix} 0 & -1 & 0 & 0 \\ -a^2 & 0 & \lambda\gamma c^2 & 0 \\ 0 & 0 & 0 & -1 \\ d^2 & 0 & -b^2 & 0 \end{pmatrix} Y = 0$$

where the <u>ℓth</u> column of Y is $y_{1\ell} = r\varphi_{1\ell}$, $y_{2\ell} = (r\,\varphi_{1\ell})'$, $y_{3\ell} = r\,\varphi_{2\ell}$, and $y_{4\ell} = (r\,\varphi_{2\ell})'$, $(\varphi_{1\ell}, \varphi_{2\ell})$, $\ell = 1, 2, 3, 4$, being four pairs of functions satisfying system 3.2. Where also on the <u>hth</u> interval the step function $(a^2, b^2, c^2, d^2) = (a_h^2, b_h^2, c_h^2, d_h^2) =$

$\left(\frac{k_{1h}}{D_{1h}}, \frac{k_{2h}}{D_{2h}}, \frac{k_{2h}}{D_{1h}}, \frac{k_{1h}}{D_{2h}}\right)$, and γ is the step function in system 3.2. Note that $a^2b^2 = c^2d^2$. The boundary conditions become

$$(1) \qquad y_1(0) = y_3(0) = 0$$

3.4

$$(2) \qquad y_1(r_s) = y_3(r_s) = 0 \quad .$$

The new interface conditions are found by examining the transformation

$$Y = R\,\Phi = \begin{pmatrix} r & 0 & 0 & 0 \\ 1 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 1 & r \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_1' \\ \varphi_2 \\ \varphi_2' \end{pmatrix} \quad .$$

Thus one has

$$Y(r_j^+) = R(r_j)\,\varphi(r_j^+) = R(r_j)\,A_j\,\varphi(r_j^-) = R(r_j)\,A_j\,R^I(r_j)\,Y(r_j^-)$$

and on substituting for $R(r_j)\,A_j\,R^I(r_j) = B_j$

3.5 $$Y(r_j^+) = B_j\,Y(r_j^-) \quad , \qquad j = 1, \ldots, n-1 \; .$$

Each matrix B_j , written in 2×2 blocks, has the form

$$\begin{pmatrix} B_{1j} & 0 \\ 0 & B_{2j} \end{pmatrix}$$

where

$$B_{ij} = \begin{pmatrix} 1 + \frac{1}{r_j}\frac{D_{ij}}{h_{ij}} & -\frac{D_{ij}}{h_{ij}} \\ \frac{1}{r_j} + \frac{1}{r_j^2}\frac{D_{ij}}{h_{ij}} - \frac{1}{r_j}\frac{D_{ij}}{D_{ij+1}} & -\frac{1}{r_j}\frac{D_{ij}}{h_{ij}} + \frac{D_{ij}}{D_{ij+1}} \end{pmatrix} \quad i = 1, 2 \ .$$

For each λ , let $U(r, \lambda)$ be the non-singular Γ-solution of 3.3 and 3.5 which has the form

$$3.6 \qquad U(r,\lambda) = U_{[i]}(r, \lambda) \prod_{i-1}^{1} B_j \, U_j(r_j, \lambda) \quad ,$$

where $[i]$ denotes the smallest i such that $r < r_i$, $0 = r_0 < r_1 < \ldots < r_n = r_s$ and where $U_j(r, \lambda)$ is on the <u>jth</u> region, $r_{j-1} \leqslant r \leqslant r_j$, an absolutely continuous solution of $Y' + P(r, \lambda)Y = 0$ and $Y(r_{j-1}, \lambda) = E$.

