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RADIATION FLUXES IN A CYLINDRICAL CAVITY RESULTING FROM AN IDEALIZED ENTERING CURRENT

C. D. Zerby

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C. D. Zerby

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RADIATION FLUXES IN A CYLINDRICAL CAVITY RESULTING FROM AN IDEALIZED ENTERING CURRENT

Abstract

The equations for the contribution to the radiation flux in a cylindrical cavity coming from the current entering through the curved surface are developed for two cases. In one case the cylinder is of finite length and the flux is obtained along the centerline. In the other case the cylinder extends to infinity in one direction from the plane in which the flux is determined along the radial from the center. In each case the entering current is idealized to have an angular distribution with an axis of symmetry which is not necessarily along the normal to the surface.

In many experiments involving cylindrical cavities it is possible to obtain the radiation flux throughout the cavity; however, it is difficult to interpret the data to obtain the angular distribution of the incoming current when there is a need for such information. The inverse situation can also arise, but the problem of converting from current to flux is about as difficult as the other problem.

It is the purpose of this report to develop some equations which will give the flux in the cavity resulting from an idealized entering current. A plot of the equations can then be compared with flux measurements to obtain the ideal current which most closely resembles the one in the experiment. In all cases the development will be limited to the flux contribution from the radiation entering the curved surfaces only. In all cases it is assumed that the entering current is constant over the entire curved surface.

The geometry and variables to be considered are given in Fig. 1. In Fig. 1 the vector $\overrightarrow{BC}$ lies along the normal to the surface and the vector $\overrightarrow{BA}$ is in the plane containing the surface normal and the axis of the cylinder.

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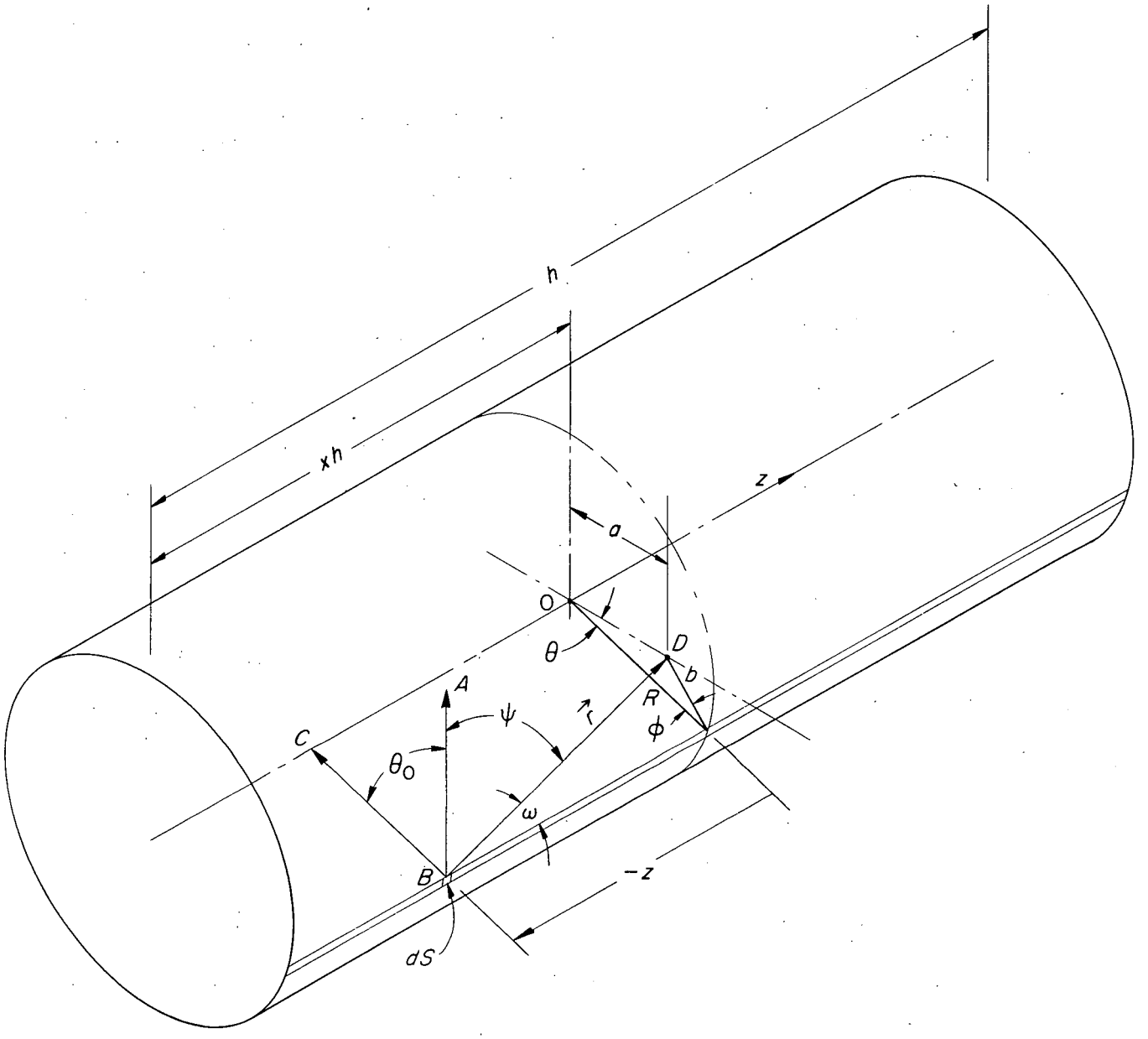

