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ON THE FUNDAMENTAL EIGENVALUE OF A SOLID

Herbert Goertzel

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MATHEMATICS PANEL

ON THE FUNDAMENTAL EIGENVALUE OF A SOLID

Herbert Goertzel

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CHAPTER I

INTRODUCTION

Consider a plane region D as the equilibrium position of a homogeneous and uniformly stretched elastic membrane fixed along the boundary, C, of D. If λ is the frequency of the gravest proper tone of the membrane and if a factor which depends on the physical properties of the membrane is dropped, then λ is a purely geometric quantity.[1] The problem of determining λ is one of the classical problems of mathematical physics. It may be stated as follows.

Given a plane region D it is required to find the smallest positive constant λ^2 for which there exists a function f defined on D such that f is continuous, f vanishes on the boundary, C, of D and f satisfies the differential equation

$$\nabla^2 f + \lambda^2 f = 0 \quad ,$$

on D, ∇^2 being the Laplace operator. There is no need, however, to require D to be a plane region. If D is a region in Euclidean n-space, λ^2 may be defined in an analogous fashion.

In Chapter II several properties of λ^2 are discussed. The major tool of the chapter is a geometric operation called Steiner symmetrization

---

[1] G. Polya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, (Princeton, Princeton University Press, 1951), p. 2.

invented by Jacob Steiner. This operation is shown to transform a region D into another region whose fundamental eigenvalue is not greater than that of D and whose volume is equal to that of D . Using this result it is shown, for a large class of regions, that of all such regions with a given n-dimensional volume the hypersphere has the smallest fundamental eigenvalue. Other properties of the fundamental eigenvalue are discussed though in less detail.

Chapter III deals with the numerical results obtained in the solution of specific problems of the above type. The results of Chapter II are used in obtaining bounds for the fundamental eigenvalue. A method of computing the fundamental eigenvalue in a case where the analytic solution is not known is described and the results of the computation are given.

If the membrane is not homogeneous but is composed of two homogeneous portions the fundamental eigenvalue may be defined in a similar fashion. Two specific cases are discussed in some detail, the case of concentric spheres and the case of three semi-infinite slabs. For these cases the fundamental eigenvalue was computed as a function of the relative size of the regions and of the matching constant at the interface.

CHAPTER II

A MINIMAL PROPERTY OF THE SPHERE

In this chapter several properties of Steiner symmetrization and the fundamental eigenvalue of a solid are developed. They, in turn, lead to a minimal property of the sphere which is stated and proven in Section VII.

Section I. Notation

Throughout this chapter and the next the following notation is used.

I.1.1 E_n represents an Euclidean space of n dimensions.

I.1.2 $X = (x_1, \ldots, x_n)$ represents a point in E_n with rectangular coordinates $x_1, \ldots, x_n$.

If $f(x_1, \ldots, x_n)$ is a function defined in E_n, and if $1 \leqslant i \leqslant n$, then

I.1.3 f_{x_i} represents $\dfrac{\partial f}{\partial x_i}$,

I.1.4 $(\nabla f)^2$ represents $\sum_1^n f_{x_i}^2$,

I.1.5 $\nabla^2 f$ represents $\sum_1^n (f_{x_i})_{x_i}$

I.1.6 $\int_B f \, dV_n$ represents $\int \ldots \int_B f \, dx_1 \ldots dx_n$.

If B is a body in E_n (cf. Definition V.1), then

I.1.7 $V_n(B)$ represents the volume of B,

I.1.8 $A_{n-1}(B)$ represents the Minkowski surface area of B (cf. Definition V.6), and

I.1.9 $B^* = S_{E_{n-1}}(B)$ denotes that B^* is the image of B under Steiner symmetrization (cf. Definition V.2) with respect to E_{n-1}, an $(n-1)$-dimensional subspace of E_n.

If B is a solid in E_n (cf. Definition II.1) then

I.1.10 $\mu(B)$ represents the fundamental eigenvalue of B (cf. Definition II.3).

I.1.11 $X = \hat{x}\left[x \text{ has property } P\right]$ signifies that X is the set of all elements, x, such that x has property P. If X and Y are pointsets in E_n, then;

I.1.12 $x \in X$ signifies that x is an element of X,

I.1.13 $x \notin X$ signifies that x is not an element of X.

I.1.14 $Y \subset X$ signifies Y is a subset of X,

I.1.15 $Y \not\subset X$ signifies Y is not a subset of X,

I.1.16 $X \cap Y = \hat{z} \left[z \epsilon X \text{ and } z \epsilon Y \right]$,

I.1.17 $X \cup Y = \hat{z} \left[z \epsilon X \text{ or } z \epsilon Y \right]$,

I.1.18 $X - Y = \hat{z} \left[z \epsilon X \text{ and } z \notin Y \right]$, and

I.1.19 $d(X, Y)$ denotes the distance between X and Y .

Section II. The Equivalence of the Variational and the Boundary Value Problem Definitions of the Fundamental Eigenvalue of a Solid

For the purposes of this paper the most useful definition of the fundamental eigenvalue is the one provided by the Calculus of Variations. In practise, however, the more usual definition is that of a boundary value problem. This section establishes, to the extent needed for the later sections of this chapter, the equivalence of these definitions.

Definition II.1. By a solid is meant a closed set B such that the interior of B is non-empty and connected and whose boundary is a surface[1] S consisting of a finite number of surfaces each of which is piecewise smooth.

Definition II.2. A function f is piecewise smooth on a solid B means f is continuous and there exists a finite number of smooth surfaces S_k which subdivide B into a finite number of sub-solids B_j such that f_{x_i} , for $1 \leqslant i \leqslant n$, is continuous on the interiors of the B_j's and coincide with a continuous function on each B_j .

---

[1] Tibor Rado, Length and Area, (American Mathematical Society, Colloq. Vol. XXX), Chapt. 3, pp. 116-133.

<u>Definition II.3</u>. If B is a solid in E_n with surface S, then by the fundamental eigenvalue of B one means the smallest positive number, $\mu(B)$, for which there exists a function f defined on B such that

(i) f is piecewise smooth on B,

(ii) f vanishes on S, and

(iii) in the interior of B, f satisfies the differential equation

II.3.1 $$\nabla^2 f + \mu(B) f = 0 ,$$

then f is called an eigenfunction associated with $\mu(B)$.

<u>Definition II.4</u>. If B is a solid in E_n with surface S and if

II.4.1 $$F = \hat{f} \left[f \text{ is piecewise smooth on } B , \; f(S) = 0 , \text{ and } \int_B f^2 \, dV_n \text{ is positive} \right] ,$$

then $\lambda(B)$ is defined by

II.4.2 $$\lambda(B) = \inf_{f \in F} \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} .$$

Theorem II.5. If B is a solid in E_n, then

II.5.1 $$\lambda(B) = \mu(B) \quad .$$

Proof. In the well-known book by Courant and Hilbert[2], the above result is proven in detail in E_2 and the authors state[3] that their proof generalizes directly to E_3 and generalizes to E_n, $n > 3$, if the Euler equation of the variational problem fulfills certain conditions[4]. If

II.5.2 $$E(v) = \int_B (p\,(\nabla v)^2 + 2\sum_1^n a_i\, v_{x_i} + qv^2)\, dV_n \quad ,$$

is the variational integral, where $p > 0$, $q > 0$, the a_i's have first and second derivatives, and p has continuous first, second, and third derivatives, then the Euler equation is

II.5.3 $$L(u) = \sum_1^n (pu_{x_i})_{x_i} - q^*u + \lambda(B)u = 0 \quad ,$$

where

II.5.4 $$q^* = q - \sum_1^n a_i \quad .$$

---

[2] R. Courant and D. Hilbert, Methoden der Mathematischen Physik, (New York: Interscience Publishers, Inc., 1943) Vol. 2, pp. 471-508.

[3] Ibid., p. 471.

[4] Ibid., p. 499.

The requirement which II.5.3 must satisfy in order that the proof generalize to n dimensions is

II.5.5 $$q^* \equiv 0 \quad .$$

Since, for this theorem, $p = 1$, $a_i = 0$ for all i , and $q > 0$, this condition is fulfilled and II.5.2 becomes

II.5.6 $$E(v) = \int_B (\nabla v)^2 \, dV_n \quad .$$

Thus the equivalence of Definitions II.3 and II.4 has been established.

<u>Corollary II.6</u>. If B is a solid in E_n with surface S and if f is an eigenfunction associated with $\mu(B)$, then

II.6.1 $$\mu(B) = \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} \quad .$$

<u>Proof</u>. Applying Green's theorem to the function $f\nabla^2 f$ yields

II.6.2 $$\int_B (\nabla f)^2 \, dV_n = - \int_B f\nabla^2 f \, dV_n + \int_S f \cdot f_n \, dS \ ,$$

where f_n is the normal derivative of f and dS is the element of surface area. Since f is an eigenfunction associated with $\mu(B)$, f vanishes on S . Therefore

II.6.3 $$\int_S f \cdot f_n \, dS = 0 \quad .$$

Hence II.6.2 becomes

II.6.4 $$\int_B (\nabla f)^2 \, dV_n = - \int_B f \nabla^2 f \, dV_n \quad .$$

But, since f is an eigenfunction associated with $\mu(B)$, then

II.6.5 $$\nabla^2 f + \mu(B) f = 0 \quad .$$

Multiplying II.6.5 by f and rearranging gives

II.6.6 $$-f \nabla^2 f = \mu(B) f^2 \quad ,$$

and integrating II.6.6 over B yields

II.6.7 $$- \int_B f \nabla^2 f \, dV_n = \mu(B) \int_B f^2 \, dV_n \quad .$$

Substituting II.6.4 in II.6.7 and dividing gives

II.6.8 $$\mu(B) = \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} \quad .$$

Q.E.D.

Section III. Approximation of a Continuous Function with Piecewise Continuous First Derivatives by a Polyhedral Function

Since Steiner symmetrization may introduce singularities, the image of a solid under symmetrization may no longer be a solid, as the example at the end of Section V shows. However, the image of a polyhedron is a polyhedron. Therefore, it is shown that the following sections may be limited to the discussion of polyhedral functions with no loss of generality.

Lemma III.1. If K_n is a hypercube, in E_n, with center C_n, then there exist $2^{n-1}n!$ simplexes in E_n with disjoint interiors and equal volumes such that K_n is their union and C_n is their intersection and a vertex of each simplex.

Proof. The lemma is true for $n = 2$, since, by constructing the lines from C_2 to the vertices of K_2, K_2 is divided into four triangles (simplexes in E_2) whose interiors are disjoint and whose areas are equal. Clearly, K_2 is their union and C_2 is their intersection, C_2 is a vertex of each triangle, and $2^1 \cdot 2! = 4$.

The proof then proceeds by induction. Assume the lemma is true for $n = p - 1 > 1$. K_p has $2p$ faces and each face is a hypercube of dimension $(p - 1)$. Therefore, each face can be decomposed into $2^{p-2}(p - 1)!$ simplexes. Construct all the E_{p-1} which contain C_p and an edge (an E_{p-2} bounding a face of the simplex). The E_{p-1}'s divide K_p into simplexes. There are $2^{p-2}(p - 1)!$ simplexes in each face and $2p$ faces, and so there are $2p\left[2^{p-2}(p - 1)!\right] = 2^{p-1}p!$

simplexes constructed above. Clearly, C_p is a vertex of each simplex, they are disjoint, K_p is their union and C_p is their intersection. Also, by Cavalieri's Principle every simplex has the same volume. Thus the induction is complete.

Q.E.D.

Definiton III.2. A function p, defined in E_n is a polyhedral function if and only if there exists a subdivision of E_n into simplexes such that p is a linear function on each simplex, p is continuous in E_n, and every bounded region of E_n contains only a finite number of simplexes.

Theorem III.3.[5] Let B be a solid in E_n with surface S and let $P(B)$ be the class of polyhedral functions which vanish in some neighborhood of S. If $F(B) = \hat{f}\left[f \text{ is defined and piecewise smooth in } B \text{ and } f(S) = 0\right]$, then for any $\epsilon > 0$ there is a $p \epsilon P(B)$ such that

$$\text{III.3.1} \qquad \frac{\int_B (\nabla p)^2 \, dV_n}{\int_B p^2 \, dV_n} < \inf_{f \epsilon F(B)} \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} + \epsilon .$$

Proof. Clearly, $P(B) \subset F(B)$ and thus one need only prove that given $\epsilon > 0$ and $f \epsilon F(B)$, there exist a $p \epsilon P(B)$ such that

$$\text{III.3.2} \qquad \frac{\int_B (\nabla p)^2 \, dV_n}{\int_B p^2 \, dV_n} < \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} + \epsilon .$$

---

[5] The theorem and proof given above are due to Professor Walter Snyder.

