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ON PSEUDO-INVERSES OF LINEAR TRANSFORMATIONS IN BANACH SPACES

R. D. Sheffield

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MATHEMATICS PANEL

ON PSEUDO-INVERSES OF LINEAR TRANSFORMATIONS IN BANACH SPACES

Roy Dexter Sheffield

DATE ISSUED JUL 19 1956

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INTRODUCTION

A singular linear transformation L does not have an inverse. Nevertheless, it is always possible to find a partial or pseudo-inverse M of L, that is, a linear transformation M such that $LML = L$. If L^{-1} exists, of course, then L^{-1} is a pseudo-inverse, for $LL^{-1}L = L$.

From the equation $LML = L$ one can deduce an important application of the pseudo-inverse. Suppose that the linear equation

$$Lx = y_o \quad ,$$

in which y_o denotes a given vector, is known to be consistent; that is, there exists x_o such that $Lx_o = y_o$. Then since

$$LM\, y_o = LML\, x_o = Lx_o = y_o \quad ,$$

it follows that My_o is a solution of the given linear equation.

A pseudo-inverse is also useful in solving the homogeneous equation $Lx = 0$. Since L is singular, $I - ML \neq 0$. If z is in the domain of L, then

$$L(I - ML)z = Lz - LMLz = 0 \quad ;$$

in other words, $x = (I - ML)z$ is a solution of the homogeneous equation $Lx = 0$. In fact, $I - ML$ projects X onto the null space of L.

In the finite-dimensional case L is a matrix, and an explicit calculation is presented (Chapter II) for a pseudo-inverse M of L. This procedure is suitable for machine programming and leads to an M with the desirable feature that the non-zero columns of $I - ML$ form a basis for the null space of L.

K. O. Friedrichs [5] (1) is credited with coining the term pseudo-inverse; he applied the notion to various problems in applied mathematics. H. Hamburger [7] called M a partial inverse and applied it in his study of non-symmetric operators in Hilbert space. As an algebraic concept with perhaps a different name, the pseudo-inverse appears in several works on linear algebra. The basic algebraic discussion of this topic which best suits the needs of this paper is given by Reinhold Baer [1]. The available material on linear transformations in Banach spaces is extensive (cf. [2], [8], [13]).

Chapter I presents notation and basic definitions.

In Chapter II the algebraic properties of singular linear transformation are investigated. Particular use is made of the notions of a complementary transformation and of an expanded range. Chapter II concludes with the finite-dimensional calculation mentioned earlier.

Since linear transformations in infinite-dimensional linear topological spaces are not necessarily continuous, one is led to investigate properties of a linear transformation which assure the existence of

---

(1) Numbers in brackets refer to the bibliography at the end of this paper.

a continuous pseudo-inverse. This investigation for normed spaces constitutes the major part of Chapter III.

In Chapter IV it is shown that under proper conditions (slightly more general than those used by R. G. Bartle in [3]) a pseudo-inverse can be used to prove the existence and uniqueness of the solution of certain nonlinear functional equations. As an illustration of the theory, an existence and uniqueness theorem is given for the nonlinear differential equation

$$x^{(n)}(t) = f\left[\alpha, t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right] .$$

An abstract application of the theory is presented in Chapter V. A generalization of the Riesz-Schauder spectral theory of compact linear transformations is developed in a manner somewhat analogous to the classical method but more nearly along lines suggested by Chapters II and III of this paper. This study was suggested by a recent paper of L. M. Graves [6] ; however, a deviation in an early definition (cf . 501) eventually forced the investigation into a different vein, leading to the notion of a pseudo-resolvent.

CHAPTER I

FORMAL PRELIMINARIES

Notation

X , with elements x , denotes a Banach space; that is, X is a normed linear space, and in the topology induced by the norm, X is a complete space. X^* denotes the Banach space of all bounded linear functionals $x^*(x)$ defined on X .

The notation $\{x;\ P,\ Q,\ \ldots\}$ indicates the set of all elements x with properties P, Q , et cetera. That S is a proper subset of T is indicated by $S \subset T$. For a subset S of X , $S^{\perp} = \{x^*;\ x^* \in X^*,\ x^*(x) = 0 \text{ for all } x \in S\}$; similarly for $T^{\perp}$, $T \subseteq X^*$.

When S is a linear subspace of X , the symbol X/S is used to denote the set of equivalence classes obtained by defining x_1 to be equivalent to x_2 if and only if $x_1 - x_2 \in S$. Under the usual definitions, X/S is a linear space; the elements of X/S are called cosets of S . If S is a closed linear subspace of X , and if the norm of a coset $\xi \in X/S$ is defined by

$$\|\xi\| = \inf_{x \in \xi} \|x\| ,$$

it is known [12, Chapter 6, page 99] that X/S is a Banach space.

It is known [1, Chapter 1, page 12] that every linear subspace S of the space X has a complementary subspace $T \subseteq X$. This means

that an arbitrary element $x \in X$ has a unique decomposition as $x = s + t$, with $s \in S$ and $t \in T$; or equivalently, that $S \cap T = 0$ (the zero vector) and $S + T = X$. It should be noted that a complement is ordinarily not unique.

Let X and Y denote two Banach spaces and $y = L(x)$ a linear transformation, not necessarily bounded, with domain $D(L) \subseteq X$ and range $R(L) \subseteq Y$. $N(L) = \{x;\ x \in D(L),\ Lx = 0\}$ is called the null space of the linear transformation L. Evidently $N(L)$ is a linear subspace of $D(L)$, and if L is bounded, then $N(L)$ is a closed linear subspace of $D(L)$. If L is bounded, its domain of definition can be extended to the boundary of $D(L)$ by defining for an arbitrary boundary point x_o,

$$L(x_o) = \lim_{n \to \infty} L(x_n) \quad ;$$

where x_n is any sequence contained in $D(L)$ and converging to x_o. When L is bounded, it can therefore be assumed without loss of generality that $D(L)$ is a closed linear subspace of X; moreover, since a closed linear subspace of a Banach space is a Banach space, it can and will be assumed that $D(L) = X$. This convention is used also whenever the discussion is purely algebraic in nature.

In conformity with modern terminology (Dieudonne [4], Yood [11]), when L is bounded and when $R(L)$ is a closed linear subspace of Y, L is termed a homomorphism. F. J. Murray [10] has constructed a closed

linear subspace for which there does not exist a bounded projection of the whole space onto the subspace (cf. 3.11 below). Thus when L is a homomorphism and $R(L)$ has the property that there exists a bounded projection of Y onto $R(L)$, L is termed a P-homomorphism. L is an isomorphism if L is a homomorphism and the correspondence between $D(L)$ and $R(L)$ is one-to-one.

For any transformation L, the restriction of L to a subspace $S \subseteq D(L)$ is denoted by L_S. In particular I_S denotes the identity transformation on S.

The equality of two transformations K and L means that $D(K) = D(L)$ and $K(x) = L(x)$ for all x in $D(L)$.

Some Basic Concepts

1.1 Definition. A transformation J is said to be an inverse of a linear transformation K provided that $D(J) = R(K)$, and

$$JK = I_{D(K)}, \qquad KJ = I_{R(K)}.$$

It follows immediately that $J = K^{-1}$ exists if and only if $K(x) = 0$ implies $x = 0$. If K^{-1} exists, it is unique; moreover, it is linear.

1.2 Corollary. If K is an isomorphism, then $J = K^{-1}$ exists and is also an isomorphism.

Proof. It is clear that J exists and it follows from [12, Theorem 1, page 162, and Theorem 6, page 163] that J is bounded.

The following definition introduces the basic concept investigated in this paper.

1.3 Definition. A linear transformation M is said to be a pseudo-inverse of a linear transformation L provided

$$LML = L \quad ;$$

that is, $LML(x) = L(x)$ for each $x \in D(L)$.

It should be observed that the definition requires $D(M)$ to contain $R(L)$ and $D(L)$ to contain $R(ML)$.

1.4 Corollary. If L^{-1} exists, then L^{-1} is a pseudo-inverse; and if M is any pseudo-inverse, then $M = L^{-1}$.

Proof. Since $L\left[D(L)\right] = R(L)$, it follows, if M is a pseudo-inverse, that $L\,M(y) = y$ for arbitrary $y \in R(L)$; therefore, $LM = I_{R(L)}$. The application of L^{-1} to the equation $LML(x) = L(x)$ gives $ML(x) = x$ for each $x \in D(L)$, and thus $ML = I_{D(L)}$. An appeal to Definition 1.1 and the remarks following it completes the proof. It will be seen that M is not unique unless L^{-1} exists.

In contrast with the definition of an inverse, from which linearity of the inverse can be proved, the definition of a pseudo-inverse prescribes linearity as a preferential condition. That this condition is independent of the condition $LML = L$ will be shown in the next chapter.

CHAPTER II

ALGEBRAIC DISCUSSION OF THE PSEUDO-INVERSE

Some Basic Properties

In this section some elementary lemmas and theorems are established for reference.

2.1 Lemma. $LML = L$ if and only if $(LM)_{R(L)} = I_{R(L)}$.

Proof. That the second property follows from the first was shown in Corollary 1.4. Conversely, if $(LM)_{R(L)} = I_{R(L)}$, then clearly $LML = I_{R(L)} L = L$.

2.2 Lemma. If $LML = L$, then $(ML)_{R(ML)} = I_{R(ML)}$.

Proof. Clearly $R(ML) \subseteq X = D(L)$. Suppose $z \in R(ML)$. Then $z = MLx$ for some $x \in X$; hence, $MLz = ML(MLx) = M(LMLx) = MLx = z$.

The converse of 2.2 is valid if $R(ML) = X$. Then $ML = I_X$, $LML = L$, and by 2.1, $LM = I_{R(L)}$. Thus one can state the following theorem.

2.3 Theorem. A linear transformation L has an inverse if and only if it has a pseudo-inverse M, with domain $R(L)$, whose range is X.

2.4 Lemma. If M is a pseudo-inverse of L, then $R(ML)$ is a complement of $N(L)$ in $X = D(L)$.

Proof. For arbitrary $x \in X$, $MLx \in R(ML)$, and since $LML = L$, $x - MLx \in N(L)$; therefore, since $x = MLx + (I - ML)x$, $X = R(ML) + N(L)$. If, moreover, $x \in R(ML) \cap N(L)$, then $x = MLx'$ for some $x' \in X$, and $Lx = 0$; hence, $0 = Lx = LMLx = Lx'$, and since M is linear, $M(Lx') = x = 0$.

2.5 <u>Lemma</u>. $LML = L$ if and only if $R(ML)$ contains one and only one element z from each coset C of $N(L)$ in X and $Lz = L(C)$.

Proof. Let $LML = L$ and let C be an arbitrary coset of $N(L)$; suppose z_1 and z_2 are elements of C and both are contained in $R(ML)$. Let $z_1 = My_1$ and $z_2 = My_2$. Since $z_1 - z_2 \in N(L)$, $L(My_1 - My_2) = 0$; that is, $LMy_1 = LMy_2$. Since $LM = I_{R(L)}$, $y_1 = y_2$; and therefore $z_1 = z_2$. On the other hand $R(ML)$ does contain at least one element z for which $Lz = L(C)$. In fact, if $y = L(C)$, $z = My$ is this element. Since $LMy = y = L(C)$, $My \in C$; and since $My = MLx$ for some $x \in X$, $My \in R(ML)$. Therefore $My \in C \cap R(ML)$ and the first assertion is proved. Conversely, if one has a prescription which selects one and only one element z from each coset C and $Lz = L(C) = y$, one defines M by the rule: $My = z$. Then $LM = I_{R(L)}$ and $LML = L$, but this M may be nonlinear.