Fig. 1. Coordinate System for Determining the Radiation Flux in a Cylindrical Cavity.

The vector $\overrightarrow{BA}$ lies on the axis of symmetry of the radiation current coming from the surface element ds. In addition, the following definitions apply:

R = radius of the cylinder,

h = length of the cylinder,

x = fraction of cylinder length, h, to the plane containing the point of detection,

a = radial distance to the point of detection,

z = distance measured along the length of the cylinder from the plane containing the point of detection,

$\vec{r}$ = vector from the surface element ds to the point of detection,

ω = angle between the z direction and the vector $\vec{r}$,

θ_o = angle between the surface normal and the axis of symmetry of the entering current,

ψ = angle between the vector $\vec{r}$ and the axis of symmetry of the entering current,

ϕ = azimuthal angle about the z direction measured from the surface normal at the element of area ds,

$P(\vec{\Omega})$ = particles per unit solid angle in direction $\vec{\Omega}$ per unit surface area per sec,

the variables b and θ and the point of detection D are shown in Fig. 1

The flux at point D is given by .

$$\Phi = 2 \int_{\theta=0}^{\pi} \int_{z=-xh}^{h(1-x)} \frac{P(\vec{\Omega})R d\theta dz}{r^2} , \qquad (1)$$

However, the most desirable form for Eq. 1 is obtained with the angles ω and ϕ as the variables of integration. Since the transformation has been described before,[1] only the results will be given here

$$\Phi = 2 \int_{\phi=0}^{\sin^{-1}k} \frac{d\phi}{\sqrt{k^2 - \sin^2\phi}} \left\{ \int_{\omega=\tan^{-1}\frac{B_1}{px}}^{\tan^{-1}\frac{B_1}{p(x-1)}} P(\vec{\Omega})d\omega + \int_{\omega=\tan^{-1}\frac{B_2}{px}}^{\tan^{-1}\frac{B_2}{p(x-1)}} P(\vec{\Omega})d\omega \right\} \quad (2)$$

where

$$k = a/R \ , \quad (3)$$

$$p = h/R \ , \quad (4)$$

$$B_1 = \cos\phi - \sqrt{k^2 - \sin^2\phi} \ , \quad (5)$$

$$B_2 = \cos\phi + \sqrt{k^2 - \sin^2\phi} \ . \quad (6)$$

Turning to the idealized entering current, $P(\vec{\Omega})$, the assumption will be made that

$$P(\vec{\Omega}) = K_j \cos^j \psi \ , \quad \psi \leq \frac{\pi}{2} \ , \ j = 0,1,2,3,\ldots.$$

$$= 0 \quad , \quad \psi > \frac{\pi}{2} \ , \quad (7)$$

where K_j is the normalizing constant. Equation 7 can be transformed to depend on the variables ω and ϕ by rotation of the axis of symmetry to give

$$P(\vec{\Omega}) = K_j \ (\cos\theta_o \ \sin\omega \cos\phi + \sin\theta_o \ \cos\omega)^j, \ \frac{-\tan\theta_o}{\tan\omega} \leq \cos\phi,$$

$$= 0 \quad , \ \frac{-\tan\theta_o}{\tan\omega} > \cos\phi \quad (8)$$

---

1. C. D. Zerby, A Monte Carlo Method of Calculating the Response of a Point Detector at An Arbitrary Position Inside a Cylindrical Shield, ORNL-2105 (June, 1956).