For any $\omega > 0$ and $f \epsilon F(B)$ define

III.3.3 $$f_\omega(X) = \max\left[0, f(X) - \omega\right] + \min\left[0, f(X) + \omega\right]$$

for $X \epsilon B$ and

III.3.4 $$f_\omega(X) = 0 \quad \text{for} \quad X \notin B \quad .$$

Then f_ω is defined and continuous in E_n and vanishes in some neighborhood of S. Let $K_\omega = \overset{\triangle}{X}\left[|f(X)| = \omega\right]$. Since $\omega_1 \neq \omega_2$ implies $K_{\omega_1} \bigcap K_{\omega_2}$ is empty, $|K_\omega| > 0$ for at most a countable set of values of ω. Requiring that $|K_\omega| = 0$ it follows that f_ω exists almost everywhere and

III.3.5 $$\int_B (\nabla f_\omega)^2 \, dV_n \leqslant \int_B (\nabla f)^2 \, dV_n \quad .$$

Indeed, if $\left\{S_i\right\}_{i=1}^{k}$ are the surfaces on which the discontinuities of the partial derivatives of f are distributed and if G is any neighborhood of S on which f_ω vanishes identically then on any component, C, of the open set D,

III.3.6 $$D = (E_n - K_\omega - \bigcup_{i=1}^{k} S_i) \bigcup G \quad ,$$

f_ω is either identically zero, identically $f + \omega$, or identically

$f - \omega$. Thus, either $(\nabla f_\omega)^2 = (\nabla f)^2$ or $(\nabla f_\omega)^2 = 0$ on C and since $|K_\omega| + \sum_{i=1}^{k} |S_i| = 0$, III.3.5 follows. Letting $\omega \longrightarrow 0$, but always requiring $|K_\omega| = 0$, $\int_B f_\omega^2 \, dV_n \longrightarrow \int_B f^2 \, dV_n$. Thus for some $\omega > 0$, $|K_\omega| = 0$,

$$\text{III.3.7} \qquad \frac{\int_B (\nabla f_\omega)^2 \, dV_n}{\int_B (f_\omega)^2 \, dV_n} < \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} + \frac{\epsilon}{2} \quad .$$

Clearly, f_ω has partial derivatives on D and these are uniformly bounded. Indeed, if $|f_{\omega_{x_1}}(X_n)| > n$, then $X_n \epsilon B - G$ and so the limit points of $\{X_n\}_1^\infty$ are also in the closed set $B - G$, thus contradicting the hypothesis that f is piecewise smooth since $|f_{\omega_{x_1}}(X_n)| = |f_{x_1}(X_n)|$, $n = 1, 2, \ldots$. Let M be a bound for all the partial derivatives of f_ω on D . Then f_ω satisfies a Lipschitz condition. For, consider a line segment, ℓ , with endpoints $X' = (x_1', \ldots, x_n')$ and $X'' = (x_1'', \ldots, x_n'')$ with $x_i' = x_i''$, $i = 2, 3, \ldots, n$. Suppose that, with the possible exception of the endpoints, ℓ is in C , a component of D . Then $|\Delta f_\omega| \leqslant M|\Delta x_1|$ by the mean value theorem. Now, assume ℓ intersects $\bigcup_{i=1}^{k} S_i$ in only a finite number of points but doesn't intersect K_ω , except possibly at the endpoints. On each segment of the subdivision of ℓ by the points of $\ell \cap (\bigcup_{i=1}^{k} S_i)$ the preceding result holds and, by summation, it holds on ℓ . Next, suppose ℓ intersects $\bigcup_{i=1}^{k} S_i$ infinitely often but doesn't intersect K_ω , except possibly at the

endpoints. Delete a small neighborhood of each endpoint, if necessary, so that the shortened segment, ℓ^*, may be translated parallel to itself and remain in $E_n - K_\omega$. This translation is to be made so that, in the new position, the translated segment intersects $\bigcup_{i=1}^{k} S_i$ at most a finite number of times. To see that this is possible, let $x_j = f_{ij}(U)$, $j = 1, \ldots, n$ be the parametric representation of S_i, $i = 1, \ldots, k$, with $U = (u_1, \ldots, u_{n-1})$ varying over a topological $(n - 1)$ - sphere, D_j, of E_{n-1}. Each f_{ij} is of class C'. Consider the transformation $x_j = f_{ij}(U)$, $j = 2, \ldots, n$ with i fixed. Let $N_i(x_2, \ldots, x_n)$ be the number of distinct points, U, in D_i such that $x_j = f_{ij}(U)$. Then N_i is defined in E_{n-1} but vanishes outside some bounded region. Banach has shown[6] that

$$\text{III.3.8} \qquad \int_{E_{n-1}} N_i \, dV_{n-1} = \int_{D_i} \left| \frac{\partial(x_2, \ldots, x_n)}{\partial(u_1, \ldots, u_{n-1}} \right| dV_{n-1} .$$

Since this integral is finite it follows that N_i is finite almost everywhere in E_{n-1}. Let $I_i = (\widehat{x_2, \ldots, x_n})$ $\left[N_i(x_2, \ldots, x_n) = +\infty\right]$ and let $I = \bigcup_{i=1}^{k} I_i$. Then I is a set of $(n - 1)$-dimensional Lebesgue measure zero and thus ℓ^* can be shifted as little as desired to a position where the translated segment, ℓ_T^*, intersects $\bigcup_{i=1}^{k} S_i$ only a finite number of times and does not intersect K_ω. For ℓ_T^*,

---

[6] Banach, S., Fund. Math., 7, 225-236.

$|\Delta f_\omega| \leqslant M|\Delta x_1|$ and by continuity this inequality holds for ℓ^*. Again by continuity the same inequality must hold for ℓ. Thus $|\Delta f_\omega| \leqslant |\Delta x_1| M$ on ℓ if ℓ intersects K_ω only at the endpoints, if at all. But if one or more of the points of K_ω lie interior to ℓ, then f_ω vanishes at these points. Let X_1 be the point of $K_\omega \cap \ell$ closest to X' and let X_2 be the point of $K_\omega \cap \ell$ closest to X''. Then $|\Delta f_\omega|_\ell \leqslant |\Delta f_\omega|_{X'}^{X_1} + |\Delta f_\omega|_{X_2}^{X''}$, since $f_\omega(X_i) = 0$, $i = 1, 2$. But $|\Delta f_\omega|_{X'}^{X''} \leqslant M|X_1 - X'| + M|X_2 + X''| \leqslant M|\Delta x_1|$. Thus the proof is complete for x_1 and similarly for the other coordinates.

Now, let $S^\theta = \hat{X}\left[d(S; X) < \theta\right]$ and let $\theta > 0$ be taken such that f_ω vanishes on $S^{5\theta\sqrt{n}}$. Let

$$\text{III.3.9} \qquad g_\nu(X) = \frac{1}{2^n \nu^n} \int_{-\nu}^{\nu} \cdots \int_{-\nu}^{\nu} f_\omega(X + Z)\, dz_1 \cdots dz_n$$

for $0 < \nu < \theta$. Then $g_\nu(X)$ is continuous in E_n and vanishes in $S^{3\theta\sqrt{n}}$. Moreover, g_ν is of class C'. Since f_ω is continuous, $g_\nu(X) \longrightarrow f_\omega(X)$ as $\nu \longrightarrow 0$ and since the functions vanish outside a bounded region and are uniformly bounded

$$\int_B g_\nu^2\, dV_n \longrightarrow \int_B f_\omega^2\, dV_n \quad \text{as} \quad \nu \longrightarrow 0 .$$

At any point $X \in (E_n - K_\omega - \bigcup_{i=1}^{k} S_i) \cup G$, $\dfrac{\partial g_\nu}{\partial x_j} \longrightarrow \dfrac{\partial f_\omega}{\partial x_j}$ and since these functions are also uniformly bounded and vanish outside a

bounded region $\int_B (\nabla g_\nu)^2 \, dV_n \longrightarrow \int_B (\nabla f_\omega)^2 \, dV_n$. Thus for ω and ν sufficiently small

III.3.10
$$\frac{\int_B (\nabla g_\nu)^2 \, dV_n}{\int_B (g_\nu)^2 \, dV_n} < \frac{\int_B (\nabla f_\omega)^2 \, dV_n}{\int_B (f_\omega)^2 \, dV_n} + \frac{\epsilon}{4}$$

$$< \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B (f)^2 \, dV_n} + \frac{3\epsilon}{4} .$$

Take a cubical net in E_n of mesh width r . By Lemma III.1 each cube can be divided into $2^{n-1}n!$ simplexes having a common vertex at the center of the cube and having disjoint interiors. Let X be any point of a simplex with vertices $V^1, \ldots, V^{n+1}$. If e^i is the vector from V^{n+1} to V^i , and

III.3.11
$$X = \sum_{i=1}^{n} \beta_i e^i ,$$

then $\sum_{i=1}^{n} \beta_i \leqslant 1$ and the (β_i) are linear functions of X , say

III.3.12
$$\beta_i = \sum_{k=1}^{n} \alpha_i^k x_k .$$

Let $e_1^i, \ldots, e_n^i$ be the components of e^i . Then

III.3.13
$$\frac{\partial X_\ell}{\partial x_k} = \delta_k^\ell = \sum_{i=1}^{n} e_\ell^i \frac{\partial \beta_i}{\partial x_k} ,$$

where $\delta_k^\ell = \begin{Bmatrix} 0 & \text{for} & \ell \neq k \\ 1 & \text{for} & \ell = k \end{Bmatrix}$ (the Kronecker delta). From the construction of the simplexes in Lemma III.1 it is clear that each vertex, except the center of the cube, is the vertex of a simplex in at least two hypercubes. Therefore, since g_ν is continuous, if $g_\nu(C)$ is the value of g_ν at the center of the hypercube and if g_ν^i is the value of g_ν at V^i, then

III.3.14 $$p_r(X) = g_\nu(C) + \sum_{i=1}^{n} (g_\nu^i - g_\nu(C))\, \beta_i$$

is a polyhedral function, and

III.3.15 $$\left.\frac{\partial p_r}{\partial x_k}\right|_X = \sum_{i=1}^{n} \left.\frac{\partial \beta_i}{\partial x_k}\right|_X \left(\sum_{\ell=1}^{n} \left[\left.\frac{\partial g_\nu}{\partial x_\ell}\right|_C + \epsilon_{\ell i} \right] e_\ell^i \right),$$

where $\epsilon_{\ell i}$ are infinitesimals with respect to r. Then

III.3.17 $$\left.\frac{\partial p_r}{\partial x_k}\right|_X = \left.\frac{\partial g_\nu}{\partial x_k}\right|_C + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \epsilon_{i\ell}\, e_\ell^i \left.\frac{\partial \beta_i}{\partial x_k}\right|_X .$$

By III.3.13, using Cramer's rule and the Hadamard bound for a determinant,

III.3.18 $$\left| \frac{\partial \beta_i}{\partial x_k} \right| = \left| \frac{e_{\ell *}^i}{e_\ell^i} \right| < \frac{2^{n-1} n!\, r^{n-1} (n-1)^{\frac{n-1}{2}}}{r^n},$$

where $\left| e_{\ell *}^i \right|$ is the determinant $\left| e_\ell^i \right|$ with the <u>ith</u> row and the <u>kth</u>

column all zeros except for the i , kth element which is one. Therefore

III.3.19 $$\left|\epsilon_{i\ell}\right| \left| e^{i}_{\ell} \right| \left|\frac{\partial \beta_i}{\partial x_k}\right| < \left|\epsilon_{i\ell}\right| 2^{n-1} n! \, (n-1) \frac{n-1}{2} .$$

But $2^{n-1} n!(n-1) \frac{n-1}{2}$ is independent of r and $\lim_{r \to 0} \epsilon_{ik} = 0$. Thus, for r sufficiently small, III.3.17 implies

$$\lim_{r \to 0} \left.\frac{\partial p_r}{\partial x_k}\right|_X = \left.\frac{\partial g_\nu}{\partial x_k}\right|_C .$$

Therefore $\int_B (\nabla p_r)^2 \, dV_n \longrightarrow \int_B (\nabla g_\nu)^2 \, dV_n$ as $r \longrightarrow 0$ and, clearly,

$$\int_B p_r^2 \, dV_n \longrightarrow \int_B g_\nu^2 \, dV_n \quad \text{as} \quad r \longrightarrow 0 .$$

Thus, for sufficiently small r ,

III.3.20 $$\frac{\int_B (\nabla p_r)^2 \, dV_n}{\int_B (p_r)^2 \, dV_n} < \frac{\int_B (\nabla g_\nu)^2 \, dV_n}{\int_B (g_\nu)^2 \, dV_n} + \frac{\epsilon}{4}$$

$$< \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} + \epsilon .$$

Q.E.D.

Section IV. Some Properties of the Fundamental Eigenvalue of a Solid

In this section some elementary properties of the fundamental eigenvalue are given, some to be used in later sections of this chapter, others to be used in the next chapter.

Theorem IV.1. If B and B' are two solids in E_n such that $B' \subset B$ and $B - B'$ consists of a finite number of solids, then

IV.1.1 $$\mu(B) \leqslant \mu(B') \quad .$$

Proof. Let F be the class of functions defined in II.4.1 and let F' be the class similarly defined when B' is substituted for B. Let

IV.1.2 $$\bar{F} = \overset{\triangle}{f} \left[\bar{f} \epsilon F \text{ and } \bar{f} \equiv 0 \text{ on } B - B' \right] \quad .$$

By Theorem II.5,

IV.1.3 $$\mu(B') = \min_{f' \epsilon F'} \frac{\int_{B'} (\nabla f')^2 \, dV_n}{\int_{B'} (f')^2 \, dV_n} ,$$

and so, obviously,

IV.1.4 $$\mu(B') = \min_{\bar{f} \epsilon \bar{F}} \frac{\int_{B} (\nabla \bar{f})^2 \, dV_n}{\int_{B} \bar{f}^{\,2} \, dV_n} \quad .$$

Similarly,

IV.1.5 $$\mu(B) = \min_{f \in F} \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} .$$

Clearly, $\bar{F} \subset F$, and, therefore

IV.1.6 $$\min_{\bar{f} \in \bar{F}} \frac{\int_B (\nabla \bar{f})^2 \, dV_n}{\int_B \bar{f}^2 \, dV_n} \geqslant \min_{f \in F} \frac{\int_B (\nabla f)^2 \, dV_n}{\int_B f^2 \, dV_n} .$$

That is,

IV.1.7 $$\mu(B') \geqslant \mu(B) .$$

Q.E.D.

Theorem IV.2. Given a solid B in E_n with surface S , there exist an eigenfunction associated with $\mu(B)$ which does not change sign on B .

