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ON THE SOLUTION OF A FREDHOLM INTEGRAL EQUATION

C. L. Gerberich

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MATHEMATICS PANEL

ON THE SOLUTION OF A FREDHOLM INTEGRAL EQUATION

Carl L. Gerberich

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CHAPTER I

INTRODUCTION

The solution of a Fredholm integral equation such as

$$f(x) = s(x) + \lambda\int_a^b K(x,y)f(y)dy \tag{I.1.1}$$

is often obtained by the method of successive approximations. Provided that the integral $\int_a^b K(x,y)u_0(y)dy$ exists, the right member of I.1.1 is considered as an operator T, which associates with a function u_0 an image function u_1 defined by

$$u_1(x) = s(x) + \lambda\int_a^b K(x,y)u_0(y)dy = Tu_0(x). \tag{I.1.2}$$

It is the object of this thesis to develop a technique for finding upper and lower bounds for the solution of I.1.1. If a sequence $\{u_n\}$ is defined inductively with u_n being the image of the function u_{n-1} under T, then with suitable conditions on the kernel function $\lambda K(x,y)$, the sequence $\{u_n\}$ converges to a limit function which is a solution of I.1.1. With the added condition that the kernel be nonnegative, it can be shown that the image of an upper (lower) bound is an upper (lower) bound. Hence if the initial function u_0 is an upper (lower) bound then all the elements of the sequence $\{u_n\}$ are upper (lower) bounds. The technique developed in this thesis, which is referred to as the method of upper and lower functions, develops methods of producing upper and lower bounds which are easily recognized as such.

<u>Definition I.2.</u> A function $g(x)$ is called an upper function if for a nonnegative kernel

I.2.1 $$g(x) \geqslant g_1(x) \qquad \text{for all } x \in [a,b]$$

where

I.2.2 $$g_1(x) = s(x) + \lambda\int_a^b K(x,y)g(y)dy.$$

<u>Definition I.3.</u> A function $h(x)$ is called a lower function if for a nonnegative kernel

I.3.1 $$h(x) \leqslant h_1(x) \qquad \text{for all } x \in [a,b]$$

where

I.3.2 $$h_1(x) = s(x) + \lambda\int_a^b K(x,y)h(y)dy.$$

Later it is shown that with certain conditions on the kernel the upper (lower) functions are upper (lower) bounds for the solution. This thesis develops methods which enable the choice of the initial functions as well as their iteration to be performed on a digital computer.

In Chapter II the technique of upper and lower functions is developed for the integral equations whose kernels obey the following restrictions:

$$(1.)\quad \lambda K(x,y) \geqslant 0 \qquad x,y \in [a,b]$$

$$(2.)\quad \lambda \int_a^b K(x,y) \leqslant \rho < 1 \qquad x \in [a,b]\ ,\ \rho > 0.$$

As in many applications of numerical methods, the original equation is replaced by a system of linear equations. Therefore in Chapter II the technique of upper and lower functions is extended to a system of linear equations.

In Chapter III an effort was made to relax somewhat the restrictions on the kernels. Chapter III also contains a section on a method to speed the convergence of the sequence. This method is an analog of the Gauss-Seidel method for linear algebraic equations.

Some numerical examples have been calculated on the Oak Ridge National Laboratory's digital computer, the Oracle, and in Chapter IV these examples are discussed. All the methods developed here are adapted to use on a digital computer, and the problem is coded in such a way as to insure that the computed functions are rigorous upper and lower bounds for the solution of the integral equation. This is achieved by handling rounding errors in such a way as consistently to increase upper bounds or decrease lower bounds.

CHAPTER II

BASIC PROPERTIES OF UPPER AND LOWER FUNCTIONS

In this chapter some of the basic properties of upper and lower functions are discussed. There follow some definitions and restrictions needed in order to complete the proofs of these properties.

<u>Definition II.1.</u> Let $\mathcal{L}$ be the class of real-valued functions $L(x,y)$ defined and bounded on $a \leqslant x,y \leqslant b$ and satisfying the two conditions:

II.1.1 for all $[\alpha,\beta] \subseteq [a,b]$ and all $x \in [a,b]$ the integral $\int_\alpha^\beta L(x,y)dy$ exists in the sense of Riemann and is Riemann integrable as a function of x in $[a,b]$;

II.1.2 for every $L \in \mathcal{L}$ the $\sup\limits_{x \in [a,b]} \int_a^b |L(x,y)|dy < 1$.

The notation $\mathcal{L}^+$ ($\mathcal{L}^-$) denotes the class of nonnegative (nonpositive) functions belonging to $\mathcal{L}$.

<u>Definition II.2.</u> The class of real-valued functions defined on $[a,b]$ and integrable in the sense of Riemann will be denoted by R.

<u>Definition II.3.</u> For each $f \in R$ a norm is defined by

$$||f|| = \sup_{x \in [a,b]} |f(x)|.$$

It is well known that R, so normed, is a Banach space.

<u>Theorem II.4.</u> If $f \in R$ then $\int_a^b L(x,y)f(y)dy$ exists for every $x \in [a,b]$ and is Riemann integrable in $[a,b]$.

<u>Proof:</u>[1] If f is a step function defined on $[a,b]$ the assertion is an immediate consequence of condition II.1.1. For any $f \in R$ and $x \in [a,b]$, define $g(x) = \int_a^b L(x,y)f(y)dy$. The integral exists since R is closed under multiplication. The set of points x where the oscillation of f exceeds $\frac{1}{n}$ has Jordan content 0. Thus there exists a finite set of disjoint intervals $\left\{I_j\right\}_{j=1}^m$ whose interiors cover the above set and such that the sum of their lengths is less than $\frac{1}{n}$. The complement of the set $\bigcup_{j=1}^m I_j^0$ contains no points where the oscillation exceeds $\frac{1}{n}$, and thus about each point in the complement a neighborhood may be taken in which the oscillation of f is less than $\frac{2}{n}$. By the Heïne-Borel theorem a finite number of such neighborhoods cover the complement. Thus the complement of $\bigcup_{j=1}^m I_j^0$ can be subdivided into a finite number of non-overlapping, closed intervals, each of which lies in one of the neighborhoods. Since the oscillation of f on any such interval is less than $\frac{2}{n}$ a constant may be chosen such that f does not differ from the constant in this interval by more than $\frac{1}{n}$. Thus, a step function h_n may be defined such that $\left|f(x) - h_n(x)\right| < \frac{1}{n}$ for $x \notin \bigcup_{j=1}^m I_j^0$ and with $\left|f(x) - h_n(x)\right| \leqslant 2\left\|f\right\|$ for $x \in [a,b]$. Let $g_n(x) = \int_a^b L(x,y)h_n(y)dy$. Then if N is a bound for $\left|L\right|$,

$$\left|g(x) - g_n(x)\right| \leqslant \int_a^b \left|L(x,y)\right| \left|f(y) - h_n(y)\right| dy \leqslant \frac{2\|f\|N}{n} + \frac{(b-a)N}{n}$$

and thus g_n converges uniformly to g on $[a,b]$. It follows that $g \in R$.

---

[1]The proof of this theorem was suggested by Dr. W. S. Snyder.

Thus if $s \in R$ then the operator T defined in I.1.2 may be considered to have domain R, and its range is a subset of R.

Theorem II.5. If $\lambda K \in \mathcal{L}$ and if $s \in R$ then the operator T defined on R by I.1.2 is totally convergent. That is, for any $u_0 \in R$ the sequence $\left\{u_n\right\}_{n=0}^{\infty}$ defined by $u_{n+1} = Tu_n$, $n \geqslant 0$ converges uniformly to some function $f \in R$, and f is the unique fixed element in R for the transformation $\mathcal{L}$.

The proof of the uniform convergence of the sequence $\left\{u_n\right\}$ and the uniqueness of the solution are standard in the literature. See Lovitt p. 15 [4].[2]

An interesting sub-class of $\mathcal{L}$ is made up of functions whose discontinuities are regularly distributed in the rectangle $a \leqslant x,y \leqslant b$ as defined by Bôcher, p. 3 [1]. For all kernels in this sub-class and $f \in R$ the integral $\int_a^b K(x,y)f(y)dy$ is a continuous function of x. Throughout this chapter it is assumed, unless the contrary is explicitly stated, that the function $\lambda K \in \mathcal{L}^+$.

Section I. Properties of Upper and Lower Functions for a Fredholm Integral Equation

In this section a criterion is developed by which certain upper (lower) bounds for the solution of the integral equation I.1.1 may be recognized provided $\lambda K \in \mathcal{L}^+$. These upper (lower) functions have certain properties which make them useful for the numerical solution of the integral equation. Such properties are developed and studied in this chapter.

---

[2]References are to the bibliography at the end of the thesis.

In general, the theorems are given for both upper and lower functions, but since the proofs of the theorems for the lower functions follow the proofs for the upper functions, with only minor changes, the proof is given only for the upper functions.

Theorem II.6. The property of being an upper (lower) function is hereditary.

Proof: By definition

II.6.1 $$g_{n+1} = Tg_n \text{ and } n = 0,1$$

and

II.6.2 $$g_0(x) \geqslant g_1(x) \text{ for all } x \in [a,b] .$$

Thus

II.6.3 $$g_1(x) - g_2(x) = \lambda \int_a^b K(x,y) \left[g_0(y) - g_1(y) \right] dy.$$

Using the assumptions $\lambda K \in \mathcal{Z}^+$ and $g_0(x) \geqslant g_1(x)$ it is clear that

II.6.4 $$g_1(x) - g_2(x) \geqslant 0 \text{ for all } x \in [a,b] .$$

Hence $g_1(x) \geqslant g_2(x)$ for all $x \in [a,b]$. Thus any iterate of an upper (lower) function is an upper (lower) function.

Theorem II.7. If $g(x)$ is an upper function then $g(x) \geqslant f(x)$, and if $h(x)$ is a lower function then $h(x) \leqslant f(x)$ for all $x \in [a,b]$ where $f(x)$ is the solution of equation I.1.1.

Proof: Since g is an upper function the sequence $\{g_n\}$ defined by I.1.2 is monotone decreasing. Let C be a bound for $|g_1(x) - g_0(x)|$ for $x \in [a,b]$. Then by induction

II.7.1 $$\left| g_{n+1}(x) - g_n(x) \right| = \left| \lambda \int_a^b K(x,y) \left[g_n(y) - g_{n-1}(y) \right] dy \right| \leqslant C\rho^n.$$

It follows that the sequence $\{g_n\}$ converges uniformly on $[a,b]$ and that the limit function is a solution, unique in the class R, of equation I.1.1. Hence $g_n(x) \geqslant f(x)$ for x in $[a,b]$.

<u>Theorem II.8.</u> An error estimate for an upper (lower) function is

II.8.1
$$\|g_0 - f\| \leqslant \left\| \frac{g_0(x) - g_1(x)}{1 - \lambda\int_a^b K(x,y)dy} \right\| .$$

<u>Proof:</u> If $g_0(x)$ is an upper function $g_0(x) \geqslant g_1(x)$, but $g_1(x) + \gamma \geqslant g_0(x)$ if $\gamma \geqslant \|g_0 - g_1\|$. By definition

II.8.2
$$g_0(x) - f(x) = \left[g_0(x) - g_1(x)\right] + \left[g_1(x) - f(x)\right]$$

$$= \left[g_0(x) - g_1(x)\right] + \lambda\int_a^b K(x,y)\left[g_0(y) - f(y)\right]dy$$

$$\leqslant \left[g_0(x) - g_1(x)\right] + \lambda\|g_0 - f\| \int_a^b K(x,y)dy.$$

For any given $\epsilon > 0$ there exists an x_0 such that

$$g_0(x_0) - f(x_0) \geqslant \|g_0 - f\| - \epsilon.$$

Hence

II.8.3

$$\|g_0 - f\| - \epsilon \leqslant g_0(x_0) - f(x_0) \leqslant \left[g_0(x_0) - g_1(x_0)\right] + \lambda \|g_0 - f\| \int_a^b K(x_0,y)dy.$$

Thus

II.8.4
$$\|g_0 - f\| \leqslant \frac{g_0(x_0) - g_1(x_0) + \epsilon}{1 - \lambda\int_a^b K(x_0,y)dy} \leqslant \left\| \frac{g_0(x) - g_1(x)}{1 - \lambda\int_a^b K(x,y)dy} \right\| + \frac{\epsilon}{1 - \rho} .$$

As $\epsilon \longrightarrow 0$ II.8.4 approaches II.8.1.

<u>Theorem II.9.</u> A lower bound for the error estimate for any iterate of an upper (lower) function is

II.9.1 $$\left\|g_0 - f\right\| \geqslant \sigma\left[1 + \left\|\lambda\int_a^b K(x,y)dy\right\|\right]$$

where $\sigma = \inf_{[a,b]} \left|g_0(x) - g_1(x)\right|$.

<u>Proof:</u> By definition

II.9.2 $$\left\|g_0 - f\right\| \geqslant \left\|\sigma + g_1 - f\right\| \geqslant \left\|\sigma + \lambda\int_a^b K(x,y)\left[g_0(y) - f(y)\right] dy\right\|.$$

Since $g_0(y) - f(y) \geqslant g_0(y) - g_1(y) \geqslant \sigma$ and $\lambda\int_a^b K(x,y)dy \geqslant 0$ for all x, it follows that

II.9.3 $$\left\|g_0 - f\right\| \geqslant \left\|\sigma + \sigma\lambda\int_a^b K(x,y)dy\right\| = \sigma\left\|1 + \lambda\int_a^b K(x,y)dy\right\|.$$

Thus

II.9.4 $$\left\|g_0 - f\right\| \geqslant \sigma\left[1 + \left\|\lambda\int_a^b K(x,y)dy\right\|\right].$$

<u>Theorem II.10.</u> Any function p(x) that lies entirely between an upper (lower) function and its iterate is an upper (lower) function.

<u>Proof:</u> Given $g_0(x) \geqslant p(x) \geqslant g_1(x)$, then

II.10.1

$$p_1(x) = s(x) + \lambda\int_a^b K(x,y)p(y)dy \leqslant s(x) + \lambda\int_a^b K(x,y)g_0(y)dy = g_1(x) \geqslant p(x).$$

Thus $p_1(x) \leqslant p(x)$ and hence p(x) is an upper function.

Theorem II.11. If $g_0(x)$ is an upper (lower) function then $\left[g_0(x) + \delta\right]$ is an upper (lower) function where δ is any nonnegative (nonpositive) constant.

Proof: By definition

$$\text{II.11.1} \quad \left[g_0(x) + \delta\right]_1 = s(x) + \lambda\int_a^b K(x,y)\left[g_0(y) + \delta\right] dy = g_1(x) + \delta\lambda\int_a^b K(x,y)dy$$

$$\leqslant g_1(x) + \delta \leqslant g_0(x) + \delta .$$

Thus $\left[g_0(x) + \delta\right]_1 \leqslant \left[g_0(x) + \delta\right]$, and hence $\left[g_0(x) + \delta\right]$ is an upper function.

Section II. Existence of Upper and Lower Functions for Integral Equations

For this technique to be used with advantage it is desirable to show not only that upper and lower functions exist and exist arbitrarily close to the solution but also to find methods for producing them. One simple method is to take $g_0(x)$ to be a constant and then find the requirements for g_0 in order to make it an upper (lower) function.

Theorem II.12. The constant function with value g_0 is an upper function if

$$\text{II.12.1} \qquad g_0 \geqslant \sup_{[a,b]} \left[\frac{s(x)}{1 - \lambda\int_a^b K(x,y)dy}\right]$$

and is a lower function if

II.12.2
$$g_0 \leqslant \inf_{[a,b]} \left[\frac{s(x)}{1 - \lambda \int_a^b K(x,y)dy} \right].$$

Proof: By definition

II.12.3
$$g_1(x) = s(x) + \lambda g_0 \int_a^b K(x,y)dy \leqslant g_0.$$

Hence

II.12.4
$$g_0 \geqslant \sup_{[a,b]} \left[\frac{s(x)}{1 - \lambda \int_a^b K(x,y)dy} \right].$$

Theorem II.13. There exist upper (lower) functions that lie arbitrarily close to the solution.

Proof: Given any $\epsilon > 0$, it is to be shown that there exists an upper function, $g(x)$, such that $f(x) \leqslant g(x) \leqslant f(x) + \epsilon$. By the definition of an upper function, $f(x)$ is an upper function. Hence by theorem II.11, $f(x) + \frac{\epsilon}{2}$ is an upper function. Thus $f(x) + \frac{\epsilon}{2}$ and all its iterates lie within ϵ of the solution.

It may be of interest at this point to mention some examples of upper and lower functions. Using theorem II.12 it is easily seen that

(1) if $\sup_{[a,b]} s(x) \leqslant 0$ then $g_0 = 0$ is an upper function;

(2) if $\sup_{[a,b]} s(x) > 0$ then $g_0 = \sup_{[a,b]} \left[\frac{s(x)}{1 - \lambda \int_a^b K(x,y)dy} \right]$ is an upper function;

(3) if $\inf_{[a,b]} s(x) \geqslant 0$ then $g_0 = 0$ is a lower function;

and

(4) if $\inf_{[a,b]} s(x) < 0$ then $g_0 = \inf_{[a,b]} \left[\dfrac{s(x)}{1 - \lambda \int_a^b K(x,y)dy} \right]$ is a lower function.