With the aid of the preceding lemma, a simple example shows that there may exist nonlinear M for which $LML = L$. Let $X = D(L)$ denote the Euclidean plane with a cartesian (u, v) coordinate system and suppose L is the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} .$$

It is easily seen that $R(L)$ is the one-dimensional subspace coinciding with the line $v = u$ and that $N(L)$ is the u-axis. Hence the cosets of $N(L)$ are the lines $v = k$, and L applied to one of these lines

is just its intersection with the line $v = u$. Let G denote the graph of any continuous monotone function $f(u)$; for definiteness, suppose $f(u) = u^3$. Clearly $G \cap C$ is a single point for any coset C. Thus for $y = C \cap R(L)$, define My by the rule: $My = z = G \cap C$. Clearly, M is nonlinear; yet, $LML = L$.

2.6 Theorem. Every linear transformation L has a pseudo-inverse M.

Proof. This proof is an adaptation of the proof of [1, Proposition 2, page 179]. Since $N(L)$ is a linear subspace of $X = D(L)$, there exists a complementary subspace S of $N(L)$ in X. If x and x' are elements of S such that $Lx = Lx'$, then $(x - x') \in N(L) \cap S = 0$, and thus $x = x'$. Therefore the restriction L_S of L to S is a one-to-one and linear transformation of S onto $R(L)$ for $R(L) = L[X] = L[S \oplus N(L)] = L[S]$. Thus it is seen that L_S has a linear inverse M. An application of Lemma 2.5 shows that $LML = L$; thus M is a pseudo-inverse of L.

It is quite possible for a pseudo-inverse M to have a domain of definition which includes $R(L)$ as a subspace. If, however, the domain of M is restricted to $R(L)$, the following characterization of $\{M;\ LML = L,\ M \text{ linear}\}$ can be noted. The proof is essentially contained in 2.5 and 2.6.

2.7 Theorem. Each complement of $N(L)$ in X determines a unique pseudo-inverse of L, and conversely each pseudo-inverse M of L determines a unique complement of $N(L)$ in X; that is, there is

a one-to-one correspondence between the complements of the null space of a linear transformation and the pseudo-inverses of the transformation.

Non-Singular Pseudo-Inverses

When $R(L)$ is a proper subspace of Y, a pseudo-inverse M of L, which is defined on $R(L)$ only, has the obvious defect that it is not defined for all $y \in Y$. Moreover, if the domain of definition of M is extended to all of Y by some process, it may turn out that M is singular on some subspace of Y and that $R(M)$ is a proper subspace of X. In this section it is shown that these difficulties may be circumvented. Of fundamental importance in the discussion which follows is the notion of a projection.

2.8 Definition. If the linear space X is the direct sum of the linear subspaces S and U; that is, for every $x \in X$ there exist unique $s \in S$ and $u \in U$ such that $x = s + u$, then the transformation P defined by the rule $Px = s$ is called the projection of X on S along U.

Evidently P is linear and $P^2 = P$. Just as easily seen is the fact that $I_X - P$ is the projection of X on U along S. Lemmas 2.2 and 2.5 show that ML and $I_X - ML$ are projections of X on $R(ML)$ and $N(L)$ respectively.

2.9 Lemma. Any pseudo-inverse M of L can be extended from $R(L)$ to all of Y.

Proof. Let Z denote some complement of $R(L)$ in Y and let Q denote the projection of Y onto $R(L)$ along Z. MQ is then

defined for all $y \in Y$ and is identical with M on $R(L)$.

2.10. Theorem. If there exists a one-to-one and linear transformation A on some complement of $R(L)$ onto $N(L)$, then there exists a pseudo-inverse of L which is non-singular and is defined for all $y \in Y$.

Proof. Let Z denote the complement of $R(L)$ on which the one-to-one and linear transformation A is defined, and let Q denote the projection of Y onto $R(L)$ along Z . $I_Y - Q$ is then the projection of Y onto Z along $R(L)$. By 2.6, there exists an M defined on $R(L)$ such that $LM = I_{R(L)}$. It is to be shown that the transformation

$$M_1 = MQ + A(I_Y - Q)$$

has the properties stated in the theorem. It is evident that M_1 is linear and that M_1 is defined for all $y \in Y$. Suppose $M_1 y = 0$ for some $y \in Y$. Let $y = r + z$, $r \in R(L)$, $z \in Z$, denote the unique decomposition of y ; then, $M_1 y = 0$ implies that

$$\begin{aligned} \left[MQ + A(I_Y - Q)\right](r + z) &= MQr + A(I_Y - Q)z \\ &= Mr + Az = 0 \quad . \end{aligned}$$

Since $r \in R(L)$, $Mr \in R(ML)$, and by hypothesis $Az \in N(L)$; therefore, by 2.4, $Mr = -Az = 0$. It follows from the non-singularity of M on $R(L)$ and A on Z that $r = z = 0$, and hence that $y = 0$. Since for

$r \in R(L)$, $M_1 r = Mr$, it follows that $LM_1 r = LMr = r$, and thus that M_1 is a pseudo-inverse of L .

Complementary Transformations

In many linear problems in which the linear transformation L is singular, particularly in differential equations, physical considerations impose supplementary conditions on the solution of the linear problem. Since the solution of any consistent linear equation $Lx = r_0$ can be written as

$$x = x_p + x_h \quad ;$$

in which x_p is a particular solution and x_h is the complete solution of $Lx = 0$, it follows that these supplementary conditions are conditions on $x_h \in N(L)$. If the conditions are linear and determine x_h uniquely, one interprets them as a non-singular linear transformation $K_{N(L)}$ on $N(L)$ into some appropriate linear space. The two transformations, L and $K_{N(L)}$, are then composed in some manner to form a non-singular linear transformation T . In this section it will be shown that T can be defined in such a way that T^{-1} is given by the natural composition of the pseudo-inverses of L and $K_{N(L)}$.

2.11 Definition. If X_1 and X_2 are linear spaces, then $X_3 = X_1 \oplus X_2$ is defined by:

(1) $$X_3 = \left\{x_3;\ x_3 = (x_1, x_2)\ ,\ x_1 \in X_1\ ,\ x_2 \in X_2\right\} \quad ;$$

(2) if $x_3 = (x_1, x_2)$ and $x_3' = (x_1', x_2')$, then

$$x_3 + x_3' = (x_1 + x_1', x_2 + x_2') \quad ,$$

and for any scalar a ;

(3) $$a(x_1, x_2) = (ax_1, ax_2) \quad .$$

X_3 is called the direct sum of X_1 and X_2 , and is evidently a linear space. The subspaces $X_1 \oplus 0$ and $0 \oplus X_2$ of X_3 may and will be identified with X_1 and X_2 respectively.

2.12 Definition. Two linear transformations, L and K , each with domain X and range in Y , are said to be complementary provided that

$$N(L) \cap N(K) = 0 \qquad \text{and} \qquad N(L) \oplus N(K) = X \quad .$$

For $Y = X$ it is clear that two projections P and $I_X - P$ have the properties of the definition.

2.13 Definition. If L and K are two linear transformations, each with domain X , and range contained in Y , then the transformation

$$x \longrightarrow (Lx, Kx) = (L, K)x$$

on X into $Y \oplus Y$ is called the direct sum of L and K .

It follows directly from the definition that (L, K) is linear

and that (L, K) agrees with L on $N(K)$ and with K on $N(L)$. If $R(L)$ and $R(K)$ are complementary subspaces in a linear space V, in which addition between elements has been defined, $(L, K)x$ may and will be written as

$$(L, K)x = Lx + Kx \quad .$$

2.14 Theorem. The direct sum of two complementary transformations is non-singular; moreover, the inverse of this non-singular transformation can be represented as the direct sum of pseudo-inverses of the complementary transformations.

Proof. Let $T = (L, K)$ denote the direct sum of the complementary transformations L and K. Then if $(L, K)x = 0 = (0, 0)$, it follows that $x \in N(L) \cap N(K) = 0$, and therefore that T is non-singular. By 2.6, 2.7, and 2.9, there exists a unique pseudo-inverse M of L for which $D(M) = Y = R(L) \oplus R(K)$ and $R(M) = N(K)$. Similarly, there exists a unique pseudo-inverse J of K for which $D(J) = Y$ and $R(J) = N(L)$. To see that $T^{-1} = (M, J)$ one observes that for arbitrary $x \in X$, x has a unique decomposition $x = s + u$, $s \in N(K)$, $u \in N(L)$, and that

$$\begin{aligned}(M, J)\,(L, K)\,(s + u) &= (M, J)\,(Ls, Ku) \\ &= MLs + JKu \\ &= s + u \quad ;\end{aligned}$$

similarly, $(L, K)(M, J)y = y$ for arbitrary $y \in Y$. This proves the assertion.

2.15. <u>Theorem</u>. Every linear transformation has a complement.

Proof. Let L, mapping X onto $R(L)$, denote the linear transformation of the theorem and let $K_{N(L)}$ denote some one-to-one and linear transformation of $N(L)$ onto a linear space Z. A $K_{N(L)}$ certainly exists; for example, let $Z = N(L)$ and let $K_{N(L)}$ be the identity. Let M denote a pseudo-inverse of L, and then consider the linear transformation

$$K = K_{N(L)}(I_X - ML) \quad .$$

If Y denotes the linear space $R(L) \oplus Z$, both L and K map X into Y. By 2.4, $S = R(ML)$ is a complement of $N(L)$ in X and by 2.2, $MLs = s$ for arbitrary $s \in S$. Thus for $s \in S$,

$$Ks = K_{N(L)}(I_X - ML)s = K_{N(L)}(s - s) = 0 \quad ;$$

therefore, $s \in N(K)$ and $S \subseteq N(K)$. If $x \in N(K)$, and if $x = s+u$, $s \in S$, $u \in N(L)$, is the unique decomposition of x, it follows since $Ks = 0$ that

$$0 = Kx = K_{N(L)}(I_X - ML)(s + u)$$

$$= K_{N(L)} u \quad .$$

Since, by hypothesis, $K_{N(L)}$ is non-singular on $N(L)$, it follows that $u = 0$, and therefore that $x = s \in S$. Hence $N(K) \subseteq S$, and this together with the other inclusion shows that $S = N(K)$. $N(L)$ and $N(K)$ are complementary and thus the requirements of Definition 2.12 are satisfied.

2.16 <u>Corollary</u>. Every linear transformation has a non-singular pseudo-inverse whose range is X.

Proof. 2.10 proves this when the hypotheses stated there are fullfilled. Otherwise, the construction of 2.15 is used, and then 2.14 is applied.