Proof. Let f be any eigenfunction associated with $\mu(B)$. If h is any function in the class F defined by II.4.1 let

IV.2.1 $$Q(h) = \frac{\int_B (\nabla h)^2 \, dV_n}{\int_B h^2 \, dV_n} .$$

Let $g = |f|$. Then g is Holder continuous[7] in B . Let J be a

[7] Schauder, Math. Zeits. 38, No. 2, pp. 257-282 (1934).

continuous function[8] such that

IV.2.2 $$\nabla^2 J = g \quad ,$$

and $J(S) = 0$. Then

IV.2.3 $$\mu(B) = Q(f) = Q(g) \quad ,$$

and

IV.2.4 $$\left[\frac{Q(J)}{Q(g)}\right]^2 = \frac{\left(\int_B (\nabla J)^2 \, dV_n\right)^2 \left(\int_B g^2 \, dV_n\right)^2}{\left(\int_B J^2 \, dV_n\right)^2 \left(\int_B (\nabla g)^2 \, dV_n\right)^2} \quad .$$

Applying Green's theorem and IV.2.2 gives

$$\left[\frac{Q(J)}{Q(g)}\right]^2 = \frac{\left(\int_B J \nabla^2 J \, dV_n\right)^2 \left(\int_B g \nabla^2 J \, dV_n\right)^2}{\left(\int_B J^2 \, dV_n\right)^2 \left(\int_B (\nabla g)^2 \, dV_n\right)^2}$$

IV.2.5 $$= \frac{\left(\int_B g \cdot J \, dV_n\right)^2 \left(\int_B \nabla J \cdot \nabla g \, dV_n\right)^2}{\left(\int_B J^2 \, dV_n\right)^2 \left(\int_B (\nabla g)^2 \, dV_n\right)^2}$$

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[8] Lichtenstein, Encykl. der Wissen., Vol. 2, part 3, pp. 1277-1344.

$$= \frac{Q(J) \left(\int_B g \circ J \, dV_n\right)^2}{Q(g) \left(\int_B J^2 \, dV_n\right) \left(\int_B g^2 \, dV_n\right)} \times$$

IV.2.5

$$\frac{\left(\int_B \nabla J \cdot \nabla g \, dV_n\right)^2}{\left(\int_B (\nabla J)^2 \, dV_n\right) \left(\int_B (\nabla g)^2 \, dV_n\right)} .$$

By the Schwartz inequality,

IV.2.6 $$\left(\int_B g \circ J \, dV_n\right)^2 \leqslant \left(\int_B J^2 \, dV_n\right) \left(\int_B g^2 \, dV_n\right) ,$$

and

IV.2.7 $$\left(\int_B \nabla J \circ \nabla g \, dV_n\right)^2 \leqslant \left(\int_B (\nabla J)^2 \, dV_n\right) \left(\int_B (\nabla g)^2 \, dV_n\right) .$$

Equality is attained if and only if J is a multiple of g. If the inequality holds then IV.2.5, IV.2.6, and IV.2.7 yield

IV.2.8 $$\left[\frac{Q(J)}{Q(g)}\right]^2 < \frac{Q(J)}{Q(g)} .$$

That is

IV.2.9 $$Q(J) < Q(g) = Q(f) = \mu(B) .$$

But $J \epsilon F$, and

IV.2.10 $$Q(f) = \mu(B) = \min_{h \epsilon F} Q(h) ,$$

which is a contradiction. Thus the equality must hold and J is a multiple of g and thus of constant sign.

Q.E.D.

<u>Theorem IV.3</u>. Given a solid B in E_n and a constant $k \neq 0$, if T is the similarity transformation which takes the point $X = (x_1, \ldots, x_n)$ into the point $Y = (kx_1, \ldots, kx_n)$, and if $B' = T(B)$, then

IV.3.1 $$\mu(B') = \frac{\mu(B)}{k^2} .$$

<u>Proof</u>. Let f be an eigenfunction associated with $\mu(B)$ and let $y_i = kx_i$ for $1 \leqslant i \leqslant n$. Let

IV.3.2 $$\nabla_y^2 = \sum_1^n \frac{\partial^2}{\partial y_i^2}$$

Then

IV.3.3 $$\nabla^2 = k^2 \nabla_y^2 .$$

Let $g(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$, then

IV.3.4 $$\nabla^2 g + \frac{\mu(B)}{k^2} g = 0 ,$$

and

IV.3.5 $$\mu(B') \leqslant \frac{\mu(B)}{k^2} .$$

By symmetry

IV.3.6 $$\mu(B) \leqslant k^2 \mu(B') ,$$

hence

IV.3.7 $$\mu(B) = k^2 \mu(B') .$$

Q.E.D.

Section V. Some Properties of Steiner Symmetrization in E_n

Since Steiner symmetrization is a major tool in the last sections of this chapter, those properties of this operation which are needed in these sections are developed here.

Definition V.1. By a body in E_n is meant a bounded, connected, measurable, point set with interior points.

Definition V.2.[9] If K is a body in E_n and if $X = (x_1, \ldots, x_n)$ is a point in E_n, then, by the Steiner symmetrization of K with

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[9] Dinghas, A., Math. Nachr., 2, 108-109 (1948).

respect to the space $x_n = 0$, one means the following operation:

Let $C(x_1, \ldots, x_n)$ be the characteristic function of K and let A be any line in E_n which is orthogonal to the E_{n-1} space $x_n = 0$ and which intersects K . Let

V.2.1 $$\ell(x_n) = \int_{-\infty}^{x_n} C(x_1, \ldots, x_{n-1}, u)du \quad .$$

Relate a point $X = (x_1, \ldots, x_{n-1}, x_n)$ of A to another point $\overline{X} = (x_1, \ldots, x_{n-1}, \overline{x}_n)$ of A by

V.2.2 $$\overline{x}_n = \ell(x_n) - \frac{1}{2}\ell(\infty) \quad .$$

Let

V.2.3 $$\overline{K} = \overset{\triangle}{\overline{X}} \left[\overline{X} = (x_1, \ldots, x_{n-1}, \overline{x}_n) \text{ and there exists a point } X = (x_1, \ldots, x_n) \epsilon K \text{ such that } \overline{x}_n = \ell(x_n) - \frac{1}{2}\ell(\infty)\right] \quad .$$

Then $\overline{K}$ is called the image of K under Steiner symmetrization with respect to the E_{n-1} , $x_n = 0$. In symbols, $\overline{K} = S_{x_n = 0}(K)$.

Since K is a body, $C(X)$ is a measurable function of $(x_1, \ldots, x_{n-1})$. Therefore, by Fubini's theorem

V.2.4 $$G(x_1, \ldots, x_{n-1}) = \frac{1}{2}\int_{-\infty}^{\infty} C(x_1, \ldots, x_{n-1}, u)du$$

is a measurable function of $(x_1, \ldots, x_{n-1})$. Hence the ordinate set of G is measurable and therefore $\bar{K}$ is measurable. Clearly $\bar{K}$ is bounded, connected, and has interior points. Therefore $\bar{K}$ is a body.

A more intuitive formulation of Steiner symmetrization for three dimensional solids is given by Polya and Szegő.[10] An obvious generalization of their definition is given below.

Definition V.3. The Steiner symmetrization of a solid B in E_n with respect to some $E_{n-1} \subset E_n$ changes B into the solid B^* in E_n characterized as follows:

(i) B^* is symmetric with respect to E_{n-1}.

(ii) If ℓ is any line in E_n orthogonal to E_{n-1}, and if ℓ intersects one of the solids B and B^*, then it intersects the other and the two intersections have the same linear measure.

(iii) The intersection of ℓ with B^* is connected (the intersection of ℓ with B may have infinitely many components).

Definition V.4. If K is a body in E_n, then

V.4.1 $$K^h = \bigcup_X \left[X \in E_n \text{, and } d(X, K) < h \right] .$$

The following theorem is proved by A. Dinghas.[11]

Theorem V.5. If K is a body in E_n and if $\bar{K} = S_{E_{n-1}}(K)$ for

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[10] G. Polya and G. Szegő, Isoperimetric Inequalities in Mathematical Physics, (Princeton, Princeton University Press, 1951), p. 5

[11] Dinghas, A., Op. Cit., p. 109.

some $E_{n-1} \subset E_n$, then

V.5.1 $$V_n(K) = V_n(\overline{K}) \quad ,$$

and

V.5.2 $$V_n(K^h) \geqslant V_n(\overline{K}^h) \quad .$$

<u>Definition V.6.</u>[12] If K is a body in E_n , then the Minkowski surface area of K is defined by

V.6.1 $$A_{n-1}(K) = \lim_{h \to 0} \frac{V_n(K^h) - V_n(K)}{h} \quad .$$

<u>Corollary V.7.</u> If K is a body in E_n and if $\overline{K} = S_{E_{n-1}}(K)$ for some $E_{n-1} \subset E_n$, then

V.7.1 $$A_{n-1}(K) \geqslant A_{n-1}(\overline{K}) \quad .$$

<u>Proof.</u> By definition V.6

V.6.1 $$A_{n-1}(K) = \lim_{h \to 0} \frac{V_n(K^h) - V_n(K)}{h}$$

and

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[12] <u>Ibid.</u>, p. 110.

V.7.2 $$A_{n-1}(\overline{K}) = \lim_{h \to 0} \frac{V_n(\overline{K}^h) - V_n(\overline{K})}{h} .$$

Theorem V.5 states

V.7.3 $$V_n(K) = V_n(\overline{K})$$

and

V.7.4 $$V_n(K^h) \geqslant V_n(\overline{K}^h) .$$

Therefore

V.7.5 $$A_{n-1}(K) \geqslant A_{n-1}(\overline{K}) .$$

Q.E.D.

The following theorem demonstrates that the symmetrization of a body in E_n simultaneously produces symmetrizations in certain sub-spaces of $n - 1$ dimensions, and thus in certain sub-spaces of dimension k for any k less than n and greater than one.

<u>Theorem V.8.</u> If

(i) $X = (x_1, \ldots, x_n)$ is a point of E_n ,

(ii) $E_{n-1} \subset E_n$ is the space $x_n = 0$,

(iii) $\overline{E_{n-1}} \subset E_n$ is the space $x_1 = 0$,

(iv) $E_{n-2} \subset E_n$ is the space $x_1 = x_n = 0$,

(v) B is a body in E_n ,

(vi) $B^* = S_{\overline{E_{n-1}}}(B)$,

(vii) $D = B \cap E_{n-1}$, and

(viii) $D^* = B^* \cap E_{n-1}$,

then

V.8.1 $$D^* = S_{E_{n-2}}(D) \quad .$$

<u>Proof</u>. Hypotheses (vii) and (viii) imply $D \subset E_{n-1}$, and $D^* \subset E_{n-1}$. If A is any line in E_{n-1} orthogonal to E_{n-2} , then A is orthogonal to $\overline{E_{n-1}}$. Therefore the mapping of A onto itself in Definition V.2 is identical for the symmetrization of B with respect to $\overline{E_{n-1}}$ and the symmetrization of D with respect to E_{n-2} . Thus $E_{n-1} \cap B^* = S_{E_{n-2}}(D)$. But $D^* = E_{n-1} \cap B^*$ and therefore $D^* = S_{E_{n-2}} \cap (D)$.

Q.E.D.

<u>Theorem V.9</u>. If

(i) $X = (x_1, \ldots, x_{n+1})$ is a point of E_{n+1} ,

(ii) $E_n \subset E_{n+1}$ is the space $x_{n+1} = 0$,

(iii) $\overline{E_n} \subset E_{n+1}$ is the space $x_1 = 0$,

(iv) $E_{n-1} \subset E_{n+1}$ is the space $x_1 = x_{n+1} = 0$,

(v) D is a solid in E_n ,

(vi) $D^* = S_{E_{n-1}}(D)$,

(vii) $f(x_1, \ldots, x_n)$ is a continuous, non-negative, single valued, function defined on D which vanishes on the surface of D ,

(viii) B is the solid in E_{n+1} defined by $0 \leqslant x_{n+1} \leqslant f(x_1, \ldots, x_n)$, where $(x_1, \ldots, x_n) \epsilon D$,

(ix) $B^* = S_{\overline{E_n}}(B)$,

(x) $f^*(x_1, \ldots, x_n)$ is the function for which $0 \leqslant x_{n+1} \leqslant f^*(x_1, \ldots, x_n)$ defines B^* , and

(xi) $G(t)$ is any continuous, monotone, function of t ,

then

V.9.1 $$\int_D G\left[f(x_1, \ldots, x_n)\right] dV_n = \int_{D^*} G\left[f^*(x_1, \ldots, x_n)\right] dV_n .$$

<u>Proof</u>. Theorem V.5 and hypothesis (ix) imply

V.9.2 $$V_{n+1}(B) = V_{n+1}(B^*) .$$

Let $D(t)$ and $D^*(t)$ be the intersections of B and B^* , respectively, with the space $x_{n+1} = t$, and let $A(t)$ and $A^*(t)$ be the orthogonal projections of $D(t)$ and $D^*(t)$, respectively, on E_n . Theorem V.8 implies

V.9.3 $$D^*(t) = S_{E_{n-1}(t)}(D(t)) ,$$

where $E_{n-1}(t)$ is the space in E_n defined by $x_1 = 0$ and $x_{n+1} = t$. Theorem V.4 then implies

V.9.4 $$V_n\left[D(t)\right] = V_n\left[D^*(t)\right] ,$$

and, therefore,

V.9.5 $$V_n\left[A(t)\right] = V_n\left[A^*(t)\right] .$$

Let $\bar{t}$ be the maximum value of f on D. Subdivide the interval $< 0 , t >$ of the x_{n+1} axis into m equal intervals of width Δt. Let

V.9.6 $$\Delta V_n\left[A(t)\right] = V_n\left[A(t) - A(t + \Delta t)\right] .$$

Then

V.9.7 $$\Delta V_n\left[A(t)\right] = V_n\left\{\hat{X}\left[X = (x_1, \ldots, x_n) \in D \text{ and } G(t) \leqslant G\left(f(X)\right) \leqslant G(t + \Delta t)\right]\right\} .$$

Similarly, if

V.9.8 $$\Delta V_n\left[A^*(t)\right] = V_n\left[A^*(t) - A^*(t + \Delta t)\right] ,$$

V.9.9 $$\triangle V_n \left[A^*(t)\right] = V_n \left\{ \hat{X} \left[X = (x_1, \ldots, x_n) \epsilon D^* \text{ and } G(t) \leqslant G\left(f^*(X)\right) \leqslant G(t + \triangle t) \right] \right\} .$$

Let K be defined by

V.9.10 $$0 \leqslant x_{n+1} \leqslant G\left(f(X)\right) ,$$

where $X = (x_1, \ldots, x_n) \epsilon D$, and let K^* be defined by

V.9.11 $$0 \leqslant x_{n+1} \leqslant G\left(f^*(\overline{X})\right) ,$$

where $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n) \epsilon D^*$.

