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A MONTE CARLO METHOD OF CALCULATING THE RESPONSE OF A POINT DETECTOR AT AN ARBITRARY POSITION INSIDE A CYLINDRICAL SHIELD

C. D. Zerby

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A MONTE CARLO METHOD OF CALCULATING THE RESPONSE OF A POINT DETECTOR AT AN ARBITRARY POSITION INSIDE A CYLINDRICAL SHIELD

Abstract

A Monte Carlo method of calculating the response of a point detector at an arbitrary position inside a cylindrical shield is described. The general procedure is developed to obtain the maximum amount of information from each history and to minimize the statistical error.

The primary purpose of this report is to describe a Monte Carlo method of calculating the response of a point detector located at an arbitrary position inside a cylindrical shield. The secondary purpose, however, is instructive since several techniques will be described to improve the statistical estimates over the straight analogue procedure. In general the attempt has been to get the most out of each randomly generated particle history.

The approach taken in writing this report is to the reader who is not too familiar with the Monte Carlo method; however, some basic knowledge is assumed. The reader who is thoroughly familiar with the many "tricks" employed in Monte Carlo calculations will find nothing new, although the application made of the techniques may be of some interest.

In this problem there is only interest in the detector response resulting from the radiation coming through the curved surface of the shield. The contribution to the detector response resulting from radiation coming through the end shields is assumed to be known or to be computable.

In the particular problem of interest the radiation incident on the shield is uniformly distributed over the outside surface, and the angular distribution of the incident radiation will have cylindrical symmetry with the axis of symmetry parallel to the axis of the cylinder. The cylindrical cavity inside the shield is to be considered a void.

In Fig. 1 a cross section of the cylindrical shield is shown. If the circular ends of the shield are of sufficient thickness, t, then the radiation entering the cavity through the curved surface, A, B will be distributed almost uniformly over that surface. Making the assumption of uniformity of the entering current over the surface A, B simplifies the problem to a considerable extent so it will be made here. The simplification leads to considering the cylinder to be of infinite length in which case the contribution to the radiation entering the cavity is the same from each source element of area on the outside of the shield. In this case the incident radiation need be sampled at only one point on the outside surface and all penetrations into the cavity can be gathered to make up the angular distribution of the current into the cavity. A qualitative description of the procedure can be had by referring to Fig. 2 where a cross section of the cylinder is shown with three typical particle penetrations in the cavity resulting from a source at point A. The procedure is then to take the coordinates of the three particles and rotate and translate them in such a way that the particle paths have a common point of intersection with the inside surface and in this way make up the angular distribution of the entering current as shown in Fig. 3. In practice the direction at which the particle passes through the surface is referred to a coordinate system with the normal to the surface at the point of passage as one of the

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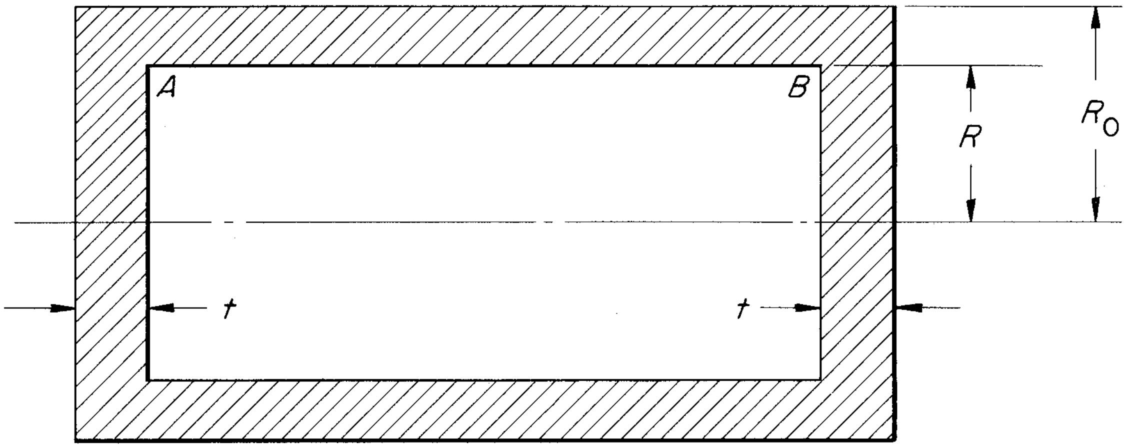

Fig. 1. Cross Section of Cylindrical Shield.