It may be noted that in (1) and (3) the first iterate of g_0 will be $s(x)$.

Section III. Approximation of Integral Equations by a System of Linear Equations

As in many numerical methods it is sometimes advantageous to approximate the original integral equation by a set of linear equations. This section discusses one method of doing this.

Theorem II.14. Under the condition that λK is a member of $\mathcal{Z}$ the solution of the integral equation depends continuously on the kernel and the source function. More precisely, if $f(x) = s(x) + \lambda \int_a^b K(x,y)f(y)dy$ and $g(x) = S(x) + \lambda \int_a^b H(x,y)g(y)dy$, then $\|f - g\|$ converges to zero as $\|s - S\|$ and $\left\| \lambda \int_a^b |K(x,y) - H(x,y)| dy \right\|$ converge to zero.

Proof: Let $\lambda K(x,y)$ be approximated by $\lambda H(x,y)$ and $s(x)$ by $S(x)$ with

II.14.1 $$f(x) = s(x) + \lambda \int_a^b K(x,y)f(y)dy$$

and

II.14.2 $$g(x) = S(x) + \lambda \int_a^b H(x,y)g(y)dy.$$

Then

II.14.3 $$f(x) - g(x) = s(x) - S(x) + \lambda \int_a^b K(x,y)f(y)dy - \lambda \int_a^b H(x,y)g(y)dy$$

$$= s(x) - S(x) + \lambda \int_a^b K(x,y)\Big[f(y) - g(y)\Big] dy + \lambda \int_a^b \Big[K(x,y) - H(x,y)\Big] g(y)dy.$$

Since $|\lambda| \int_a^b |K(x,y)|\, dy \leq \rho < 1$ it follows that given τ such that $\rho < \tau < 1$ then $|\lambda| \int_a^b |H(x,y)|\, dy \leq \tau < 1$ if $|\lambda| \int_a^b |K(x,y) - H(x,y)|\, dy \leq \tau - \rho$. The function $g(x)$ is defined as the solution of an integral equation II.14.2, and thus by theorem II.12 it follows that

II.14.4 $$\|g\| \leq \left\| \frac{S}{1 - \tau} \right\|.$$

Substituting this information into II.14.3 one finds that

II.14.5
$$\|f - g\| \leq \|s - S\| + \|f - g\| \rho + \left\| \frac{S}{1 - \tau} \right\| |\lambda| \int_a^b |H(x,y) - K(x,y)|\, dy.$$

Thus

II.14.6 $$\|f - g\| \leq \frac{\|s - S\|}{1 - \rho} + \left[\frac{1}{1 - \rho} \right] \left\| \frac{S}{1 - \tau} \right\| |\lambda| \int_a^b |H(x,y) - K(x,y)|\, dy.$$

Therefore for any $\epsilon > 0$ and $\rho < \tau < 1$ by making $|\lambda| \int_a^b |K(x,y) - H(x,y)|\, dy$ less than the minimum of $\tau - \rho$ and $\frac{(1 - \rho)}{2} \frac{(1 - \tau)}{\|S\|} \epsilon$ and making $\|s - S\|$ less than $\frac{\epsilon}{2}$, it follows that $\|f - g\| < \epsilon$.

An example of such a modified equation is that obtained by approximating the kernel and the source function by step functions. That is, divide $[a,b]$ in equal intervals of length δ and let

II.15.1 $$k_{ij}(x) = \frac{\lambda}{\delta^2} \int_{i\delta+a}^{i\delta+\delta+a} \int_{j\delta+a}^{j\delta+\delta+a} K(\xi, y)\, d\xi\, dy \text{ for } i\delta+a \leq x \leq i\delta + \delta + a$$

$$= 0 \text{ for all other } x,$$

and

II.15.2 $$h_{ij}(y) = 1 \text{ for } j\delta + a \leqslant y \leqslant j\delta + \delta + a$$

$$= 0 \text{ for all other } y.$$

The notation δ^j will be used to represent $a + j\delta$. Then define

II.15.3 $$H(x,y) = \sum_{ij} h_{ij}(y)k_{ij}(x),$$

and let

II.15.4 $$g(x) = s(x) + \lambda\int_a^b H(x,y)g(y)dy.$$

Then

II.15.5 $$g(x) = s(x) + \sum_{ij} k_{ij}(x)\int_{\delta^j}^{\delta^{j+1}} g(y)dy = s(x) + \delta \sum_j k_{ij}(x)g_j$$

where $g_j = \frac{1}{\delta}\int_{\delta^j}^{\delta^{j+1}} g(y)dy$. By integrating II.15.5 with respect to x from $m\delta$ to $m\delta+\delta$

II.15.6 $$\delta g_m = \int_{\delta^m}^{\delta^{m+1}} g(x)dx = \int_{\delta^m}^{\delta^{m+1}} s(x)dx + \int_{\delta^m}^{\delta^{m+1}} \delta \sum k_{ij}(x)g_j dx$$

$$= \delta s_m + \delta \sum_{ij} g_j \int_{\delta^m}^{\delta^{m+1}} k_{ij}(x)dx$$

$$= \delta s_m + \delta \sum_j g_j \int_{\delta^m}^{\delta^{m+1}} k_{mj}(x)dx$$

$$= \delta s_m + \delta^2 \sum_j g_j k_{mj}.$$

Hence

II.15.7 $$g_m = s_m + \delta \sum k_{mj}g_j.$$

Thus by the introduction of the degenerate kernel, $H(x,y)$, the integral

equation

II.15.8 $$f(x) = s(x) + \lambda\int_a^b K(x,y)f(y)dy$$

may be replaced by the matrix equation

II.15.9 $$\hat{g} = \hat{s} + K\hat{g}.$$

Henceforth if a symbol w is referred to as the vector w, what is meant is a finite vector w $(w_1, w_4 \ldots w_p)$, but if w is referred to as the function or step function w then what is meant is a step function defined by the vector w such that

II.15.10 $w(x) = w_1$ for $\delta^i \leqslant x < \delta^{i+1}$, and $i = 0, 1, \ldots, p - 1$.

It might be well to note that the matrix equation II.15.9 is equivalent to the integral equation

II.15.11 $$g(x) = S(x) + \lambda\int_a^b H(x,y)g(y)dy$$

where $S(x)$ is the step function defined by the vector $\hat{s}$ and $H(x,y) = k_{ij}$ for $\delta^i \leqslant x < \delta^{i+1}$ and $\delta^j \leqslant y < \delta^{j+1}$.

In order to determine if the replacement of II.15.8 by II.15.9 is a reasonable one it is necessary to show when g approximates f to the desired degree.

Using equation II.14.3 $f(x) - g(x) = s(x) - S(x) +$

$\lambda\int_a^b K(x,y)\left[f(y) - g(y)\right] dy + \lambda\int_a^b\left[K(x,y) - H(x,y)\right] g(y)dy$ where $f(x)$, $s(x)$ and $K(x,y)$ are defined in equation II.15.8 and $g(x)$, $S(x)$, and $H(x,y)$ are defined in equation II.15.11. Thus

II.15.12

$$\left\|f - g\right\| \leqslant \left\|s - S\right\| + \left\|f - g\right\|\rho + \left\|\frac{S}{1 - \tau}\right\|\left[\left|\lambda\right|\int_a^b \left|K(x,y) - H(x,y)\right| dy\right].$$

Hence $\left\|f - g\right\| \leqslant \frac{\|s - S\|}{1 - \rho} + \frac{1}{1 - \rho}\left\|\frac{S}{1 - \tau}\right\| \left[\left|\lambda\right|\int_a^b \left|K(x,y) - H(x,y)\right| dy\right]$.

For the case of a continuous source function, $s(x)$, it is possible for any given ϵ to choose a δ_1 such that $\left|s(x) - S_i\right| < \frac{\epsilon(1 - \rho)}{2}$ for $\delta_1^i \leqslant x \leqslant \delta_1^{i+1}$.

If the only discontinuities of s are at mesh points and are simple discontinuities and if s and S are suitably defined at these points then $\|s - S\|$ can still be made arbitrarily small. Take any τ such that $\rho < \tau < 1$. By definition $K(x,y) - H(x,y)$ within the rectangles $\delta^i \leqslant x < \delta^{i+1}$, and $\delta^j \leqslant y \leqslant \delta^{j+1}$ is equal to $K(x,y) - k_{ij}$. Hence

II.15.13 $$K(x,y) - k_{ij} = K(x,y) - \frac{1}{\delta^2}\int_{\delta^i}^{\delta^{i+1}}\int_{\delta^j}^{\delta^{j+1}} K(x,y)dxdy.$$

Thus

II.15.14 $$K(x,y) - K_M \leqslant K(x,y) - H(x,y) \leqslant K(x,y) - K_m$$

where K_m is the minimum and K_M the maximum of $K(x,y)$ within the rectangle. If K is continuous then for given ϵ there exists a δ_2 such that $\left|K(x_1,y_1) - K(x_2,y_2)\right| < \min\left[\frac{\epsilon(1-\tau)(1-\rho)}{2(b-a)|\lambda|\,\|S\|}, \tau - \rho\right]$ for any x_1, y_1 and x_2, y_2 contained in the rectangle, and therefore

II.15.15 $$|K(x,y) - H(x,y)| < \frac{\epsilon(1-\tau)(1-\rho)}{2(b-a)\lambda\,||S||},$$

and

II.15.16 $$|\lambda|\int_a^b |K(x,y) - H(x,y)|dy < \frac{\epsilon(1-\tau)(1-\rho)}{2||S||}.$$

Thus for continuous source and kernel functions $||f - g||$ will be less than ϵ if δ is taken smaller than δ_1, δ_2. The arguments to follow are still valid in regard to upper and lower bounds for the more general kernels as defined at the beginning of this chapter and for more general source functions, but it may be impossible for g to approximate f to the desired degree if the approximations used are required to be step functions.

Section IV. The Upper and Lower Functions for a System of Linear Equations

It has been shown that the integral equation

II.16.1 $$f(x) = s(x) + \lambda\int_a^b K(x,y)f(y)dy$$

can be approximated by the matrix equation

II.16.2 $$\hat{g} = \hat{s} + \mathcal{K}\hat{g}.$$

Also discussed was the relation between II.16.2 and an integral equation

II.16.3 $$g(x) = S(x) + \lambda\int_a^b H(x,y)g(y)dy$$

where $S(x)$ is the step function defined by the vector $\hat{s}$ and $H(x,y) = k_{ij}$ for $\delta^i \leq x < \delta^{i+1}$ and $\delta^j \leq y < \delta^{j+1}$. Thus in the ith interval

II.16.4 $$g(x) = S_i + \lambda\sum_{j=0}^{p-1} k_{ij}\int_{a+j\delta}^{a+\delta j+\delta} g(y)dy.$$

Hence $g(x)$ is a constant in the interval, and it can easily be shown that $g(x)$ is thus the step function defined by the vector $\hat{g}$. It can also be shown that the upper and lower functions of II.16.3 are equal to the step functions defined by the vectors of the operator

II.16.5 $$\hat{U}_1 = \hat{s} + K\hat{U}_0.$$

Thus the two systems are equivalent, and the theorems given for upper and lower functions for the operator T can be carried over directly to upper and lower functions as defined by the integral equation of the type II.16.3 and hence to the upper and lower functions defined for the solution vectors of the matrix equation II.16.2. The conditions on the matrix K become

(1) $k_{ij} \geqslant 0$ for all i and j

(2) $\sum_{j=0}^{p-1} k_{ij} \leqslant \rho < 1$ for all i.

Similarly any matrix equation of the form II.16.2 is equivalent to an integral equation of the form II.16.3 having a degenerate kernel. If the matrix K satisfies conditions (1) and (2) then the technique of upper and lower functions for integral equations can be extended to apply to the matrix equation. Throughout the remainder of this section, it is assumed that the matrix K satisfies conditions (1) and (2). For a given matrix equation $\hat{f} = \hat{s} + K\hat{f}$ an operator can be defined by

II.16.6 $$\hat{g}^1 = \hat{s} + K\hat{g}^0,$$

and the technique of upper and lower functions for integral equations applies to the matrix equation.

<u>Definition II.17.</u> A function $g_0(x)$ is called an upper function for

the equation II.16.6 and $\hat{g}_0$ an upper vector if

II.17.1 $g_0(x) \geqslant g_1(x)$ for all $x \in [a,b]$ where $g_1(x)$ is defined by the vector

II.17.2 $$\hat{g}^1 = \hat{s} + K\,\hat{g}^0.$$

<u>Definition II.18.</u> A function $h_0(x)$ is called a lower function for the equation II.16.6 and $\hat{h}^0$ a lower vector if

II.18.1 $$h_0(x) \leqslant h_1(x) \text{ for all } x \in [a,b]$$

where $h_1(x)$ is defined by the vector

II.18.2 $$\hat{h}^1 = \hat{s} + K\,\hat{h}^0.$$

Since the $g_n(x)$'s $\left(h_n(x)\text{'s}\right)$ are also upper (lower) functions for an integral equation the theorems concerning upper and lower vectors of equation II.16.6 follow directly from the proofs given for integral equations.

<u>Theorem II.19.</u> The property of being an upper (lower) function for the operator II.17.2 is hereditary.

<u>Theorem II.20.</u> If $g(x)$ is an upper function defined by the upper vector $\hat{g}$ and $h(x)$ is a lower function defined by the lower vector $\hat{h}$, then $g(x) \geqslant f(x) \geqslant h(x)$ where $f(x)$ is the function defined by the solution vector $\hat{f}$ of II.16.2.

<u>Theorem II.21.</u> An error estimate for any function defined by an iterate of an upper (lower) vector is

II.21.1 $$\sup_{[a,b]} \left|g(x) - f(x)\right| \leqslant \sup_m \left| \frac{g_m^0 - g_m^1}{1 - \sum_{j=1}^{\rho} k_{mj}} \right|$$

where the notation g_m^i means the mth component of the vector g^i.

Theorem II.22. A lower bound for the error estimate for any function defined by the iterate of an upper (lower) vector is

II.22.1 $$\sup\left|g(x) - f(x)\right| \geqslant \left|\zeta\left(1 + \sup_m \sum_{j=1}^{p} k_{mj}\right)\right|$$

where $\zeta = \min_m \left|g_m^0 - g_m^1\right|$.

Section V. The Use of Upper and Lower Vectors to Calculate Upper and Lower Bounds on the Solution of a Fredholm Integral Equation

It has been shown that the integral equation

II.23.1 $$f(x) = s(x) + \lambda \int_a^b K(x,y)f(y)dy$$

can be replaced by the system of linear equations

II.23.2 $$\hat{g} = \hat{s} + K\hat{g}.$$

To obtain upper and lower bounds on f(x), the system of equation II.23.2 is replaced by the pair of equations

II.23.3 $$\hat{\bar{g}} = \hat{\bar{s}} + \bar{K}\hat{\bar{g}}$$

and

II.23.4 $$\hat{\underline{g}} = \hat{\underline{s}} + \underline{K}\hat{\underline{g}}$$

where $\overline{h_i}$ ($\underline{h_i}$) is the supremum (infimum) of h(x) in the ith interval, and $\overline{k_{ij}}$ ($\underline{k_{ij}}$) is the supremum (infimum) of $\lambda \int_{\delta^j}^{\delta^{j+1}} K(x,y)dy$ in the ith interval.

Theorem II.24. The solution of II.23.3 is an upper bound for the

solution of II.23.2, and the solution of II.23.4 is a lower bound for II.23.3 under the restrictions on K.

<u>Proof</u>: By definition

II.24.1 $$g_m - \overline{g_m} = (S_m - \overline{S_m}) + \sum_{j=0}^{p-1} k_{mj} g_j - \sum_{j=0}^{p-1} \overline{k_{mj}}\,\overline{g_j}$$

$$\leqslant \sum_{j=0}^{p-1} k_{mj} g_j - \sum_{j=0}^{p-1} \overline{k_{mj}}\,\overline{g_j}$$

$$\leqslant \sum_{j=0}^{p-1} k_{mj}(g_j - \overline{g_j})$$

$$\leqslant \sup\,(\hat{g} - \hat{\bar{g}}) \sum_{j=0}^{p-1} k_{mj}.$$

Let q be the value of m such that $\sup_m (\hat{g}_m - \hat{\bar{g}}_m) = g_q - \overline{g_q}$. Hence

II.24.2 $$\left[1 - \sum_{j=0}^{p-1} k_{mj}\right]\left[g_q - \overline{g_q}\right] \leqslant 0.$$

Since $\sum_{j=0}^{p-1} k_{mj} \leqslant \rho < 1$ for all i it follows that

II.24.3 $$g_q - \overline{g_q} = \sup_m \left[\hat{g}_m - \hat{\bar{g}}_m\right] \leqslant 0.$$

Thus $$\overline{g_m} \geqslant g_m \text{ for } m = 1, \ldots, p.$$

In actual practice the integral equation is thus replaced by both II.23.3 and II.23.4. An initial upper (lower) vector u_0 is chosen with the aid of theorem II.12 assuming that u_0 is a constant. This upper (lower)

vector is then iterated by II.23.3 (II.23.4) until $\left\| g_{n+1} - g_n \right\| < \epsilon$ for some given $\epsilon > 0$. The solution of the integral equation is then bounded by the "limit" functions for the two sequences.