V.9.12 $$V_{n+1}(K) = \int_D G\left(f(X)\right) dV_n ,$$

and

V.9.13 $$V_{n+1}(K^*) = \int_{D^*} G\left(f^*(\overline{X})\right) dV_n .$$

If S_m is the subdivision of the interval $< 0 , t >$ of the x_{n+1} axis into m equal intervals of width $\triangle t_m$, then

V.9.14 $$\sum_{i=0}^{m-1} G(t_i) \triangle V_n \left(A(t_i)\right) \leqslant V_{n+1}(K) \leqslant$$

V.9.14 $$\leqslant \sum_{i=0}^{m-1} G(t_i + \Delta t_m)\, \Delta V_n\, A(t_i) \quad ,$$

and

V.9.15 $$\sum_{i=0}^{m-1} G(t_i)\, \Delta V_n\big(A^*(t_i)\big) \leqslant V_{n+1}(K^*) \leqslant$$

$$\sum_{i=0}^{m-1} G(t_i + \Delta t_m)\, \Delta V_n\big(A^*(t_i)\big) \quad .$$

Since $G(t)$ is continuous and $\Delta V_n\big(A(t)\big)$ and $\Delta V_n\big(A^*(t)\big)$ are of bounded variation, then

V.9.16 $$\lim_{m \to \infty} \sum_{S_m} G(t)\, \Delta V_n\big(A(t)\big) = \lim_{m \to \infty} \sum_{S_m} G(t+\Delta t)$$

$$\Delta V_n\big(A(t)\big) = \int_0^{\bar{t}} G(t)\, dV_n\big(A(t)\big) \quad ,$$

and

V.9.17 $$\lim_{m \to \infty} \sum_{S_m} G(t)\, \Delta V_n\big(A^*(t)\big) = \lim_{m \to \infty} \sum_{S_m} G(t+\Delta t)\Delta V_n\big(A^*(t)\big)$$

$$= \int_0^{\bar{t}} G(t)\, dV_n\big(A^*(t)\big) \quad .$$

But V.9.5 implies

V.9.18 $$dV_n\big(A(t)\big) = dV_n\big(A^*(t)\big) .$$

Therefore

V.9.19 $$\int_0^{\bar{t}} G(t)\, dV_n\big(A(t)\big) = \int_0^{\bar{t}} G(t)\, dV_n\big(A^*(t)\big) .$$

Applying V.9.16, V.9.17, and V.9.19 to V.9.14 and V.9.15, yields

V.9.20 $$V_{n+1}(K) = \int_0^{\bar{t}} G(t)\, dV_n\big(A(t)\big) = \int_0^{\bar{t}} G(t)\, dV_n\big(A^*(t)\big)$$

$$= V_{n+1}(K^*) .$$

That is, by V.9.12 and V.9.13,

V.9.21 $$\int_D G\Big(f(x_1, \ldots, x_n)\Big)\, dV_n = \int_{D^*} G\Big(f^*(x_1, \ldots, x_n)\Big)\, dV_n .$$

Q.E.D.

<u>Example V.10</u>. Consider a square, S, with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$ and $(2, 0)$. Its corners can be rounded in such a way that the boundary curve of S is of class C^∞. For example, to round the corner at $(0, 0)$ consider the curve defined on $0 \leqslant x \leqslant 1/2$ by $y = f(x)$, where

V.10.1 $$f(x) = e^{\frac{1}{x-1/2}} \left(\frac{1}{\log x} + \frac{1}{2} e^2\right) .$$

This curve fits the boundary of S continuously at $(0, 1/2)$ and $(1/2, 0)$ as do all its derivatives. Let S' be the result of rounding all the corners of S in a similar manner. Then the boundary of S' is of class C^{∞}. Let S'' be the same as S' except the line segment $y = 2$ for $1/2 \leqslant x \leqslant 3/2$ is replaced by $y = f(x)$ for $1/2 \leqslant x \leqslant 3/2$ and

$$\text{V.10.2} \qquad f(x) = e^{\frac{1}{(1/2-x)(3/2-x)}} \sin^2 \frac{\pi}{(1/2-x)} + 2 \quad .$$

The boundary of S'' is also of class C^{∞}.

Let $\bar{S}$ be the symmetrization of S'' with respect to the y-axis. The boundary curve of $\bar{S}$ has a discontinuity in the first derivative for each value of $y > 2$ at which the boundary curve of S'' has a relative maximum. Since there are an infinite number of such values of y, the boundary curve of $\bar{S}$ is not even piecewise smooth.

Section VI. The Effect of Steiner Symmetrization on the Fundamental Eigenvalue of a Solid

The result of the final section of this chapter will depend upon the properties of Steiner symmetrization. Therefore the effect of this operation on the quantity under discussion, the fundamental eigenvalue, is established here. The proof of the following theorem, for E_2, is sketched by Polya and Szegö.[13]

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[13] G. Polya and G. Szegö, op. cit., pp. 89-91 and pp. 182-186.

<u>Theorem VI.1.</u> If D is a solid in E_n and if $D^* = S_{E_{n-1}}(D)$, where $E_{n-1} \subset E_n$, then

VI.1.1 $$\mu(D) \geqslant \mu(D^*) \quad .$$

<u>Proof.</u> Let $E_n \subset E_{n+1}$ and choose a rectangular coordinate system in E_{n+1} such that E_n is the space $x_{n+1} = 0$ and E_{n-1} is the space $x_1 = x_{n+1} = 0$. Let $\overline{E_n}$ be the space $x_1 = 0$. Let $p(x_1, \ldots, x_n)$ be a polyhedral function defined in E_n, which is non-negative on D, and vanishes on $E_n - D$. Consider the polyhedron defined by

VI.1.2 $$0 \leqslant x_{n+1} \leqslant p(x_1, \ldots, x_n) \quad ,$$

where $(x_1, \ldots, x_n) \epsilon D$. This solid will be referred to as the "polyhedron" B. Let $B^* = S_{\overline{E_n}}(B)$ and let $D^* = B^* \cap E_n$. Since $D = B \cap E_n$, Theorem V.8 implies

VI.1.3 $$D^* = S_{E_{n-1}}(D) \quad .$$

Since B^* is a polyhedron there exists a polyhedral function p^* such that B^* is defined by

VI.1.4 $$0 \leqslant x_{n+1} \leqslant p^*(x_1, \ldots, x_n) \quad ,$$

where $(x_1, \ldots, x_n) \in D^*$. Since $B^* = S_{\overline{E_n}}(B)$, Theorem V.9, VI.1.2, and V.1.4 yield

VI.1.5 $$\int_D p^2 \, dV_n = \int_D (p^*)^2 \, dV_n \quad .$$

Since B and B^* are polyhedra,

VI.1.6 $$A_n(B) = \int_D \left[1 + (\nabla p)^2\right]^{\frac{1}{2}} dV_n + V_n(D)$$

and

VI.1.7 $$A_n(B^*) = \int_{D^*} \left[1 + (\nabla p^*)^2\right]^{\frac{1}{2}} dV_n + V_n(D^*).$$ [14]

Theorem V.5 and Corollary V.7, with VI.1.3, imply

VI.1.8 $$\int_D \left[1 + (\nabla p)^2\right]^{\frac{1}{2}} dV_n \geqslant \int_{D^*} \left[1 + (\nabla p^*)^2\right]^{\frac{1}{2}} dV_n \quad .$$

Let ϵ be any positive constant and let C be the polyhedron defined by replacing p by ϵp in VI.1.2. Similarly, let C^* be the polyhedron obtained by replacing p^* by ϵp^* in VI.1.4. Then

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[14] Favard, M. J., La Longeur et L'Aire d'apres Minkowski, Bull. Soc. Math. France, 61 (1933), pp. 63-84.

VI.1.9 $$C^* = S_{\overline{E_n}}(C) \quad ,$$

and $D = C \cap E_n$, and $D^* = C^* \cap E_n$. Therefore, by the preceding argument,

VI.1.10 $$\int_D \left[1 + \epsilon^2 (\nabla p)^2\right]^{\frac{1}{2}} dV_n \geqslant \int_{D^*} \left[1 + \epsilon^2 (\nabla p^*)^2\right]^{\frac{1}{2}} dV_n \quad .$$

Choosing ϵ so that $\epsilon^2(\nabla p)^2 < 1$, $\left[1 + \epsilon^2(\nabla p)^2\right]^{\frac{1}{2}}$ may be expanded in a power series in terms of $\epsilon^2(\nabla p)^2$, yielding

VI.1.11 $$\left[1 + \epsilon^2 (\nabla p)^2\right]^{\frac{1}{2}} = 1 + \frac{\epsilon^2 (\nabla p)^2}{2} + \sum_{j=2}^{\infty} \frac{\prod_{\ell=0}^{j-1} (\frac{1}{2} - \ell)}{j!} \left[\epsilon^2 (\nabla p)^2\right]^j \quad .$$

Since the series in VI.1.11 is an alternating series, the magnitude of the terms of which decrease monotonically, then

VI.1.12 $$\left[1 + \epsilon^2 (\nabla p)^2\right]^{\frac{1}{2}} = 1 + \frac{\epsilon^2 (\nabla p)^2}{2} + O(\epsilon^4) \quad ,$$

where

VI.1.13 $$\left| O(\epsilon^4) \right| < \frac{1}{8} \epsilon^4 (\nabla p)^4 \quad .$$

Since p is a polyhedral function, there exists a constant k such that

VI.1.14 $$k > \frac{1}{8} \epsilon^4 (\nabla p)^4 ,$$

for all points in D. Therefore

VI.1.15 $$\left| O(\epsilon^4) \right| < k\epsilon^4 .$$

Similarly, $\left[1 + \epsilon^2 (\nabla p^*)^2\right]^{\frac{1}{2}}$ may be expanded in a power series, yielding

VI.1.16 $$\left[1 + \epsilon^2 (\nabla p^*)\right]^{\frac{1}{2}} = 1 + \frac{\epsilon^2 (\nabla p^*)^2}{2} + O^*(\epsilon^4) ,$$

where

VI.1.17 $$\left| O^*(\epsilon^4) \right| < k^*\epsilon^4 ,$$

and k^* is some finite constant such that

VI.1.18 $$k^* > \frac{1}{8} \epsilon^4 (\nabla p^*)^4$$

for all points in D^*. Inserting VI.1.12 and VI.1.16 in VI.1.10 yields

VI.1.19 $$\int_D \left[1 + \frac{\epsilon^2}{2} (\nabla p)^2 + O(\epsilon^4)\right] dV_n \geqslant$$

$$\int_{D^*} \left[1 + \frac{\epsilon^2}{2} (\nabla p^*)^2 + O^*(\epsilon^4)\right] dV_n \quad .$$

Since $V_n(D) = V_n(D^*)$, VI.1.19 implies

VI.1.20 $$\int_D \left[\epsilon^2 (\nabla p)^2 + 2 \times O(\epsilon^4)\right] dV_n \geqslant$$

$$\int_{D^*} \left[\epsilon^2 (\nabla p^*)^2 + 2 \times O^*(\epsilon^4)\right] dV_n \quad .$$

Dividing both sides of VI.1.20 by ϵ^2 gives

VI.1.21 $$\int_D \left[(\nabla p)^2 + \frac{2 \times O(\epsilon^4)}{\epsilon^2}\right] dV_n \geqslant$$

$$\int_{D^*} \left[(\nabla p^*)^2 + \frac{2 \times O^*(\epsilon^4)}{\epsilon^2}\right] dV_n \quad .$$

Letting ϵ go to zero in VI.1.21 yields

VI.1.22 $$\int_D (\nabla p)^2 \, dV_n \geqslant \int_{D^*} (\nabla p^*)^2 \, dV_n \quad .$$

By Theorem III.3, given an $\epsilon > 0$, p may be chosen such that

VI.1.23 $$\mu(D) + \epsilon > \frac{\int_D (\nabla p)^2 \, dV_n}{\int_D p^2 \, dV_n} .$$

By VI.1.5, VI.1.22, and VI.1.23 and Theorem III.3, it follows that

VI.1.24 $$\mu(D) + \epsilon > \frac{\int_D (\nabla p)^2 \, dV_n}{\int_D p^2 \, dV_n} \geqslant \frac{\int_{D^*} (\nabla p^*)^2 \, dV_n}{\int_{D^*} (p^*)^2 \, dV_n} > \mu(D^*) .$$

But ϵ is arbitrary, and so

VI.1.25 $$\mu(D) \geqslant \mu(D^*) .$$

Q.E.D.

Section VII. A Minimal Property of the Sphere.

Lemma VII.1. If S and B are two convex solids in E_n, with surfaces s and b, and if $S^h \subset B^h$ for some $h > 0$, then $S \subset B$.