In view of the foregoing discussion it is quite easy to set up a schematic diagram showing the transformations involved in 2.10 et seq. In the following diagram the various transformations are represented by arrows which point from domain to range. The complementary transformations are represented by diagonal arrows and the vertical arrows on the right of the equalities indicate the restrictions of these transformations to the domain on which they are non-singular. The transformations that are represented by vertical arrows are non-singular and the inverse of each is represented by an adjacent (oppositely directed) arrow. A pseudo-inverse of a singular transformation is found by selecting the transformation represented by a vertical arrow whose domain coincides with the range of the singular transformation. Its extension to the whole space is represented by the diagonal arrow with the same name, but without the subscript.

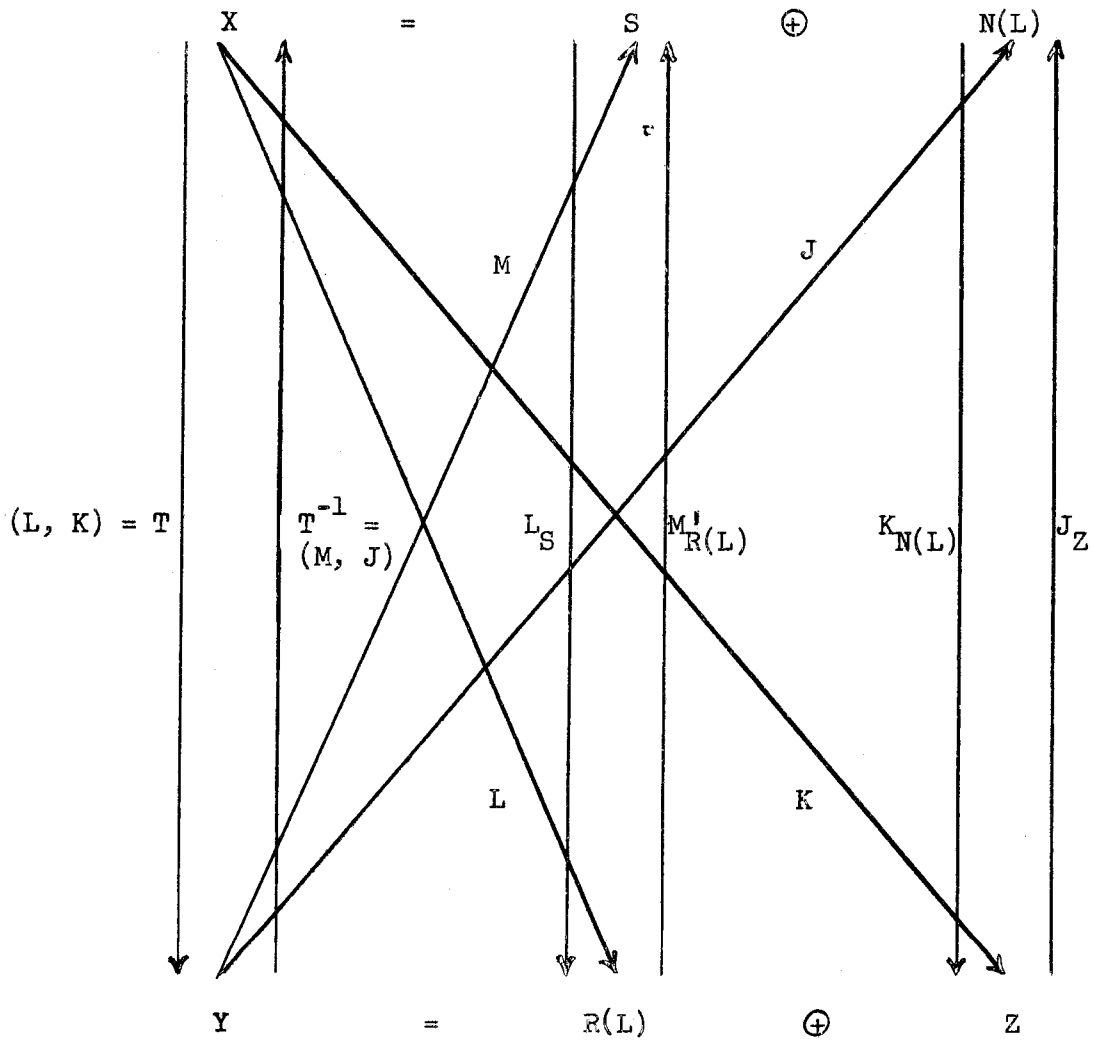

Figure 1.

In a dual manner a similar diagram may be constructed for the transformations L^* , K^* , J^* , and M^* showing their behavior on the appropriate subspaces of X^* and Y^* .

Since Y is the direct sum of $R(L)$ and Z , it follows from [12, Theorem 3, page 172] that

$$Y^* = \left[R(L)\right]^{\perp} \oplus Z^{\perp} .$$

The algebraic discussion of [12, Theorem 1, page 285] shows that

$$Z^{\perp} = \left[N(M)\right]^{\perp} = \left[R(K)\right]^{\perp} = N(K^*) = R(M^*) ,$$

and

$$R(L) = N(L^*) = R(J^*) .$$

Hence

$$Y^* = N(L^*) \oplus R(M^*) = R(J^*) \oplus N(K^*)$$

and dually,

$$X^* = N(M^*) \oplus R(L^*) = R(K^*) \oplus N(J^*) .$$

The following diagram is then read in the same way as that in Figure 1.

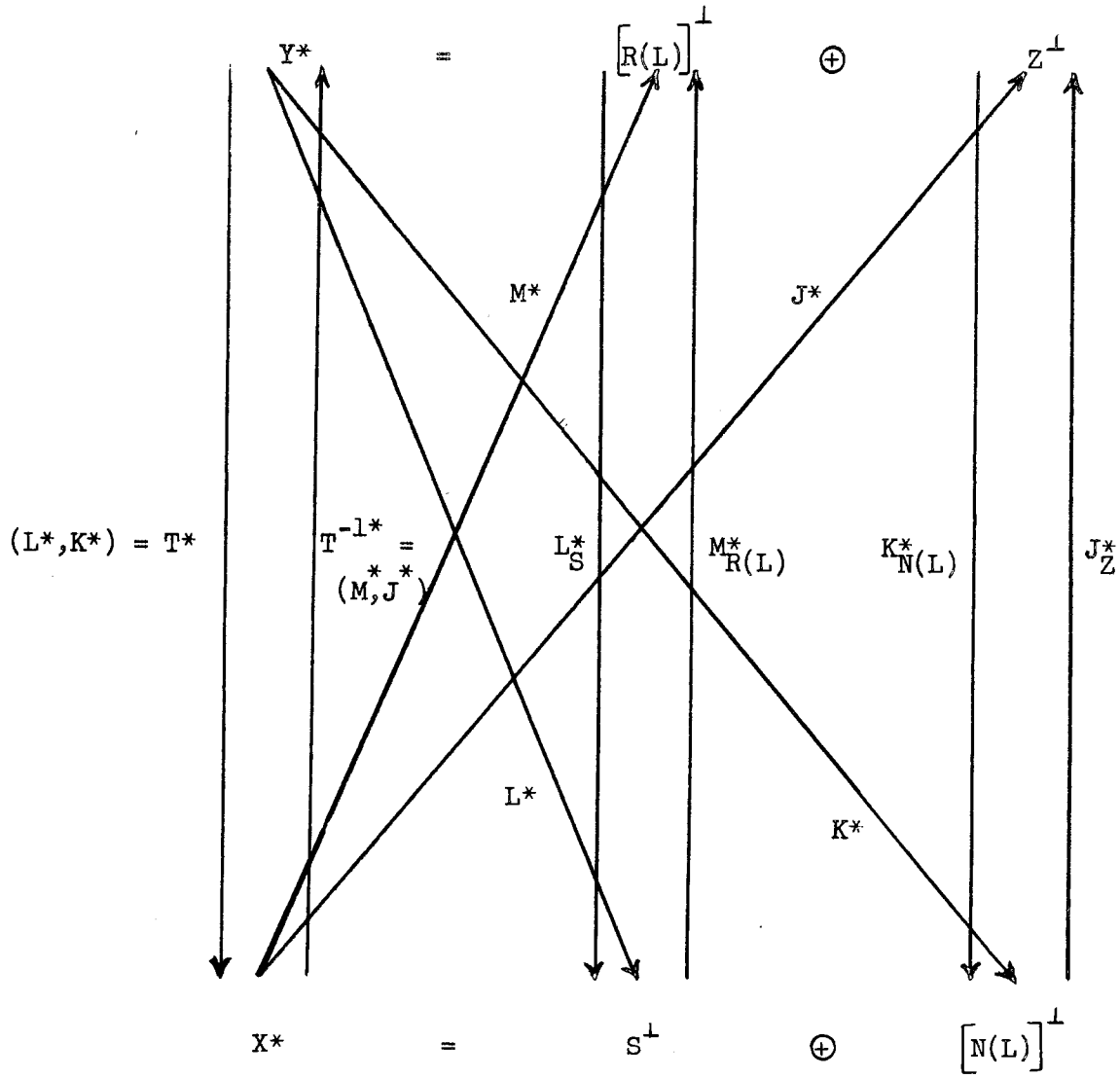

Figure 2.

An Application in Finite-Dimensional Space

An explicit procedure is presented here for computing a pseudo-inverse M of a given $n \times n$ singular matrix L. For a given consistent linear equation

$$Lx = y_o \quad ,$$

as was noted in the introduction, My_o is a particular solution, and it will be shown that the calculation gives simultaneously a complete solution of the homogeneous equation

$$Lx = 0 \quad .$$

The computational scheme uses nothing more than elementary row operations on L; and for a singular matrix L, the number of computations required is less than that needed for a non-singular L. Thus the procedure could be readily adapted for machine calculation.

The process makes use of the Hermite canonical form of an $n \times n$ matrix.

2.17 Definition. A matrix $H = |h_{ij}|$ is said to be in Hermite canonical form provided

(1) $h_{ij} = 0$, whenever $i > j$;

(2) $h_{ii} = 0$ or 1;

(3) if $h_{ii} = 0$, then $h_{ik} = 0$, $(k = 1, 2, \ldots, n)$; and

(4) if $h_{ii} = 1$, then $h_{ji} = 0$, $(j = 1, 2, \ldots, n, j \neq i)$.

2.18 Theorem. For a given $n \times n$ matrix L , there exists a non-singular matrix M which is a product of elementary matrices such that $ML = H$ is in Hermite canonical form.

A proof of this theorem is given in many of the standard texts; see, for example [9, Theorem 18, page 35].

2.19 Lemma. If H is in Hermite canonical form, then $H^2 = H$.

Proof. Let a_{ik} denote an arbitrary element of H^2 . By 2.17 (1), $a_{ik} = \sum_{j=1}^{n} h_{ij} h_{jk}$ vanishes if $i > k$ and reduces, if $i \leqslant k$, to $\sum_{j=i}^{k} h_{ij} h_{jk}$. By 2.17 (2), $h_{ii} = 0$ or 1 ; and if $h_{ii} = 0$, then by 2.17 (3), $h_{ij} = 0$ for all j and thus $a_{ik} = 0 = h_{ik}$. If $h_{ii} = 1$, then

$$a_{ik} = h_{ik} + \sum_{j=i+1}^{k} h_{ij} h_{jk} .$$

Now whenever $h_{ij} \neq 0$ for some $j > i$, it follows from 2.17 (4) that $h_{jj} = 0$, and hence from (3) that $h_{jm} = 0$ for all m and in particular for $m = k$. Hence, in any case, $a_{ik} = h_{ik}$ and $H^2 = H$.