<u>Theorem II.25.</u> If g is an upper (lower) vector for the system II.23.3 (II.23.4) then the step function g is an upper (lower) function for the equation II.23.1.

<u>Proof:</u> Let g be an upper vector. Thus

$$\text{II.25.1} \quad g_m \geqslant \overline{s_m} + \sum_{j=0}^{p-1} \overline{k_{mj}}\, g_j \qquad \text{for } \delta^m \leqslant x\ \delta^{m+1} \quad m = 0, 1, \ldots, p-1.$$

Now let $x \in [a,b]$ belong to the mth interval of the subdivision. Then substituting g into the right member of II.23.1 one sees that its transform at x satisfies the inequality

$$\text{II.25.2} \quad Tg(x) = s(x) + \lambda \int_a^b K(x,y)g(y)dy \leqslant \overline{s_m} + \sum_{j=0}^{p-1} g_j \int_{\delta^j}^{\delta^{j+1}} K(x,y)dy$$

$$\leqslant \overline{s_m} + \sum_{j=0}^{p-1} g_j\, \overline{k_{mj}} \leqslant g_m.$$

Since m is arbitrary, x is an arbitrary point in $[a,b]$, and it follows that g is an upper function for II.23.1.

CHAPTER III

EXPANSION OF THE BASIC THEORY

In this chapter an effort is made to extend the upper and lower function technique to a greater variety of Fredholm integral equations and to modify the technique in order to speed the convergence. The first problem is to lessen the restrictions placed on the kernel in Chapter II.

Section I. Nonpositive Kernels

The restriction

III.1.1 $$\lambda K(x,y) \geqslant 0 \qquad \text{for } x \in [a,b]$$

was placed on the kernel in Chapter II in order to insure the hereditary property of the upper or lower functions. It is quite simple in theory to extend the technique to include kernels of the form

III.1.2 $$\lambda K(x,y) \leqslant 0 \qquad \text{for } x \in [a,b] .$$

If the integral equation is iterated analytically the equation becomes

III.1.3 $$f(x) = s(x) + \lambda\int_a^b K(x,y)\left[s(y) + \lambda\int_a^b K(y,t)f(t)dt\right]dy$$

$$= s(x) + \lambda\int_a^b K(x,y)s(y)dy + \lambda^2\int_a^b K(x,y)\left[\int_a^b K(y,t)f(t)dt\right]dy$$

$$= s(x) + \lambda\int_a^b K(x,y)s(y)dy + \lambda^2\int_a^b\left[\int_a^b K(x,y)K(y,t)dy\right]f(t)dt.$$

Thus by setting

III.1.4 $$\mathcal{S}(x) = s(x) + \lambda\int_a^b K(x,y)s(y)dy$$

and

III.1.5 $$\mathcal{K}(x,t) = \lambda\int_a^b K(x,y)K(y,t)dy$$

the equation becomes

III.1.6 $$f(x) = \mathcal{S}(x) + \lambda\int_a^b \mathcal{K}(x,t)f(t)dt$$

where $\lambda\,\mathcal{K}(x,t) \geqslant 0$.

The integral equation III.1.6 is now one of those handled in Chapter II provided $\sup_{[a,b]} \lambda\int_a^b \mathcal{K}(x,t)dt < 1$. Therefore if the integration is possible in a closed form and if the original kernel satisfies the conditions

(1) $$\lambda^2\int_a^b\int_a^b K(x,y)K(y,t)dydt < 1 \qquad \text{for } x \in [a,b]$$

and

(2) $$\lambda K(x,y) \leqslant 0$$

then the methods of Chapter II apply to the iterated equation.

Section II. Method of Wiarda

The other major restriction on the kernel is that

III.2.1 $$\lambda\int_a^b K(x,y)dy \leqslant \rho < 1 \qquad \text{for } x \in [a,b] .$$

This restriction insures the total convergence of the Neumann series. In papers by L. B. Rall [5] and H. Bückner [2, 3] a special method of successive approximations for Fredholm integral equations is

introduced. Instead of using the operator $Tg(x) = s(x) + \lambda\int_a^b K(x,y)g(y)dy$ the Wiarda operator T_θ, defined by

$$\text{III.2.2} \quad f_1(x) = T_\theta f_0(x) = (1 - \theta)f_0(x) + \theta\lambda\int_a^b K(x,y)f_0(y)dy + \theta s(x)$$

is used with θ a number such that $0 < \theta < 1$. A set of necessary and sufficient conditions for total convergence is

$$\text{III.2.3} \qquad \left|\theta + (1 - \theta)\lambda_k^{-1}\lambda\right| < 1 \qquad (k = 1, 2, \ldots)$$

where λ_k is the kth eigenvalue.

<u>Theorem III.3.</u> An upper (lower) function for the operator T is an upper (lower) function for T_θ and conversely.

<u>Proof:</u> Let $g(x)$ be an upper function for T, thus

$$\text{III.3.1} \qquad Tg(x) = g_1(x) = s(x) + \lambda\int_a^b K(x,y)g(y)dy \leqslant g(x)$$

and let

$$\text{III.3.2} \quad T_\theta g = g_2(x) = (1 - \theta)g(x) + \theta\left[s(x) + \lambda\int_a^b K(x,y)g(y)dy\right]$$

$$= (1 - \theta)g(x) + \theta g_1(x) \leqslant (1 - \theta)g(x) + \theta g(x) \leqslant g(x).$$

Thus $T_\theta g \leqslant g(x)$. Let $q(x)$ be an upper function for T_θ. Therefore

$$\text{III.3.3} \quad T_\theta q(x) = q_1(x) = (1 - \theta)q(x) + \theta s(x) + \theta\lambda\int_a^b K(x,y)q(y)dy \leqslant q(x).$$

By the definition of T

$$\text{III.3.4} \qquad Tq(x) = q_2(x) = s(x) + \lambda\int_a^b K(x,y)q(y)dy.$$

Thus it follows that

III.3.5 $$q(x) \geqslant (1 - \theta)q(x) + \theta s(x) + \theta\lambda\int_a^b K(x,y)q(y)dy$$

$$= (1 - \theta)q(x) + \theta\left[s(x) + \lambda\int_a^b K(x,y)q(y)dy\right]$$

$$= (1 - \theta)q(x) + \theta\left[q_2(x)\right]$$

or

III.3.6 $$\theta q(x) \geqslant \theta q_2(x).$$

Hence $q(x) \geqslant q_2(x) = Tq(x)$ proving that $q(x)$ is an upper (lower) function of T if and only if it is an upper (lower) function of T_θ.

<u>Theorem III.4</u>. The property of being an upper (lower) function for T_θ is hereditary.

<u>Proof</u>: By definition of T_θ

III.4.1 $$T_\theta g_0(x) = g_1(x) = (1 - \theta)g_0(x) + \theta\left[s(x) + \lambda\int_a^b K(x,y)g_0(y)\right]dy.$$

Assume that

III.4.2 $$g_0(x) \geqslant g_1(x) \qquad \text{for } x \in [a,b].$$

Hence by definition

III.4.3 $$g_2(x) - g_1(x) = \left[1 - \theta\right]\left[g_1(x) - g_0(x)\right]$$

$$+ \theta\lambda\int_a^b K(x,y)\left[g_1(y) - g_0(y)\right]dy \leqslant 0,$$

and thus $g_1(x) \geqslant g_2(x)$.

In a like manner most of the theorems in Chapter II carry over to the Wiarda operator (T_θ). Since an upper (lower) function in one system is an upper (lower) function in the other, the discussion of the existence of upper (lower) functions for T directly implies the existence of upper (lower) functions for T_θ. The greatest advantage in the Wiarda operator is that the sequence $\{u_n\}$ now converges for classes of kernels that do not satisfy the restrictions on the kernel for the integral operator T. Note that for $\theta = 1$ the two operators are identical.

Section III. The Gauss-Seidel Method

The number of iterations needed for convergence increases quite rapidly if θ is required to be small. Dr. A. S. Householder suggested that the Gauss-Seidel method might decrease the number of iterations required. Basically this method replaces the matrix operator

III.5.1 $$\hat{f}_n = \hat{s} + K\hat{f}_{n-1}$$

by the operator

III.5.2 $$\hat{f}_n = U\hat{f}_n + L\hat{f}_{n-1} + \hat{s}$$

where $U = (u_{ij})$ and $L = (\ell_{ij})$ with

$$u_{ij} = 0 \quad \text{for } i \leqslant j \qquad \text{and} \qquad \ell_{ij} = 0 \quad \text{for } i > j$$

$$= k_{ij} \quad \text{for } i > j \qquad\qquad\qquad = k_{ij} \quad \text{for } i \leqslant j.$$

Hence $U + L = K$.

It was felt that it would be interesting if a type of Gauss-Seidel method could be applied to the integral equation operator T. In the

following discussion the Gauss-Seidel operator T_G is defined by

III.5.3 $$T_G g_0(x) = g_1(x) = h_{p-1}(x)$$

where

$$\begin{aligned} h_n(x) &= h_{n-1}(x) && \text{for } a \leqslant x \leqslant \delta^n \\ &= Th_{n-1}(x) && \text{for } \delta^n \leqslant x < \delta^{n+1} \\ &= g_0(x) && \text{for } \delta^{n+1} \leqslant x \leqslant b, \end{aligned}$$

and by definition $h_{-1}(x) = g_n(x)$.

<u>Theorem III.6.</u> If $g(x)$ is a function such that $Tg(x) \leqslant g(x)$ $\big(Tg(x) \geqslant g(x)\big)$ or $T_G g(x) \leqslant g(x)$ $\big(T_G g(x) \geqslant g(x)\big)$ and $g(x) = h_{-1}(x)$ then $h_n(x) \geqslant h_{n+1}(x)$ $\big(h_n(x) \leqslant h_{n+1}(x)\big)$ for all $x \in [a,b]$ and $n = -1, 0, 1, \ldots, p-1$.

<u>Proof:</u> Given that $Tg(x) \leqslant g(x)$ then by definition

III.6.1 $$\begin{aligned} h_0(x) &= Th_{-1}(x) && \text{for } a \leqslant x < \delta \\ &= g(x) && \text{for } \delta \leqslant x \leqslant b. \end{aligned}$$

Thus either $h_0(x) = Th_{-1}(x) = Tg(x) \leqslant g(x) = h_{-1}(x)$ or $h_0(x) = g(x) = h_{-1}(x)$, and therefore $h_0(x) \leqslant h_{-1}(x)$ for all $x \in [a,b]$. Assume that $h_{q-1}(x) \geqslant h_q(x)$ for $q = 0, 1, \ldots, n$. Then

III.6.2 $$\begin{aligned} h_{n+1}(x) &= h_n(x) && \text{for } a < x < \delta^{n+1} \\ &= Th_n(x) \leqslant Th_{n-1}(x) \leqslant Tg(x) \leqslant g(x) = h_n(x) \\ & && \text{for } \delta^{n+1} \leqslant x < \delta^{n+2} \\ &= g(x) = h_n(x) && \text{for } \delta^{n+2} \leqslant x \leqslant b. \end{aligned}$$

Hence $h_{n+1}(x) \leqslant h_n(x)$, and thus by induction $h_n(x) \geqslant h_{n+1}(x)$ for $x \in [a,b]$ and for $n = -1, 0, 1, 2, \ldots, p-2$ if $T_G g(x) \leqslant g(x)$. Should $T_G g(x)$ be less than $g(x)$ for $x \in [a,b]$ then by definition

III.6.3 $$T_G g(x) = h_{p-1}(x) \leqslant g(x).$$

Then for any $n = -1, 0, 1, \ldots, p-1$

III.6.4 $$\begin{aligned} h_{n+1}(x) &= h_n(x) && \text{for } a \leqslant x < \delta^{n+1} \\ &= Th_n(x) = h_{p-1}(x) \leqslant h_n(x) = g(x) && \text{for } \delta^{n+1} \leqslant x < \delta^{n+2} \\ &= g(x) = h_n(x) && \text{for } \delta^{n+2} \leqslant x \leqslant b. \end{aligned}$$

Thus for an $n = -1, 0, 1, 2, \ldots, p-2$ it follows that $h_{n+1}(x) \leqslant h_n(x)$ for $x \in [a,b]$.

<u>Theorem III.7.</u> An upper (lower) function for the operator T is an upper (lower) function for the operator T_G.

<u>Proof:</u> By definition

III.7.1 $$Tg(x) = g_1(x) = s(x) + \lambda \int_a^b K(x,y)g(y)dy \leqslant g(x).$$

Then

$$\begin{aligned} h_0(x) &= Th_{-1}(x) && \text{for } a \leqslant x \leqslant \delta^1 \\ &= g(x) && \text{for } \delta^1 \leqslant x \leqslant b. \end{aligned}$$

Since $Th_{-1}(x) = Tg(x) = g_1(x) \leqslant g(x)$ it follows that $h_0(x) \leqslant g(x)$. Assume that $h_{n-1}(x) \leqslant g(x)$. By definition

III.7.2 $h_n(x) = h_{n-1}(x) \leqslant g(x)$ for $a \leqslant x < \delta^n$

$$= Th_{n-1}(x) = s(x) + \lambda\int_a^b K(x,y)h_{n-1}(y)dy$$

$$\leqslant s(x) + \lambda\int_a^b K(x,y)g(y)dy = g_1(x) \leqslant g(x) \quad \text{for } \delta^n \leqslant x < \delta^{n+1}$$

$$= g(x) \quad \text{for } \delta^{n+1} \leqslant x \leqslant b.$$

Thus $h_n(x) \leqslant g(x)$ for $(n = 0, 1, \ldots, p - 1)$, and since $T_G g(x) = h_{p-1}(x)$ it follows that $T_G g(x) \leqslant g(x)$.

<u>Theorem III.8.</u> If $g_1(x)$ is an image of $g(x)$ under the operator T_G such that $g_1(x) \leqslant g(x)$ $\big(g_1(x) \geqslant g(x)\big)$ then $Tg_1(x) \leqslant g_1(x)$ $\big(Tg_1(x) \geqslant g_1(x)\big)$.

<u>Proof:</u> By definition

III.8.1 $$Tg_1(x) = Th_{p-1}(x).$$

For any interval $\delta^n \leqslant x \leqslant \delta^{n+1}$ it follows that

III.8.2 $$Tg_1(x) = Th_{p-1}(x) \leqslant Th_{n-1}(x) = h_n(x) = h_{p-1}(x) = g_1(x).$$

Since this was any interval, $Tg_1(x) \leqslant g_1(x)$ for $a \leqslant x \leqslant b$.

<u>Theorem III.9.</u> Given any two functions f and g such that $f(x) \geqslant g(x)$ for $x \in [a,b]$ then $T_G g(x) \geqslant T_G f(x)$.

<u>Proof:</u> Let $g(x) = h_{-1}(x)$ and $f(x) = k_{-1}(x)$. Since the operator T is monotone,

III.9.1 $h_0(x) = Th_{-1}(x) \geqslant Tk_{-1}(x) = k_0(x)$ for $a \leqslant x < \delta^1$

$$= k_0(x) \quad \text{for } \delta^1 \leqslant x \leqslant b.$$

Thus $h_0(x) \geqslant k_0(x)$ for all $x \in [a,b]$. Then assume $h_n(x) > k_n(x)$ for all

$x \in [a,b]$. By definition

III.9.2 $\quad h_{n+1}(x) = h_n(x) \geqslant k_n(x) = k_{n+1}(x) \qquad$ for $a \leqslant x < \delta^{n+1}$

$$= Th_n(x) \geqslant Tk_n(x) = k_{n+1}(x) \qquad \text{for } \delta^{n+1} \leqslant x < \delta^{n+2}$$

$$= g(x) \geqslant f(x) = k_{n+1}(x) \qquad \text{for } \delta^{n+2} \leqslant x < b.$$

Hence $h_{n+1}(x) \geqslant k_{n+1}(x)$ for all $x \in [a,b]$, and $n = -2, -1, 0, 1, \ldots, p-2$. Then $T_G g(x) = h_{p-1}(x) \geqslant k_{p-1}(x) = T_G f(x)$.

<u>Corollary III.10.</u> The property of being an upper (lower) function for the operator T_G is hereditary.

<u>Proof:</u> Given that $T_G g(x) \leqslant g(x)$. Then by setting $T_G g(x) = f(x)$ in theorem III.9, it follows that if $T_G g(x) = f(x) \leqslant g(x)$ then $T_G(T_G g(x)) = T_G f(x) \leqslant T_G g(x)$. Hence the image of an upper function for the operator T_G is an upper function for the operator T_G.

As in the case of the Wiarda operator most of the theorems for upper and lower functions follow directly. Since an upper (lower) function for T is an upper (lower) function for T_G (cf. theorem III.7) the existence of upper (lower) functions for T implies the existence of upper (lower) functions for T_G.