Proof. Assume $S \not\subset B$. Then there exists a point $X \epsilon (S - B)$. Let

VII.1.1 $$D = d(X, B) = d(X, Q) ,$$

where $Q \epsilon B$. Clearly $Q \epsilon b$. Let L be the line such that $Q \epsilon L$ and $X \epsilon L$ and let E_{n-1} be the (n-1)-space such that $E_{n-1} \subset E_n$, $Q \epsilon E_{n-1}$, and L is orthogonal to E_{n-1}. Let $T(X, D)$ be the sphere in E_n

of radius D and center X. Then $T(X, D)$ is tangent to E_{n-1} and on the same side of E_{n-1} as X. Thus E_{n-1} seperates X and B. For if E_{n-1} did not seperate X and B, then there exists a point $Y \epsilon B$ such that Y is on the same side of E_{n-1} as X. But then the line segment between Y and Q must be contained in B and must intersect $T(X, D)$. Then $d(X, B) < D$, which is a contradiction. If $Z \epsilon L$ and Z is on the same side of E_{n-1} as X, let $n(Z) = d(Q, Z)$. Then $n(X) = D$. Let s^h and b^h be the boundaries of S^h and B^h, and let L' be the part of L on the same side of E_{n-1} as X. Let $Y = L' \cap b^h$. Then $n(Y) = h$. Let $Z = L' \cap s^h$. Since $X \epsilon S$, $n(Z) \geqslant n(X) + h$. This is a contradiction since $h > 0$. Therefore $S - B$ is empty, and thus $S \subset B$.

Q.E.D.

It has been shown by R. Courant[15] that of all simply connected plane domains, with piecewise analytic boundary curve of a given length, the circle has the smallest fundamental eigenvalue. G. Faber[16] has obtained the stronger result that of all simply connected plane domains of a given area and with a piecewise analytic boundary curve the circle has the smallest fundamental eigenvalue. Unfortunately their proofs depend strongly on the use of the theory of an analytic function of a complex variable and therefore do not generalize directly to n-space.

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[15] R. Courant, Math. Zeits., 1, pp. 321-328.

[16] G. Faber, Sitzs. Bayer. Akad. Wissen., (1923), pp. 169-172.

However, using Steiner symmetrization and the preceding theorems it is shown, in Theorem VII.3, that of all solids of a given volume, which can be made convex by a finite number of symmetrizations, the sphere has the smallest fundamental eigenvalue. The following result is proved by A. Dinghas.[17]

Lemma VII.2. If K is a body in E_n and S is a sphere in E_n such that

VII.2.1 $$V_n(K) = V_n(S) \quad ,$$

and if $\bar{K} = \hat{K}^*$ [K^* is the image of K after a finite number of Steiner symmetrizations], then given numbers h and k such that $h > k > 0$ there exists a $K^* \epsilon \bar{K}$ such that

VII.2.2 $$S^{h-k} \subset (S \cap K^*)^h \quad .$$

Theorem VII.3. Given a solid B in E_n, if

VII.3.1 $$\bar{B} = \hat{B}^* \quad [B^* \text{ is the convex image of } B \text{ after a finite number of Steiner Symmetrizations}] \quad ,$$

is not empty, and if S is a sphere in E_n such that

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[17] A. Dinghas, op. cit., p. 111. See also Math. Rev., 11, p. 386 which is a review by P. Scherk of this article.

VII.3.2 $$V_n(B) = V_n(S) \quad ,$$

then

VII.3.3 $$\mu(B) \geqslant \mu(S) \quad ,$$

and given an $\epsilon > 0$ there exists a $B^* \epsilon \bar{B}$ such that

VII.3.4 $$\mu(B^*) - \epsilon < \mu(S) \quad .$$

<u>Proof</u>. Note that, under symmetrization, any image of a convex solid is again convex. If $h > k > 0$, then by Lemma VII.2 there exists a $B^* \epsilon \bar{B}$ such that

VII.3.5 $$S^{h-k} \subset (S \cap B^*)^h \quad ,$$

and obviously

VII.3.6 $$(S \cap B^*)^h \subset B^{*h} \quad .$$

If D is a solid with surface C, let

VII.3.7 $$D^{-\ell} = \hat{X} \left[X \epsilon D \text{ and } d(X, C) \geqslant \ell \right] \quad .$$

Since B^* is convex, by Lemma VII.2,

VII.3.8 $$S^{-k} \subset B^* .$$

By Theorem V.5, $V_n(S^{-k}) < V_n(B^*)$, and $B^* - S^{-k}$ is not empty. Let X be a point of $B^* - S^{-k}$ such that $d(X, S^{-k}) = \max_{Y \in B^*} d(Y, S^{-k})$, and let $\nu = d(X, S^{-k})$. Let G be the cone with vertex X each ray of which is tangent to S^{-k} and let K be the solid consisting of the part of G between X and S^{-k}. Since B^* is convex

VII.3.9 $$K \subset B^* ,$$

and, therefore,

VII.3.10 $$V_n(B^*) > V_n(S^{-k}) + V_n(K) .$$

Since k is arbitrary and $V_n(B) = V(S)$, it follows that as $k \longrightarrow 0+$, $V_n(K) \longrightarrow 0$, and this implies $\nu \longrightarrow 0$. Therefore given an $\epsilon > 0$,

VII.3.11 $$S^{-\epsilon} \subset B^* \subset S^{+\epsilon} .$$

But, Theorem VI.1 implies

VII.3.12 $$\mu(B) \geqslant \mu(B^*) ,$$

and Theorem IV.1 implies

VII.3.13 $$\mu(S^{-\epsilon}) \geqslant \mu(B^*) \geqslant \mu(S^{+\epsilon}) \quad .$$

Therefore, by Theorem IV.3

VII.3.14 $$\mu(B) \geqslant \mu(B^*) \geqslant \frac{\mu(S)}{(1+\epsilon)^2} \quad ,$$

and since ϵ is arbitrary,

VII.3.15 $$\mu(B) \geqslant \mu(S) \quad .$$

Moreover, VII.3.13 implies

VII.3.16 $$\mu(B^*) \leqslant \frac{\mu(S)}{(1-\epsilon)^2} \quad .$$

Thus, for sufficiently small h and k, Dinghas' Lemma (Lemma VII.2) furnishes a $B^* \epsilon \overline{B}$ such that the fundamental eigenvalue of B^* is arbitrarily close to the fundamental eigenvalue of a sphere of equal volume.

Q.E.D.

CHAPTER III

SOME NUMERICAL RESULTS FOR ONE AND TWO REGION PROBLEMS

Section I. Some Examples of Steiner Symmetrization

In this section, all work is done in the Euclidean plane. By using some of the results of Chapter II and the known fundamental eigenvalues of certain plane regions, bounds are obtained for the fundamental eigenvalue of a rhombus with diameters a and $\frac{\sqrt{3}}{2}a$. By similar techniques, bounds can be obtained for the fundamental eigenvalue of many plane regions.

Table I is, to the best of the author's knowledge, a complete listing of those fundamental eigenvalues of plane regions which are known exactly. This tabulation is given by G. Polya and G. Szegő.[18]

Table I

	Region	Fundamental Eigenvalue
(1)	Circle[19], radius a.	$\frac{j_o^2}{a^2}$
(2)	Square, side a.	$\frac{2\pi^2}{a^2}$
(3)	Rectangle, sides a and b.	$\frac{\pi^2(a^2 + b^2)}{a^2 b^2}$

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[18] G. Polya and G. Szegő, op. cit., pp. 251-274. The fundamental eigenvalue is denoted by Λ^2 in their notation.

[19] j_o is the first zero of the J_o Bessel function, $\doteq 2.4048$.

(4) Semicircle[20], radius a . $\frac{j_1^2}{a^2}$

(5) Sector of a circle, radius a and angle (of opening) γ . $\frac{j_1^2 \pi^2}{\gamma^2 a^2}$

(6) Equilateral triangle, side a . $\frac{16\pi^2}{3a^2}$

(7) 45°, 45°, 90° triangle, sides a , a , $\sqrt{2}a$. $\frac{5\pi^2}{a^2}$

(8) 30°, 60°, 90° triangle, sides a , $a/2$, $\sqrt{3}/2\ a$. $\frac{112\pi^2}{9a^2}$

<u>Example I.1</u>. Let T be an equilateral triangle with side a and let L be a line parallel to one of the sides of T . Let

I.1.1 $$R = S_L(T) \quad .$$

Then R is a rhombus with diameters a and $\frac{\sqrt{3}}{2}a$. Let M be a line perpendicular to one of the sides of R . Then, if

I.1.2 $$Q = S_M(R) \quad ,$$

Q is a rectangle with sides $\sqrt{\frac{3}{7}}\,a$ and $\sqrt{\frac{7}{4}}\,a$. Table I yields

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[20] j_1 is the first zero of the J_1 Bessel Function, $\doteq 3.8317$.

I.1.3 $$\mu(T) = \frac{16\pi^2}{3a^2} ,$$

and

I.1.4 $$\mu(Q) = \frac{97}{7}\left[\frac{\pi^2}{3a^2}\right] .$$

By Theorem IV.1 of Chapter II,

I.1.5 $$\mu(T) \geqslant \mu(R) \geqslant \mu(Q) .$$

Therefore

I.1.6 $$16\left(\frac{\pi^2}{3a^2}\right) \geqslant \mu(R) \geqslant \frac{97}{7}\left(\frac{\pi^2}{3a^2}\right) .$$

That is

I.1.7 $$\frac{52.64}{a^2} > \mu(R) > \frac{45.58}{a^2} .$$

One is tempted to find a much more general class of transformations which will provide similar bounds on the fundamental eigenvalue. It is clear from the arguments of the preceding chapter that the fundamental eigenvalue is rather intimately related to the volume to surface area ratio. Unfortunately, as the following example shows, this relationship is not simply a proportionality.

Example I.2. In the Euclidean plane, the volume of a solid is the plane area and the surface area of a solid is its perimeter. If two solids can be constructed which have equal areas and equal perimeters but different fundamental eigenvalues, then the following is true. There exist three solids, F_1 , F_2 , and F_3 , such that

I.2.1 $$V_2(F_1) = V_2(F_2) = V_2(F_3) ,$$

I.2.2 $$A_1(F_2) \geqslant A_1(F_1) \geqslant A_1(F_3) ,$$

I.2.3 $$\mu(F_2) > \mu(F_1)$$

and

I.2.4 $$\mu(F_3) > \mu(F_1) .$$

Let T be an equilateral triangle of side a and let R be a rectangle with sides $\frac{a}{4}(3 - \sqrt{9 - 4\sqrt{3}})$ and $\frac{a}{4}(3 + \sqrt{9 - 4\sqrt{3}})$. Then

I.2.5 $$V_2(R) = V_2(T) ,$$

I.2.6 $$A_1(R) = A_1(T) ,$$

but

I.2.7 $$\mu(R) = \frac{\pi^2}{3a^2}(36 - 8\sqrt{3}) > \mu(T) = 16(\frac{\pi^2}{3a^2}) .$$

Therefore let F_1 be the equilateral triangle T, let F_2 be a rectangle with sides $\frac{a\sqrt{3}}{5}$ and $\frac{5a}{4}$, and let F_3 be a rectangle with sides $\frac{\sqrt{3}}{4}$ a and a. Then

I.2.8 $$V_2(F_1) = V_2(F_2) = V_2(F_3) = \frac{a^2\sqrt{3}}{4} ,$$

I.2.9 $$A_1(F_2) = a(\frac{2\sqrt{3}}{5} + \frac{5}{2}) > A_1(F_1) = 3a >$$

$$> A_1(F_3) = (2 + \frac{\sqrt{3}}{2})a ,$$

I.2.10 $$\mu(F_1) = \frac{16}{3}\frac{\pi^2}{a^2} < \mu(F_2) = \frac{673}{60}\frac{\pi^2}{a^2} ,$$

and

I.2.11 $$\mu(F_1) = \frac{16}{3}\frac{\pi^2}{a^2} < \mu(F_3) = \frac{19}{3}\frac{\pi^2}{a^2} .$$

Section II. An Approximate Computation of the Fundamental Eigenvalues of Certain Circular Segments

Let S be a semicircle of radius sixteen, where the origin is at the midpoint of the diameter of S. Let

II.1.1 $$S_k = \widehat{(x, y)} \left[(x, y) \epsilon S \text{ and } x \geqslant k \right] .$$

The problem is to obtain the fundamental eigenvalue of S_k for $0 \leqslant k \leqslant 5$. The solution of this problem is known only when $k = 0$, i.e. when the region is a semicircle. The differential equation which an eigenfunction f must satisfy is

II.1.2 $$f_{xx} + f_{yy} + \mu(S_k)f = 0 .$$

To obtain approximate values of $\mu(S_k)$ recourse was had to numerical techniques. It was noted from the symmetry of S_k about the line $y = 0$, that if f were an eigenfunction then

II.1.3 $$f(x, y) = f(x, -y) .$$

A mesh was then placed on the square with vertices (0, 0), (0, 16), (16,0), and (16, 16) with the mesh points being the points in the square with integers for coordinates. Figure I is a diagram of the mesh employed. While not all of these points lie in S_k it was convenient for computational purposes to retain all the points of the mesh.

The differential equation (II.1.2) was then replaced by a difference equation which gave rise to a set of two hundred and seventy-two

simultaneous equations of the form

II.1.4 $$-4\omega_{i,j} + \omega_{i+1,j} + \omega_{i-1,j} + \omega_{i,j+1} + \omega_{i,j-1} = -\mu(S_k)\,\omega_{i,j} \quad ,$$

where $\omega_{i,j}$ is the value of the function at the point (i, j) .