2.20 Theorem. If M is the product of the elementary matrices which reduce L to H , then M is a non-singular pseudo-inverse of L .

Proof. By hypothesis $ML = H$, and since M is a product of elementary matrices, M^{-1} exists; therefore $L = M^{-1} H$. Hence

$$LML = M^{-1} HM M^{-1} H = M^{-1} H^2 = M^{-1} H = L .$$

M can be computed by performing the elementary row operations which reduce L to H on the identity matrix. If, however, one is interested only in the solution of $Lx = y_o$, it is not necessary to compute M. To solve the problem one proceeds in the following manner. Form the $n \times (n + 1)$ matrix $L^+ = (L, y_o)$, and then perform the elementary row operations on L^+ which reduce L to H. The resulting matrix ML^+ is of the form $(H, M y_o)$, where H is of the form

$$\begin{bmatrix} 1 & a & 0 & b & d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Since the only non-zero rows of H are those with a 1 in the diagonal, it follows that the rank of L, which equals the rank of H, is given by the number of 1's along the diagonal. Moreover, any column which has a 1 in the diagonal is a unit vector by 2.17 (4). Since the $(n+1)$st column of L^+ becomes My_o as L is reduced to H and since by 2.1 and 2.2 L and M are full inverses of each other on $R(L)$ and $R(ML)$, it follows that the equation $Lx = y_o$ is consistent if and only if My_o has non-zero components only in the rows in which H has 1's. In this case, My_o is a particular solution of $Lx = y_o$. If $H = ML$ has r 1's down the diagonal, then ML contains r unit column vectors and hence $I - ML = I - H$ contains exactly $n - r$ non-zero columns. Since $(I - ML)u = u$ for arbitrary $u \in N(L)$, it follows that these columns

of $I - ML$ span $N(L)$; and since $N(L)$ is of dimension $n - r$, it follows even that they are linearly independent. Thus the complete solution of $Lx = y_o$ is given by

$$x = My_o + (I - ML)D \quad ,$$

in which D is an arbitrary diagonal matrix containing only $n - r$ effective parameters since $I - ML$ has only $n - r$ non-zero columns.

CHAPTER III

PSEUDO-INVERSES IN NORMED SPACES

Since the statements that were made in the previous chapter concerning the pseudo-inverse were purely algebraic in nature, they are not sufficient whenever approximations are employed. Then continuity requirements are of paramount importance, and in this chapter conditions are given which assure the boundedness (continuity) of the pseudo-inverse. First the case in which no continuity requirements are placed on L is discussed. Before undertaking this investigation, a definition is needed.

3.1 Definition. If S is any subset of $D(L)$, then L is said to be bounded below on S provided there exists a positive real number b such that

$$\|Lx\| \geqslant b\,\|x\| \quad ,$$

for all $x \in S$.

3.2 Theorem. A linear transformation L has a bounded pseudo-inverse M if and only if L is bounded below on some complement of $N(L)$ in $D(L)$. Cf. [12, Theorem 3, page 163].

Proof. Suppose M is a bounded pseudo-inverse of L. By 2.4, $R(ML)$ is a complement of $N(L)$ in $D(L)$ and by 2.2, $x = MLx$ for arbitrary $x \in R(ML)$. Hence

$$\|x\| = \|MLx\| \leqslant \|M\|\,\|Lx\| \quad ;$$

thus

$$\|Lx\| \geqslant \|M\|^{-1} \|x\|$$

for all $x \in R(ML)$, and $b = \|M\|^{-1}$ is the required number. For the converse, suppose S is the complement of $N(L)$ in $D(L)$ on which L is bounded below. By Theorem 2.6, there exists a pseudo-inverse M of L defined on $R(L)$ into $D(L)$. By 2.1, $(LM)_{R(ML)} = I_{R(L)}$. Thus for y arbitrary in $R(L)$,

$$\|My\| = \|x\| \leqslant b^{-1} \|Lx\| = b^{-1} \|y\| \quad ;$$

therefore,

$$\|M\| \leqslant b^{-1} \quad ,$$

and thus M is bounded.

<u>Henceforth</u>, unless specified to the contrary, it is assumed that L is a <u>bounded</u> linear transformation. Thus, as indicated in Chapter I, it is assumed that $D(L) = X$.

3.3 <u>Definition</u>. If L_1 and L_2 are linear transformations with domains X_1 and X_2 respectively, and if $X_1 \subset X_2$, then L_2 is said to be an extension of L_1 provided $L_1 x = L_2 x$ for all $x \in X_1$.

3.4 <u>Lemma</u>. If L_2 is an extension of L_1 and if M_2 is a pseudo-inverse of L_2, then M_2 is a pseudo-inverse of L_1.

Proof. Since $S_2 = M_2 L_2 (X_2)$ is a complement of $N(L_2)$ in X_2, it follows that $S_1 = X_1 \cap S_2$ is a complement of $N(L_1) = X_1 \cap N(L_2)$ in X_1. Let $x_1 = s_1 + u_1$, $s_1 \in S_1$, $u_1 \in N(L_1)$, denote an arbitrary element of X_1. Then since $M_2 L_2 s_1 = s_1$, one observes that

$$L_1 M_2 L_1 x_1 = L_1 M_2 L_1 s_1 = L_1 M_2 L_2 s_1 = L_1 s_1 = L_1 x_1 \quad .$$

3.5 <u>Theorem</u>. L has a bounded pseudo-inverse M only if L^* has a bounded pseudo-inverse J^*.

Proof. Suppose M is bounded and linear and that $LML = L$. M^* is defined for all linear functionals $x^* \in R(L^*)$ and by [12, Theorem 1, page 169], M^* is linear and $\|M^*\| = \|M\|$. Hence for an arbitrary $y^* \in Y^*$,

$$L^* y^*(x) = y^*(Lx) = y^*(LMLx) = L^* y^*(MLx)$$

$$= M^* L^* y^*(Lx) = L^* M^* L^* y^*(x) \quad ;$$

therefore,

$$L^* M^* L^* = L^* \quad .$$

Thus $J^* = M^*$ is a bounded pseudo-inverse of L^*. As for the converse, by [12, Theorem 3, page 170], L^{**} on $X^{**} \supseteq X$ is an extension of L and $\|L^{**}\| = \|L\|$. By the proof just given, $L^{**} J^{**} L^{**} = L^{**}$;

hence, by 3.4, $M = J^{**}$ is a bounded pseudo-inverse of L.

The theorem that was just stated applies to the general case; that is, it applies when $R(L)$ is a proper subspace of the Banach space Y. If $R(L) = Y$, the result can be sharpened.

3.6 Corollary. If the range of L is the Banach space Y, then $M^*_{R(L^*)} = (L^*)^{-1}$.

Proof. By [12, Theorem 4, page 287], L^* has an inverse, and by Corollary 1.4, it must be $M^*_{R(L^*)}$.

3.7 Lemma. For a bounded linear transformation L, let

$$I(L, y) = \inf \left\{ \|x\| ; \ Lx = y \right\}$$

for each y, and

$$s(L) = \sup \left\{ I(L, y); \ y \in R(L) , \ \|y\| = 1 \right\} .$$

Then $R(L)$ is closed in Y if and only if $s(L)$ is finite.

Proof. This is Lemma 6 in [6, page 143].

3.8 Theorem. If L is a bounded linear transformation, the following propositions are equivalent.

(a) L has a bounded pseudo-inverse.

(b) L has a bounded right inverse.

(c) The number $s(L)$ of Lemma 3.7 is finite.

(d) L is a homomorphism; that is, $R(L)$ is a closed linear subspace of Y.

(e) $X/N(L)$ is isomorphic with $R(L)$.

Proof. That (a) and (b) are equivalent is merely a restatement of 2.1 for the insertion of the condition on boundedness does not alter the argument in any respect. That (c) and (d) are equivalent is Lemma 3.7.

(b) implies (d). $R(L)$ is clearly a linear subspace of Y so it is sufficient to show closure. Suppose $y_n \longrightarrow y_o$ with $y_n \in R(L)$ for all n . Then $z_n = My_n$ is a Cauchy sequence for, since M is bounded,

$$\|z_n - z_m\| = \|M(y_n - y_m)\| \leqslant \|M\| \, \|y_n - y_m\| \quad ;$$

and the latter expression tends to zero as n and m increase without bound. Hence by the completeness of X , there exists a $z_o \in X$ such that $z_n \longrightarrow z_o$; therefore, by the continuity of L , $Lz_n \longrightarrow Lz_o$. Since $Lz_n = LMy_n = y_n$, $Lz_n \longrightarrow y_o$; therefore, by the uniqueness of the limit of a convergent sequence, $y_o = Lz_o$ and so $R(L)$ is closed.

(d) implies (e). By assumption $R(L)$ is a closed linear subspace of a Banach space and is therefore a Banach space also. As noted in the introduction, since $N(L)$ is a closed linear subspace of X , $X/N(L)$ is a Banach space. For purposes of this theorem let $\{x\}$ denote that element of $X/N(L)$ containing the element $x \in X$. On $X/N(L)$ define a transformation T by the rule

$$T\{x\} = Lz = y \quad , \qquad z \in \{x\}$$

noting that y is independent of which $z \in \{x\}$ is used. Since $T\{x\} = 0$ only for $\{x\} = \{0\} = N(L)$, and since L is linear onto $R(L)$, it follows that T is one-to-one onto $R(L)$ and linear. Hence there remains to be shown only the continuity of T and of T^{-1}. Let $\{x\} \in X/N(L)$ be arbitrary. Then $T\{x\} = Lz$ and hence

$$\|T\{x\}\| = \|Lz\| \leqslant \|L\| \, \|z\| ,$$

for all $z \in \{x\}$. Since $\|\{x\}\| = \inf \left\{ \|z\| ; \; z \in \{x\} \right\}$, it follows that

$$\|T\{x\}\| \leqslant \|L\| \, \|\{x\}\| .$$

Since $\{x\}$ is arbitrary, it follows even that $\|T\| \leqslant \|L\|$. Thus T is an isomorphism; hence, by Corollary 1.2, T^{-1} is also bounded.

(e) implies (a). Let T denote the assumed isomorphism. By the definition of an isomorphism, $R(L)$ is a closed subspace of Y. Theorem 2.6 shows that there exists a linear M on $R(L)$ into X such that $LML = L$. It must be shown that M is bounded. Let $S = R(ML)$ denote the complement of $N(L)$ on which M takes its assigned values. Since $R(L)$ is a closed set, and since S is the complete inverse image of the closed set $R(L)$ under the continuous transformation L_S, S is a closed complement of $N(L)$ in X. By Lemma 2.5, S contains one and only one element from each coset of $X/N(L)$. Therefore

defining for each $z \in S$,

$$U(z) = \{z\} \quad , \quad \{z\} \in X/N(L) \quad ;$$

one sees immediately that U is linear. U is one-to-one for $U(z) = \{0\} \equiv N(L)$ if and only if $z = 0$; moreover, U is bounded for

$$\|U(z)\| = \|\{z\}\| = \inf\left\{\|x\| \; ; \; x \in \{z\}\right\} \leqslant \|z\| .$$

Thus U is also an isomorphism and hence U^{-1} is bounded. Since T was defined by the rule

$$T\{z\} = Lx = y \quad ; \quad x \in \{z\} \quad ,$$

it follows that

$$y = LMy = TUMy \quad .$$

Hence

$$M = U^{-1} T^{-1} \quad ,$$

and since U^{-1} and T^{-1} are bounded, M is bounded. The cycle of implications is now complete and the theorem is proved.