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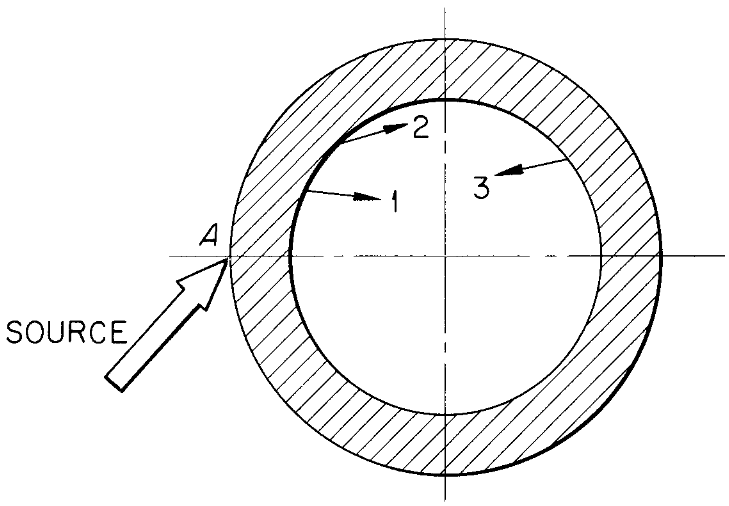

Fig. 2. Cross Section of Cylindrical Shield Showing Three Typical Particle Penetrations into the Cavity.

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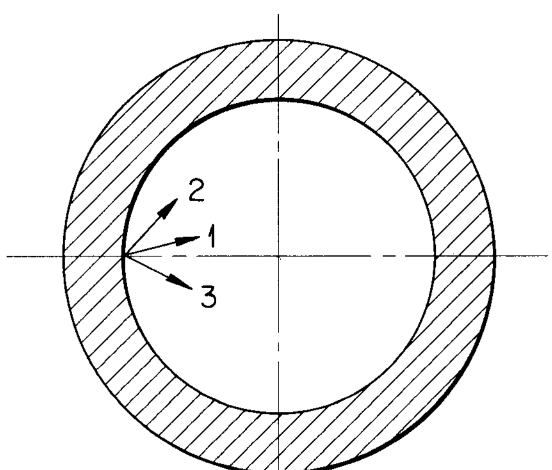

Fig. 3. Cross Section of Cylindrical Shield Showing Angular Distribution of Three Typical Particle Penetrations into the Cavity.

cartesian coordinates and axis of the cylinder as another. (This will be called the normal coordinate system in what follows.) No record is kept of its position relative to the source.

The particle current into the cavity can be calculated as follows:

$$T = \frac{\sum_i n_i(E_i)}{n_T(E_o)} \quad \frac{R}{R_o}, \tag{1}$$

where

E_o = source energy of the particles (Mev),

E_i = energy of the i^{th} particle that passes into the cavity (Mev),

$n_i(E_i)$*= 1,

$n_T(E_o)$ = total number of source particles,

R_o = outside radius of the cylinder (cm),

R = inside radius of the cylinder (cm),

T = particle current into the cavity at the inside surface normalized to one source particle per cm^2 at the outside surface.

The energy current is given by

$$K = \frac{\sum_i E_i \; n_i(E_i)}{E_o \; n_T(E_o)} \quad \frac{R}{R_o}, \tag{2}$$

where

K = energy current into the cavity at the inside surface normalized to unit source energy per cm^2 at the outside surface.

*The function $n_i(E_i)$ is introduced to indicate the values of the parameters associated with the i^{th} particle at the time it is counted. It always has the value one since it represents one particle.

The previous discussion appealed to the straight analogue technique for its understanding; however, improvement can be made on this simple method by statistical estimation of the integral representation of the penetrations into the cavity. Consider the definition

$\Phi(\vec{r},E,\vec{\Omega})d\vec{r}$ = number of particles per unit solid angle about the detector $\vec{\Omega}$ per unit energy at energy E after making a collision in the element of volume $d\vec{r}$.

Then the number of particles penetrating the inside surface is

$$L = \int_E \int_{\vec{\Omega}} \int_{\vec{r}} \Phi(\vec{r},E,\vec{\Omega})\, e^{-\Sigma_t(E)S(\vec{\Omega})}\, d\vec{\Omega}\, d\vec{r}\, dE, \tag{3}$$

where

$S(\vec{\Omega})$ = distance from position $\vec{r}$ to the inside surface along the direction $\vec{\Omega}$,

= ∞ if $\vec{\Omega}$ does not intercept the inside surface when extended,

$\Sigma_t(E)$ = total macroscopic cross section at energy E.

Clearly an estimate of L is

$$L = \sum_i \sum_j \frac{n_i(\vec{r}_{ij},E_{ij},\vec{\Omega}_{ij})}{n_T(E_o)}\, e^{-\Sigma_t(E_{ij})S(\vec{\Omega}_{ij})} = \sum_i \sum_j L_{ij}, \tag{4}$$

where

E_{ij} = energy of the i^{th} particle after the j^{th} collision,

$\vec{\Omega}_{ij}$ = the direction taken by the i^{th} particle after the j^{th} collision,

$\vec{r}_{ij}$ = the position of the i^{th} particle at the j^{th} collision,

$n_i(\vec{r}_{ij},E_{ij},\vec{\Omega}_{ij})^* = 1,$

*See footnote on page 4.

and L is normalized to one source particle and L_{ij} is defined in the equation.

To obtain the current into the cavity solve

$$T = \sum_i \sum_j L_{ij} \frac{R}{R_o} ,$$

where T is defined as before.

With this method of statistical estimation many of the collisions of each particle history will contribute to the estimate of T. It is obvious that this will give a better estimate of T over the estimate given in Eq. 1.