CHAPTER IV

NUMERICAL EXAMPLES

In order to illustrate the applications of some of the methods an approximate solution was sought for the integral equation

IV.1.1 $$f(x) = x^2 + \lambda\int_0^1 K(x,y)f(y)dy$$

where

IV.1.2 $$K(x,y) = x(1 - y) \qquad \text{for } 0 \leqslant x \leqslant y \leqslant 1$$

$$= y(1 - x) \qquad \text{for } 0 \leqslant y \leqslant x \leqslant 1.$$

This is the same example used by L. B. Rall [5]. The method used was that of section V, Chapter II. The integral equation was replaced by the pair of equations

IV.1.3 $$\hat{\bar{g}} = \hat{\bar{s}} + \bar{K}\hat{\bar{g}}$$

and

IV.1.4 $$\hat{\underline{g}} = \hat{\underline{s}} + \underline{K}\hat{\underline{g}}$$

where $\overline{s_i}$ ($\underline{s_i}$) is the supremum (infimum) of $s(x)$ in the ith interval and $\overline{k_{ij}}$ ($\underline{k_{ij}}$) is the supremum (infimum) of $\lambda\int_{\delta i}^{\delta j+1} K(x,y)dy$ for x in the ith interval. Thus if x_i is the value of x at $a + \delta i$ then $\overline{s_i} = (x_{i+1})^2$ and $\underline{s_i} = (x_i)^2$. To determine the value of k_{ij}, three cases are to be considered:

(1) $$k_{ij} = \lambda\int_{x_j}^{x_{j+1}} K(x,y)dy = \frac{\lambda x}{2}\left[(1 - x_j^2) - (1 - x_{j+1})^2\right] \qquad \text{for } j < i\ ;$$

(2) $$k_{ij} = \lambda\int_{x_j}^{x_{j+1}} K(x,y)dy = \frac{\lambda}{2}(x^2 - x_j^2)(1 - x) + \frac{\lambda x}{2}\left[(1 - x)^2 - (1 - x_{j+1})^2\right]$$

$$\text{for } i = j;$$

(3) $k_{ij} = \lambda\int_{x_j}^{x_{j+1}} K(x,y)dy = \frac{\lambda x}{2}\left[(1 - x_j)^2 - (1 - x_{j+1})^2\right]$ for $i < j$.

Since $\overline{k_{ij}} = \sup \lambda\int_{x_j}^{x_{j+1}} K(x,y)dy$ for $x_i \leqslant x \leqslant x_{i+1}$

(1) $\overline{k_{ij}} = \frac{\lambda}{2}(x_{j+1}^2 - x_j^2)(1 - x_i)$ for $i > j$ (maximum at $x = x_i$);

(2) $\overline{k_{ij}} = \frac{\lambda}{8}\left[x_i^2 - x_{i+1}^2 + 2x_{i+1}\right]^2 - \frac{x_i^2}{2}$ for $i = j$ (maximum at $x = (1 - \bar{x})x_{i+1} - \bar{x}x_i$ where $\bar{x} = (1/2)(x_i + x_{i+1})$);

(3) $\overline{k_{ij}} = \frac{\lambda}{2}(x_{i+1})\left[(1 - x_j)^2 - (1 - x_{j+1})^2\right]$ for $i < j$ (maximum at $x = x_{i+1}$).

By definition of $\underline{k_{ij}} = \inf \int_{x_j}^{x_{j+1}} K(x,y)dy$ for $x_i \leqslant x \leqslant x_{i+1}$ it follows that

(1) $\underline{k_{ij}} = \frac{\lambda}{2}(1 - x_{i+1})(x_{j+1}^2 - x_j^2)$ for $i > j$ (minimum at $x = x_{i+1}$);

(2) $\underline{k_{ii}} = \frac{\lambda x_i}{2}\left[(1 - x_i)^2 - (1 - x_{i+1})^2\right]$ for $i = j$ (minimum at $x = x_i$);

(3) $\underline{k_{ij}} = \frac{\lambda x_i}{2}(1 - x_j)^2 - (1 - x_{j+1})^2$ for $i < j$ (minimum at $x = x_i$).

In the case of $\overline{k_{ii}}$ ($\underline{k_{ii}}$), that is for $i = j$, a maximum (minimum) could have been gotten by finding the maximum (minimum) for each term. This, however, increases the difference between the upper and lower function, which is undesirable.

Approximate solutions were calculated for $\lambda = 1, 2, \ldots, 9$. The graphs of the solutions are found in Figures 1 through 9. In all cases the initial function considered was taken to be a constant as prescribed

by Theorem II.8 and the functions iterated until $\left\|g_n - g_{n-1}\right\| < 2^{-36}$. It should be noted that $\left\|\lambda\int_a^b K(x,y)dy\right\| = \frac{\lambda}{8}$ and thus for cases $\lambda = 8$ and 9, $\lambda K \notin \mathcal{L}^+$. For these cases the convergence of the Wiarda operator was needed. The eigenvalues for this example are $\lambda_k = k^2\pi^2$ for $k = 1, 2, \ldots$ and it follows by III.2.3 that the Wiarda operator is totally convergent for all θ in $0 < \theta < 1$.

In Figures 10, 11, and 12, the graphs represent the solution of the systems IV.1.3 and IV.1.4 for various interval sizes. The interval sizes are determined by taking $p = 10$, 100, 200, and 500. For $p = 200$ and 500 and for the chosen scale the upper and lower solutions are not distinguishable.

The results of applying the method in a case where the solution was not continuous are shown in Figures 13 and 14. The function chosen was defined by

IV.1.5
$$f(x) = 2x \qquad \text{for } 0 \leqslant x < \tfrac{1}{2}$$
$$= 2x - 1 \qquad \text{for } \tfrac{1}{2} \leqslant x \leqslant 1.$$

By substituting IV.1.5 into equation IV.1.1 a new source term, $s(x)$, was calculated.

IV.1.6
$$s(x) = \frac{x}{24}\left[8x^2 + 43\right] \qquad \text{for } 0 \leqslant x < \tfrac{1}{2}$$
$$= \frac{1}{24}\left[8x^3 - 12x^2 + 55x - 27\right] \qquad \text{for } \tfrac{1}{2} \leqslant x \leqslant 1.$$

Care was used in defining $s(x)$ at $\frac{1}{2}$, so that $s(x)$ was single valued throughout the interval $[0, 1]$. An interval size was also used that

caused a mesh point to fall at $\frac{1}{2}$. Note also that $\overline{s_i}$ ($\underline{s_i}$) is defined as maximum (minimum) in the interval $x_i \leqslant x < x_{i+1}$ for this example.

In order to study the behavior of the method for a symmetric source function another source term was chosen to be $s(x) = \frac{x}{2}(1 - x)$. The solution of this example is symmetric about $\frac{1}{2}$ with a maximum at that point. Figures 15, 16, and 17 are graphs of the approximate solutions of this integral equation obtained for various interval sizes.

For nonpositive kernels, the equation

IV.1.7 $$f(x) = 1 - \int_0^1 xyf(y)dy$$

was used. Thus $\lambda K(x,y) = -xy$. By iterating analytically once

IV.1.8 $$f(x) = 1 - \int_0^1 xt \left[1 - \int_0^1 tyf(y)dy\right] dt$$

$$= 1 - \int_0^1 xtdt + \int_0^1 xt \left[\quad tyf(y)dy\right] dt$$

$$= 1 - \int_0^1 xtdt + \int_0^1 \left[\int_0^1 xt^2ydt\right] f(y)dy.$$

By setting

IV.1.9 $$s(x) = 1 - \int_0^1 xtdt = 1 - \frac{x}{2}$$

and

IV.1.10 $$K(x,y) = \int_0^1 xt^2ydt = \frac{xy}{3}$$

a new integral equation

IV.1.11 $$f(x) = (1 - \frac{x}{2}) + \int_0^1 \frac{xy}{3} f(y)dy$$

is formed. The approximate solutions of IV.1.11 are shown in figures 18 and 19 for various interval sizes.

In the paper by Rall [5] p. 981 two estimates for the error are given as

E1 $$\|x - x_n\| \leqslant (1 - \mu)^n \| x - x_0\|$$

and

E2 $$\|x - x_n\| \leqslant \frac{1 - \mu}{\mu} \|x_n - x_{n-1}\| .$$

Where x is the solution of the integral equation, x_n is the nth iterate function, and μ is a function of a bound on the integral operator. From these equations it was possible to calculate the number of iterations that were needed to insure a given accuracy. In figure 20, curve (2) is a graph of the number of iterations plotted against λ for $\|x_n - x_{n-1}\| < 2^{-36}$. Curve (1) is the actual number of iterations required in the numerical examples for the corresponding integral equation. The number of iterations used by the Gauss-Seidel operator is shown as curve (3). It will be noted that the use of the Gauss-Seidel operator greatly decreases the number of iterations for values of λ near the smallest eigenvalue $\lambda_1 = \pi^2$. It seems on the whole that the extra precision acquired by the use of upper (lower) functions in the iterative procedure does not greatly increase the number of iterations required.

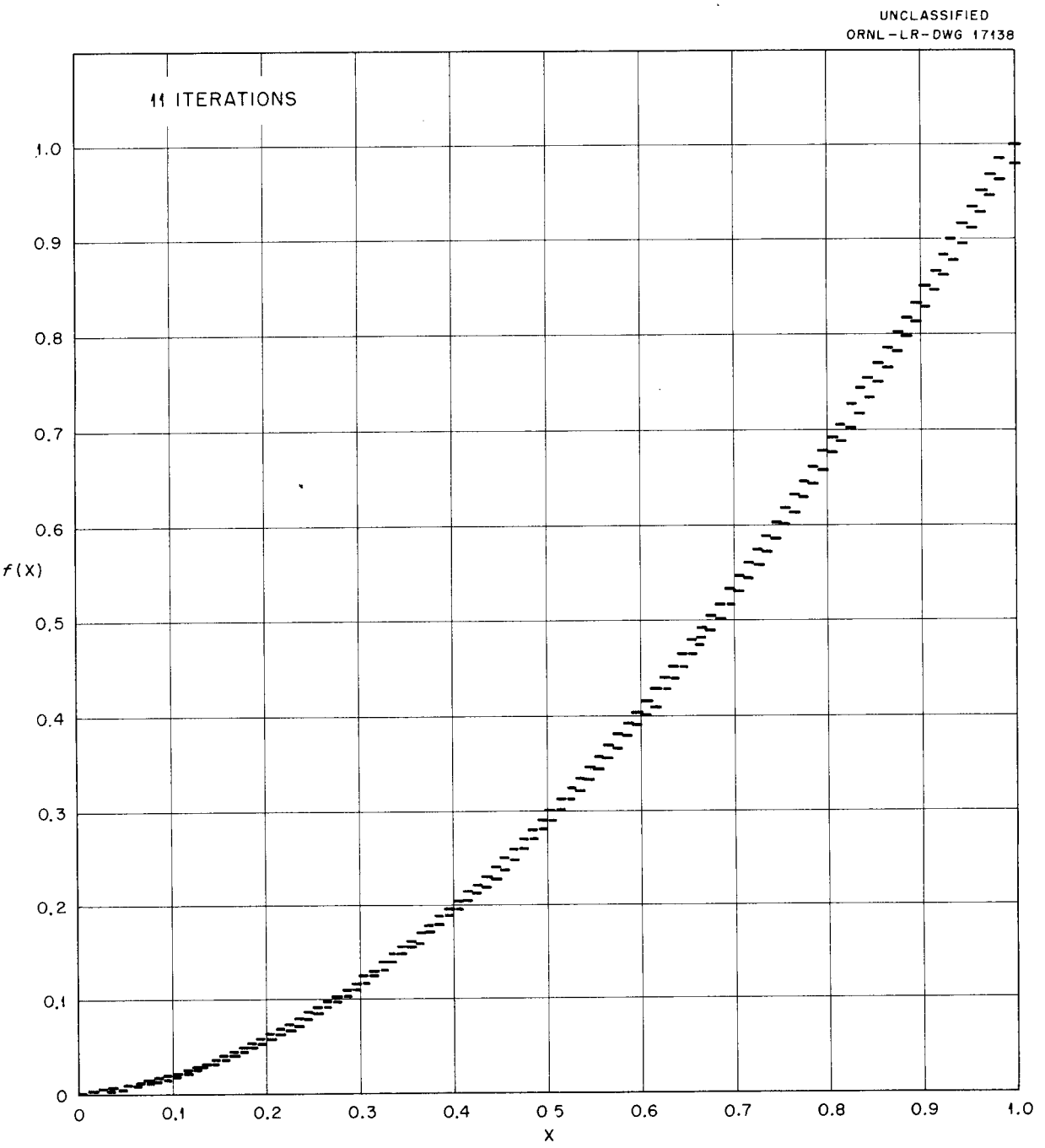

Fig. 1. Example One — $\lambda = 1$, $P = 100$.

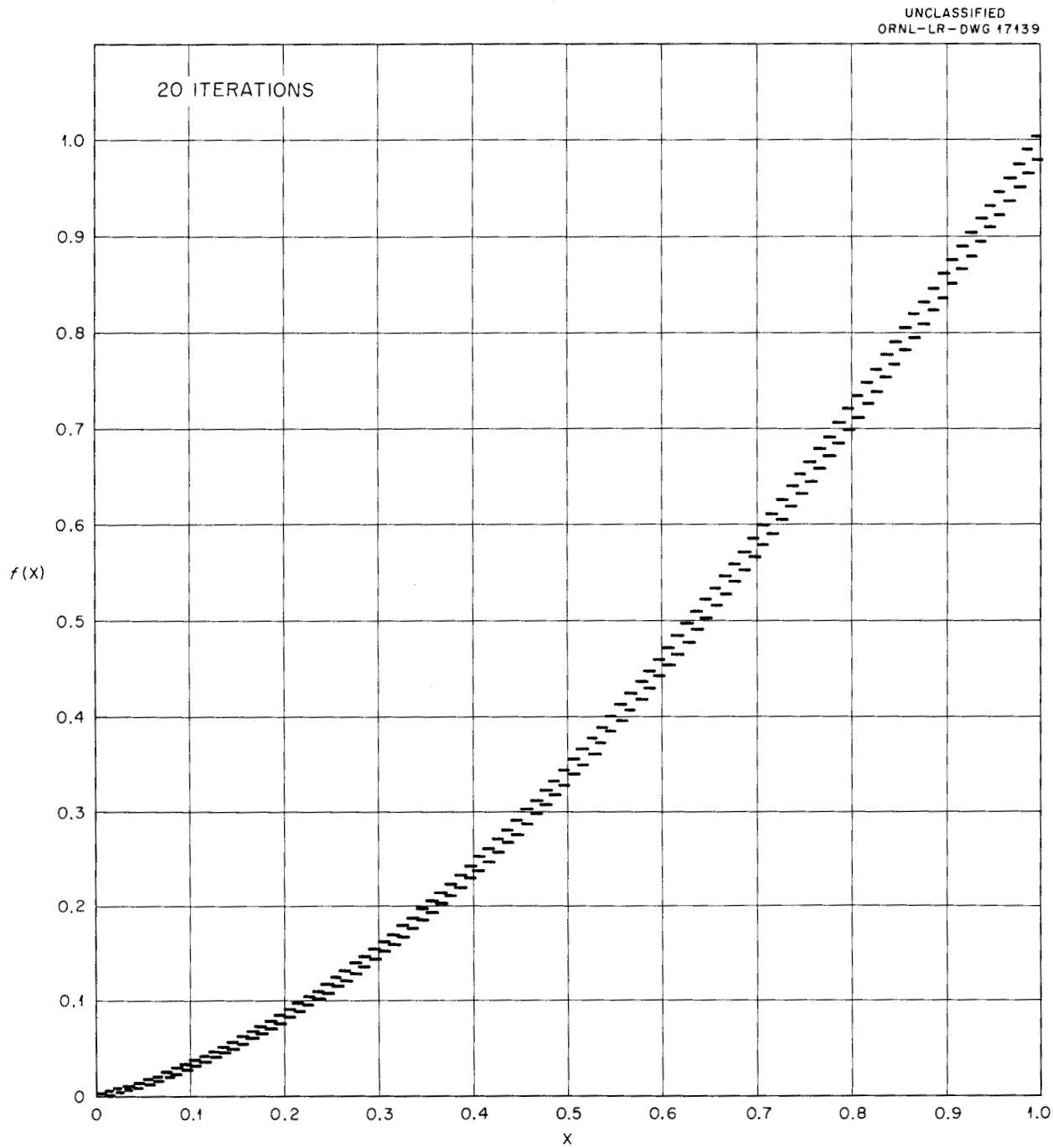

Fig. 2. Example One — $\lambda = 2$, $P = 100$.