Let

$$\alpha_j^2 = \frac{a_j^2 + b_j^2}{2} + \sqrt{\left(\frac{a_j^2 - b_j^2}{2}\right)^2 + \lambda \gamma_j a_j^2 b_j^2} \quad ,$$

$$\beta_j^2 = \frac{a_j^2 + b_j^2}{2} - \sqrt{\left(\frac{a_j^2 - b_j^2}{2}\right) + \lambda \gamma_j a_j^2 b_j^2} \quad ,$$

and

$$S_{j\sigma} = \begin{bmatrix} \cosh \sigma_j (r - r_{j-1}) & \dfrac{\sinh \sigma_j (r - r_{j-1})}{\sigma_j} \\ \sigma_j \sinh \sigma_j (r - r_{j-1}) & \cosh \sigma_j (r - r_{j-1}) \end{bmatrix} .$$

Then $U_j(r, \lambda)$ has the form

$$3.7 \quad U_j(r,\lambda) = \frac{1}{\alpha_j^2 - \beta_j^2} \begin{bmatrix} (\alpha_j^2 - a_j^2) S_{j\beta} - (\beta_j^2 - a_j^2) S_{j\alpha} & \lambda \gamma_j \beta_j^2 (S_j - S_{j\alpha}) \\ d^2 (S_{j\beta} - S_{j\alpha}) & (\alpha_j^2 - b_j^2) S_{j\beta} - (\beta^2 - b_j^2) S_{j\alpha} \end{bmatrix} .$$

The general Γ-solution of 3.3 and 3.5 is of the form

$$3.8 \qquad Y(r, \lambda) = U(r, \lambda) C \quad ,$$

where C is an arbitrary matrix. If each column of $Y(r, \lambda)$ is to be a solution to 3.3, 3.4 and 3.5, then boundary condition 3.4 (1) gives that the first row and third row of C must be (0000) because $U(0, \lambda) = E$. The second boundary condition gives the two equations

$$U^1(r_s, \lambda) = (0000)$$

and

$$U^3(r_s, \lambda) = (0000) \quad ,$$

where U^i denotes the <u>ith</u> row of the matrix $U(r_s, \lambda)$. Considering both conditions gives a set of equations

$$3.9 \qquad \begin{aligned} u_{12}(r_s, \lambda)\, C_{2j} + u_{14}(r_s, \lambda)\, C_{4j} &= 0 \\ u_{32}(r_s, \lambda)\, C_{2j} + u_{34}(r_s, \lambda)\, C_{4j} &= 0 \quad , \end{aligned} \qquad j = 1, 2, 3, 4$$

which have non trivial solutions if and only if the determinant

$$3.10 \qquad \begin{vmatrix} u_{12}(r_s, \lambda) & u_{14}(r_s, \lambda) \\ u_{32}(r_s, \lambda) & u_{34}(r_s, \lambda) \end{vmatrix} = 0 \quad .$$

Furthermore, if not all u_{ij} of this determinant are zero, then the solutions are linearly dependent, i.e., scalar multiples of the solution (C_{21}, C_{41}) .

The equation 3.10 is then the characteristic equation of the system 3.3, 3.4 and 3.5 for the parameter λ . One observes that this

characteristic equation involves functions of each interface r_i and each coefficient $(a_i^2, b_i^2, c_i^2$ and $d_i^2)$ and the step function γ for $i = 1, 2, \ldots, n-1$.

One may also arrive at equation 3.3 if one considers a diffusion problem on an infinite composite slab of finite thickness made up of n slabs of different materials. The only changes necessary to consider this problem from what has been established for the spherical problem would be in the interface conditions. The new interface conditions would be

$$I_i(Y) = Y(x_i^+) - \underline{B}_i\, Y(x_i^-) = 0 \tag{3.11}$$

where

$$\underline{B}_i = \begin{pmatrix} 1 & -\dfrac{D_{1i}}{h_{1i}} & 0 & 0 \\ 0 & \dfrac{D_{1i}}{D_{1i+1}} & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{D_{2i}}{h_{2i}} \\ 0 & 0 & 0 & \dfrac{D_{2i}}{D_{2i+1}} \end{pmatrix} . \tag{3.12}$$

The form of the general Γ-solution 3.6 would only be changed in that the B_i's would be replaced by $\underline{B}_i$'s of 3.12. The characteristic

equation again may be found as before. In the next section, conditions are found under which a system of the type 3.3, 3.4 and 3.5 is self-adjoint and definitely self-adjoint as defined in Section 1.