In practice each value L_{ij} is treated as a particle penetrating into the cavity and its coordinates are obtained relative to the normal coordinates. It should be noted that in this scheme a particle that passes through the inside surface during its history does not make a contribution to T.

An additional improvement can be had by letting the particle scatter at every collision in its history and weighting it accordingly by its probability of survival. The survival probability is given by

$$W_{k+1} = \frac{\Sigma_s(E_k)}{\Sigma_T(E_k)} , \tag{6}$$

where

E_k = energy of the particle after the k^{th} collision,

$\Sigma_s(E_k)$ = macroscopic scattering cross section at energy E_k,

$\Sigma_T(E_k)$ = macroscopic total cross section at energy E_k,

W_{k+1} = probability of surviving the $k + 1^{st}$ collision.

To include this process, rewrite Eq. 5 as follows:

$$T = \sum_i \sum_j \prod_{k=0}^{j} W_{ik} \frac{n_i(\vec{r}_{ij}, E_{ij}, \vec{\Omega}_{ij})}{n_T(E_o)} e^{-\Sigma_t(E_{ij})S(\vec{\Omega}_{ij})} = \sum_i \sum_j T_{ij} , \quad (7)$$

where

W_{ik} = probability that the i^{th} particle will survive the k^{th} collision,

and T_{ij} is defined in the equation.

By referring each "fractional particle" T_{ij} to the normal coordinates and adding those that enter a particular solid angle and energy interval, the complete energy spectrum and angular distribution of the current into the cavity can be obtained.

The detector response at a position inside the cavity can be obtained by statistical estimation of another integral which will be developed next.

In this problem the angular distribution of the source radiation has cylindrical symmetry as stated before, and because of the source symmetry, the angular distribution of the radiation current into the cavity also has cylindrical symmetry. In addition, it is assumed that the current is uniform over the inside surface of the cavity. Using these two conditions the integral representation of the detector response is easily developed.

The geometry and variables to be considered are shown in Fig. 4 which represents the cylindrical cavity in the shield. In Fig. 4 the vector $\vec{BC}$ lies along the normal to the surface at the element of surface area ds. In addition, the following definitions apply:

R = radius of the cavity,

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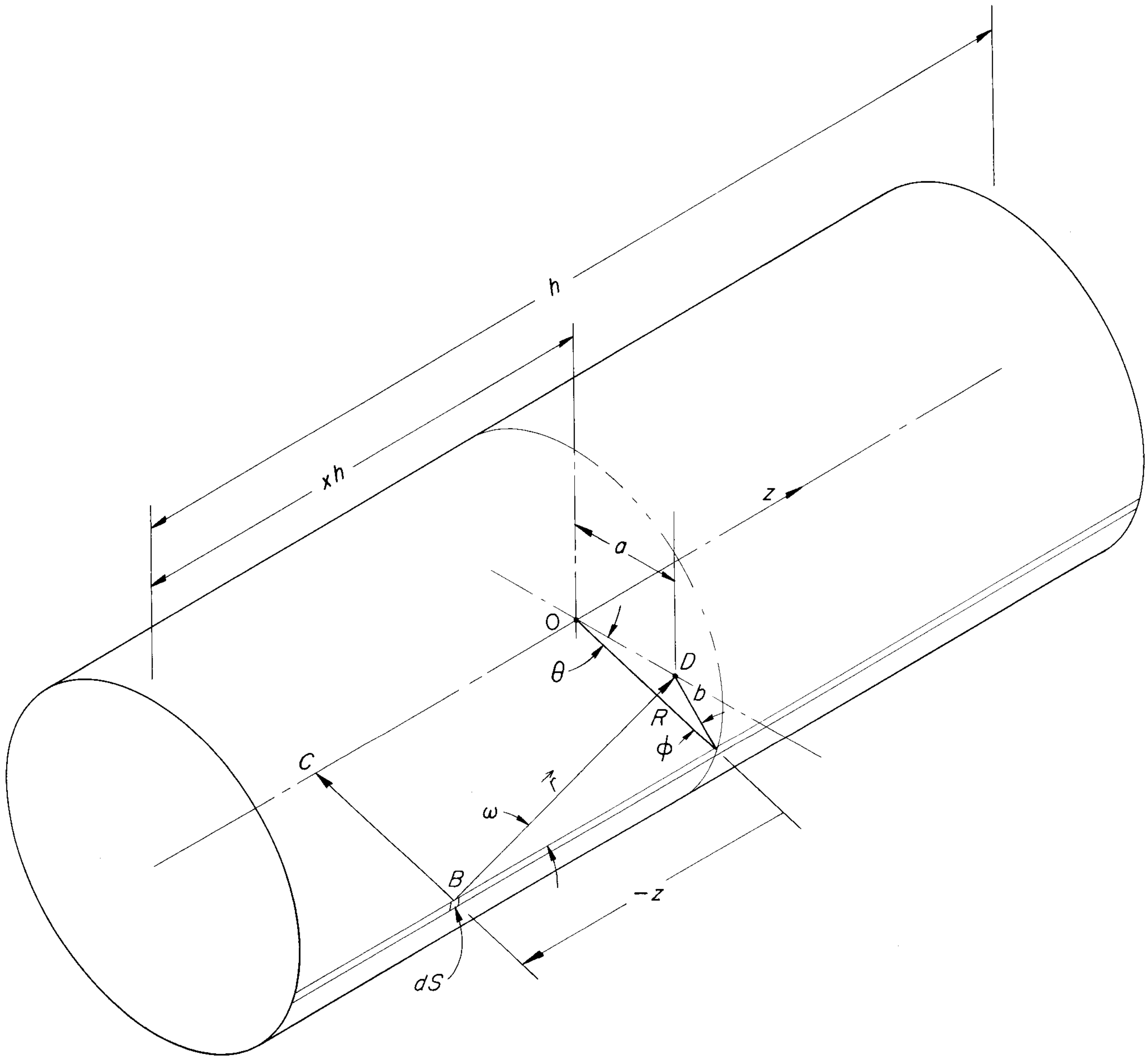