One gets an interesting corollary from the above discussion when L is a compact linear transformation; that is, when L maps bounded sets into compact sets. (Cf. [12], Chapter 11, pages 311-369).

3.9 Corollary. If L is a compact linear transformation, $R(L)$ is closed if and only if $R(L)$ is of finite dimension.

Proof. Since by [12, Theorem 1, page 311], a compact linear transformation is bounded, Theorem 3.8 applies. Assume $R(L)$ is closed. Then by 3.8 (d) and (b), there exists a bounded linear transformation M such that $LM = I_{R(L)}$. Since L is compact and M is bounded, by [12, Theorem 10, page 315], $LM = I_{R(L)}$ is a compact transformation. But if an identity transformation is compact, its domain, $R(L)$, is of finite dimension. The converse is easy for every finite dimensional linear space is closed.

3.10 Corollary. If K is a compact linear transformation on X to X, then for any complex number c;

$$L_c = I - cK$$

has a bounded pseudo-inverse.

Proof. By [12, Theorem 2, page 332], $R(L_c)$ is closed in X, and the result follows from 3.8 (d and a).

In view of Theorem 3.8 and the results of the previous chapter, it would appear that if L is a homomorphism, then a bounded non-singular pseudo-inverse whose range is X could be found. As it will be shown

presently, this is indeed the case when the Banach space Y is constructed from two other Banach spaces by a prescribed rule. If, however, the space Y must be considered as prescribed initially, other conditions must be assumed. The technique employed in 2.9 and 2.10 was to extend M by means of a projection. That this leads to topological complications is evident from the following lemma.

3.11 Lemma. If S is a closed linear subspace of a Banach space Y, then there exists a bounded projection of Y on S if and only if there exists a closed linear subspace T of Y which is complementary to S in Y.

The proof is given by F. J. Murray in [10, Lemma 1.1.1, page 138]. To show that this lemma is non-trivial, he constructs a closed linear subspace of the Banach space ℓ_p, and then proves that this subspace does not have a closed complement.

3.12 Theorem. If L is a homomorphism and if there exists an isomorphism A defined on some closed complement Z of $R(L)$ in Y onto $N(L)$, then there exists a bounded pseudo-inverse of L which is non-singular and is defined for all $y \in Y$.

Proof. As in 2.10, let Q and $I_Y - Q$ denote the projections of Y onto $R(L)$ and Z respectively, and let M denote a bounded pseudo-inverse with domain $R(L)$. Define, as before,

$$M_1 = MQ + A(I_Y - Q) .$$

By 3.8, M is bounded, and by 3.11, Q and $I_Y - Q$ are bounded; therefore, M_1 is bounded and the remainder of the assertions follow from 2.10.

Since a linear subspace of Y is closed if and only if it is complete, it follows that if L is a homomorphism, then $R(L)$ is a Banach space. Thus for any Banach space Z isomorphic with $N(L)$, it is useful to consider the linear space $Y_1 = R(L) \oplus Z$, as defined in 2.11. As shown in [12, page 127], if for $y_1 = (r, z) \in R(L) \oplus Z$, $\|y\|$ is defined by

$$\|y\| = \left(\|r\|^p + \|z\|^p\right)^{1/p}, \quad 1 \leqslant p \leqslant \infty ,$$

then Y_1 is a Banach space. It should be noted that for any choice of p in this definition, if $r \in R(L)$ and $z \in Z$, then $\|r, 0\|_{Y_1} = \|r\|_{R(L)}$ and $\|(0, z)\|_{Y_1} = \|z\|_Z$. Therefore, since the metric topology induced by this norm reduces to the metric topology on $R(L)$ and on Z for points in these spaces, the space Y_1 may and will be called, following Zaanen, the direct sum of $R(L)$ and Z.

3.13 <u>Theorem</u>. Every homomorphism L has a non-singular pseudo-inverse which is bounded and whose range is X.

Proof. By 3.8, L has a bounded pseudo-inverse and by applying the argument of 2.15, it follows even that L has a bounded complement K. It is supposed that $Y_1 = R(L) \oplus Z$ is normed by the process just given, and thus it follows that $T = (L, K)$ is bounded for

$$\|Tx\| = \|(L, K)x\| = \left[\|Lx\|^p + \|Kx\|^p\right]^{1/p}$$

$$= \left[\|Lx\|^p + \|K_{N(L)}(I_X - ML)x\|^p\right]^{1/p}$$

$$\leqslant \left[\left(\|L\|^p + \|K_{N(L)}(I_X - ML)\|^p\right)\|x\|^p\right]^{1/p}$$

$$= \left[\|L\|^p + \|K\|^p\right]^{1/p}\|x\| \quad .$$

By 2.14, T^{-1} exists; and since T is an isomorphism of X onto Y_1, it follows from 1.2 that T^{-1} is bounded. Since T^{-1} is a pseudo-inverse of L whose range is X (cf. Fig. 1), the proof is complete.

3.14 <u>Corollary</u>. If $R(L)$ and Z are complementary subspaces of X, and if $\|I_X - T\| < 1$, then a non-singular pseudo-inverse T^{-1} of L is given by

$$T^{-1} = I_X + \sum_{n=1}^{\infty}\left[I_X - (L, K)\right]^n \quad .$$

The proof of the above formula, given in [8, Theorem 5.2.1, page 92], is just a verification that $T^{-1}T = TT^{-1} = I_X$ and that the series converges. The implied convergence is convergence in norm for the linear space of bounded linear transformations with domain X and range in X. Expressed in terms of the metric topology of X it means, for the sequence of partial sums determining T^{-1}, that for a given number $a > 0$ one can find an index N_a such that

$$\|S_n(x) - S_m(x)\| < a\|x\| ,$$

whenever m and n exceed N_a .

An Application

In many problems involving a bounded linear transformation L , the problem itself suggests what Banach space X one should employ as $D(L)$. If some pseudo-inverse M of L can be found, then, as noted in the introduction, if $x \in X$, then $(I_X - ML)x$ is a solution of the homogeneous equation $Lx = 0$. Since it is often impractical to apply $I_X - ML$ to points of X indiscriminately, one looks for additional properties of X which might be of assistance in characterizing $N(L)$.

Suppose X has a basis x_α $(\alpha \in A)$, and that a bounded pseudo-inverse M of L has been found. Although the vectors $y_\alpha = (I_X - ML)x_\alpha$ may not be independent, they span $N(L)$;for, if $u = \sum_\alpha u_\alpha x_\alpha$ then $u = (I_X - ML)u = \sum u_\alpha (I - ML)x_\alpha$.

Of greater practical importance is the following:

3.15 Theorem. Suppose that $[x_n]$, $(n = 1, 2, 3, \ldots)$, is a countable set of points dense in X . If L and a pseudo-inverse M of L are bounded, then the set of points $[(I_X - ML)\, x_n]$ is dense in $N(L)$.

Proof. Let $u \in N(L)$ be arbitrary and let s denote some arbitrary but fixed element in $R(ML)$. Then $x = s + u$ is uniquely determined, and by hypothesis for an arbitrary number $a > 0$, there exists $x_{n_a} \in [x_n]$ such that

$$\|x_{n_a} - x\| < a \, \|I_X - ML\|^{-1} .$$

Hence, since $(I_X - ML)x = u$,

$$\|(I_X - ML)\, x_{n_a} - u\| = \|(I_X - ML)\, x_{n_a} - (I_X - ML)x\|$$

$$\leqslant \|I_X - ML\| \, \|x_{n_a} - x\| < a .$$

The Weierstrass approximation theorem assures the separability of the Banach space C_n with the uniform norm.

More general applications are discussed in the following two chapters.

CHAPTER IV

ON SOLUTIONS OF A FUNCTIONAL EQUATION

A Fixed Point Theorem

The topological character of a pseudo-inverse is very important when some iteration scheme or approximation method is being employed to find a solution of a given equation. The equation considered in this chapter is

$$G(x, y) \equiv L(x) + F(x, y) = 0 \quad ;$$

in which L denotes a homomorphism of a Banach space X onto a Banach space V, $F(x, y)$ denotes the nonlinear part of $G(x, y)$ and $F(x, y)$ maps the product space $X \times Y$, where Y is a metric space, into V. Explicit hypotheses will be given later.

In order to study the solutions of the equation $G(x, y) = 0$, one lets M denote a pseudo-inverse of L, chooses $u \in N(L)$, and shows that the equation

$$x + M F_y(x) \text{[1]} = u$$

---

[1] The notation $F_y(x)$ is used to indicate that y is a fixed quantity; that is, $F_y(x) = F(x, y)$ with y fixed.

has a unique solution $x = f_y(u)$. This is accomplished, with suitable restrictions, by means of a fixed point theorem. Then for the same y, the pair $(x, y) = \left[f_y(u), y\right]$ is shown to be a solution of $G(x, y) = 0$ if and only if the pair satisfies a certain consistency condition.

The following fundamental existence and uniqueness theorem is the basic tool used in this chapter. Numerous proofs (based on an iterative process) are in the literature [5, Theorem 6.1, page 63].

4.1 Theorem. Let X be a Banach space, let Y be a metric space, and suppose that:

(H1) $\varphi(x, y)$ is a continuous function on $S_x(d)$[2] $\times Y$ into X; there exists a number $\theta = \theta(y)$ such that $0 < \theta < 1$ and, if $y \in Y$ and $x, x' \in S_x(d)$, then

$$\|\varphi_y(x) - \varphi_y(x')\| \leqslant \theta \|x - x'\| .$$

(H2) There exists a number a, $0 < a < 1$, and a set $Y_o \subseteq Y$ such that for $x \in S_x(d)$ and $y \in Y_o$,

$$\|\varphi_y(x)\| \leqslant ad .$$

---

[2] Closed spheres about the point p in the space X of radius d are denoted by $S_x(p, d) = \left\{x; \|x - p\| \leqslant d\right\}$. If p is the origin, it is omitted from the notation.

(C1) Then for each $u \in S_x\left[(1 - a)d\right]$ and for $y \in Y_o$, the equation

$$x + \varphi_y(x) = u$$

has a unique solution $x = f_y(u)$.

(C2) The function $f_y(u)$ is continuous on $S_x\left[(1-a)d\right] \times Y_o$ to $S_x(d)$ uniformly on $S_x\left[(1 - a)d\right]$.