The boundary conditions for the difference equation are

II.1.5 $$\omega_{i,j} = 0 \qquad \text{for } i \leqslant k \ ,$$

II.1.6 $$\omega_{i,j} = 0 \qquad \text{for } i^2 + j^2 \geqslant 256 \ ,$$

and

II.1.7 $$\omega_{i,j} = \omega_{i,-j} \quad .$$

However, II.1.4 and II.1.6 would provide a poor approximation of the boundary. Therefore, II.1.4 was replaced by

II.1.8 $$-4\omega_{i,j} + \frac{\frac{1}{A}\,\omega_{i+1,j} + \frac{1}{B}\,\omega_{i,j+1} + \omega_{i-1,j} + \omega_{i,j-1}}{2 + \frac{1}{A} + \frac{1}{B}} = \mu(S_k)\omega_{i,j}$$

where A is the distance from (i, j) to the boundary in the x direction, or the distance to the next point in the x direction, which

ever is smaller, and B is the distance from (i, j) to the boundary in the y direction, or the distance to the next point in the y direction, which ever is smaller.

The matrix D of the coefficients of II.1.8 was found. Thus the approximate problem was to find the smallest eigenvalue of D. If $-\mu$ is the smallest eigenvalue of D and $-\lambda$ is the largest eigenvalue of D then, if $m > |\lambda|$ and ω is the largest root of $I + \frac{1}{m} D = E$, the eigenvectors of E and D are the same and

II.1.9 $$\mu = (1 - \omega)m \quad .$$

The largest root of E was computed by the Aitken[21,22] Iteration process. Figure 2 shows μ plotted as a function of k. The exact value for the semicircle is given and indicates, roughly, the error in the computed values. All values have been normalized to those for the case where the radius of S is one. From Chapter II, it is clear that a simple multiplication by 256 performs this normalization. Figure 3 shows the estimate of μ as a function of the number of iterations performed. Figure 4 is a plot of the level lines of the eigenfunction obtained by interpolating between the values of the eigenvector components.

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[21] A. C. Aitken, Proc. Roy. Soc. Edinburgh, 57 (1937) pp. 269-304.

[22] H. Hotelling, Ann. Math. Stat., 14 (1945), pp. 1-34.

Section III. Two Boundary Problems

Consider a pair of solids T and R in E_n with surfaces t and r such that R is contained in the interior of T. Let

III.1.1 $$S = (T - R) \cup r \quad ,$$

and then

III.1.2 $$s = t \cup r \quad .$$

Let

III.1.3 $$F = \hat{f} \left[f \text{ is a continuous function of } x_1, \ldots, x_n \text{ and a parameter } \lambda \text{, with piecewise continuous first derivatives with respect to } X \text{, and defined on } R \right] ,$$

and let

III.1.4 $$G = \hat{g} \left[g \text{ is a continuous function of } x_1, \ldots, x_n \text{ and a parameter } \lambda \text{, with piecewise continuous first derivatives with respect to } X \text{, defined on } S \text{ and which vanishes on } s \right] .$$

The two boundary problem consists of finding a function $f \epsilon F$ and a function $g \epsilon G$ such that

III.1.5 $$\nabla^2 f + \lambda^2 f = 0 \quad ,$$

in R ,

III.1.6 $$\nabla^2 + \lambda^2 g = 0 \quad ,$$

in S ,

III.1.7 $$f(X) = g(X) \quad ,$$

where $X \epsilon r$, and

III.1.8 $$k \left. \frac{\partial f}{\partial n_+} \right|_X = \left. \frac{\partial g}{\partial n_-} \right|_X \quad ,$$

where $\frac{\partial}{\partial n_+}$ is the outer normal derivative and $\frac{\partial}{\partial n_-}$ is the inner normal derivative, k is a constant, and III.1.7 and III.1.8 are satisfied for all $X \epsilon r$.

Assume $f \epsilon F$ and $g \epsilon G$ satisfy III.1.5 and III.1.6 respectively. Then, if a and b are arbitrary constants, $af \epsilon F$, $bg \epsilon G$, af satisfies III.1.5 and bg satisfies III.1.6. If af and bg are to be solutions of the two boundary problem they must satisfy III.1.7 and

III.1.8 for some value of λ . That is

III.1.9 $$af(X) = bg(X) \quad ,$$

and

III.1.10 $$ka \left.\frac{\partial f}{\partial n_+}\right|_X = b \left.\frac{\partial f}{\partial n_-}\right|_X \quad ,$$

for all $X \epsilon r$. For III.1.9 and III.1.10 to have a non-trivial solution, it is necessary that

III.1.11 $$\begin{vmatrix} f(X) & g(X) \\ k \left.\frac{\partial f}{\partial n_+}\right|_X & \left.\frac{\partial g}{\partial n_-}\right|_X \end{vmatrix} = 0 \quad ,$$

for all $X \epsilon r$. That is

III.1.12 $$f(X) \left.\frac{\partial g}{\partial n_-}\right|_X - kg(X) \left.\frac{\partial f}{\partial n_+}\right|_X = 0 \quad ,$$

for all $X \epsilon r$. Then III.1.12 is an equation in λ and is called the characteristic equation of the two boundary problem. It is clear that the square of the roots of this equation are the eigenvalues of the two boundary problem and it is assumed that the smallest non-zero

root of this equation is the smallest (fundamental) eigenvalue of the two boundary problem. Once λ is determined a and b are determined by III.1.9 and III.1.10 and af and bg form an eigenfunction associated with the fundamental eigenvalue. As in the one boundary problem, any multiple of an eigenfunction is again an eigenfunction.

Example III.2. If R is a sphere of radius a, $0 < a < 1$, with center at the origin and if S is the spherical shell of outer radius one and inner radius a, then the corresponding two boundary problem has been solved. Changing to spherical coordinates it is clear there will be no angular dependence in the eigenfunction. Indeed, if there were, then by rotating any eigenfunction one obtains a non-denumerable number of independent eigenfunctions belonging to the eigenvalue. Since each eigenvalue has only a finite multiplicity, this is clearly impossible. Thus, III.1.5 and III.1.6 become

III.2.1 $$f_{rr} + \frac{2}{r} f_r + \lambda^2 f = 0 \quad ,$$

and

III.2.2 $$g_{rr} + \frac{2}{r} g_r + \lambda^2 g = 0 \quad .$$

The interface conditions III.1.7 and III.1.8 become

III.2.3 $$f(a) = g(a) \quad ,$$

and

III.2.4 $$k\, f_r\big|_a = g_r\big|_a \quad .$$

Let

III.2.5 $$f(r) = \frac{\sin \lambda r}{r} \quad ,$$

and

III.2.6 $$g(r) = \frac{\sin \lambda(1 - r)}{r} \quad .$$

Then, $f \epsilon F$, $g \epsilon G$, f satisfies III.2.1 and g satisfies III.2.2. The characteristic equation is

III.2.7 $$(1 - k) \sin \lambda a \sin \lambda(1 - a) + a\lambda \left[\sin \lambda a \cos \lambda(1 - a) + k \cos \lambda a \sin \lambda(1 - a)\right] = 0 \quad .$$

It is clear that III.2.7 is an equation in $|\lambda|$. For $k = 1$, the problem reduces to a one region problem since the differential equation is satisfied at the interface. In this case, III.2.7 becomes

III.2.8 $$a\lambda \left[\sin \lambda a \cos \lambda(1 - a) + \cos \lambda a \sin \lambda(1 - a)\right] = 0 .$$

That is

III.2.9 $$\sin \lambda\left[a + (1 - a)\right] = \sin \lambda = 0 .$$

Therefore $\lambda = \pi$ is the smallest positive root which is exactly as expected.

Using Newton's method of approximating the roots of an equation, λ was computed for various values of k and a. Figure 5 shows λ as a function of a for five values of k. The slight non-linearity in the curve for $k = 1$ gives some indication of the amount of error in the computation. Figure 6 shows λ as a function of k for four values of a. Unfortunately, the Newton approximation did not converge for certain combinations of k and a, and therefore the range in a was cut.

<u>Example III.3</u>. Relaxing the condition that R and S be solids, consider next the situation where S is a pair of semi-infinite slabs and R is a semi-infinite slab between the two slabs of S. Specifically, R is the slab in E_3 defined by

III.3.1 $$-a \leqslant x \leqslant a ,$$

where $0 < a < 1$, and S is the pair of slabs defined by

III.3.2 $$-1 \leqslant x \leqslant -a \quad ,$$

and

III.3.3 $$a \leqslant x \leqslant 1 \quad .$$

The eigenfunction will, as in the previous case, be independent of two of the three variables. Therefore, III.1.5 and III.1.6 become

III.3.4 $$f_{xx} + \lambda^2 f = 0 \quad ,$$

and

III.3.5 $$g_{xx} + \lambda^2 g = 0 \quad .$$

The interface conditions III.1.7 and III.1.8 become

III.3.6 $$f(a) = g(a) \quad ,$$

III.3.7 $$f(-a) = g(-a) \quad ,$$

III.3.8 $$k\, f_x\big|_a = g_x\big|_a \quad ,$$

and

III.3.9 $$k\, f_x\big|_{-a} = g_x\big|_{-a} \quad .$$

Let

III.3.10 $$f(x) = \cos \lambda x$$

and let

III.3.11 $$g(x) = \sin \lambda (1 - |x|) \quad .$$

Then $f \epsilon F$, $g \epsilon G$, f satisfies III.3.4, and g satisfies III.3.5. Since f and g are both even functions of x , III.3.7 and III.3.9 may be omitted. The characteristic equation is then

III.3.12 $$k \sin \lambda a \sin \lambda(1 - a) - \cos \lambda a \cos \lambda(1 - a) = 0 \; .$$

Again, if $k = 1$, this becomes a single region problem. In this case III.3.12 becomes

III.3.13 $$\cos \lambda(a + (1 - a)) = \cos \lambda a = 0 \quad .$$

That is $\lambda = \frac{\pi}{2}$, the expected result.

The first positive root of III.3.12 was computed for various values of k and a by Newton's approximation. It is clear, as in the previous example, the characteristic equation is in $|\lambda|$. The characteristic equation in this example is also symmetric about $a = \frac{1}{2}$. Figure 7 shows λ as a function of a for twelve values of k, and Figure 8 shows λ as a function of k for five values of a. By the symmetry mentioned above, Figure 8 actually shows λ as a function of k for nine values of a.