Fig. 4. Coordinate System for Determining the Detector Response in the Cavity of a Cylindrical Shield.

h = length of the cavity,

x = fractional part of length h to the plane containing the point of detection,

a = radial distance to the point of detection,

z = distance measured along the length of the cavity from the plane containing the point of detection,

$\vec{r}$ = vector from surface element ds to point of detection,

ω = polar angle between the z direction and the vector $\vec{r}$,

ϕ = azimuthal angle about the z direction measured from the normal at the element of area ds,

$P(E,\vec{\Omega})$ = particles per unit solid angle in direction $\vec{\Omega}$ per unit energy at energy E per unit surface area,

$K(E)$ = response of the detector to a particle with energy E.

The distance b and the angle θ are best described in Fig. 4.

The detector response at position D in Fig. 4 is given by

$$D = 2 \int_{E=0}^{\infty} \int_{\theta=0}^{\pi} \int_{z=-xh}^{h(1-x)} \frac{K(E)P(E,\vec{\Omega})Rd\theta dz dE}{r^2}, \tag{8}$$

where integration is over only one half the surface and multiplied by 2 to give the total detector response. Changing variables of integration by letting

$$\tan\omega = -\frac{z}{b} \tag{9}$$

and then using the relation $r^2\sin^2\omega = b^2$, Eq. 8 becomes

$$D = 2 \int_{E=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\omega=\tan^{-1}\frac{b}{xh}}^{\tan^{-1}\frac{b}{h(x-1)}} \frac{K(E)P(E,\vec{\Omega})Rd\theta d\omega dE}{b} . \tag{10}$$

The next step is to change the variable of integration from θ to ϕ. Starting with the law of sines

$$\frac{R}{\sin(\pi - \theta - \phi)} = \frac{a}{\sin\phi} , \tag{11}$$

by reduction one obtains

$$\frac{R}{a} = \sin\theta \operatorname{ctn}\phi + \cos\theta , \tag{12}$$

and differentiation of Eq. 12 gives

$$(\operatorname{ctn}\theta \operatorname{ctn}\phi - 1)d\theta = \frac{d\phi}{\sin^2\phi} . \tag{13}$$

Substituting into Eq. 13 for $\operatorname{ctn}\phi$ obtained from Eq. 12 gives the following equation after some reduction:

$$\frac{1}{\sin^2\theta}\left(\frac{R}{a}\cos\theta - 1\right)d\theta = \frac{d\phi}{\sin^2\phi} . \tag{14}$$

Using the relation

$$\frac{b}{\sin\theta} = \frac{a}{\sin\phi} , \tag{15}$$

Eq. 14 becomes

$$\frac{a^2}{b^2}\left(\frac{R}{a}\cos\theta - 1\right)d\theta = d\phi . \tag{16}$$

From the law of cosines one obtains

$$b^2 = R^2 + a^2 - 2aR\cos\theta\,, \tag{17}$$

which can be solved for the $\cos\theta$ and substituted into Eq. 16 to obtain

$$d\theta = \frac{2b^2\, d\phi}{R^2 - a^2 - b^2} \tag{18}$$

where b is a function of ϕ through the relation

$$a^2 = R^2 + b^2 - 2Rb\cos\phi \tag{19}$$

or

$$b = R\left\{\cos\phi \pm \sqrt{\frac{a^2}{R^2} - \sin^2\phi}\right\}. \tag{20}$$