(C3) For d finite, $f_y(u)$ is the uniform limit of the sequence of functions defined by

$$f_y^{(0)}(u) = u \ , \quad f_y^{(1)}(u) = u - \varphi_y(u) \ , \ \ldots \ ,$$

$$f_y^{(n+1)}(u) = u - \varphi_y\left[f_y^{(n)}(u)\right] \ , \quad \ldots \quad .$$

Solutions of the Functional Equation

The basic result of this section is the following theorem, which is a generalization of Theorem 1.5 of the paper of R. G. Bartle [3]. In his treatment of the equation $L(x) + F(x, y) = 0$, he assumes that L is a homomorphism and that the null spaces of L and L^* have equal and finite dimension. Here, the latter condition is dropped and it is assumed only that L is a P-homomorphism. The finite dimensionality of $N(L^*)$ is sufficient to show that L is a P-homomorphism; however, this condition is not a necessary one.

4.2 <u>Theorem</u>. Suppose X and V are Banach spaces and that Y is a metric space. Let $G(x, y)$ be expressed in the form

$$G(x, y) = L(x) + F(x, y) \quad ,$$

where

(H1) L is a P-homomorphism of X into V with pseudo-inverse M (cf.3.8).

(H2) $F(x, y)$ is defined for all $(x, y) \in S_x(d) \times Y$, $F(x, y) \in V$, and

(α) $F(x, y)$ is continuous on $S_x(d) \times Y$;

(β) if $y \in Y$ and $x', x \in S_x(d)$, then for some number $\theta = \theta(y)$ such that $0 < \theta < 1$,

$$\| MF_y(x') - MF_y(x) \| \leqslant \theta \| x' - x \| ;$$

(γ) there exists a number a, $0 < a < 1$, and a set $Y_o \subseteq Y$ such that for $y \in Y_o$ and $x \in S_x(d)$,

$$\| MF_y(x) \| \leqslant ad \quad .$$

(C1) Then, if $y \in Y_o$ and $u \in S_u\left[(1 - a)d\right] = S_x\left[(1 - a)d\right] \cap N(L)$, then the equation

$$x + M F_y(x) = u$$

has a unique solution $x = g_y(u) \in S_x(d)$.

(C2) The function $g_y(u)$ is continuous on $S_u\left[(1 - a)d\right] \times Y_o$ to $S_x(d)$ uniformly on $S_u\left[(1 - a)d\right]$.

(C3) For d finite, $g_y(u)$ is the uniform limit of the sequence of functions defined by

$$g_y^{(0)}(u) = u\ , \quad g_y^{(1)}(u) = u - M\left[F_y(u)\right], \ \ldots\ ,$$

$$g_y^{(n+1)}(u) = u - M\left\{F_y\left[g_y^{(n)}(u)\right]\right\}, \ \ldots\ .$$

(C4) There exists a bounded projection Q of V onto $R(L)$ and the pair $\left[g_y(u), y\right]$ satisfies the equation $G(x, y) = 0$ if and only if

$$\left(I_V - Q\right)\left\{F_y\left[g_y(u)\right]\right\} = 0 \ .$$

Proof. By H1 and Theorem 3.8, there exists a bounded linear transformation M' defined on $R(L)$ so that $LM'L = L$. Since L is a P-homomorphism, there exists by Lemma 3.11 a bounded projection Q mapping V onto $R(L)$; hence, $M = M'Q$ is a bounded pseudo-inverse of L defined for all $v \in V$. This is the M of hypothesis (H2). It follows from (H2) that $MF_y(x)$ satisfies the hypothesis of Theorem 4.1 ; therefore, the conclusion of Theorem 4.1 gives

all the results of 4.2 except (C4). To prove C4 one observes that if (x, y) is a solution of $G(x, y) = 0$, then $F(x, y) = -L(x) \in R(L)$; hence, $Q\left[F(x, y)\right] = F(x, y)$, and therefore $(I_V - Q)\left[F(x, y)\right] = 0$. To establish the converse, suppose that $(I_V - Q)\left\{F_y\left[g_y(u)\right]\right\} = 0$. Since $LM = LM'Q = I_{R(L)}Q = Q$, $Lu = 0$, and $g_y(u) = u - M\left\{F_y\left[g_y(u)\right]\right\}$; it follows that

$$\begin{aligned} G\left[g_y(u), y\right] &= L\left\{u - M\left[F_y\left(g_y(u)\right)\right]\right\} + F_y\left[g_y(u)\right] \\ &= -Q\left[F_y\left(g_y(u)\right)\right] + F_y\left[g_y(u)\right] \\ &= (I_V - Q)\left[F_y\left(g_y(u)\right)\right] = 0 \quad . \end{aligned}$$

4.3 <u>Corollary</u>. If $y_o \in Y_o$ is such that $F_{y_o}(x)$ maps some closed sphere $S_u(d_1)$, where $d_1 \leqslant (1 - a)d$, into $R(L)$, then the equation $L(x) + F(x, y)$ has a family of solutions $\left[g_{y_o}(u), y_o\right]$ which varies continuously with u.

Proof. By (C2), $g_{y_o}(u)$ is continuous on $S_u(d_1) \times Y_o$ to $S_x(d)$, and since $(I_V - Q)v = 0$ for all $v \in R(L)$, the consistency condition is evidently satisfied.

The consistency condition of Theorem 4.2 can be replaced by an equivalent one which is quite often more practical.

4.4 <u>Corollary</u>. The pair $\left[g_y(u), y\right]$ is a solution of $L(x) + F(x, y) = 0$ if and only if

$$v^* \left\{ F \left[g_y(u), y \right] \right\} = 0$$

for all $v^* \in N(L^*)$.

Proof. The assertion follows from the observation that $(I_V - Q) \left\{ F_y \left[g_y(u), y \right] \right\} = 0$ if and only if $F_y \left[g_y(u), y \right] \in R(L)$, and since $R(L)$ is closed, by [8, Theorem 2.13.6, page 28] , $R(L) = \left\{ v;\ L^*(v^*) = 0 \text{ implies } v^*(v) = 0 \right\}$.

4.5 Corollary. If L maps X onto Y , then for each $u \in S_u(d)$, the equation $x + M F_y(x) = u$ has a unique solution $x = g_y(u)$.

Proof. In this case it follows from [12, Theorem 4, page 287] that $N(L^*)$ contains only the zero functional; therefore, the consistency condition is trivially satisfied.

Example

As an example of Theorem 4.2, consider the nonlinear differential equation

$$x^{(n)}(t) \equiv \frac{d^n x}{dt^n} = -f \left[\alpha, t, x(t), x'(t), \ldots, x^{(n-1)}(t) \right] \tag{401}$$

in which α denotes a real parameter contained in some subset E of the real line. Let C_n denote the set of functions $x(t)$ with n continuous derivatives on the closed interval $I = \left[a, b \right]$, and for $x \in C_n$, define

$$\|x\|_{C_n} = \sum_{i=0}^{n} \max_{t \in I} |x^{(i)}(t)| \quad . \tag{402}$$

One observes that the left side of (401) is a linear transformation of C_n onto C ; if f is continuous and if $x(t) \in C_n$, it follows that each of the arguments of f is continuous and therefore the function f is a continuous function of t . Thus the right side of (401) may be interpreted as a continuous nonlinear function on $C_n \times E$ to C ; in this way one is led to the following theorem.

4.6 <u>Theorem</u>. Suppose that there exists a set $E_o \subseteq E$ and a real valued function $\gamma(\alpha)$ such that $0 < \gamma(\alpha) \leqslant 1$ and if $\alpha \in E_o$, then

(1) if $x \in S_x(d)$ then $\|f_\alpha(x)\|_C \leqslant \dfrac{d}{1 + e^{b-a}}$, and

(2) if $x_1 , x_2 \in S_x(d)$ then

$$\|f_\alpha(x_1) - f_\alpha(x_2)\|_C \leqslant \frac{\gamma(\alpha)}{1 + e^{b-a}} \|x_1 - x_2\|_{C_n} \quad .$$

Then for each $u \in S_u \left(\dfrac{d}{1 + e^{b-a}}\right)$ and each $\alpha \in E_o$, equation (401) has a unique solution $x = g_\alpha(u)$.

Proof. The proof is merely a matter of showing that the hypotheses of Theorem 4.2 are fullfilled. $L \equiv \dfrac{d^n}{dt^n}$ is bounded for

$$\|Lx\|_C = \|x^{(n)}(t)\|_C = \max_{t\in I} |x^{(n)}(t)|$$

$$\leqslant \sum_{i=0}^{n} \max_{t\in I} |x^{(i)}(t)| = \|x\|_{C_n} \quad ;$$

therefore, $\|L\| \leqslant 1$. L maps C_n onto C ; for let $\varphi(t)$ C be arbitrary, then the function

$$p(t) = \int_a^t \frac{(t-u)^{n-1}}{(n-1)!} \varphi(u)\, du \tag{403}$$

is in C_n and $Lp(t) = \varphi(t)$. The necessary projection is just the identity I_C , and therefore (H1) is fullfilled. For $\varphi(t)$ arbitrary in C , let $M\varphi(t)$ be defined by (403). If $u \in N(L)$; that is, if u is a polynomial of degree at most $n - 1$, then $L(u + M\varphi(t)) = \varphi(t)$ and thus M is a pseudo-inverse of L . **Since $f_\alpha(x)$ has the prescribed continuity properties, it is necessary to verify only that $Mf_\alpha(x)$ satisfies (β) and (γ) of H2.** It simplifies matters to look first for a bound for $\|M\|$.

$$\|M\varphi\|_{C_n} = \sum_{i=0}^{n} \max_{t\in I} \left| \frac{d^i\left[M\varphi(t)\right]}{dt^i} \right|$$

$$= \sum_{i=0}^{n-1} \max_{t\in I} \left| \int_a^t \frac{(t-u)^{n-1-i}}{(n-1-i)!} \varphi(u)\, du \right| + \max_{t\in I} |\varphi(t)|$$

$$\leqslant \sum_{i=0}^{n-1} \max_{t\in I} |\varphi(t)| \left| \frac{(t-a)^{n-i}}{(n-i)!} \right| + \max_{t\in I} |\varphi(t)|$$

$$\leqslant \left[1 + \sum_{i=0}^{n-1} \frac{(b-a)^{n-i}}{(n-i)!} \right] \max_{t \in I} |\varphi(t)|$$

$$= \left[\sum_{i=0}^{n} \frac{(b-a)^{i}}{i!} \right] \max_{t \in I} |\varphi(t)|$$

$$\leqslant e^{b-a} \|\varphi\|_C \quad .$$

Hence $\|M\| \leqslant e^{b-a}$. Thus to verify that properties (β) and (γ) are satisfied, choose $\theta = \theta(\alpha) = \dfrac{e^{b-a}\gamma(\alpha)}{1 + e^{b-a}}$ for (β) and $a = \dfrac{e^{b-a}}{1 + e^{b-a}}$ for (γ) . It is evident that $0 < \theta < 1$ and $0 < a < 1$; moreover, if $\alpha \in E_o$ and $x , x_1 , x_2 \in C_n$, then

$$\|M f_\alpha(x_1) - M f_\alpha(x_2)\| \leqslant e^{b-a} \|f_\alpha(x_1) - f_\alpha(x_2)\|_C$$

$$\leqslant \frac{e^{b-a}\gamma(\alpha)}{1 + e^{b-a}} \|x_1 - x_2\|_{C_n}$$

$$= \theta \|x_1 - x_2\|_{C_n} \quad ;$$

and

$$\|M f_\alpha(x)\| \leqslant e^{b-a} \|f_\alpha(x)\|_C \leqslant \frac{e^{b-a} d}{1 + e^{b-a}} = ad \quad .$$

Therefore the conclusion of Theorem 4.2 assures the existence and uniqueness of a solution of equation (401) and Corollary 4.5 shows that the consistency condition is fullfilled.