Section 4. Definitely Self-Adjoint Cases for the Example

If one writes the interface problem of the preceding section in the form considered in Section 1 of this chapter, he has that

$$4.1 \qquad P(x) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ d^2 & 0 & -b^2 & 0 \end{pmatrix}, \qquad R(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma c^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and that the boundary conditions 3.4 can be written as

$$4.2 \qquad A\,Y(0) + B\,Y(r_s) = 0$$

where

$$4.3 \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The interface conditions 3.5 (or 3.12) are already in the form

considered in Section 1 of this chapter.

Assume (i) the ratios $\frac{D_1}{D_2}$ and $\frac{k_1}{k_2}$ are constant on $0 \leqslant r \leqslant r_s$ and (ii) $\frac{D_{1j}}{h_{1j}} = \frac{D_{2j}}{h_{2j}}$ for each j.

The object of this section is to show that under the assumptions (i) and (ii) that the interface problem 3.3, 3.4 and 3.5 is definitely self-adjoint, with the matrix K given by

$$4.4 \qquad K(r) = k(r) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -g \\ 1 & 0 & g & 0 \end{pmatrix}$$

where $k(r)$ is a step function to be determined and $g = \frac{a^2 - b^2}{d^2}$.

Note that under assumption (i) g is a constant on $0 \leqslant r \leqslant r_s$ for

$$g(x) = \frac{a^2 - b^2}{d^2} = \frac{\frac{k_1}{D_1} - \frac{k_2}{D_2}}{\frac{k_1}{D_2}} = \frac{D_2}{D_1} - \frac{k_2}{k_1} .$$

Also $K' = 0$ and simply by multiplying one may check that $P^T K + KP = 0$ and $R^T K + KR = 0$. Therefore, conditions 1.11 and 1.9 are satisfied. The assumptions (i) and (ii) give that each matrix B_j of the interface conditions has $B_{1j} = B_{2j}$ and, therefore, has the form

4.5
$$\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$$

where C is a 2×2 block. Observe that since $C^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C = |C| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ one has

4.6
$$\begin{pmatrix} C^T & 0 \\ 0 & C^T \end{pmatrix} \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & -g \\ 1 & 0 & g & 0 \end{array}\right] \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} = |C| \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & -g \\ 1 & 0 & g & 0 \end{array}\right].$$

Hence, condition 1.10 requires that

$$B_j^T K(r_j^+) B_j - K(r_j^-) = 0$$

or by 4.6

$$|B_{1j}| \quad K(r_j^+) - K(r_j^-) = 0$$

which gives that if $|B_{1j}| \, k(r_j^+) = k(r_j^-)$ then condition 1.10 is satisfied. Now let $k(r_1^-) = 1$ then $k(r)$ is determined on $c\Gamma$. Note that since $|B_{1j}| > 0$, $k(r) > 0$ on $c\Gamma$.

Now since

$$4.7 \qquad K^{I}(r) = \begin{pmatrix} 0 & -g & 0 & 1 \\ g & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{k}(r)$$

and by an observation similar to 4.6 one has that since A and B are of the form 4.5 that

$$A\, K^{I}(a)\, A^{T} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} K^{I}(a) = 0$$

and

$$B\, K^{I}(b)\, B^{T} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} K^{I}(b) = 0 \quad .$$

Hence condition 1.12 is satisfied and by Theorem 5, system 3.3, 3.4 and 3.5 is self-adjoint with the matrix K given by 4.4.

Now to show that 3.3, 3.4 and 3.5 is definitely self-adjoint, one may show that conditions (1)', (2)' and (3)' of Theorem 6 are satisfied.

Observe that

$$KR = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma k c^{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is symmetric and positive semi-definite since γ , $k(r)$, and c^2 are non-negative. Hence, condition (1)' and (2)' are satisfied.

Suppose that there exist a matrix $Y \neq 0$ satisfying $Y' + P(x)\,Y = 0$, both 3.4 and 3.5, and $RY \equiv 0$ on $0 \leqslant r \leqslant s$.