Now let $k = \frac{a}{R}$ and make the following definitions:

$$b_1 = R\left\{\cos\phi - \sqrt{k^2 - \sin^2\phi}\right\} \equiv R\,B_1 \tag{21}$$

$$b_2 = R\left\{\cos\phi + \sqrt{k^2 - \sin^2\phi}\right\} \equiv R\,B_2 \tag{22}$$

where B_1 and B_2 are defined in Eqs. 21 and 22. Making the definition

$$\frac{h}{R} = p \tag{23}$$

and using Eq. 18, one obtains for Eq. 10 the following:

$$D(x,p,k) = 2\int_{E=0}^{\infty} K(E)dE \left\{ \int_{\omega=\tan^{-1}\frac{B_1}{xp}}^{\tan^{-1}\frac{B_1}{p(x-1)}} \int_{\phi=0}^{\sin^{-1}k} P(E,\vec{\Omega})R\left(\frac{b_1}{R^2 - a^2 - b_1^2}\right) d\omega d\phi \right.$$

$$\left. + \int_{\omega=\tan^{-1}\frac{B_2}{xp}}^{\tan^{-1}\frac{B_2}{p(x-1)}} \int_{\phi=\sin^{-1}k}^{0} P(E,\vec{\Omega})R\left(\frac{b_2}{R^2 - a^2 - b_2^2}\right) d\omega d\phi \right\} . \qquad (24)$$

Reduction can be obtained by using Eqs. 21 and 22 to give

$$\frac{b_1}{R^2 - a^2 - b_1^2} = \frac{1}{R\sqrt{k^2 - \sin^2\phi}} \qquad (25)$$

and

$$\frac{b_2}{R^2 - a^2 - b_2^2} = \frac{-1}{R\sqrt{k^2 - \sin^2\phi}} \quad . \qquad (26)$$

Using Eqs. 25 and 26 in Eq. 24 gives

$$D(x,p,k) = 2\int_{E=0}^{\infty} \int_{\phi=0}^{\sin^{-1}k} \frac{K(E)d\phi dE}{\sqrt{k^2 - \sin^2\phi}} \left\{ \int_{\omega=\tan^{-1}\frac{B_1}{xp}}^{\tan^{-1}\frac{B_1}{p(x-1)}} P(E,\vec{\Omega})d\omega \right.$$

$$\left. + \int_{\omega=\tan^{-1}\frac{B_2}{xp}}^{\tan^{-1}\frac{B_2}{p(x-1)}} P(E,\vec{\Omega})d\omega \right\} , \qquad (27)$$

where the integration is over one-half the distribution and is multiplied by 2 to give the complete contribution.

Multiply numerator and denominator of Eq. 27 by $\sin \omega$ and remember $P(E,\vec{\Omega}) \sin\omega \, d\omega \, d\phi dE$ represents a number of particles. Then the statistical estimate of D is

$$D(x,p,k) = \sum_i \sum_j \frac{T_{ij}(E_{ij},\omega_{ij},|\phi_{ij}|)K(E_{ij})}{\sin \omega_{ij}\sqrt{k^2 - \sin^2\phi_{ij}}} \; (Q_1 + Q_2) \equiv \sum_i \sum_j D_{ij}, \quad (28)$$

where

$$Q_1 = 1 \quad \text{when} \begin{cases} 0 \leq \sin|\phi_{ij}| < k \\ \tan^{-1}\dfrac{B_1}{xp} \leq \omega_{ij} \leq \tan^{-1}\dfrac{B_1}{p(x-1)} \end{cases}$$

$$= 0 \text{ otherwise},$$

$$Q_2 = 1 \quad \text{when} \begin{cases} 0 \leq \sin|\phi_{ij}| < k \\ \tan^{-1}\dfrac{B_2}{xp} \leq \omega_{ij} \leq \tan^{-1}\dfrac{B_2}{p(x-1)} \end{cases}$$

$$= 0 \text{ otherwise},$$

$D(x,p,k)$ = the detector response at position located by the parameters x, p, and k and normalized to unit source strength per unit outside surface area.

The quantity D_{ij} is defined in the equation. The factor 2 is removed in Eq. 28 because T_{ij} gives an estimate of the complete current distribution and the absolute value appears on the variable ϕ_{ij} because of the cylindrical symmetry of the distribution.

With this method of computation the direction relative to the normal coordinates is established by the variables ω and ϕ as shown in Fig. 4.

A singularity appears in Eq. 28 when $\sin \phi_{ij} = k$, which cannot be avoided. It is not unusual to find such a singularity when trying to calculate a detector response when the problem is simplified by considering at least one dimension to be infinite. The infinity can be at least removed by defining the Q's to have the value one only if $\sin \phi_{ij} < k$ (see Appendix A); however, unusual statistical fluctuations in the estimate of the current distribution may still cause wide fluctuations in the estimate of D if $\sin \phi_{ij}$ is close to the value k too frequently. The best procedure is to solve simultaneously the detector response at various values of k, keeping x and p constant, and to plot the results to remove wide variations. It should be noted that it is possible through Eq. 28 to find the detector response for many different values of k, p, and x from a given set of values T_{ij}.