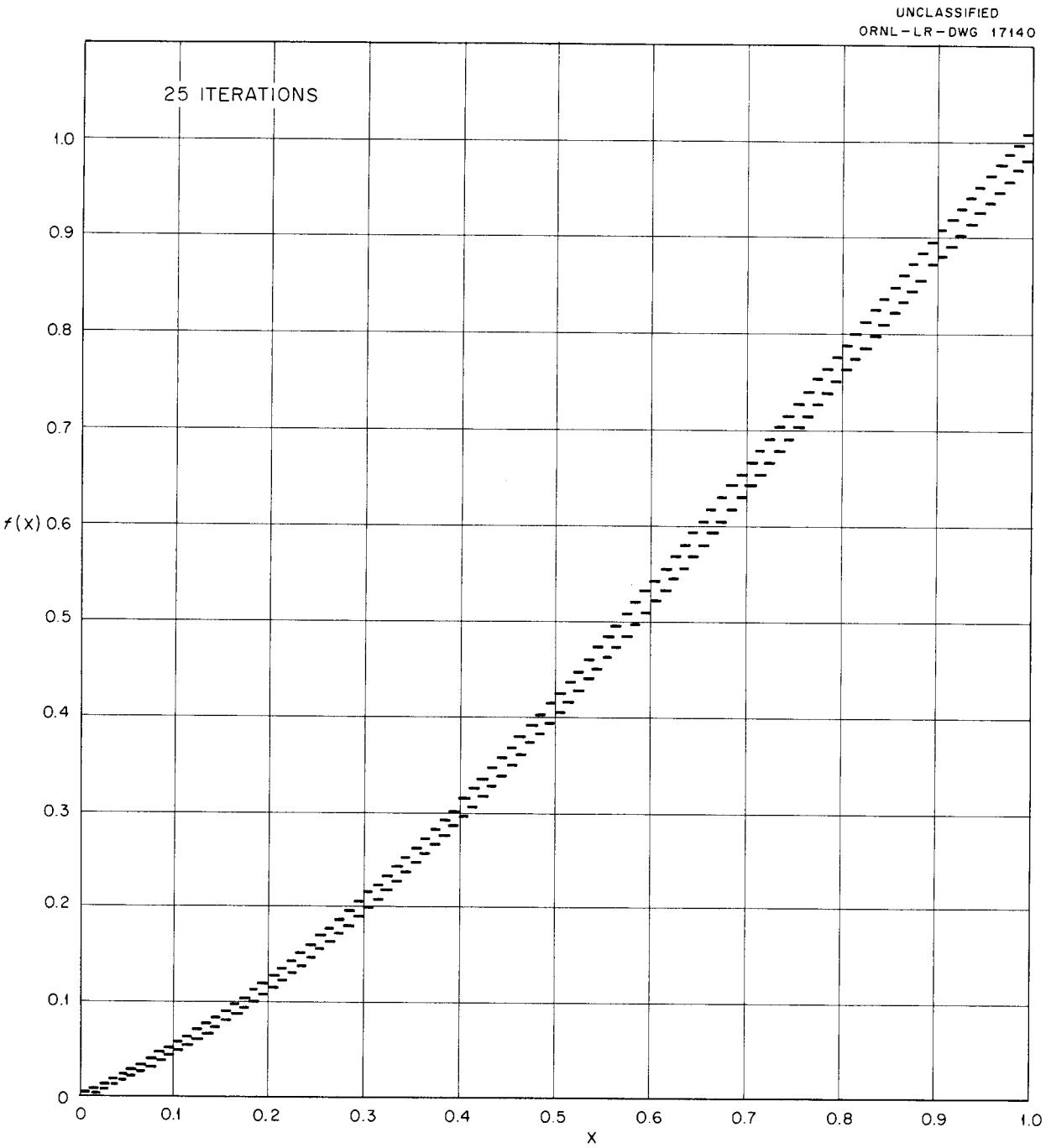

Fig. 3. Example One — $\lambda = 3$, $P = 100$.

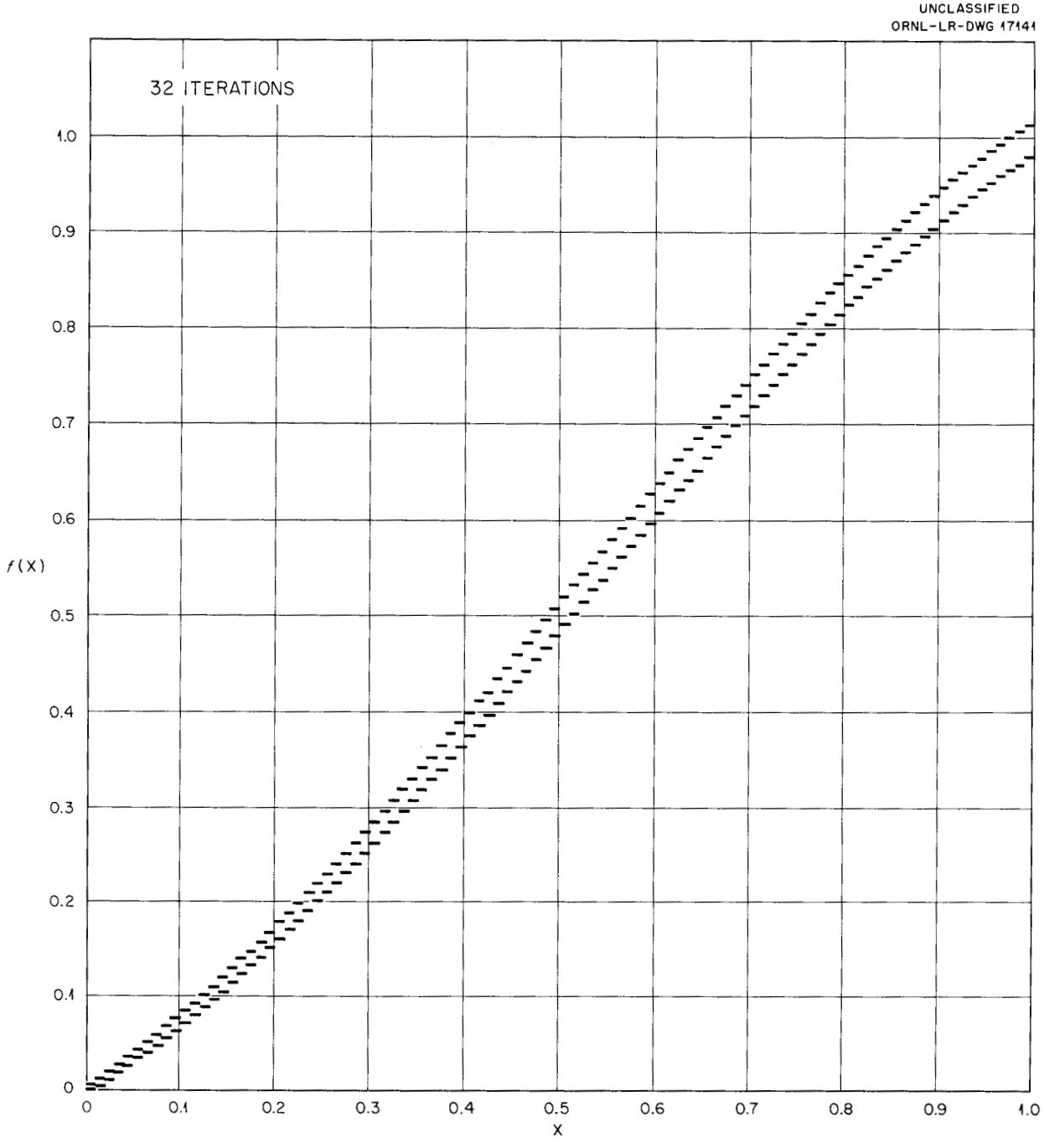

Fig. 4. Example One — $\lambda = 4$, $P = 100$.

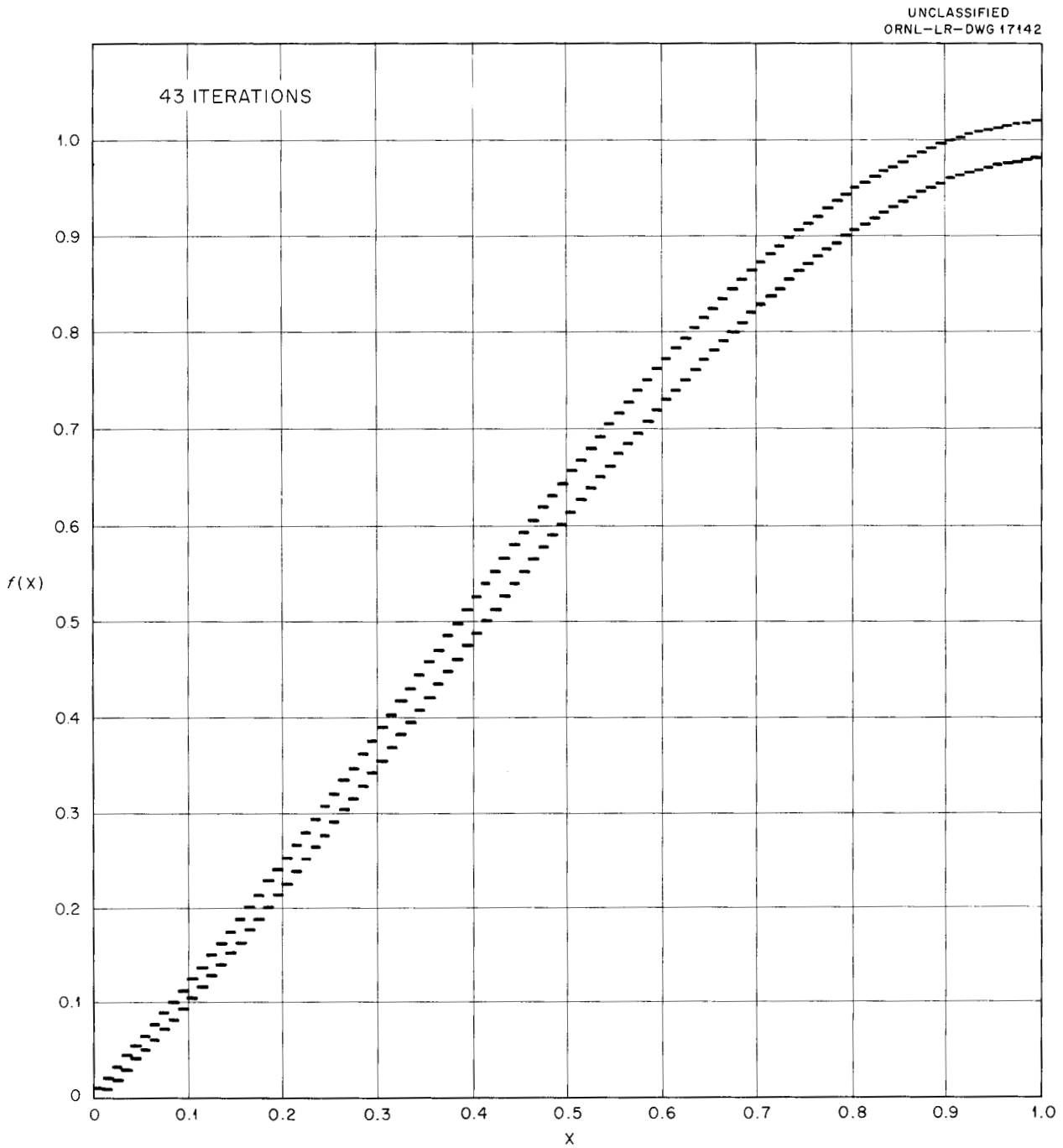

Fig. 5. Example One — $\lambda = 5$, $P = 100$.

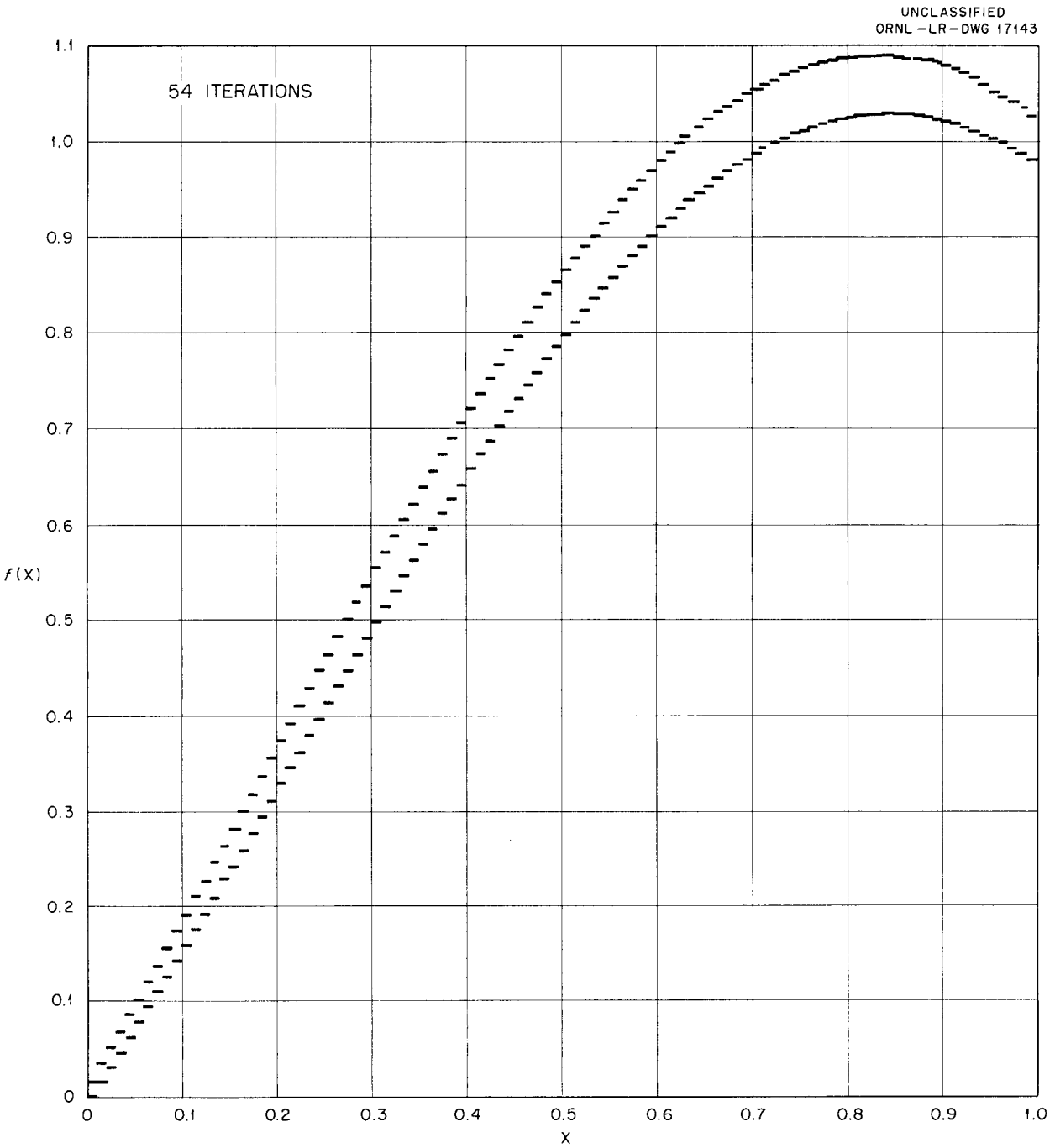

Fig. 6. Example One — $\lambda = 6$, $P = 100$.