If $RY \equiv 0$ then $y_{3j} \equiv 0$ on any region for which $\gamma \neq 0$. Since

$$Y' + \begin{pmatrix} 0 & -1 & 0 & 0 \\ -a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ d^2 & 0 & -b^2 & 0 \end{pmatrix} Y = 0$$

one has that

$$\begin{aligned} y'_{1j} &= y_{2j} \\ y'_{2j} &= a^2\, y_{1j} \\ y'_{3j} &= y_{4j} \\ y'_{4j} &= -d^2\, y_{1j} + b^2\, y_{3j} \end{aligned} \qquad j = 1, 2, 3, 4 \quad . \tag{4.8}$$

On any region where $y_{3j} \equiv 0$, the third equation above gives that $y_{4j} \equiv 0$, and the fourth equation that $y_{1j} \equiv 0$. If $y_{1j} \equiv 0$ on a region, the first equation gives that $y_{2j} \equiv 0$. Thus one has that if $RY \equiv 0$, then on any region where $\gamma \neq 0$, $Y \equiv 0$. But Y satisfies 3.5 and each B_i is non-singular. Hence $Y \equiv 0$ on $0 \leqslant r \leqslant r_s$. This is a contradiction and $RY \equiv 0$ implies that $Y \equiv 0$. Hence Condition (3)' is satisfied and by Theorem 6, system 3.3, 3.4 and 3.5 is

CHAPTER III

DIFFERENTIAL SYSTEMS WITH INTERFACE CONDITIONS AT INFINITE SUBSETS OF $[a, b]$

The methods and results of Chapters I and II are based on the assumption that the set Γ is finite. Since most of the results follow from applications of Theorem I Chapter I and since the proof of this theorem depends only on Γ being a set of measure zero, one is led quite naturally to consider the possibility of Γ being infinite.

Definition 9. The statement that either of the ordered pairs $(\Gamma, \mathcal{B})$ or $(\mathcal{B}, \Gamma)$ is a set of interface conditions on $[a, b]$ means Γ is a subset of $[a, b]$ and $\mathcal{B}$ is a set of non-singular matrices B_α such that for each x_α in Γ there is one and only one B_α in $\mathcal{B}$.

Definition 10. The statement that a matrix $W(x)$ satisfies the interface conditions $(\mathcal{B}, \Gamma)$ means that at each point x_α of Γ the right and left hand limits $W(x_\alpha^+)$ and $W(x_\alpha^-)$ exist and satisfy the relations

$$W(x_\alpha^+) - B_\alpha W(x_\alpha^-) = 0 \quad . \tag{1}$$

Definition 11. The statement that a matrix $W(x)$ satisfies the interface conditions $(\Gamma, \mathcal{B})$ means that at each point x_α of Γ the right and left limits $W(x_\alpha^+)$ and $W(x_\alpha^-)$ exist and satisfy the

definitely self-adjoint with the matrix K given by 4.4 where k is a step function which is constant on each region and satisfies

$$k(r_i^+) = \frac{1}{|B_{1i}|} k(r_i^-)$$

at each interface r_i.

If one starts out to find K by imposing conditions 1.9, 1.10, 1.11 and 1.12, he finds that on those regions for which $\gamma \neq 0$, that K must have the form 4.4 if condition 1.9 and 1.11 are satisfied. If one then assumes that K has this form on all regions, then he is led to assumptions (i) and (ii) in order that condition 1.10 may be satisfied. Clearly, it is not necessary, in general, that K have the same form on each subinterval since on these intervals for which $\gamma = 0$, the matrix $R \equiv 0$, and condition 1.11 places no restriction on K. However, if K does not have the same form on each interval, then one must concern himself with the arrangement of the regions for which $\gamma = 0$ and those for which $\gamma \neq 0$.

The assumptions (i) and (ii) while quite restrictive are plausible. In fact, the conditions that $\frac{D_1}{D_2}$ be a constant and that $\frac{D_{1j}}{h_{1j}} = \frac{D_{2j}}{h_{2j}}$ are used by Weinberg [36] and Rashevsky [20] and the condition that $\frac{k_1}{k_2}$ be a constant is just that the equilibrium constant for the reaction $S_1 \leftrightharpoons S_2$ be the same in each region.