The definition of the Q's indicate for which values of k the best estimates of D are obtained. It is obvious that the smaller the value of k the fewer values of T_{ij} that will contribute, while the larger the value of k the more values of T_{ij} that will contribute. In fact, at $k = 0$ it is not possible to get an estimate of the detector response. This means the response of the detector at k near zero should be obtained by another method. This can be done by gathering the values T_{ij} to get an estimate of $P(E,\omega,\phi)$ as defined for Eq. 27, plotting the results, and numerically performing the integration. For the case $k = 0$ Eq. 27 reduces to

$$D(x,p,k=0) = 2\pi \int_{E=0}^{\infty} \int_{\omega=\tan^{-1}\frac{1}{xp}}^{\tan^{-1}\frac{1}{p(x-1)}} K(E)P(E,\omega,\phi=0)\, d\omega dE. \qquad (29)$$

In this particular problem there are many different source distributions for which the detector response is to be calculated; therefore, it is better to solve a series of problems with a general source and integrate the separate results to give the solution for any particular source. In Fig. 5 an x,y,z coordinate system is constructed at an arbitrary position on the outside surface of the cylinder. The polar angle α and azimuthal angle ψ describe the negative direction of a particular incident particle in that coordinate system. The incoming radiation will be independent of the angle ϕ because of the cylindrical symmetry, thus, a general source that may be used is one that is monoenergetic with energy E_e and with a fixed polar angle α_e but uniformly distributed in the azimuthal angle ψ. The response D(x,p,k) of a particular detector is then calculated for a series of these general sources where the set of variables α_e and E_e are selected to span the range of interest. A smooth plot of the results of these separate calculations gives

$D(x,p,k,\alpha,E)$ = response of a detector located by the parameters x, p, and k, resulting from a unit source per unit outside surface area which has incident angle α and incident energy E and is uniformly distributed in the angle ψ.

Defining the source as

$S(\alpha,E)$ = incident particles per unit outside surface area per unit solid angle in the negative direction α per unit energy at energy E,

the response for the source $S(\alpha,E)$ is obtained by numerically performing the integral

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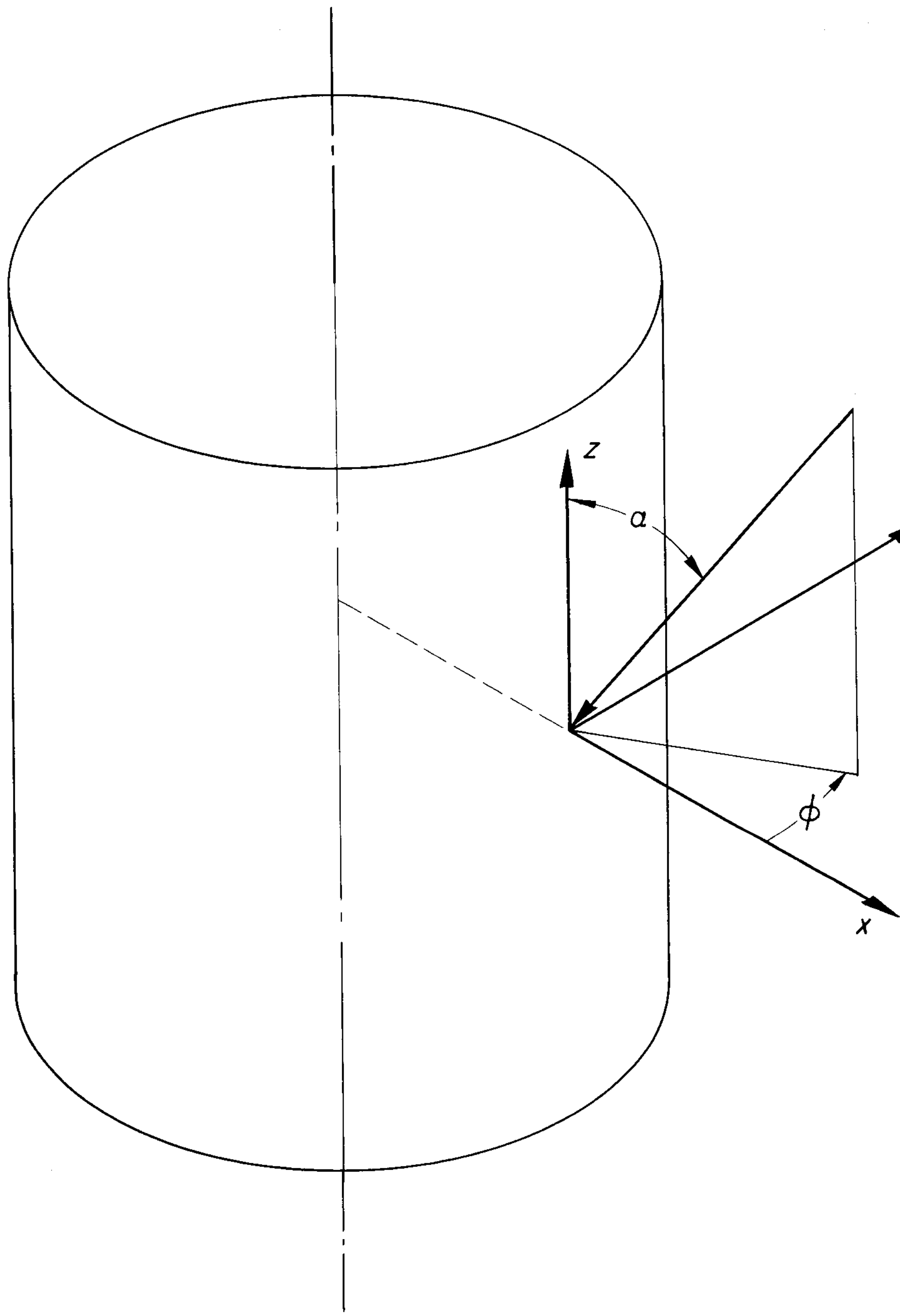