Expanding Eq. 8 in a series

$$P(\vec{\Omega}) = K_j \sum_{n=o}^{j} C\binom{j}{n} \sin^n\theta_o \cos^{j-n}\theta_o \cos^{j-n}\phi \sin^{j-n}\omega \cos^n\omega, \quad \frac{-\tan\theta_o}{\tan\omega} \leq \cos\phi$$

$$= 0 \qquad , \quad \frac{-\tan\theta_o}{\tan\omega} > \cos\phi \qquad (9)$$

where $C\binom{j}{n}$ are the binomial coefficients. Simplification is ultimately achieved if Eq. 9 is substituted into Eq. 2 and the transformation

$$\sin\phi = k \sin\alpha \qquad (10)$$

is made to obtain

$$\Phi_j = 2K_j \sum_{n=o}^{j} C\binom{j}{n} \sin^n\theta_o \cos^{j-n}\theta_o \int_{\alpha=o}^{\pi/2} (1 - k^2 \sin^2\alpha)^{\frac{j-n-1}{2}} d\alpha$$

$$x \left\{ \int_{\omega=\tan^{-1}\frac{C_1}{px}}^{Q_1} \sin^{j-n}\omega \cos^n\omega \, d\omega + \int_{\omega=\tan^{-1}\frac{C_2}{px}}^{Q_2} \sin^{j-n}\omega \cos^n\omega \, d\omega \right\} \qquad (11)$$

where

$$C_1 = \sqrt{1 - k^2 \sin^2\alpha} - k \cos\alpha,$$

$$C_2 = \sqrt{1 - k^2 \sin^2\alpha} + k \cos\alpha,$$

Q_1 = whichever is the least of $\tan^{-1} \frac{C_1}{p(x-1)}$ or $\tan^{-1}\left(\frac{-\tan\theta_o}{\sqrt{1 - k^2 \sin^2\alpha}}\right)$,

Q_2 = whichever is the least of $\tan^{-1} \frac{C_2}{p(x-1)}$ or $\tan^{-1}\left(\frac{-\tan\theta_o}{\sqrt{1 - k^2 \sin^2\alpha}}\right)$.

It is not possible to solve Eq. 11 in general; however, there are two cases and a special example of the second which lead to analytic expressions and will be treated here.

For the first case let k = 0 in Eq. 11 which will give the flux along the centerline of the cylinder, then

$$\Phi_j = 2\pi K_j \sum_{n=o}^{j} C\binom{j}{n} \sin^n\theta_o \cos^{j-n}\theta_o \int_{\omega=\mathrm{ctn}^{-1}px}^{P} \sin^{j-n}\omega \cos^n\omega \, d\omega \tag{12}$$

where

$$P = \mathrm{ctn}^{-1}p(x-1) \text{ if } \mathrm{ctn}^{-1}p(x-1) < \pi - \theta_o,$$

$$= \mathrm{ctn}^{-1}(-\mathrm{ctn}\theta_o) \text{ if } \mathrm{ctn}^{-1}p(x-1) \geq \pi - \theta_o.$$

Equation 12 can be evaluated by direct integration; however, it is better to introduce a new function defined by

$$F_{mn}(y) \equiv \int_{\mathrm{ctn}^{-1}y}^{\pi/2} \sin^m\omega \cos^n\omega \, d\omega . \tag{13}$$

Some special cases of Eq. 13 are obtained by direct integration as

$$F_{o,o}(y) = \frac{\pi}{2} - \mathrm{ctn}^{-1}y, \tag{14}$$

$$F_{m,1}(y) = \frac{1}{m+1}\left[1 - \frac{1}{(y^2+1)^{\frac{m+1}{2}}}\right], \qquad m \geq 0, \tag{15}$$

$$F_{1,n}(y) = \frac{y^{n+1}}{(n+1)(y^2+1)^{\frac{n+1}{2}}}, \qquad n \geq 0. \tag{16}$$

Recursion relations for Eq. 13 are obtained after integrating by parts to give

$$F_{m,n}(y) = \frac{1}{m+n}\left[\frac{y^{n+1}}{(y^2+1)^{\frac{m+n}{2}}} + (m-1)\,F_{m-2,n}(y)\right], \quad \begin{matrix} m \geq 2 \\ n \geq 0 \end{matrix}, \tag{17}$$

and

$$F_{m,n}(y) = \frac{1}{m+n}\left[\frac{-y^{n-1}}{(y^2+1)^{\frac{m+n}{2}}} + (n-1)\,F_{m,n-2}(y)\right], \quad \begin{matrix} m \geq 0 \\ n \geq 2 \end{matrix}. \tag{18}$$