relation

$$W(x_\alpha^+) - W(x_\alpha^-)B_\alpha = 0 \quad . \tag{2}$$

<u>Definition 12.</u> The interface conditions $(\mathcal{B}, \Gamma)$ $[\text{or } (\Gamma, \mathcal{B})]$ are said to be of Type I on $[a, b]$ if Γ is a closed and countable subset of $[a, b]$ and on $c\Gamma$ there exist a matrix $D(x)$ having the following properties:

(i) $D'(x) = 0$ <u>everywhere on</u> $c\Gamma$;

(ii) $D(x_\alpha^+)$ <u>and</u> $D(x_\alpha^-)$ <u>exist and are finite for each point</u> x_α <u>in</u> Γ;

(iii) $|D(x)| \neq 0$ <u>on</u> $c\Gamma$;

(iv) $|D(x_\alpha^+)|$ <u>and</u> $|D(x_\alpha^-)|$ <u>are both different from zero for each</u> x_α <u>in</u> Γ; <u>and</u>

(v) $Y(x) = D(x)\,W(x)$ <u>or</u> $[(v)'\ \ Z(x) = W(x)\,D(x)]$ <u>satisfies the interface conditions</u> $(\mathcal{B}, \Gamma)$ $[(\Gamma, \mathcal{B})]$ <u>if</u> $W(x)$ <u>is continuous on</u> $[a, b]$.

Examination of the $D(x)$ as defined in Section 3 of Chapter I reveals that it satisfies all of the properties (i)-(v) of Definition 12. Γ was assumed to be finite, hence, closed and countable. Therefore, the interface conditions considered in Chapters I and II are of Type I on $[a, b]$.

<u>Lemma 1.</u> If $G(x)$ is a continuous matrix and $G(x)$ is a constant on each component of $c\Gamma$, Γ a closed and countable subset

of the interval $[a, b]$, then $G(x)$ is a constant on $[a, b]$.

<u>Proof</u>. Since Γ is closed, $c\Gamma$ is the sum of a countable number of non-overlapping open intervals. On each of these intervals $G(x)$ is constant, hence $G(x)$ takes on only a countable number of values on $c\Gamma$. Since Γ is countable, it follows that $G(x)$ takes on only a countable number of values on $[a, b]$. If $G(x)$ were not a constant then $G(x_1) \neq G(x_2)$ for some pair of points (x_1, x_2) on $[a, b]$, but $G(x)$ is continuous hence $G(x)$ must take on all values between $G(x_1)$ and $G(x_2)$. This set of values is uncountable. This is a contradiction, therefore $G(x)$ is a constant.

<u>Lemma 2</u>. $(\mathcal{B}, \Gamma)$ are of Type I on $[a, b]$ if and only if $(\Gamma, \mathcal{B}^I)$ are of Type I on $[a, b]$.

The proof of this lemma consists of showing that if $S(x)$ satisfies (i)-(iv) then so does $S^I(x)$, and that if S satisfies either of the relations (1) or (2) that $S^I(x)$ satisfies the other.

By properties (i) and (iii), $S(x)$ is a constant on each component of $c\Gamma$ and $S^I(x)$ exist and satisfies properties (i) and (iii). By property (ii) and (iv) and the continuity of the function $\Phi(A) = A^I$, which is defined and continuous on the set of all non-singular matrices, $S^I(x)$ has properties (ii) and (iv).

Suppose that $S(x)$ satisfies $(\mathcal{B}, \Gamma)$. Then at each point x_α of Γ one has from (v) by letting $W(x) = E$, $S(x_\alpha^+) - B_\alpha S(x_\alpha^-) = 0$. But it has been shown that each of the matrices are non-singular. Hence $S^I(x_\alpha^+) - S^I(x_\alpha^-) B_\alpha^I = 0$.

Had one supposed that $S(x)$ satisfied $(\Gamma, \mathcal{B}^I)$, then the roles of $S^I(x)$ and $S(x)$ are interchanged. Therefore $(\Gamma, \mathcal{B}^I)$ of Type I implies $(\mathcal{B}, \Gamma)$ is of Type I.

<u>Lemma 3</u>. Every matrix $S(x)$ satisfying properties (i) and (ii) of definition 12 is Lebesgue integrable.