Fig. 5. Coordinate System for the Radiation Source on the Outside Surface of a Cylindrical Shield.

$$D_s(x,p,k) = \int_{E=0}^{\infty} \int_{\alpha=0}^{\pi} \pi S(\alpha,E) D(x,p,k,\alpha,E) \sin\alpha \, d\alpha \, dE . \tag{30}$$

Some simplification can be achieved with a reduction in the number of cases to be run if for each set of T_{ij} the detector response is obtained for a symmetrical pattern of detectors such that for every detector response $D(x,p,k,\alpha_e,E_e)$ one also obtains the response $D(1-x,p,k,\alpha_e,E_e)$. Then by the symmetry of the situation Eq. 30 can be altered as follows:

$$D_s(x,p,k) = \int_{E=0}^{\infty} \int_{\alpha=0}^{\frac{\pi}{2}} \pi S(\alpha,E) D(x,p,k,\alpha,E) \sin\alpha \, d\alpha \, dE$$

$$+ \int_{E=0}^{\infty} \int_{\alpha=\frac{\pi}{2}}^{\pi} \pi S(\alpha,E) D(1-x,p,k,\pi-\alpha,E) \sin\alpha \, d\alpha \, dE . \tag{31}$$

This means that the set of values α_e picked for the general cases need only span the interval $\left[0, \frac{\pi}{2}\right]$ which reduces the number of cases by one half.

The variance on the estimate of $D(x,p,k)$ is given by

$$V = \frac{n_T \sum_i D_i^2 - \left(\sum_i D_i\right)^2}{n_T} \tag{32}$$

where

n_T = total number of source particles,

$D_i = \sum_j D_{ij}$.

The variance on the estimate of T is given by

$$V = \frac{n_T \sum_i T_i^2 - \left(\sum_i T_i\right)^2}{n_T} , \tag{33}$$

where

$$T_i = \sum_j T_{ij} .$$

In conclusion a few statements should be made about the coordinate system used. In general it is better to generate a history relative to a fixed coordinate system which, in this case, shall be selected as the X, Y, Z coordinates shown in Fig. 6. Six numbers are carried at all times to characterize the path and position of the state of the history. The numbers are the three position coordinates x, y, z of the last collision point and the three direction cosines α, β, γ of the path after the collision with the respective coordinates X, Y, Z. Now, because the cylinder is infinitely long, the coordinate z can be set equal to zero after each collision since none of the probabilities depend on that coordinate; this avoids problems in scaling when coding the problem for machine computation.

Another point to be made is on the simplified way of handling a particle that crosses into the cavity at some time in its history. Since the cavity is void, one need not calculate any attenuation. In addition, because of the symmetry and because the absolute value of ϕ_{ij} is used in Eq. 28, it is only necessary to determine the point at which the particle leaves the inside surface with direction cosines α, β, γ, reassign the direction cosines as $-\alpha$, $-\beta$, γ, and have it reenter the shield at the point of exit.