Using the definition from Eq. 13 in Eq. 12 gives

$$\Phi_j = 2\pi K_j \sum_{n=o}^{j} C\binom{j}{n} \sin^n\theta_o \cos^{j-n}\theta_o \left[F_{j-n,n}(g) - F_{j-n,n}(S)\right] \tag{19}$$

where

$$g = px,$$

$$S = p(x-1) \text{ if } \operatorname{ctn}^{-1}p(x-1) < \pi - \theta_o,$$

$$= -\operatorname{ctn}\theta_o \text{ if } \operatorname{ctn}^{-1}p(x-1) \geq \pi - \theta_o.$$

Equation 19 is most easily solved with the use of Eqs. 14, 15, 16, 17, and 18; the resulting equations for values of j from zero to five are given in Appendix A. Graphs and tables of values computed using the equations with the value of p taken as 3.701 are also included in Appendix A.

The second case of interest occurs when $x = 1$, $p = \infty$ and $\theta_o \geq 0$ in which case Eq. 11 reduces to

$$\Phi_j = 4K_j \sum_{n=o}^{j} C\binom{j}{n} \sin^n\theta_o \cos^{j-n}\theta_o \, I_{\frac{j-n-1}{2}}(k) \frac{\Gamma\left(\frac{j-n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{j+2}{2}\right)} \tag{20}$$

where $\Gamma(n)$ = gamma function,

$$\frac{\Gamma\left(\frac{j-n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{j+2}{2}\right)} = \int_0^{\pi/2} \sin^{j-n}\omega \cos^n\omega \, d\omega,$$

and the new function $I_n(k)$ is defined by

$$I_n(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\alpha)^n \, d\alpha. \tag{21}$$

Evaluation of Eq. 20 is most easily accomplished with the use of a recursion formula for the function $I_n(k)$ developed in Appendix C and given as

$$I_n(k) = \frac{(2n-1)(2-k^2)}{2n} I_{n-1}(k) - \frac{(1-k^2)(n-1)}{n} I_{n-2}(k), \; n \neq 0. \tag{22}$$

The expression for the function $I_n(k)$ for the first few values of n of interest can be obtained directly from Eq. 21 as

$$I_{-1/2}(k) = K(k), \tag{23}$$

$$I_o(k) = \pi/2, \tag{24}$$

$$I_{1/2}(k) = E(k), \tag{25}$$

$$I_1(k) = \pi/4\,(2 - k^2), \tag{26}$$

where K(k) and E(k) are the complete elliptic integrals of the first and second kind respectively. For two values of k the function $I_n(k)$ is given by

$$I_n(0) = \frac{\pi}{2} , \tag{27}$$

$$I_n(1) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(n + \frac{1}{2}\right)}{(n + 1)} , \quad n \geq 0, \tag{28}$$

$$I_{-1/2}(1) = \infty . \tag{29}$$

Since a term having $I_{-1/2}(k)$ occurs in Eq. 20 for any value of j and all values of θ_o except $\theta_o = 0$, it is obvious that Φ_j will be infinite for k = 1 in all cases but those for which $\theta_o = 0$. This situation arises because of the idealized source term which has a finite probability of emitting radiation tangent to the interior surface. For this reason Eq. 20 will not give realistic results for values of k near one; however, they should be useful over most of the range of the parameter k.

The equations that result from Eq. 20 for j values of zero through 5 are given in Appendix B. Graphs and tables of values computed using the equations are also included in Appendix B.

A special case of Eq. 20 that leads to a simple result occurs when $\theta_o = 0$. In that case Eq. 20 reduces to

$$\Phi_j = 2 \sqrt{\pi} \; K_j \; I_{\frac{j-1}{2}}(k) \; \frac{\Gamma\left(\frac{j + 1}{2}\right)}{\Gamma\left(\frac{j + 2}{2}\right)} . \tag{30}$$

Letting $K_j = \frac{(j + 1)}{2\pi}$ so that the source is normalized to one particle per square centimeter of surface per sec and using Eq. 22 in Eq. 30 it is possible to obtain the recursion formula

$$\Phi_j = \frac{(j + 1)(j - 2)(2 - k^2)}{j(j - 1)} \Phi_{j-2} - \frac{(j + 1)(j - 3)(1 - k^2)}{j(j - 2)} \Phi_{j-4} . \qquad (31)$$

The values of the function Φ_j for the first few values of j are obtained directly from Eq. 30 as

$$\Phi_o = K(k), \qquad (32)$$

$$\Phi_1 = 1, \qquad (33)$$

$$\Phi_2 = \frac{3}{2} E(k), \qquad (34)$$

$$\Phi_3 = \frac{4}{3} (2 - k^2). \qquad (35)$$

The values of Φ_j for greater values of j are obtained from Eq. 31.