<u>Proof</u>. Property (i) implies that $S(x)$ is continuous on $c\Gamma$. Property (ii) then implies that $S(x)$ is bounded and has discontinuities of the first kind only. Such matrices are Riemann integrable and Lebesgue integrable (Hobson [10] p. 468).

<u>Lemma 4</u>. If M and N are bounded Lebesgue integrable matrices and P is Lebesgue integrable, then the matrix MPN is Lebesgue integrable.

<u>Proof</u>. Each element is of the form $\sum_{k=1}^{n} \sum_{\ell=1}^{n} m_{i\ell}\, p_{\ell k}\, n_{kj}$ where each term $m_{()}\, p_{()}\, n_{()}$ is the product of two bounded L-integrable functions and an L-integrable function.

Since the product of two L-integrable functions is L-integrable if one of them is bounded ([14] p. 127) it follows that each element is a sum of L-integrable functions, hence is L-integrable. Therefore, MPN is L-integrable.

<u>Theorem 7</u>. If $(\mathcal{B}, \Gamma)$ is of Type I on $[a, b]$ and P and Q are Lebesgue integrable matrices on $[a, b]$ then the system

$$L(Y) = Y' + PY = Q \tag{3}$$

has a Γ-solution satisfying $(\mathcal{B}, \Gamma)$.

Proof. Since $(\mathcal{B}, \Gamma)$ is of Type I, there exist a $D(x)$ with properties (i)-(v), and D^I exist and satisfies properties (i)-(iv) and (v)'. Consider the system

$$(4) \qquad \mathcal{L}(W) = W' + D^I PDW = D^I Q \quad .$$

Lemma 3 then gives that D^I and D are bounded and Lebesgue integrable. Hence Lemma 4 gives $D^I PD$ and $D^I Q$ are L-integrable. The fundamental existence theorem gives that $\mathcal{L}(W) = D^I Q$ has an absolutely continuous solution $W(x)$. Theorem 1 of Chapter I now applies to give that (3) has a Γ-solution with the character of D, i.e. $Y(x) = DW$. Property (v) gives that $Y(x)$ satisfies $(\mathcal{B}, \Gamma)$.

Theorem 8. Every Γ-solution $Y(x)$ of (3) that satisfies $(\mathcal{B}, \Gamma)$ has the character of D, i.e. is of the form $Y(x) = D(x)\,\hat{Y}(x)$ where $\hat{Y}$ is absolutely continuous on $[a, b]$.

Proof. Let $Z(x) = D^I Y$. Then since D^I satisfies $(\Gamma, \mathcal{B}^I)$ and Y satisfies $(\mathcal{B}, \Gamma)$ one has $Z(x_\alpha^+) = D^I(x_\alpha^+)\, Y(x_\alpha^+) = D^I(x_\alpha^-) B_\alpha^I B_\alpha\, Y(x_\alpha^-) = Z(x_\alpha^-)$. Let

$$\hat{Z}(x) = \begin{cases} Z(x) & \text{on } c\,\Gamma \\ Z(x_\alpha^+) & \text{on } \Gamma \end{cases}$$

and note that $\hat{Z}(x)$ is continuous, and $\mathcal{L}(\hat{Z}) = \mathcal{L}(D^I Y) = D^I L(Y) = D^I Q$ holds almost everywhere on $[a, b]$.

If $\hat{Z}(x)$ were absolutely continuous the proof would be complete. Suppose $\hat{Z}(x)$ is not absolutely continuous and let $W(x)$ be an absolutely continuous solution of $\mathcal{L}(W) = D^I Q$ and $W(c) = \hat{Z}(c)$, $a \leqslant c \leqslant b$. Let $U = W - \hat{Z}$ and note that U satisfies the system $\mathcal{L}(U) = 0$ and $U(c) = 0$. The homogeneous system $\mathcal{L}(W) = 0$ has a non-singular absolutely continuous solution $\mathcal{W}(x)$ and $\mathcal{W}^I(x)$ is a non-singular absolutely continuous solution of $\mathcal{M}(V) = V' - VD^I PD = 0$. Now consider $\mathcal{W}^I U$. One has that

$$
\begin{aligned}
(\mathcal{W}U)' &= \mathcal{W}^I U' + \mathcal{W}^{I\prime} U \\
&= \mathcal{W}^I U' + \mathcal{W}^I D^I PDU \\
&= \mathcal{W}^I \mathcal{L}(U) \\
&= 0
\end{aligned}
$$

almost everywhere on $[a, b]$.