This theorem brings out the fact that it is often not necessary to consider supplementary data in order to investigate the existence of a solution of a differential equation. It should be noted, however, that if $u \in N(L)$ is the polynomial which satisfies given initial conditions at the point $t = a$, then the existence proof given here coincides with the classical proof by successive approximations.

CHAPTER V

A GENERALIZATION OF THE RIESZ-SCHAUDER SPECTRAL THEORY

Preliminary Lemmas

Throughout this chapter it is supposed that L is a bounded linear transformation with a Banach space X as its domain and a Banach space $Y = R(L)$ as its range. Y can be different from X, equal to X or even a proper subspace of X, and L is permitted to be either singular or non-singular. It is supposed also that K is a compact linear transformation with domain X and range in Y, and then the bounded linear transformation

$$T_c = cL - K$$

is considered for an arbitrary complex number c.

The object here is to obtain results similar to those of the Riesz-Schauder theory [12, Chapter 11] of transformations of the form $T_c = cI - K$.

In a recent paper L. M. Graves [6] considered this topic and obtained some elegant results. By modifying slightly Graves' definition of a certain sequence of linear subspaces (cf. 501), it will be shown that one is led to a generalized spectral theory.

One basic theorem in the classical Riesz theory is that there exists a subspace $S_q = R(T_c^q)$, such that when T_c is restricted to S_q, T_c

is non-singular and the whole space X is the direct sum of S_q and $N(T_c^q)$. A suitable generalization of this result is found, and thus a pseudo-resolvent can be defined and some of the usual properties developed.

It is perhaps of some interest to note that even when $Y = X$ so that iteration of T_c is possible, $N(T_c^n) = N\left[(cL - K)^n\right]$ can be a proper subspace of $N(T_c^{n+1})$ for every integer n even though L has a null space of one dimension. This can be seen by means of the following example.

Let X be the real sequence space ℓ_1 and for $x = (a_1, a_2, a_3, \ldots)$, define $L(x) = (a_2, a_3, a_4, \ldots)$ and $K(x) = (a_2, 0, 0, \ldots)$. Since the range of K is finite-dimensional, K is compact. Let $c = 1$ and consider $T = L - K$. Suppose $x_k = e_i^k$ is the usual set of basis vectors; that is, $e_k^k = 1$, $e_i^k = 0$ $(i \neq k)$. It is to be shown that $T^k(x_{k+1}) = 0$, $T^{k-1}(x_{k+1}) \neq 0$, for each k . For $k = 1$, $Tx_2 = (L - K)\, x_2 = x_1 - x_1 = 0$, and for $k > 1$, $T(x_{k+1}) = L(x_{k+1}) - K(x_{k+1}) = L(x_{k+1}) = x_k$. Therefore $T^{k-1}(x_{k+1}) = x_2 \neq 0$ and $T^k(x_{k+1}) = T(x_2) = 0$. Hence $N(T) \subset N(T^2) \subset N(T^3) \subset \ldots \subset N(T^k) \ldots$, and the inclusion is proper for every integer k . That this difficulty can be overcome will be shown in what follows.

Perhaps the first question to settle is - in this generalized sense, what is an eigenvalue? In the classical theory one defines a non-zero number c to be an eigenvalue provided $T_c(x) = 0$ for some element $x \neq 0$, and then proves that c is an eigenvalue of T_c if and only if c is an eigenvalue of T_c^* [12, Theorem 13, page 338] . It is also shown

that the dimension of the null space of T_c is finite and is equal to that of T_c^*. Here, for $Y \neq X$ and $T_c = cL - K$, it can very well happen that $T_c(x) = 0$ for all x in a space of infinite dimension; yet, $T_c^*(y^*) = 0$ will imply that $y^* = 0$. With no restrictions on $N(T_c)$ this is the case when $R(T_c) = Y$; then, by Corollary 3.6, T_c^* has an inverse. When $Y \neq X$, the fundamental connection between T_c and T_c^* is given by the relation $N(T_c^*) = \left[R(T_c)\right]^{\perp}$; therefore, the theory promises to be more rewarding if the definition of an eigenvalue pertains to the range instead of the domain of T_c. The definition below is the one given by Graves and it is followed here.

5.1 <u>Definition</u>. A complex number c is said to be an eigenvalue of K provided the range of $T_c = cL - K$ is a proper subset of $Y = R(L)$.

5.2 <u>Corollary</u>. A complex number c is an eigenvalue of K if and only if $T_c^* y^* = 0$ for some $y^* \neq 0$.

Proof. This follows immediately from the fact that $R(T_c) \subset Y$ if and only if $\left[R(T_c)\right]^{\perp} = N(T_c^*)$ is not empty.

J. Dieudonné [4] and B. Yood [11] have investigated the properties of linear transformations preserved under addition of a compact transformation. Some of these results are listed here for reference.

5.3 <u>Definition</u>. A linear transformation T of X into Y is said to have

<u>Property A</u> provided T is a homomorphism and $N(T)$ is finite dimensional,

and to have

Property B provided T is a homomorphism and the factor space $Y/R(T)$ is finite dimensional.

The dimension of the null space of T is called the nullity of T and is designated by nul T. The dimension of the factor space $Y/R(T)$ is called the deficiency of T and is designated def T. A linear transformation has finite deficiency n if and only if $R(T)$ has a complementary subspace Z in Y of dimension n.

5.4 Lemma. T has property B with deficiency n if and only if T^* has property A with nullity n. [4, Théorème 3].

5.5 Lemma. If L has property A (property B) and if K is compact, then $T = L + K$ has property A (property B). [11, Theorem 3.7].

5.6 Lemma. If L has property A (property B), then for any complex number $c \neq 0$, cL has property A (property B).

Proof. $R(cL) = R(L)$ and $N(L) = N(cL)$.

5.7 Lemma. Suppose L is a homomorphism and K is compact. Let S denote a closed complement of $N(L)$ in X and let $T = L + K$. Then if W is any closed subspace of S, the restriction T_W of T to W has property A. If L_W has property B, then T_W has property B.

Proof. The first assertion follows from Lemma 5.5 and the observation that L is non-singular on W. The second follows directly from 5.5.

5.8 Lemma. If a bounded linear transformation M on Y into X has a bounded inverse and if Z is any closed subspace of Y, then $M(Z)$ is a closed subspace of $R(M)$.

Proof. M^{-1} is continuous.

The Eigenvalues of K

It is supposed now that $c \neq 0$ is an eigenvalue of K and that M is a bounded pseudo-inverse of L. An M with this property is assured by Theorem 3.8. Consider the following two sequences of linear spaces, in which the spaces S_k are subspaces of X and the spaces V_k are subspaces of Y. Let

$$
\begin{aligned}
V_1 &= T_c(X) \quad , \qquad S_1 = M(V_1) \quad , \\
V_2 &= T_c(S_1) \quad , \qquad S_2 = M(V_2) \quad , \\
&\text{- - - - - - - - - - - - - - - - - - - -} \\
V_k &= T_c(S_{k-1}) \quad , \qquad S_k = M(V_k) \quad , \\
&\text{- - - - - - - - - - - - - - - - - - - -}
\end{aligned}
\tag{501}
$$

Since this is the modification of Graves' definition referred to earlier, the sequences he defined are listed for comparison. Graves [6, page 144] defined

$$
\begin{aligned}
W_1 &= T_c(X) \\
Z_1 &= \text{the maximal subspace of } X \text{ such that} \\
&\quad L(Z_1) = W_1 \quad , \\
&\text{- -} \\
W_k &= T_c(Z_{k-1}) \text{ , and } Z_k \text{ such that} \\
L(Z_k) &= W_k \text{ , } Z_k \text{ maximal,} \\
&\text{- -}.
\end{aligned}
\tag{502}
$$

If L is non-singular, then $W_k = V_k$ and $Z_k = S_k$ for every k; however, if L is singular, it is easily seen that $Z_1 = S_1 \oplus N(L)$, $W_2 = V_2 \oplus K\left[N(L)\right]$, etc. It can be shown [6, page 145] that each Z_k has a finite-dimensional complement in X, but this is impossible for the spaces S_k when $N(L)$ is of infinite dimension, for $Z_k \supseteq S_k \oplus N(L)$ for every k. That this presents no real difficulty will be apparent in the discussion that follows.

5.9 <u>Theorem</u>. The spaces V_k and S_k are closed.

Proof. Since the restriction L_S of L to S (where $X = S \oplus N(L)$) is the bounded inverse of M, it is sufficient by 5.8 to show that the spaces V_k are closed. The proof proceeds by induction on n. Since L is a homomorphism with property B (def $L = 0$), it follows from Lemma 5.5 that T_c is a homomorphism with property B; therefore, from the definition of a homomorphism, $V_1 = R(T_c) = T_c(X)$ is closed. Assume now that V_m and S_m are closed and note that $V_m = LM(V_m) = L(S_m)$; hence, the restriction of L to S_m has property B. Therefore by Lemma 5.8, the restriction of T_c to S_m has property B. Since $V_{m+1} = (cL - K)\, S_m$, V_{m+1} is closed and the theorem is established.

It is clear from the definition that

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k \supseteq \cdots, \text{ and}$$

$$S_1 \supseteq S_2 \supseteq \cdots \supseteq S_k \supseteq \cdots .$$

It is shown next that a proper inclusion can hold for only a finite number of the indices, and that the $\supseteq$ relation is equality from that index onward.

5.10 <u>Theorem</u>. There exists a finite index $q = q(c)$ such that if $m < n \leqslant q$ then $V_m \supseteq V_n$ and $S_m \supseteq S_n$, and if $q \leqslant r$ then $V_q = V_r$ and $S_q = S_r$.

Proof. If $S_k = S_{k+1}$, then by the definition $M(V_k) = M(V_{k+1})$, and since M is non-singular, $V_k = V_{k+1}$. Thus it is sufficient to consider the S sequence only.

Since the product of a bounded transformation and a compact transformation is a compact transformation, it follows that

$$MT_c = cML - MK = cI_{S_1} - MK$$

is of the form $cI_{S_1} - K'$, with K' compact. Therefore, since $S_k = MT_c(S_{k-1}) = \dots = (MT_c)^k(X) = R\left[(MT_c)^k\right]$ the desired result can be deduced in exactly the same way as the comparable result in the classical theory. A proof for the classical case is given in [12, Theorem 7, page 335].

The above discussion shows that one may infer the validity of the next three theorems in like manner.

5.11 <u>Theorem</u>. Let $U_k = N\left[(MT_c)^k\right]$. Then $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots \subseteq U_k \subseteq \dots$, and there exists an index $p = p(c)$ such that $U_s = U_p$ for $s \geqslant p$, while U_n is a proper subspace of U_{n+1} for

$n < p$. [12, Theorem 6, page 334].