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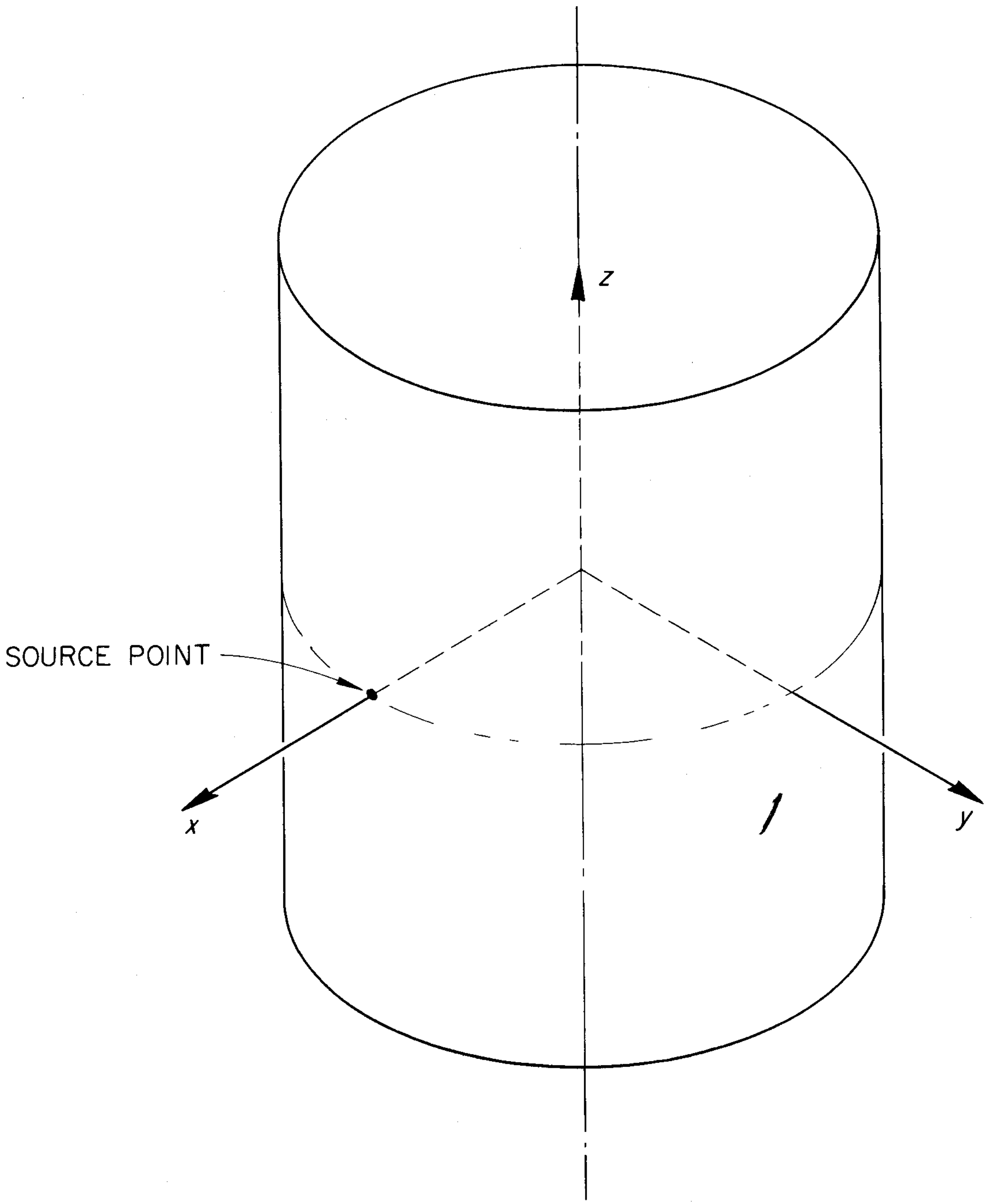

Fig. 6. Fixed Coordinate System for Tracking Random Particle Histories in a Cylindrical Shield.

Appendix A

Maximum Error Introduced by Removing Singularity in Eq. 28

In Eq. 28 there appears a singularity that can be removed if Q is defined to have the value one if

$$0 \leq \sin |\phi_{ij}| \leq k\epsilon, \tag{34}$$

where $0 < \epsilon \leq 1$. An upper limit to the error introduced can be had if $P(E,\vec{\Omega})$ is expanded in a series as

$$P(E,\vec{\Omega}) = \sum_{n=0}^{j} A_n(E) \cos^{j-n}\phi \sin^{j-n}\omega \cos^n\omega \, . \quad \begin{matrix} j \geq 1 \\ j=1,2,3,4\ldots \end{matrix} \tag{35}$$

Now substitute Eq. 35 into Eq. 27, assume the cylinder to be infinitely long, replace the upper limit of the integral over the ϕ variable by $\sin^{-1}\epsilon k$, and make the transformation

$$\sin\phi = k \operatorname{Sin}\psi \tag{36}$$

to obtain

$$D(k) = \sum_{n=0}^{j} A_n(E,\omega) \int_{n=0}^{\sin^{-1}\epsilon} (1 - k^2 \sin^2\psi)^{\frac{j-n-1}{2}} d\psi \,, \tag{37}$$

which clearly shows the dependence of D on the value of ϵ. The correct value of D is obtained if $\epsilon = 1$. Thus the fractional error is

$$\frac{\Delta D(k)}{D(k)} = \frac{\sum_{n=0}^{j} A_n(E,\omega) \int_{\sin^{-1}\epsilon}^{\pi/2} (1 - k^2 \sin^2\psi)^{\frac{j-n-1}{2}} d\psi}{\sum_{n=0}^{j} A_n(E,\omega) \int_{0}^{\pi/2} (1 - k^2 \sin^2\psi)^{\frac{j-n-1}{2}} d\psi} \, . \tag{38}$$

To obtain an upper limit of this fractional error note that

$$\cos^{j-1}\psi \leq \cos^{j-n-1}\psi \leq (1 - k^2 \sin^2\psi)^{\frac{j-n-1}{2}} \leq 1. \tag{39}$$

Using these inequalities gives an upper limit to the fractional error of

$$E = \frac{\Delta D(k)}{D(k)} = \frac{\frac{\pi}{2} - \sin^{-1}\epsilon}{\int_0^{\frac{\pi}{2}} \cos^{j-1}\psi} = \frac{\frac{\pi}{2} - \sin^{-1}\epsilon}{\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)}} . \tag{40}$$

Setting the error at 0.01 and picking the maximum value of j as 5 gives the value of ϵ as

$$\epsilon = 0.999990 \tag{41}$$

which gives a maximum value for the factor in Eq. 28 as

$$\left(\frac{1}{\sqrt{k^2 - \sin^2\phi_{ij}}}\right)_{max} = \frac{1}{k\sqrt{1 - \epsilon^2}} = \frac{223.6}{k} . \tag{42}$$