Since $\mathcal{W}^I$ and U are absolutely continuous on each component of $c\Gamma$, $\mathcal{W}^I U$ is a constant on each component of $c\Gamma$. But $\mathcal{W}^I U$ is continuous on $[a, b]$. Therefore $\mathcal{W}^I U$ is a constant matrix on $[a, b]$.

Since $U(c) = 0$, U must be the zero matrix. Therefore $\hat{Z}(x) = W(x)$ and $\hat{Z}$ is absolutely continuous. Hence Y has the character of D since $Y = D\hat{Z}$.

In the interface problems of Chapters I and II, one may replace the interface conditions $I_i(Y)$ by $(\mathcal{B}, \Gamma)$ and if $(\mathcal{B}, \Gamma)$ is of Type I on $[a, b]$ then all of the results obtained in Sections 2, 3 and 4 of Chapter I and Sections 1 and 2 of Chapter II are valid for the new interface problem. The results of Section 5 of Chapter I depend upon the finiteness of Γ hence do not extend to the more general interface problem without modification.

The condition that Γ be countable comes about for two different reasons. First by requiring that Γ-solutions have both left and right limits at each point of $[a, b]$ one implies that these limits can differ only at a countable (see Hobson [10] p. 304) subset of $[a, b]$. The second reason is found in Lemma 1. Since this lemma is not true if Γ is uncountable, Theorem 8 does not hold and as a result the interface condition would no longer determine the character of the solution.

If the condition that Γ be closed is removed, then one would have three cases to consider. Letting $\overline{\Gamma}$ denote the closure[6] of Γ, one has Case I: $\overline{\Gamma}$ is countable; Case II: $\overline{\Gamma}$ is uncountable but dense in no interval; and Case III: $\overline{\Gamma}$ is uncountable and contains some subinterval of $[a, b]$. The first case might be reduced to Type I by letting the matrix corresponding to each point of $(\overline{\Gamma} - \Gamma)$

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[6] By the closure of a point set Γ one understands a point set consisting of the points of Γ and the limit points of Γ.

be E , the identity matrix. Case II would require something more than just interface conditions to determine the character of the solutions, and Case III would require a different definition of Γ-solution since all of the components of $c\Gamma$ may now be degenerate.

SUMMARY

This paper establishes that differential systems with interface conditions may be studied by investigating an associated system without such conditions, where the matrices involved in each system have the same dimension. The association gives a one to one correspondence between the discontinuous solutions of the interface problem and the absolutely continuous solutions of the associated system. As a consequence of this correspondence, one has that theoretical results for boundary value problems with absolutely continuous solutions imply corresponding results for associated interface problems. Among the results established by this method are existence and uniqueness of the solutions, the Green's matrix and its properties and adjoint and self-adjoint relationships.

A formula (5.7) is developed for finding the general solution of homogeneous systems with interface conditions. It does not depend upon the specific form of the interface conditions, hence one may treat conditions that require some, all or none of the elements of the solution matrix Y to be continuous at the interface in exactly the same manner. The effectiveness of this approach is illustrated by solving a steady state, n-region, one substance diffusion problem and a characteristic value, or eigenvalue problem for a n-region, two substance diffusion problem. These examples also serve to illustrate that the results of this paper have applications to the composite boundary value problems of heat conduction, chemical diffusion, potential theory, vibration theory and nuclear reactor theory.

Sangren [28] has studied the single nth order equation with the point of view of developing expansion theorems and applications to the composite boundary value problems mentioned above. It is possible using the method of this paper to obtain generalizations of his results for the matrix differential systems.

Differential systems with interface conditions at a finite set of points and two point boundary conditions are a special case of the k-point boundary value problem investigated by Mansfield and others (see Whyburn [34] for references) and might have been studied from this viewpoint. However, the method used here avoids the artificial limitation that k be finite which is imposed by the methods used in these studies of the k-point boundary problem.

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