5.12 Theorem. The index $p = p(c)$ of 5.11 is equal to the index $q = q(c)$ of 5.10 [12, Theorem 7, page 335].

5.13 Theorem. The space X is the direct sum of S_q and U_q. [12, Theorem 8, page 336].

Suppose now that c_j denotes a sequence (finite or infinite) of distinct eigenvalues of K. Let N_j^* denote the null space of $T_{c_j}^*$. Since T_c has property B, by Lemma 5.4, T_c^* has property A; thus, let $\left[y_{ij}^*\right]$ $(i = 1, 2, \ldots, k_j)$ denote a basis for the finite dimensional space N_j^*.

5.14 Theorem. For any positive integer n the elements $\left[y_{ij}^*\right]$ $(i = 1, 2, \ldots, k_j;\ j = 1, 2, \ldots, n)$ are linearly independent.

Proof. The proof proceeds by induction on j. For $j = 1$ the assertion follows from the definition of the basis. Suppose that the assertion is true for $j = m$, and suppose there exist scalars a_{ij} such that the form

$$B(a, y^*) = \sum_{j=1}^{m+1} \sum_{i=1}^{k_j} a_{ij}\, y_{ij}^* = 0 .$$

Clearly $T_{c_{m+1}}^* \left[B(a, y^*)\right] = 0$, and since $T_{c_{m+1}}^*\, y_{i\,m+1}^* = 0$ for $i = 1, 2, \ldots, k_{m+1}$, it follows from the definition of T_c^* that

$$T_{c_{m+1}}^* \left(\sum_{j=1}^{m} \sum_{i=1}^{k_j} a_{ij}\, y_{ij}^* \right) = \sum_{j=1}^{m} \sum_{i=1}^{k_j} T_{c_{m+1}}^* (a_{ij}\, y_{ij}^*)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{k_j} (c_{m+1} L^* - K^*)(a_{ij}\, y_{ij}^*) = 0 .$$

Since on N_j^*, $T_{c_j}^* = 0$, and thus $c_j L^* = K^*$, it follows upon substitution that

$$\sum_{j=1}^{m} \sum_{i=1}^{k_j} (c_{m+1} - c_j) L^* (a_{ij} y_{ij}^*) =$$

$$L^* \sum_{j=1}^{m} \sum_{i=1}^{k_j} (c_{m+1} - c_j) a_{ij} y_{ij}^* = 0 .$$

Since L^* is non-singular, the form itself must vanish; therefore, by the induction hypothesis, since $c_{m+1} \neq c_j$, it follows that $a_{ij} = 0$ for $i = 1, \ldots, k_j$, and $j = 1, 2, \ldots, m$. Thus the relation

$$\sum_{j=1}^{m+1} \sum_{i=1}^{k_j} a_{ij} y_{ij}^* = \sum_{j=1}^{m} \sum_{i=1}^{k_j} a_{ij} y_{ij} + \sum_{i=1}^{k_{m+1}} a_{im+1} y_{im+1}^* = 0$$

implies all scalars a_{ij} must vanish since the elements y_{im+1}^* are a basis for the space N_{m+1}^*. This completes the proof.

5.15 <u>Theorem</u>. The number of distinct eigenvalues c_n of K is either finite or countably infinite, and in the latter case $\lim_{n \to \infty} c_n = 0$.

Proof. It is sufficient to prove that if d is an arbitrary positive number, the number of distinct eigenvalues c_n, for which $|c_n| \geqslant d$ is finite. Suppose the contrary; that is, suppose $|c_n| \geqslant d$ for an infinite number of c_n. Let Z_n^* denote the finite-dimensional linear space spanned by the spaces $N_1^*, N_2^*, \ldots, N_n^*$. Then $Z_1^* \subset Z_2^* \subset \ldots \subset Z_n^*$, where the inclusion is proper by Theorem 5.14. Hence for each integer n there exists an element $z_n^* \in Z_n^*$ such that

$\|z_n^*\| = 1$ and $\|z_n^* - z\| \geqslant 1/2$ for every $z^* \quad Z_{n-1}^*$. Since the sequence $\{z_n^*\}$ is bounded, and since $M^* K^*$ is compact, there exists a subsequence $\{z_j^*\}$ of the $\{z_n^*\}$ for which $M^* K^* z_j^*$ converges. Thus for p and q sufficiently large,

$$\|M^* K^* z_p^* - M^* K^* z_q^*\| \leqslant \epsilon < d/2 \quad . \tag{503}$$

Since by 3.6 $M^* L^* = I_{Y^*}$, and since $M^* T_c^* = cM^* L^* - M^* K^* = cI - M^* K^*$, the left member of (503) can be expressed as

$$c_p z_p^* - c_q z_q^* - M^* T_{c_p} z_p^* + M^* T_{c_q}^* z_q^* \quad .$$

If $y^* \in N_p^*$, $K^* y^* = c_p L^* y^*$; therefore, for arbitrary c, $T_c^* y^* = c L^* y^* - c_p L^* y^* = L^* \left[(c - c_p)y^*\right]$, and this expression is zero only for $c = c_p$. Thus

$$T_c^* Z_p^* \subset L^* Z_{p-1}^* \quad ,$$

and therefore the element $z^* = c_q c_p^{-1} z_q^* - c_p^{-1} M^* T_{c_p}^* z_p^* + c_p^{-1} M^* T_{c_q}^* z_q^*$ is contained in Z_{p-1}^* for $p > q$. Hence for $p > q$,

$$\|M^* K^* z_p^* - M^* K^* z_q\| = |c_p| \, \|z_p^* - z^*\| \geqslant d/2 \, , \tag{504}$$

which contradicts inequality (503) and the assertion follows.

A Pseudo-Resolvent

Suppose now that c_o is an eigenvalue of K and $q = q(c_o)$ is the index for which $S_q = S_r$, $r \geqslant q$.

5.16 Definition. The values of c for which T_c has a bounded non-singular pseudo-inverse $R_c = R(c, K)$ such that $T_c R_c = I_Y$ form the pseudo-resolvent set $r(K)$ of K . The set of all these $R(c, K)$ is called the pseudo-resolvent of K . The values of c not in $r(K)$ form the pseudo-spectrum $s(K)$ of K .

5.17 Corollary. The complex number c is contained in $r(K)$ if and only if c is not an eigenvalue of K .

Proof. If $c \in r(K)$, $T_c R_c = I_Y$ and therefore for arbitrary $y \in Y$, $T_c \left[R_c y\right] = y$; thus, the range of T_c is Y and c is not an eigenvalue. If c is not an eigenvalue, Theorem 3.8 shows that $c \in r(K)$.

5.18 Corollary. The pseudo-spectrum $s(K)$ contains at most a countable number of elements. If Y is infinite dimensional, $s(K)$ is not empty.

Proof. Theorem 5.15 and Corollary 5.17 prove the first statement. If $c = 0$, $T_o = -K$; hence, T_o is compact. Corollary 3.9 shows that T_o does not have a bounded, non-singular pseudo-inverse R_o for which $T_o R_o = I_Y$. Therefore $c = 0$ is contained in $s(K)$ and this proves the second statement.

5.19 Theorem. Let c_o denote an element of the pseudo-resolvent set and suppose that R_{c_o} is a pseudo-inverse of T_{c_o} . If $|c - c_o| <$

$(\|L\|\|R_{c_o}\|)^{-1}$, then $c \in r(K)$; that is, $r(K)$ is open.

Proof. Since

$$\|I_Y - \left[T_{c_o} - (c_o - c)\, L\right] R_{c_o}\| = \|(c_o - c)\, L\, R_{c_o}\| < 1 ,$$

and since $V_o \equiv \left[T_{c_o} - (c_o - c)L\right] R_{c_o}$ is a bounded linear transformation on Y to Y and $\|I_Y - V_o\| < 1$, it follows from Corollary 3.14 that V_o^{-1} exists, V_o^{-1} is bounded, and that

$$\begin{aligned} V_o^{-1} &= I_Y + \sum_{n=1}^{\infty} (I_Y - V_o)^n \\ &= I_Y + \sum_{n=1}^{\infty} \left[I_Y - T_{c_o} R_{c_o} + (c_o - c)\, L\, R_{c_o}\right]^n \\ &= I_Y + \sum_{n=1}^{\infty} (c_o - c)^n (L\, R_{c_o})^n \quad . \end{aligned}$$

Noting that $T_c = T_{c_o} - (c_o - c)L$, one obtains $T_c\, R_{c_o}\, V_o^{-1} = I_Y$; therefore, $R_{c_o}\, V_o^{-1}$ is a bounded, non-singular pseudo-inverse of T_c with the desired property. Explicitly,

$$R_c = R_{c_o} + R_{c_o} \sum_{n=1}^{\infty} (c_o - c)^n (L\, R_{c_o})^n \quad . \qquad (505)$$

In the classical theory the final result is a calculation of the resolvent of T_c for a number c sufficiently near an eigenvalue c_o of index q by means of a Laurent expansion about c_o . The finite sum

in this expansion is of the form

$$\sum_{k=1}^{q} - \frac{(T_{c_o})^{k-1}}{(c-c_o)^k} \quad .$$

The infinite sum is of the same form as that in (505) above, being restricted to apply on S_q , since on S_q c_o is not an eigenvalue. However, since here iteration of T is not possible, and since the null space of $T_{c_o} = c_o L - K$ may very well be infinite dimensional, a direct Laurent expansion as in the classical case is impossible. Whether or not this difficulty can be circumvented seems to be an unanswered question.

BIBLIOGRAPHY

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1. Baer, R. Linear Algebra and Projective Geometry. Academic Press, Inc., New York, 1952.

2. Banach, S. Théorie des Opérations Linéaires. Monografje Matematyczne, Warsaw, 1932.

3. Bartle, R. G. Singular points of functional equations. Trans. Amer. Math. Soc. 75, 366-384 (1953).

4. Dieudonné, J. Sur les homomorphismes d'espaces normés. Bull. Sci. Math. 67, 72-84 (1943).

5. Friedrichs, K. O. Functional Analysis. (Mimeographed Lecture Notes), New York University, 1950.

6. Graves, L. M. A generalization of the Riesz-Schauder theory of completely continuous transformations. Trans. Amer. Math. Soc. 79, 141-157 (1955).

7. Hamburger, H. Non-symmetric operators in Hilbert space. Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 67-112. Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma, 1951.

8. Hille, E. Functional Analysis and Semi-Groups. American Mathematical Society Colloquium Publications, Vol. 31. American Mathematical Society, New York, 1948.

9. MacDuffee, C. C. Vectors and Matrices. Carus Monograph Series, No. 7. Mathematical Association of America, Ithaca, N. Y., 1943.

10. Murray, F. J. On complementary manifolds and projections in spaces L_p and ℓ_p. Trans. Amer. Math. Soc. 41, 138-152 (1937).

11. Yood, B. Properties of linear transformations preserved under addition of a compact transformation. Duke Math. J. 18, 599-612 (1951).

12. Zaanen, A. C. Linear Analysis. Interscience Publishers, Inc., New York, 1953.