The ratio of the flux at k = 0 to that at k = 1 for the case $\theta_o = 0$ is obtained from Eq. 30 with the use of Eqs. 27 and 28 as

$$R_j = \frac{(\Phi_j)_{k=o}}{(\Phi_j)_{k=1}} = \frac{\sqrt{\pi}\, \Gamma\left(\frac{j + 1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)} = \frac{j - 1}{j - 2} \quad R_{j-2} \qquad (36)$$

where

$$R_o = 0, \qquad (37)$$

$$R_1 = 1, \qquad (38)$$

$$R_2 = \frac{\pi}{2}. \qquad (39)$$

Appendix A

Formulas, Graphs, and Tables for the Flux on the Centerline of a Cylinder of Finite Length

Equation 19 has been evaluated for $j = 0, 1, 2, 3, 4, 5$. The formulas are given below. In every case the following definitions apply.

$$p = \frac{h}{R},$$

$$q = px,$$

$$S = p(x - 1) \text{ if } \operatorname{ctn}^{-1} p(x - 1) < \pi - \theta_o$$

$$= -\operatorname{ctn}\theta_o \text{ if } \operatorname{ctn}^{-1} p(x - 1) \geq \pi - \theta_o.$$

Case 1: $j = 0$

$$\Phi_o = 2\pi K_o \left[\operatorname{ctn}^{-1} S - \operatorname{ctn}^{-1} q\right]$$

Case 2: $j = 1$

$$\Phi_1 = 2\pi K_1 \left\{ \cos\theta_o \left[\frac{q}{(q^2 + 1)^{1/2}} - \frac{S}{(S^2 + 1)^{1/2}} \right] + \sin\theta_o \left[\frac{1}{(S^2 + 1)^{1/2}} - \frac{1}{(q^2 + 1)^{1/2}} \right] \right\}$$

Case 3: $j = 2$

$$\Phi_2 = 2\pi K_2 \left\{ \frac{1}{2} \cos^2\theta_o \left[\frac{q}{(q^2 + 1)} - \frac{S}{(S^2 + 1)} - \operatorname{ctn}^{-1} q + \operatorname{ctn}^{-1} S \right] + \sin\theta_o \cos\theta_o \right.$$

$$\left. \times \left[\frac{1}{(S^2 + 1)} - \frac{1}{(q^2 + 1)} \right] + \frac{1}{2} \sin^2\theta_o \left[\frac{S}{(S^2 + 1)} - \frac{q}{(q^2 + 1)} - \operatorname{ctn}^{-1} q + \operatorname{ctn}^{-1} S \right] \right\}$$

Case 4: $j = 3$

$$\Phi_3 = 2\pi K_3 \left(\Phi_{3,0} + \Phi_{2,1} + \Phi_{1,2} + \Phi_{0,3}\right)$$

where

$$\Phi_{3,0} = \frac{1}{3}\cos^3\theta_o \left[\frac{g}{(g^2+1)^{3/2}} - \frac{S}{(S^2+1)^{3/2}} + \frac{2g}{(g^2+1)^{1/2}} - \frac{2S}{(S^2+1)^{1/2}}\right]$$

$$\Phi_{2,1} = \sin\theta_o \cos^2\theta_o \left[\frac{1}{(S^2+1)^{3/2}} - \frac{1}{(g^2+1)^{3/2}}\right]$$

$$\Phi_{1,2} = \sin^2\theta_o \cos\theta_o \left[\frac{g^3}{(g^2+1)^{3/2}} - \frac{S^3}{(S^2+1)^{3/2}}\right]$$

$$\Phi_{0,3} = \frac{1}{3}\sin^3\theta_o \left[\frac{S^2}{(S^2+1)^{3/2}} - \frac{g^2}{(g^2+1)^{3/2}} + \frac{2}{(S^2+1)^{1/2}} - \frac{2}{(g^2+1)^{1/2}}\right]$$