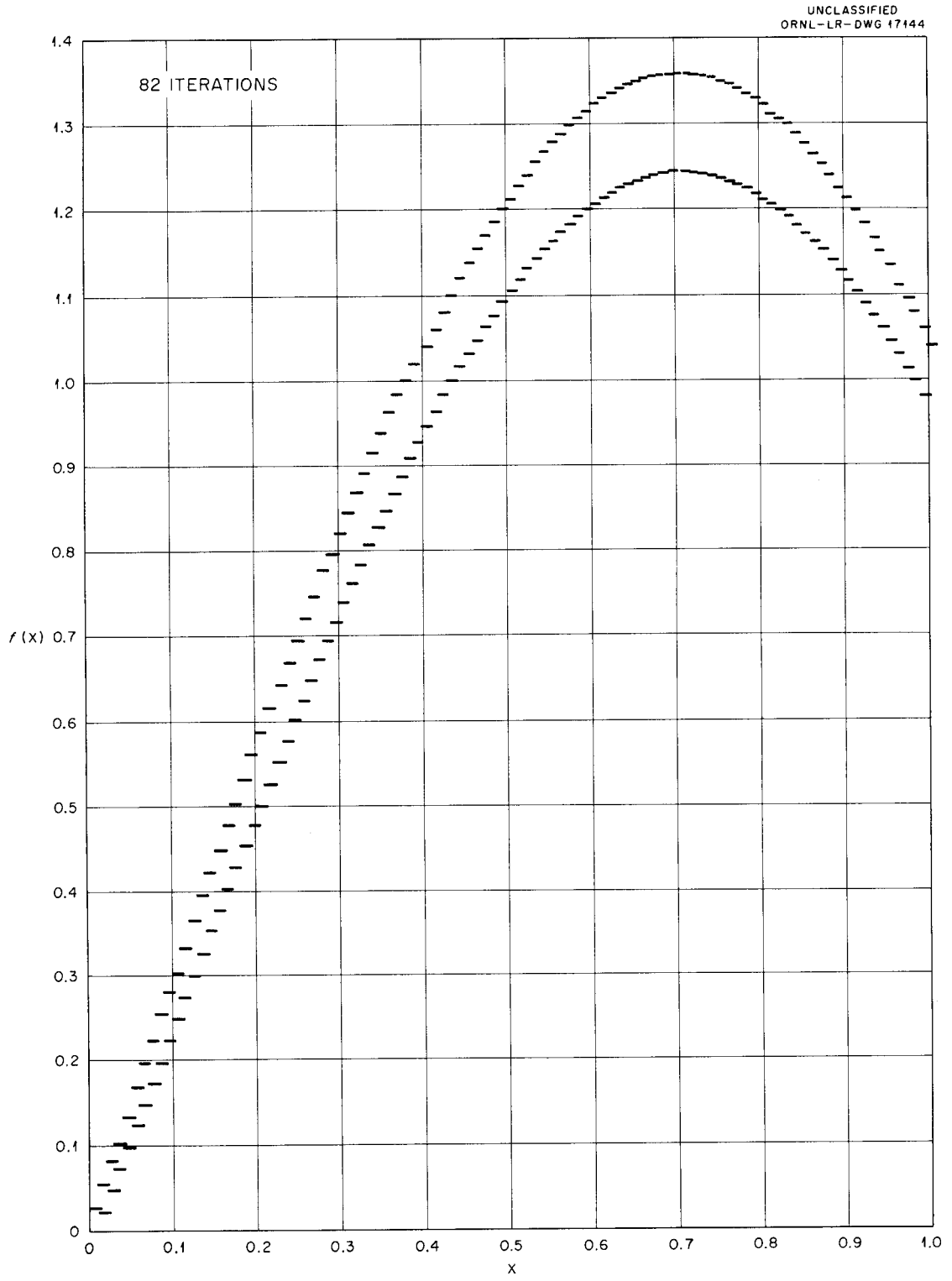

Fig. 7. Example One — $\lambda = 7$, $P = 100$

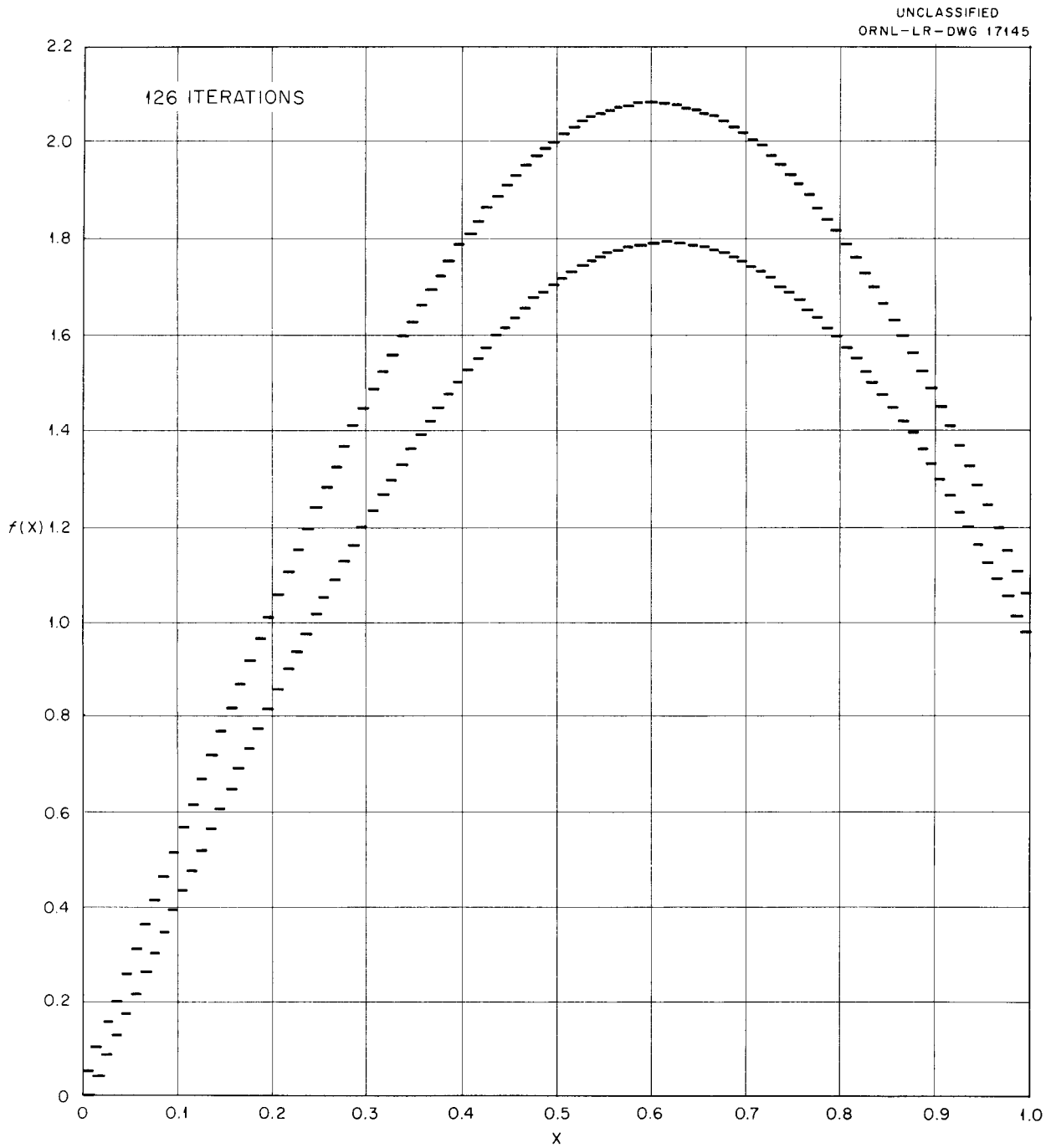

Fig. 8. Example One — $\lambda = 8$, $P = 100$.

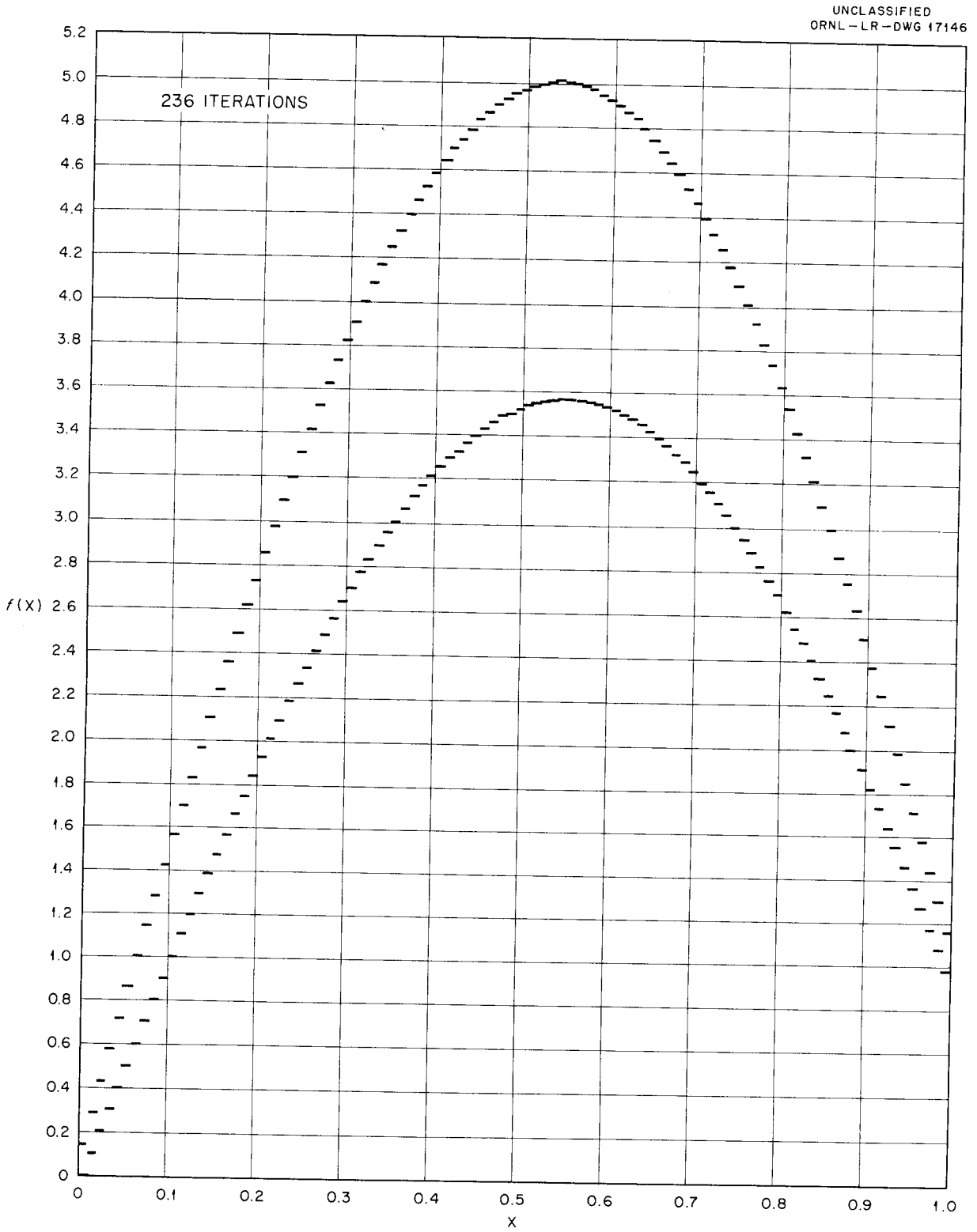

Fig. 9. Example One – $\lambda = 9$, $P = 100$.

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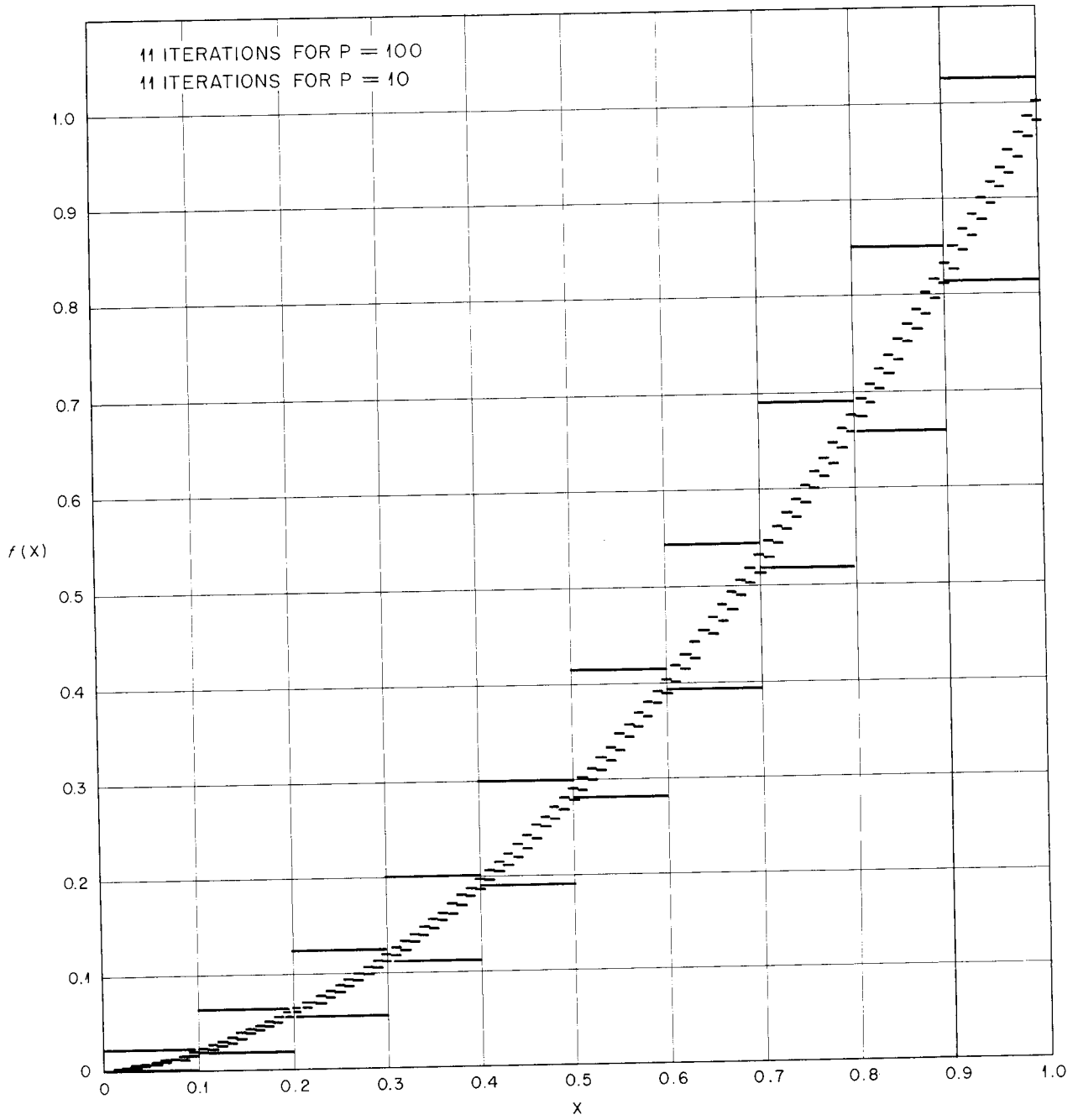

Fig. 10. Example One — $\lambda = 1$, $P = 10, 100$.

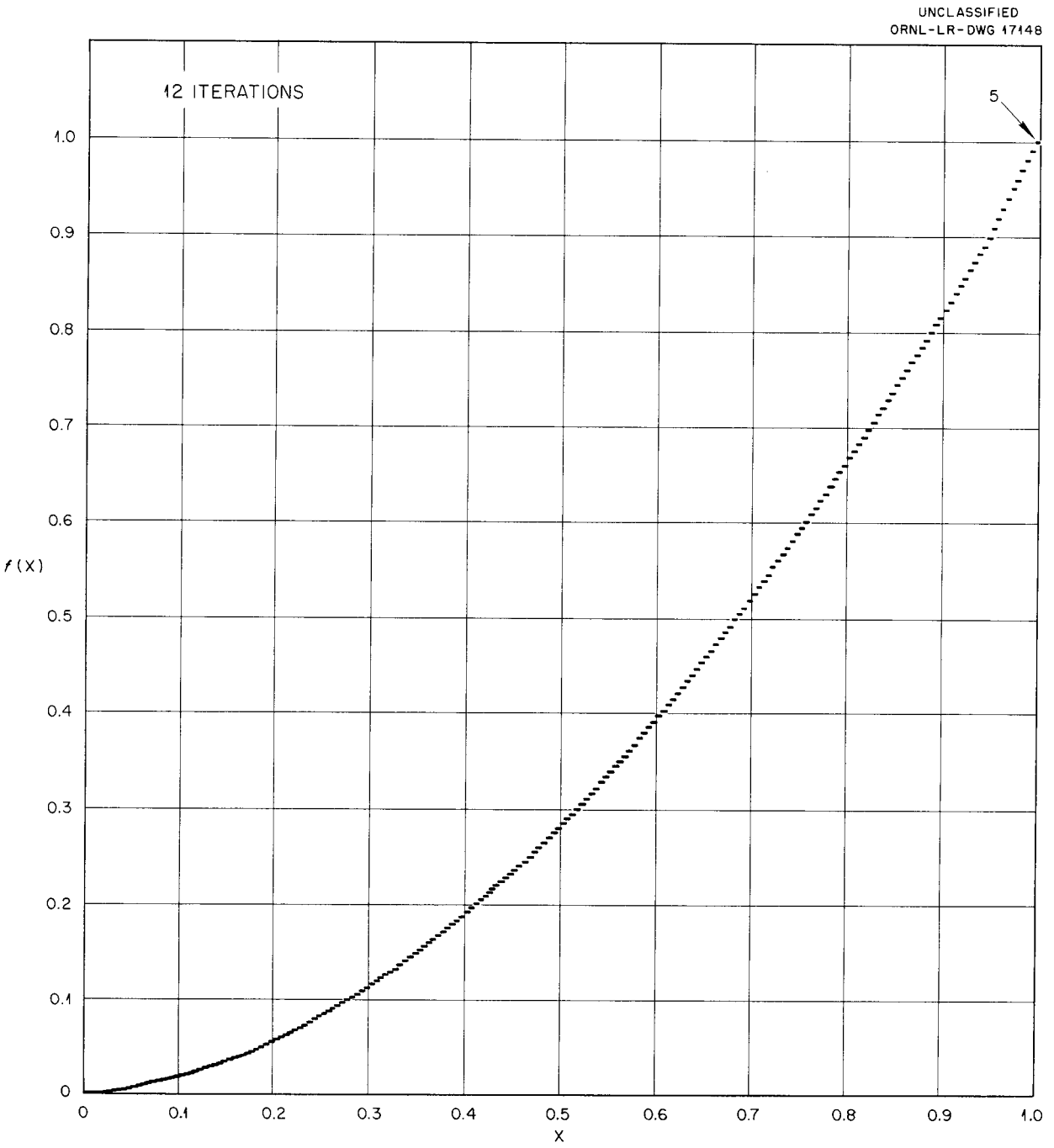

Fig. 11. Example One — $\lambda = 1$, $P = 200$.