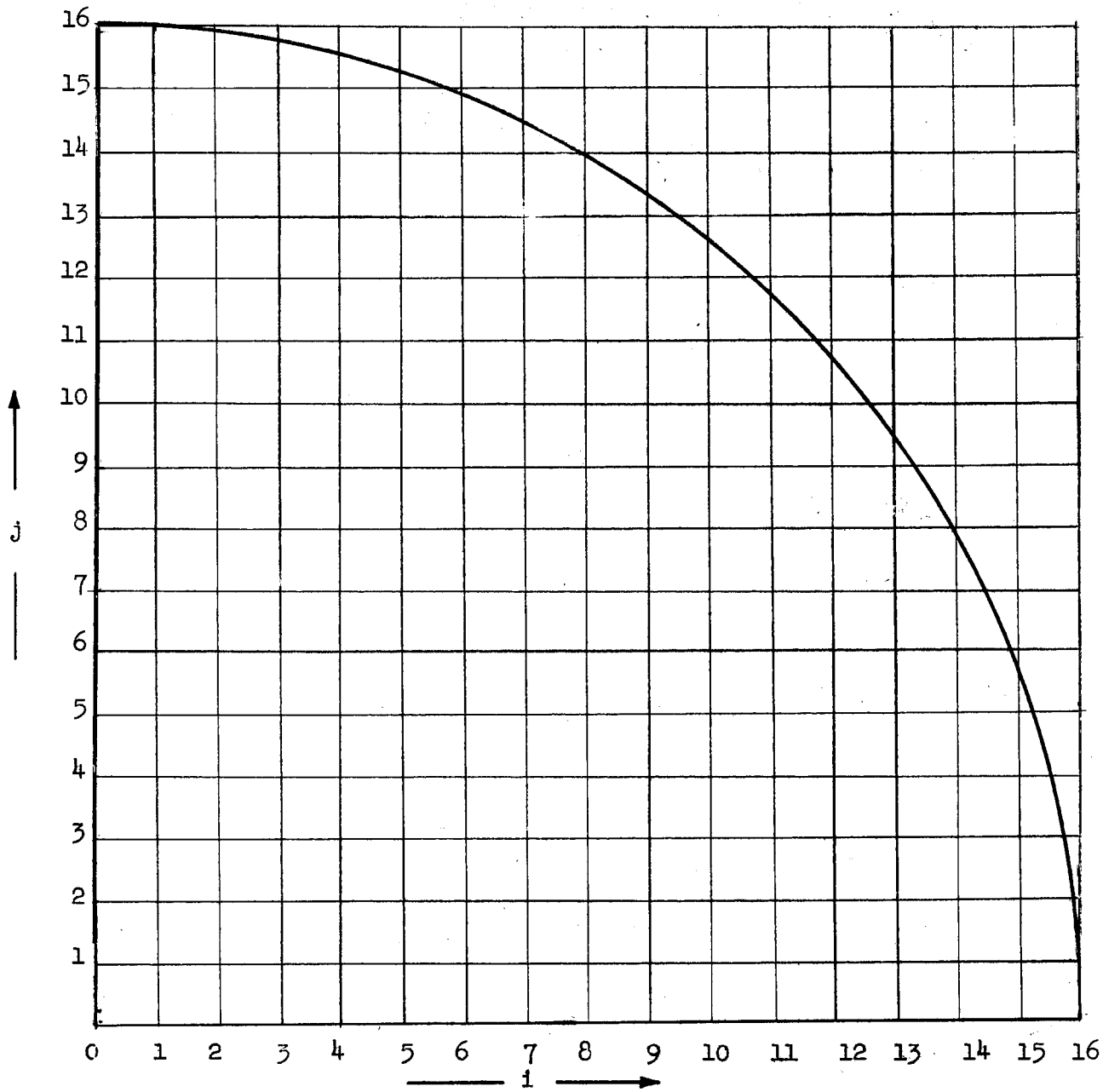

Figure 1. Mesh Points for Example II.1

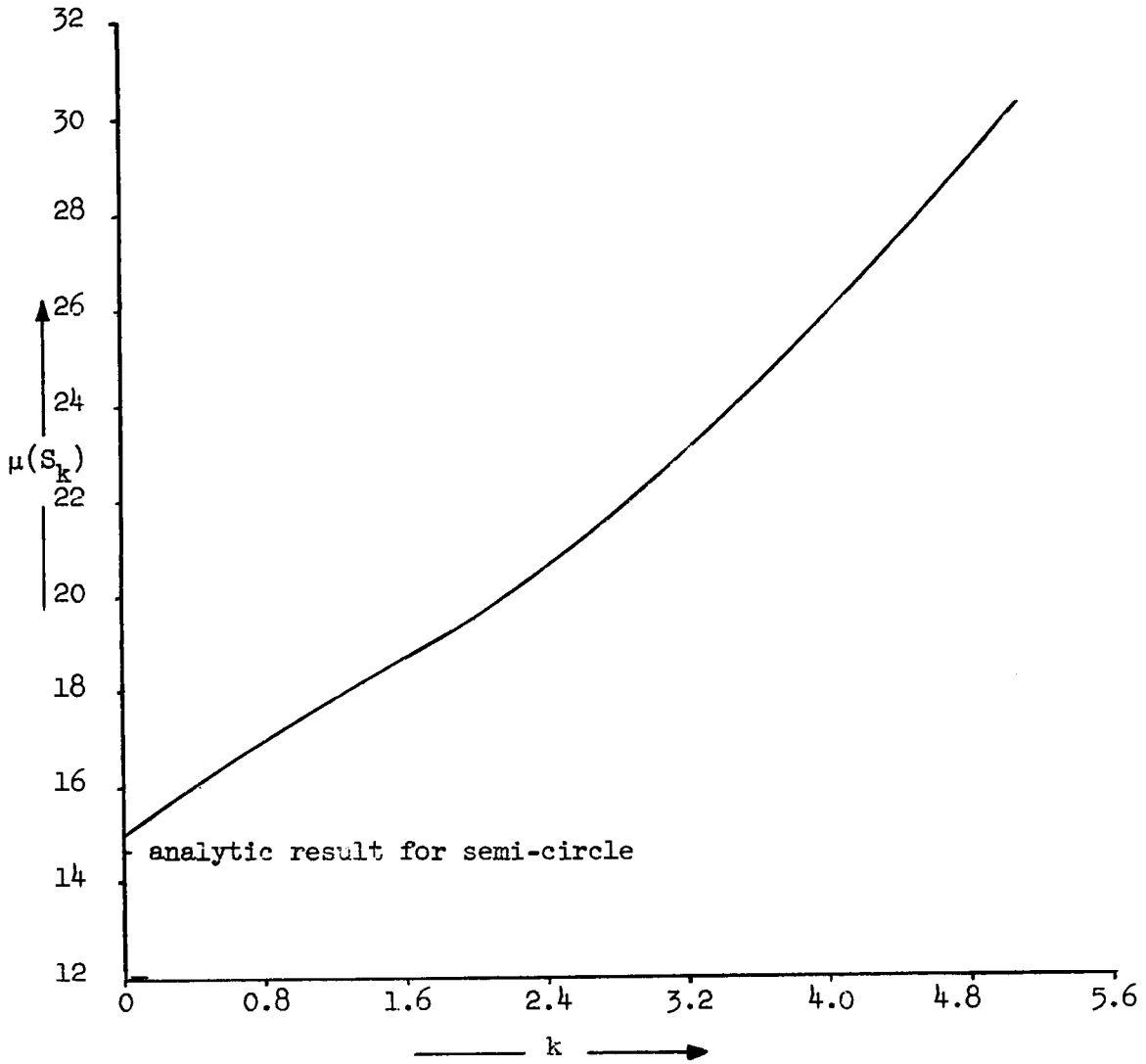

Figure 2. μ vs. k for Example II.1

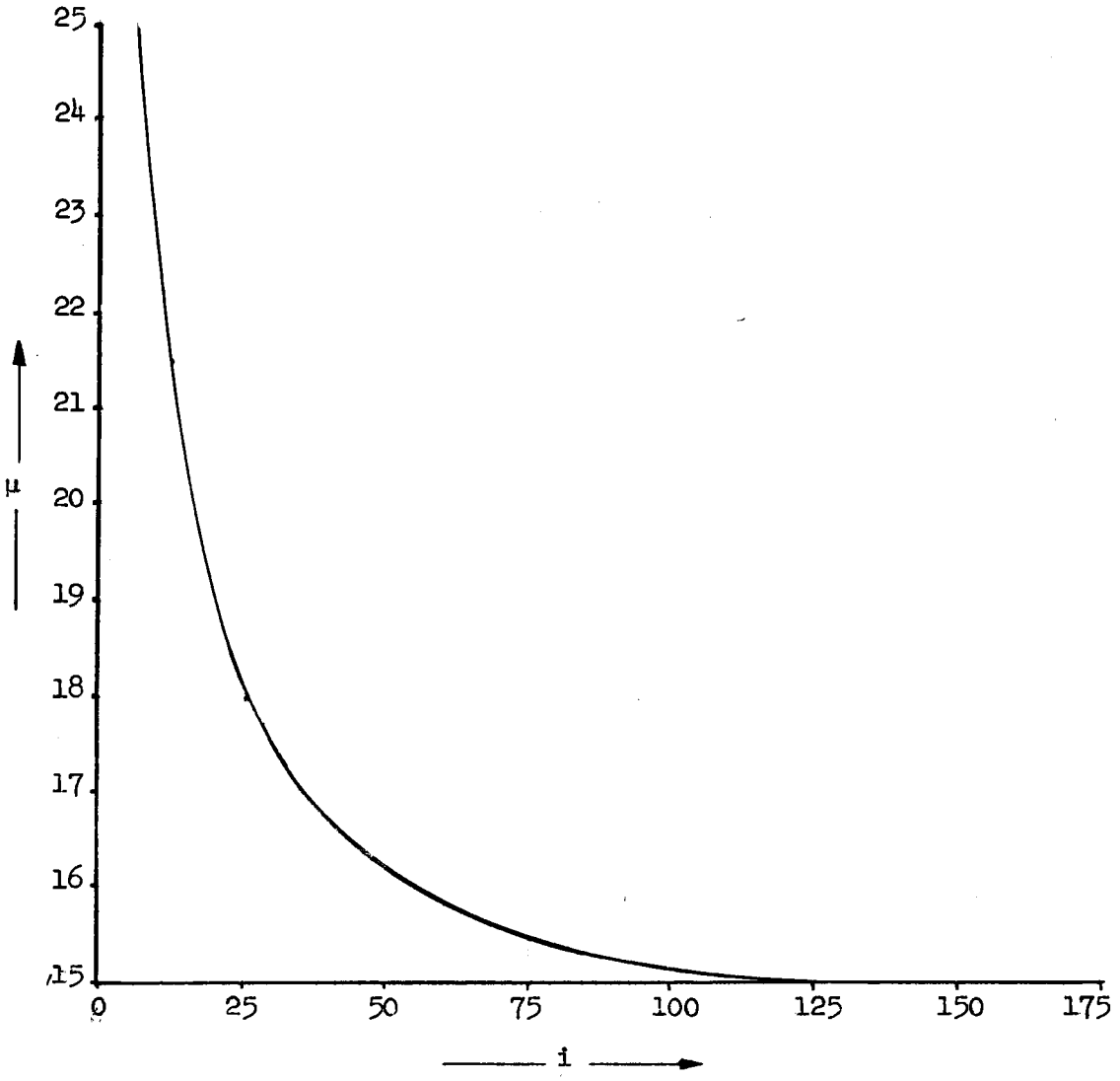

Figure 3. μ vs. Number of Iterations for Example II.1

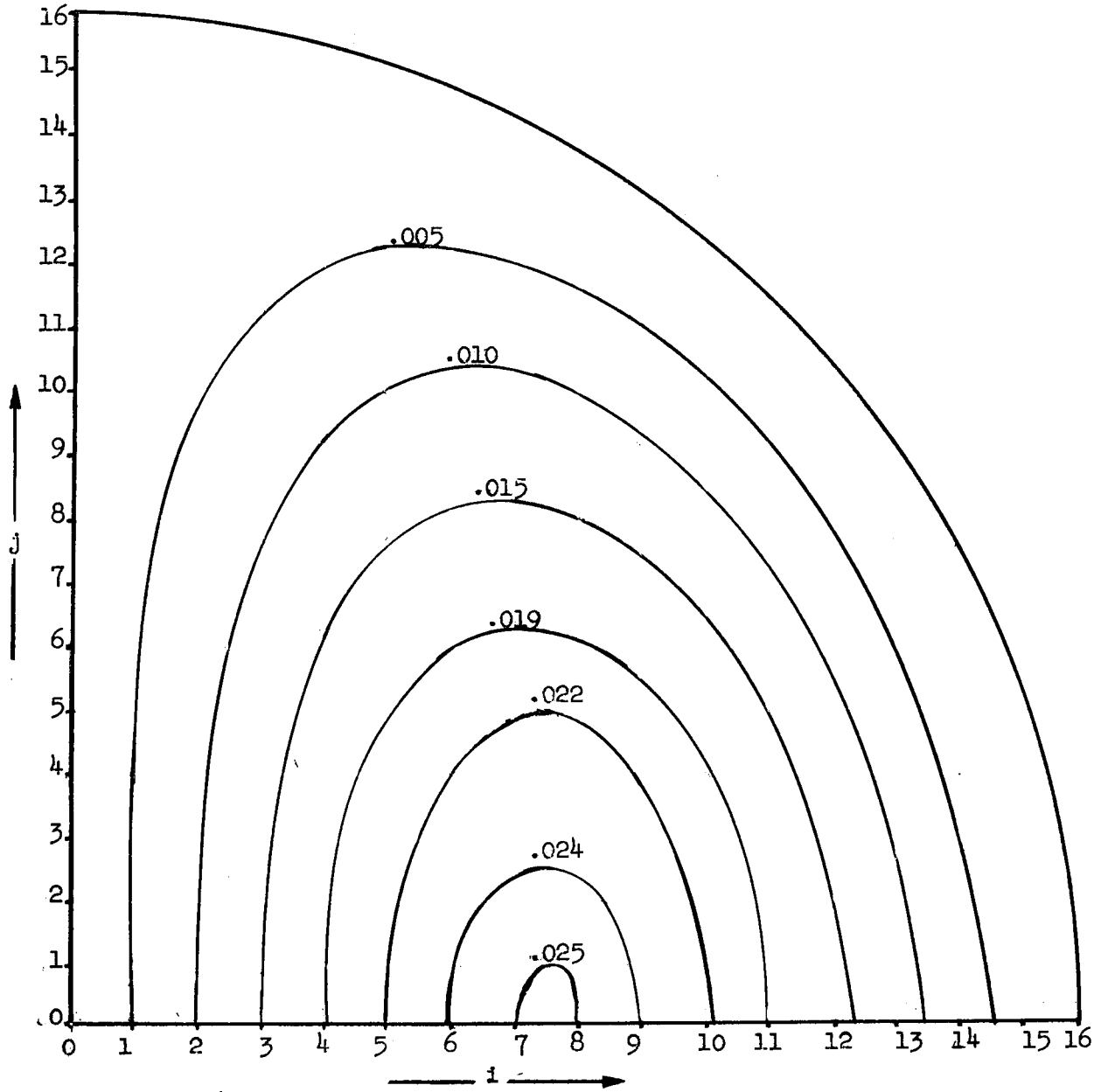

Figure 4. Lines of Constant Flux for Example II.1