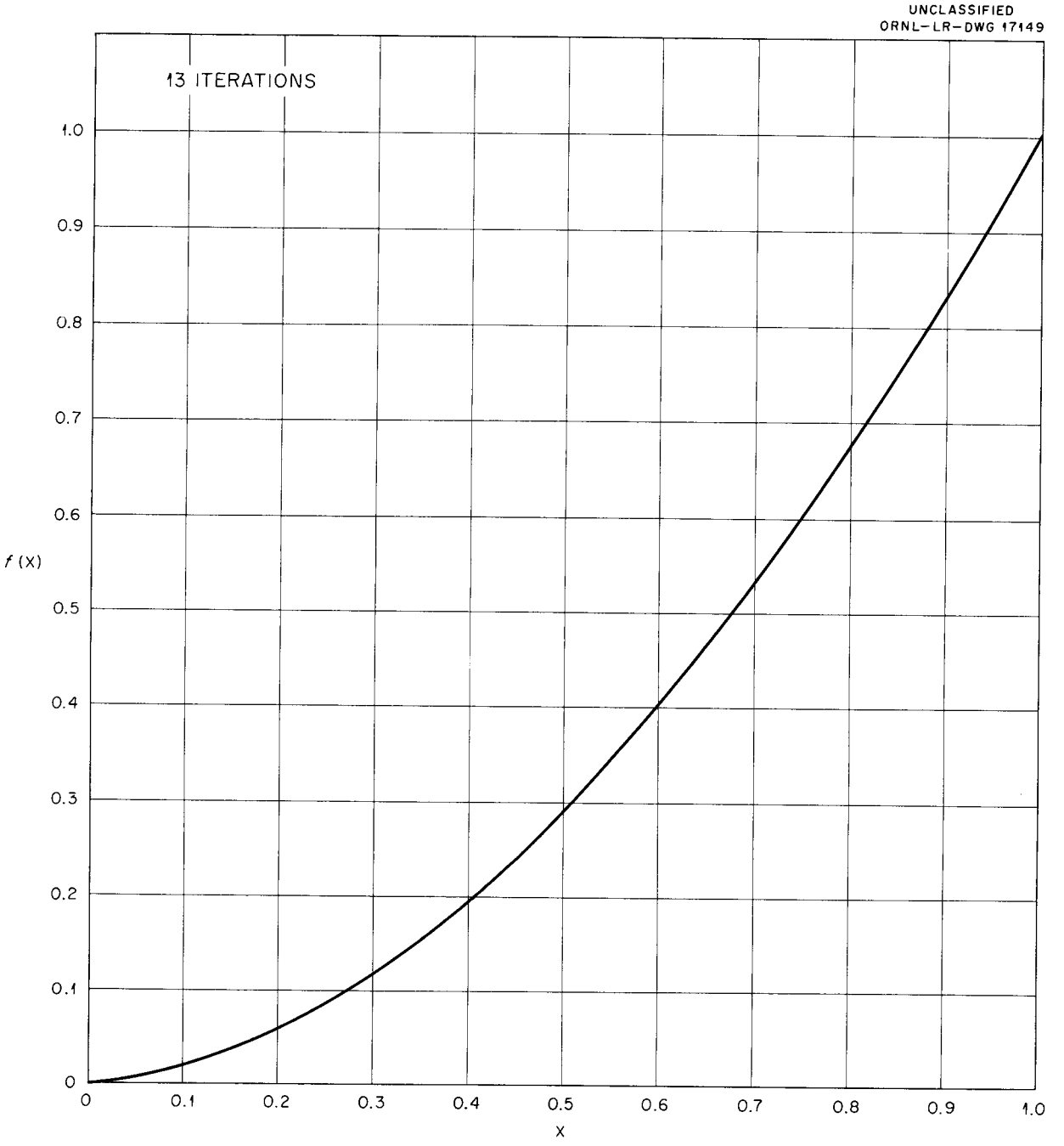

Fig. 12. Example One — $\lambda = 1$, $P = 100$.

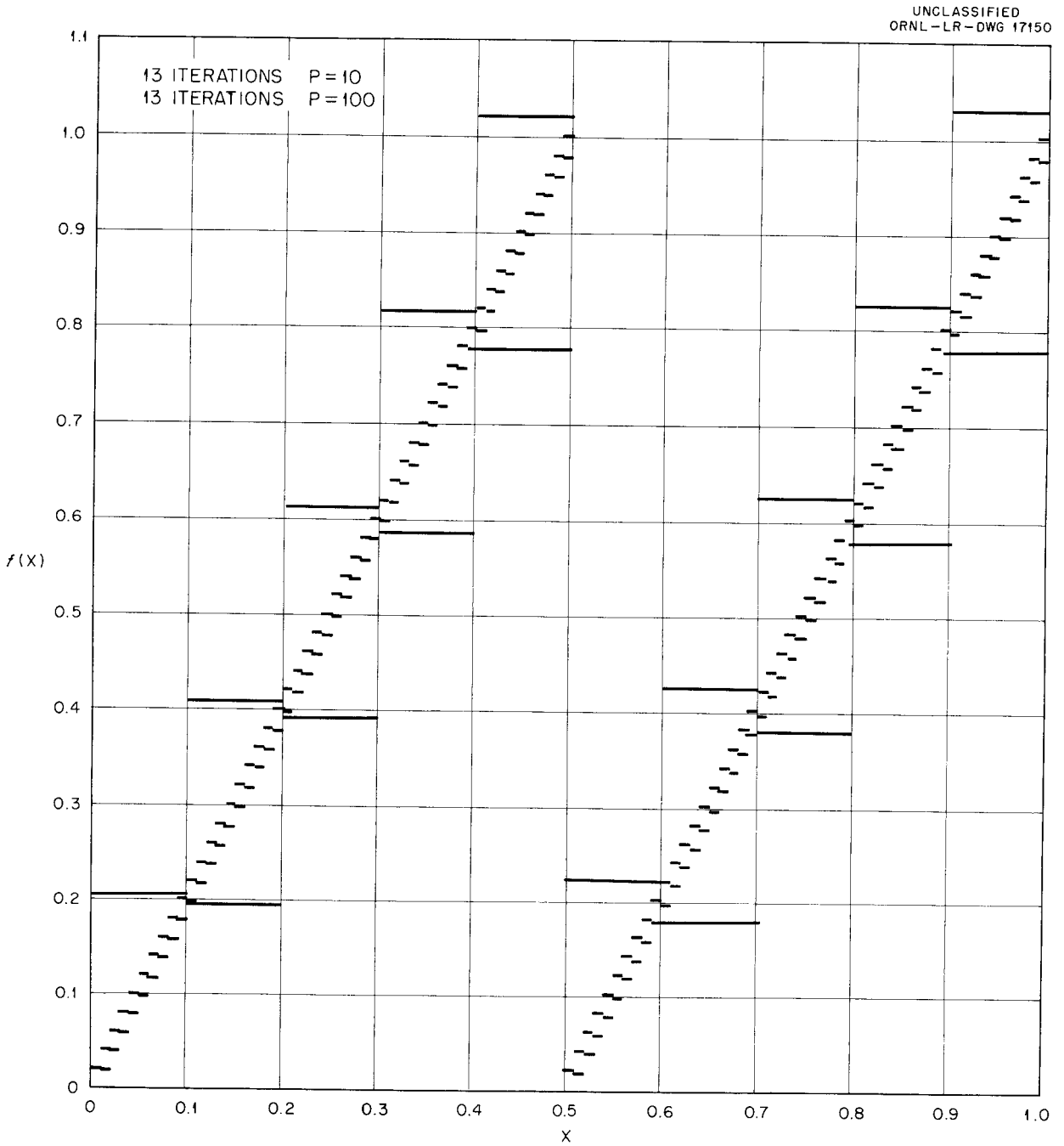

Fig. 13. Example Two — P=10, 100

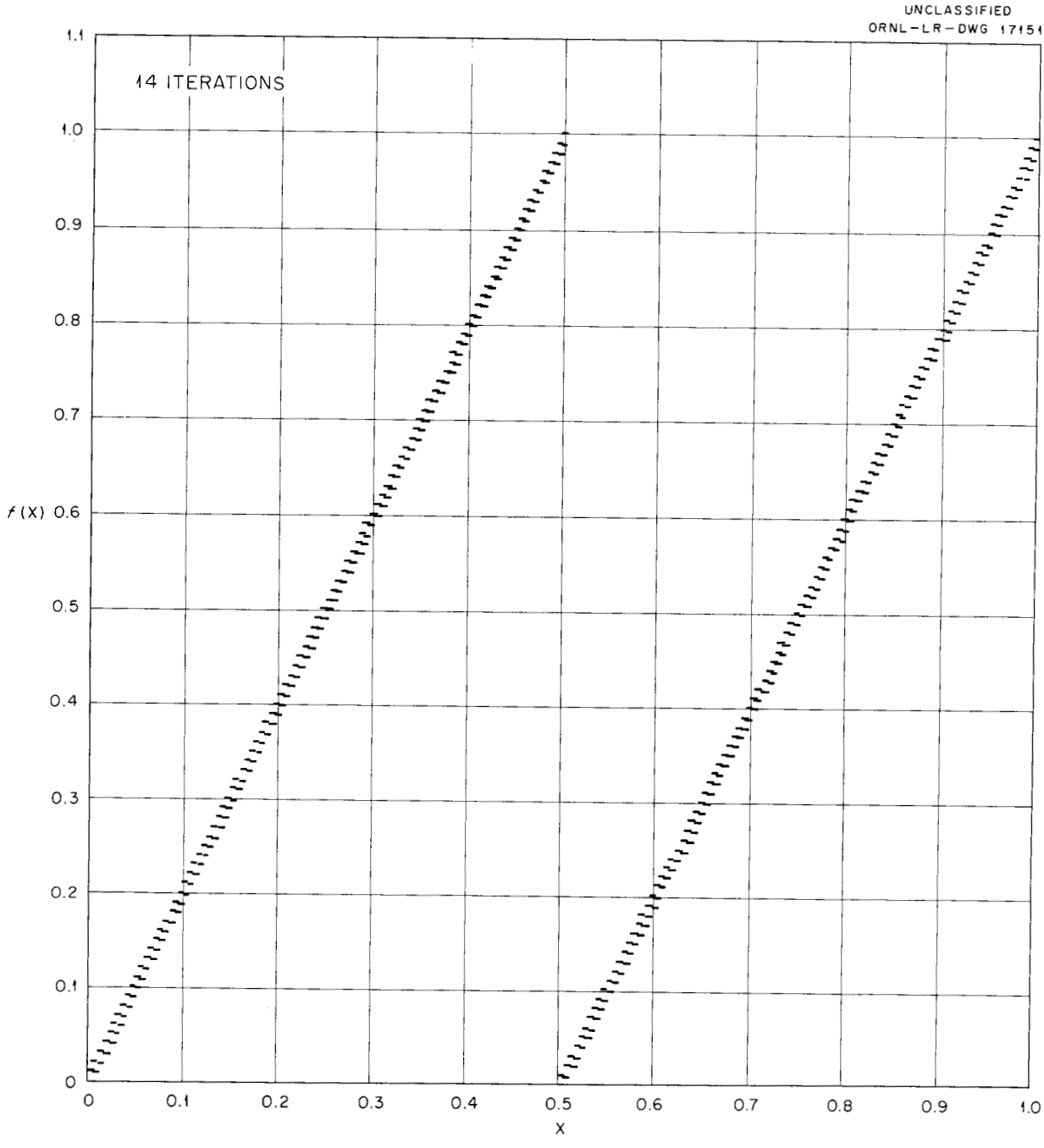

Fig. 14. Example Two – P = 200.

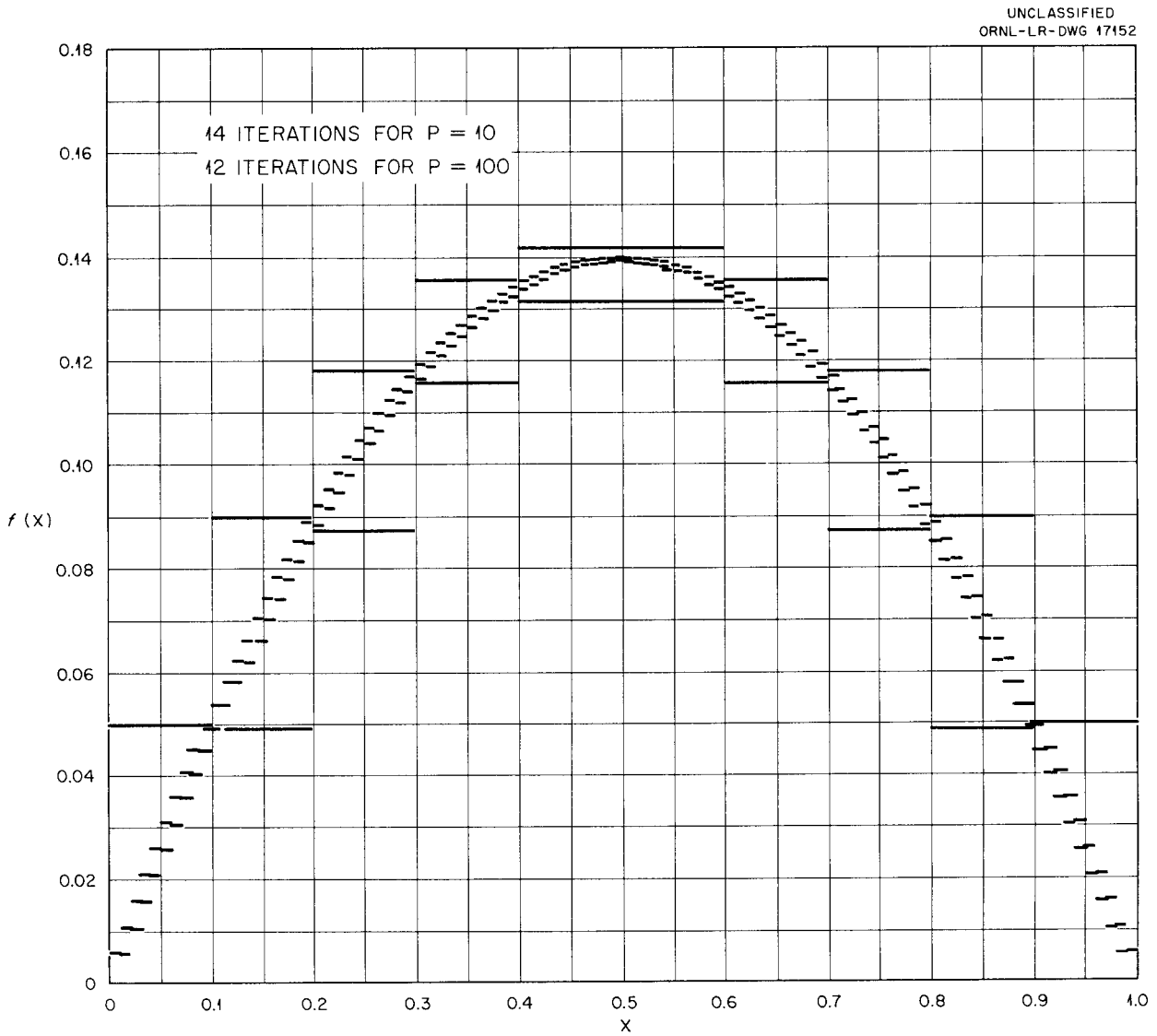

Fig. 15. Example Three – P = 10, 100.

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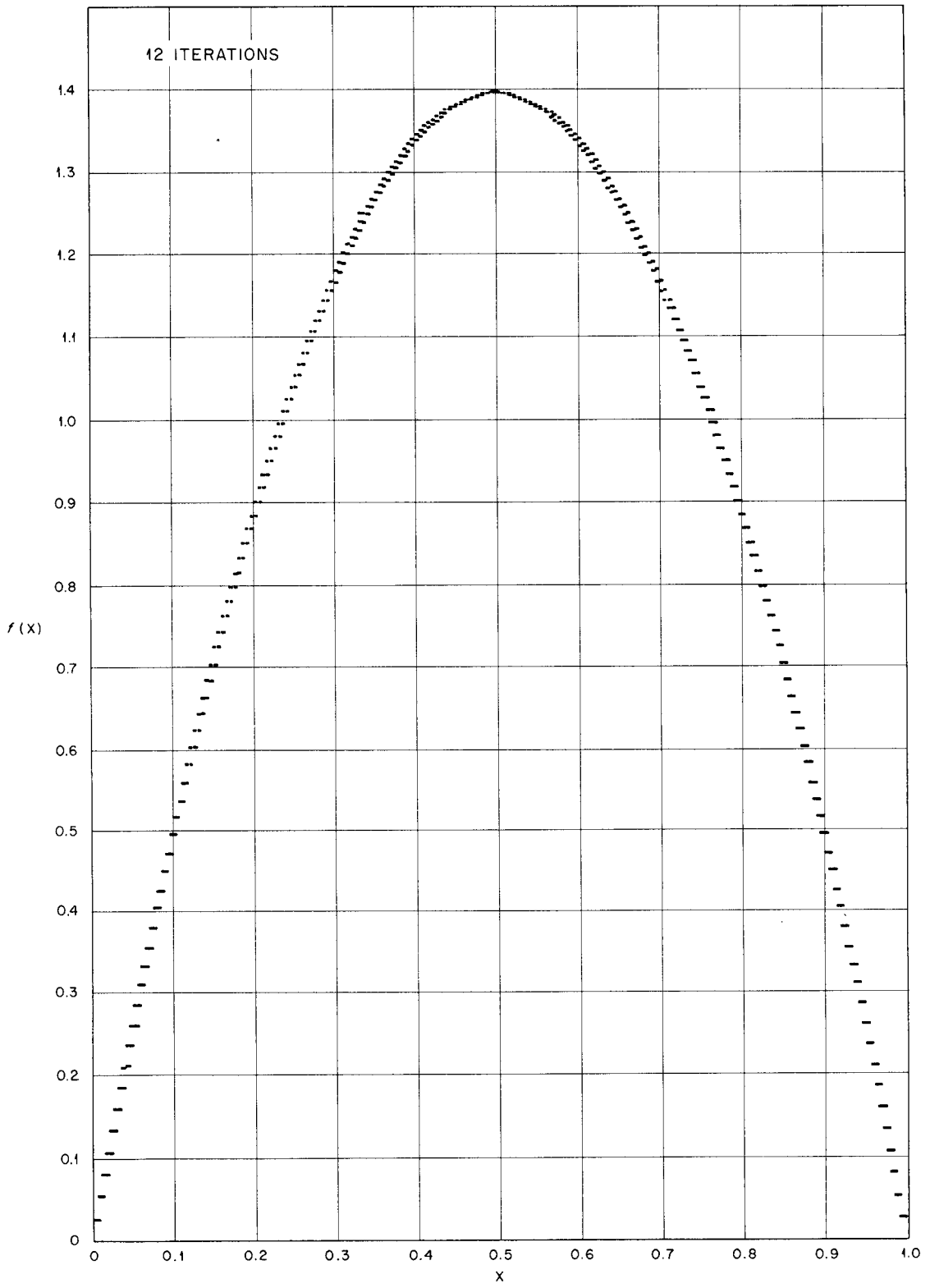

Fig. 16. Example Three — P = 200.