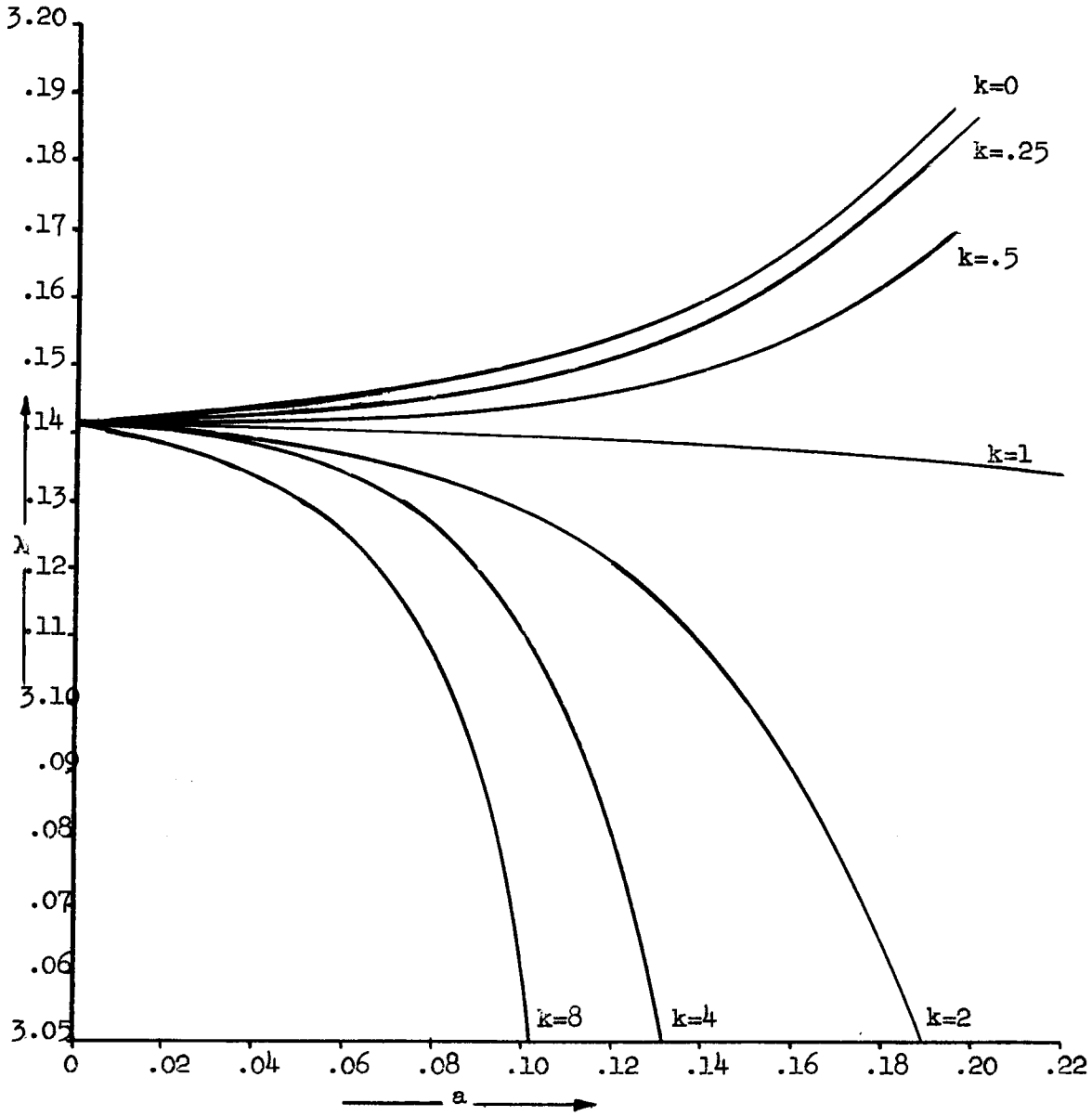

Figure 5. λ vs. a for Example III.2

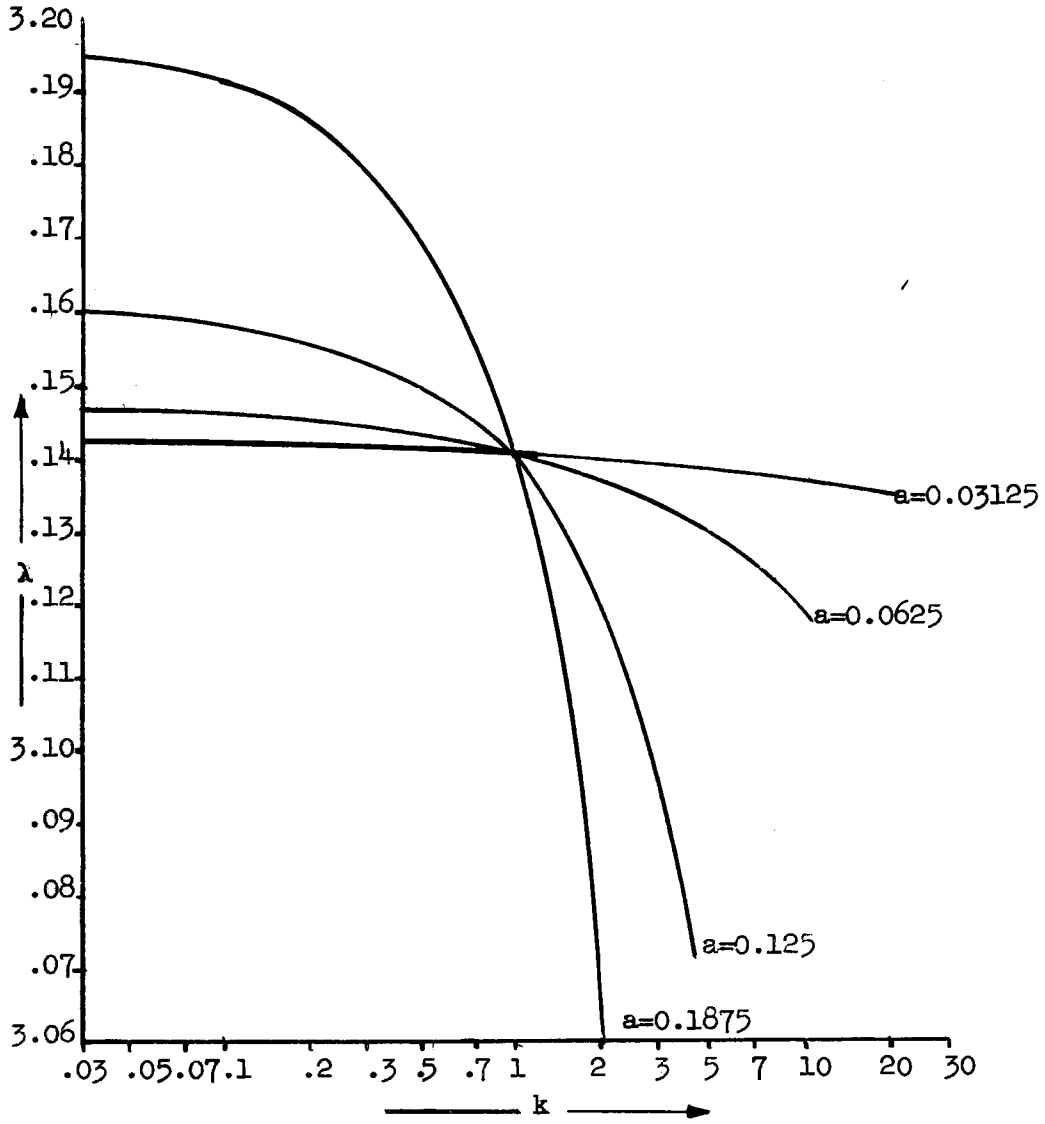

Figure 6. λ vs. k for Example III.2

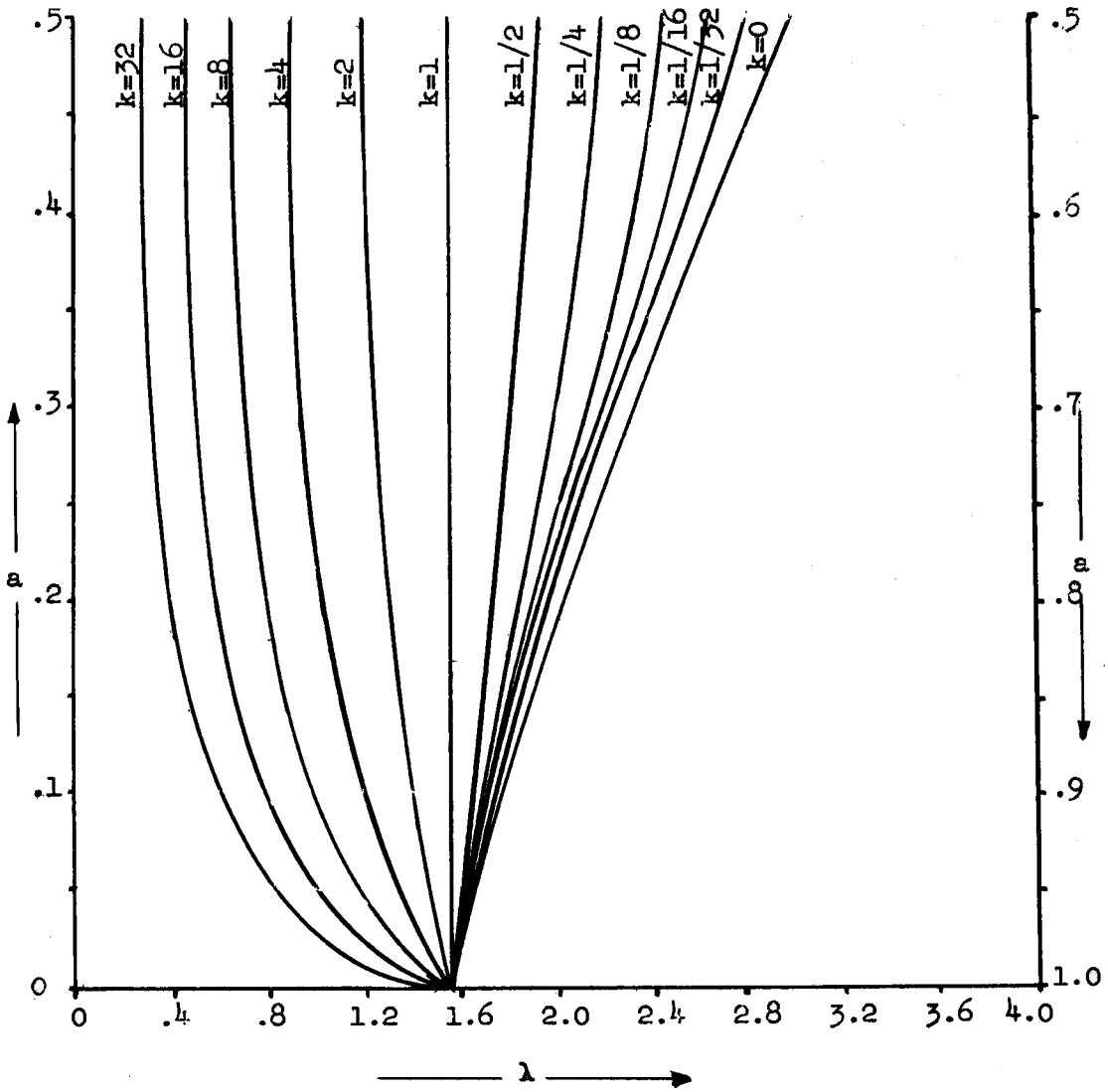

Figure 7. a vs. λ for Example III.3

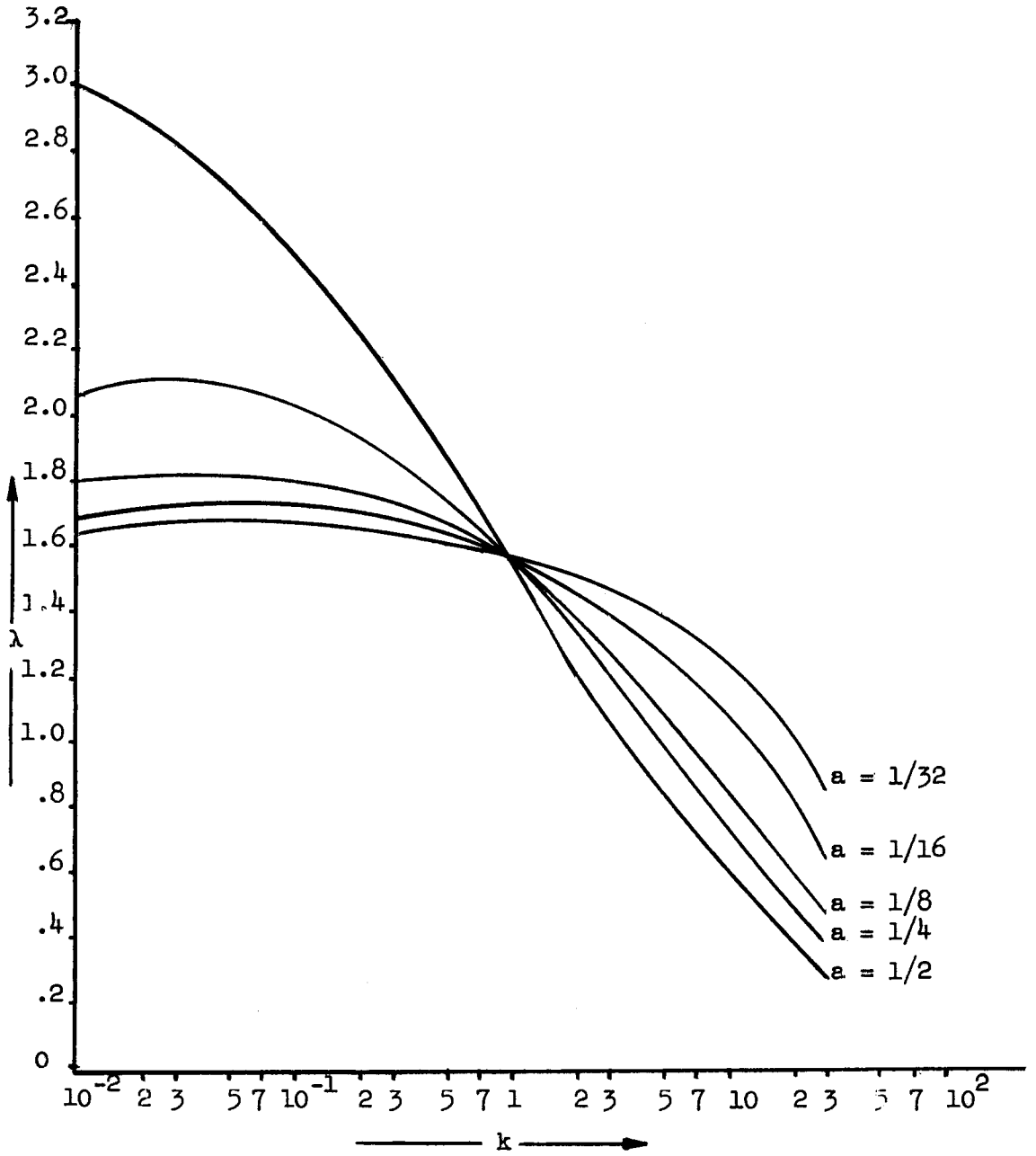

Figure 8. λ vs. k for Example III.3

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