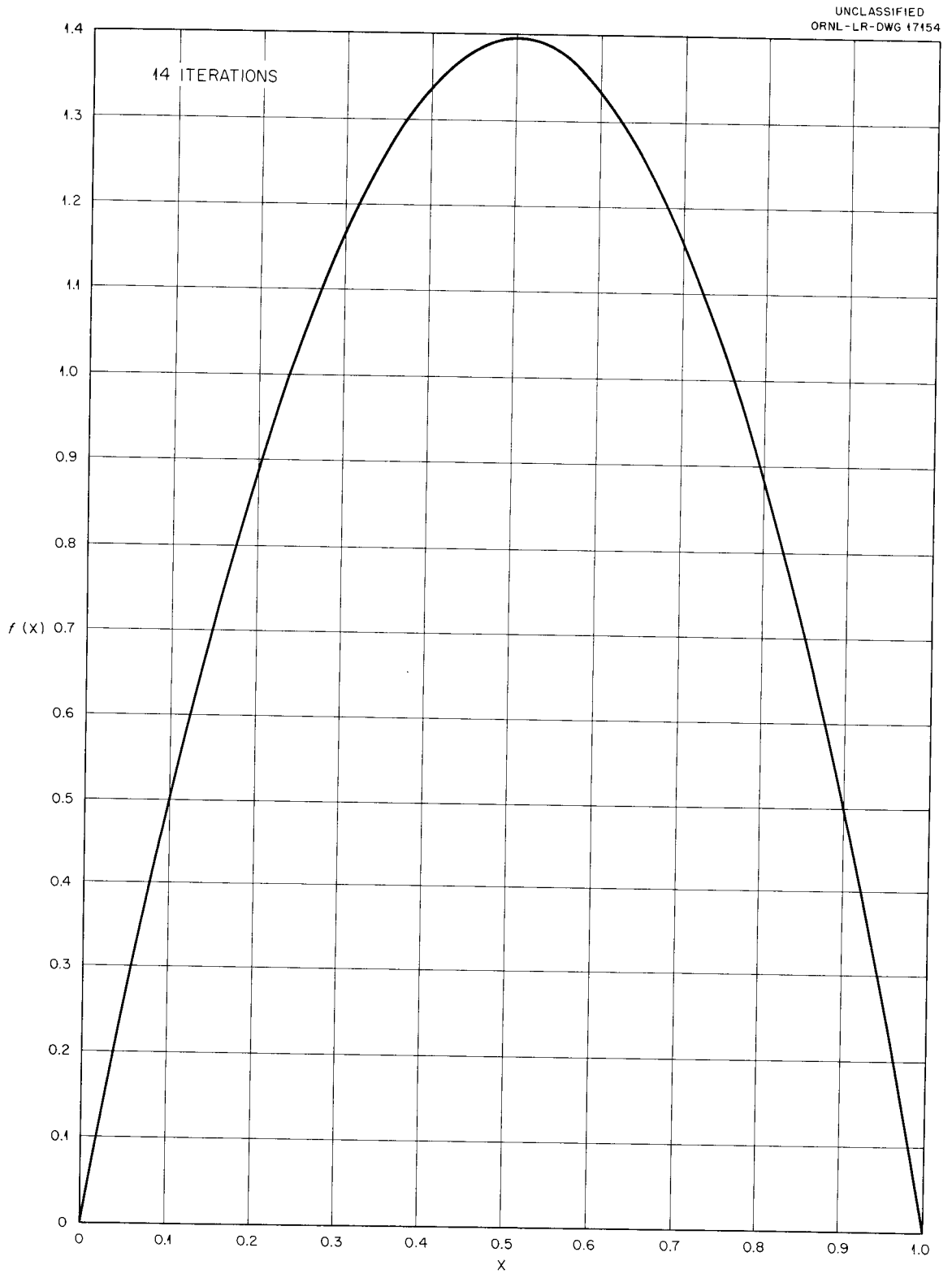

Fig. 17. Example Three — P = 500.

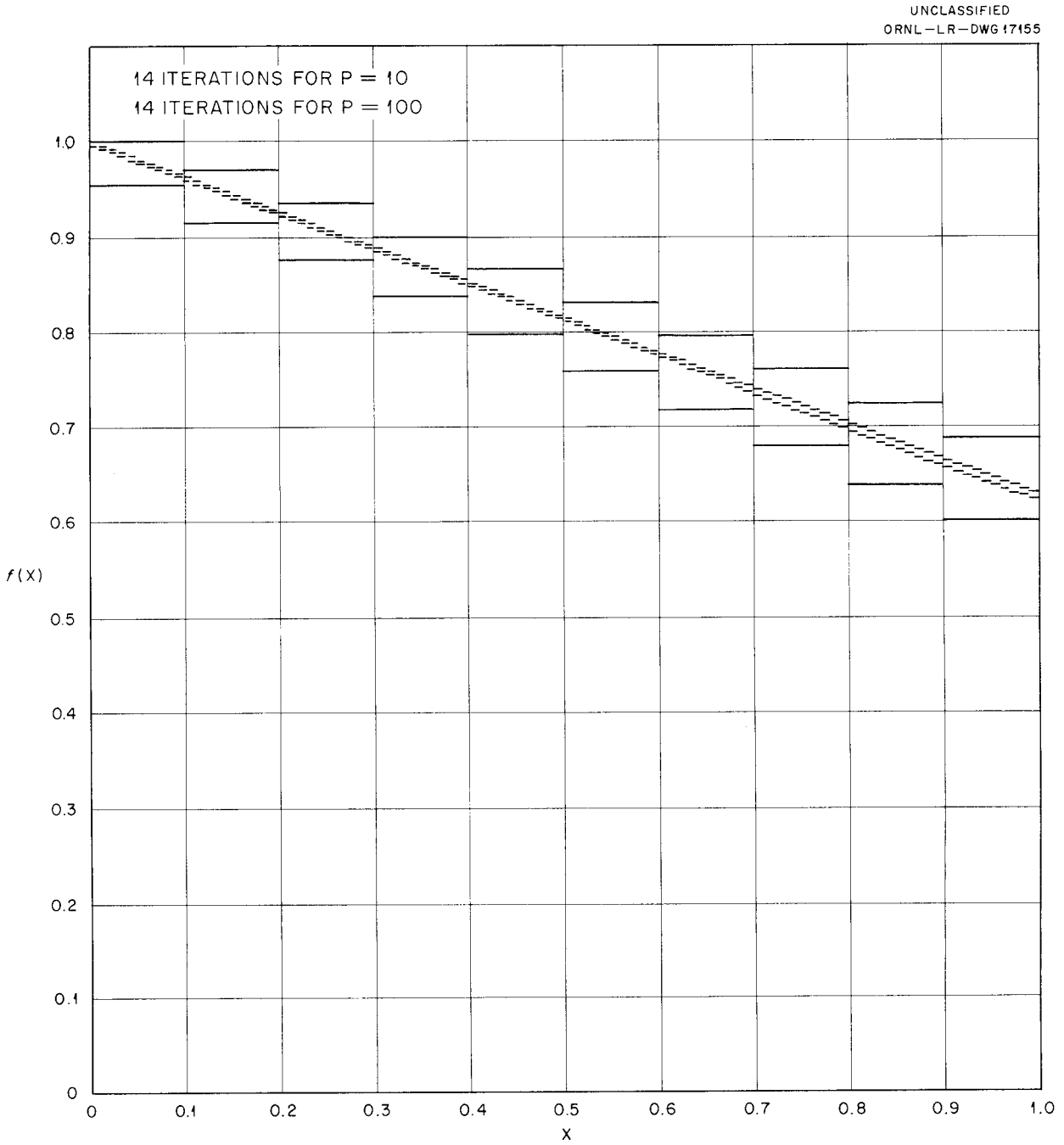

Fig. 18. Example Four — P = 10, 100.

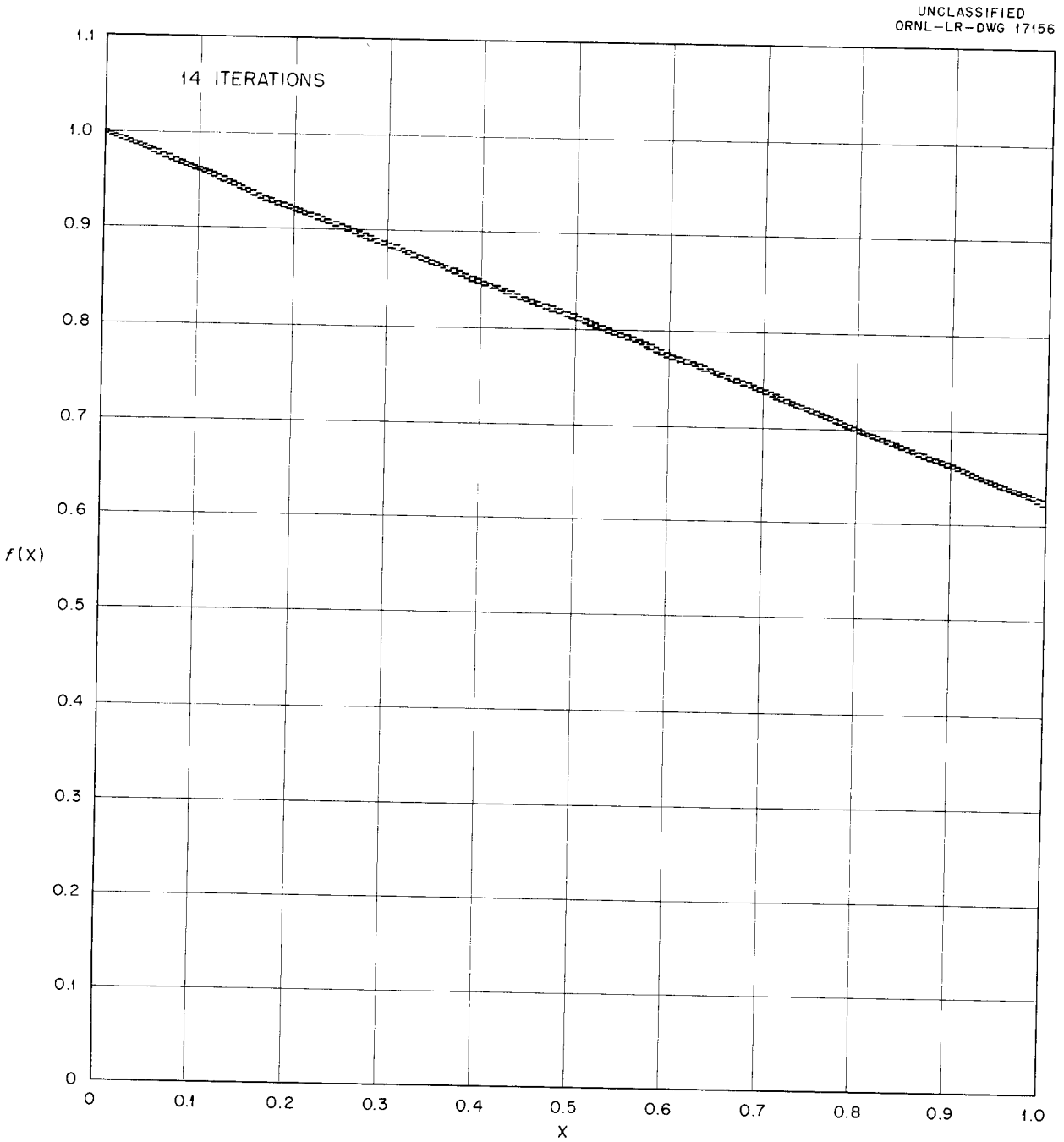

Fig. 19. Example Four — P = 200.

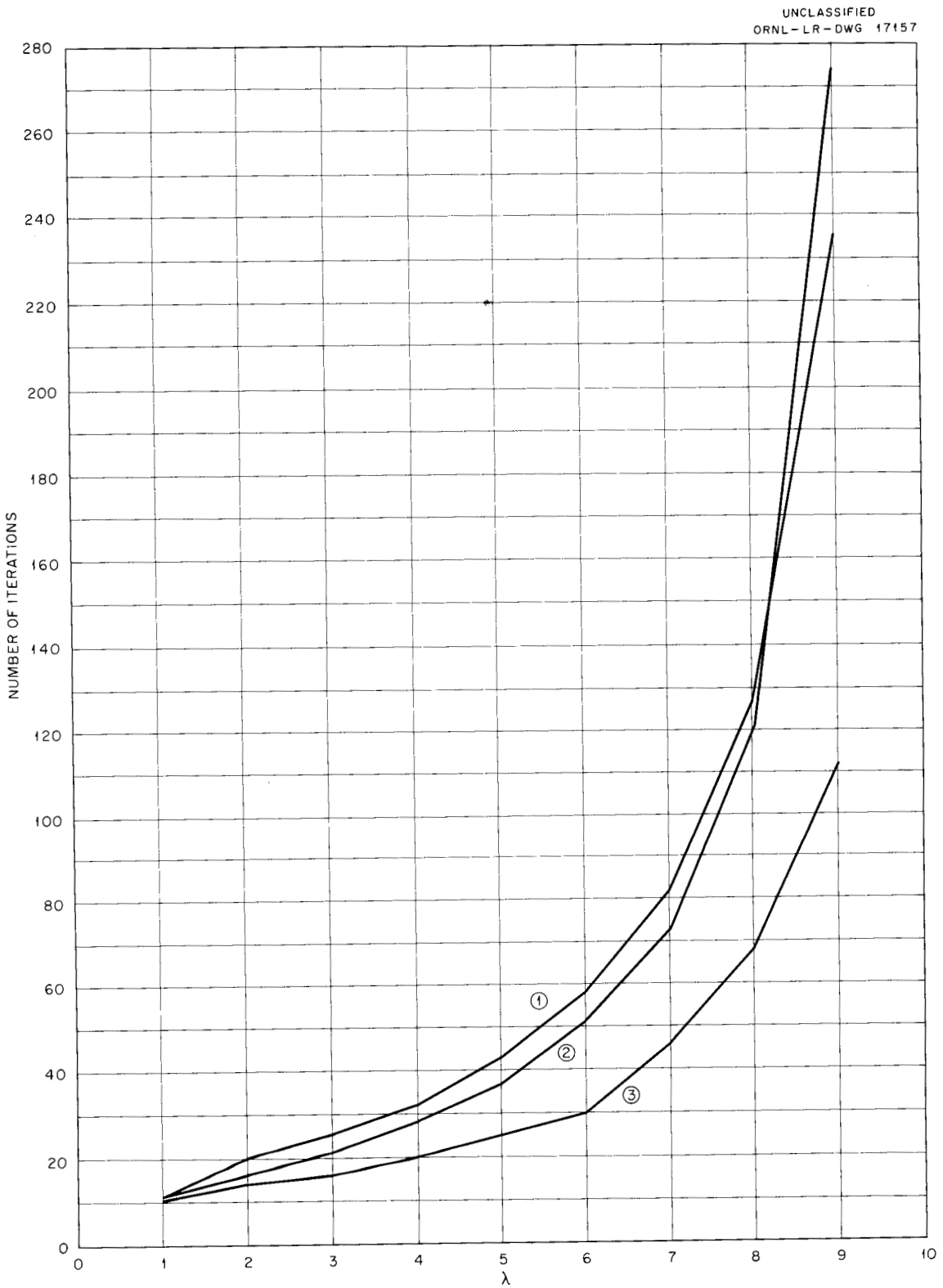

Fig. 20. Number of Iterations vs. λ for Example One.

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