Case 5: $j = 4$

$$\Phi_4 = 2\pi K_4 \left(\Phi_{4,0} + \Phi_{3,1} + \Phi_{2,2} + \Phi_{1,3} + \Phi_{0,4}\right)$$

where

$$\Phi_{4,0} = \frac{1}{4}\cos^4\theta_o \left\{\frac{g}{(g^2+1)^2} - \frac{S}{(S^2+1)^2} + \frac{3}{2}\left[\frac{g}{(g^2+1)} - \frac{S}{(S^2+1)} - \mathrm{ctn}^{-1}g + \mathrm{ctn}^{-1}S\right]\right\}$$

$$\Phi_{3,1} = \sin\theta_o \cos^3\theta_o \left[\frac{1}{(S^2+1)^2} - \frac{1}{(g^2+1)^2}\right]$$

$$\Phi_{2,2} = \frac{3}{2}\sin^2\theta_o \cos^2\theta_o \left\{ \frac{s}{(s^2+1)^2} - \frac{g}{(g^2+1)^2} + \frac{1}{2}\left[\frac{g}{(g^2+1)} - \frac{s}{(s^2+1)} - \mathrm{ctn}^{-1}g + \mathrm{ctn}^{-1}s \right] \right\}$$

$$\Phi_{1,3} = \sin^3\theta_o \cos\theta_o \left[\frac{g^4}{(g^2+1)^2} - \frac{s^4}{(s^2+1)^2} \right]$$

$$\Phi_{0,4} = \frac{1}{4}\sin^4\theta_o \left\{ \frac{s^3}{(s^2+1)^2} - \frac{g^3}{(g^2+1)^2} + \frac{3}{2}\left[\frac{s}{(s^2+1)} - \frac{g}{(g^2+1)} - \mathrm{ctn}^{-1}g + \mathrm{ctn}^{-1}s \right] \right\}$$

<u>Case 6:</u> $j = 5$

$$\Phi_5 = 2\pi K_5 \left(\Phi_{5,0} + \Phi_{4,1} + \Phi_{3,2} + \Phi_{2,3} + \Phi_{1,4} + \Phi_{0,5}\right)$$

where

$$\Phi_{5,0} = \frac{1}{5}\cos^5\theta_o$$

$$\times \left\{ \frac{g}{(g^2+1)^{5/2}} - \frac{s}{(s^2+1)^{5/2}} + \frac{4}{3}\left[\frac{g}{(g^2+1)^{3/2}} - \frac{s}{(s^2+1)^{3/2}} \right] + \frac{8}{3}\left[\frac{g}{(g^2+1)^{1/2}} - \frac{s}{(s^2+1)^{1/2}} \right] \right\}$$

$$\Phi_{4,1} = \sin\theta_o \cos^4\theta_o \left[\frac{1}{(s^2+1)^{5/2}} - \frac{1}{(g^2+1)^{5/2}} \right]$$

$$\Phi_{3,2} = 2\sin^2\theta_o \cos^3\theta_o \left\{ \frac{g^3}{(g^2+1)^{5/2}} - \frac{s^3}{(s^2+1)^{5/2}} + \frac{2}{3}\left[\frac{g^3}{(g^2+1)^{3/2}} - \frac{s^3}{(s^2+1)^{3/2}} \right] \right\}$$

$$\Phi_{2,3} = 2\sin^3\theta_o \cos^2\theta_o \left\{ \frac{s^2}{(s^2+1)^{5/2}} - \frac{g^2}{(g^2+1)^{5/2}} + \frac{2}{3}\left[\frac{1}{(s^2+1)^{3/2}} - \frac{1}{(g^2+1)^{3/2}} \right] \right\}$$

$$\Phi_{1,4} = \sin^4\theta_o \cos\theta_o \left[\frac{g^5}{(g^2+1)^{5/2}} - \frac{s^5}{(s^2+1)^{5/2}} \right]$$

$$\Phi_{0,5} = \frac{1}{5}\sin^5\theta_o$$

$$\times \left\{ \frac{s^4}{(s^2+1)^{5/2}} - \frac{g^4}{(g^2+1)^{5/2}} + \frac{4}{3}\left[\frac{s^2}{(s^2+1)^{3/2}} - \frac{g^2}{(g^2+1)^{3/2}}\right] + \frac{8}{3}\left[\frac{1}{(s^2+1)^{1/2}} - \frac{1}{(g^2+1)^{1/2}}\right]\right\}$$

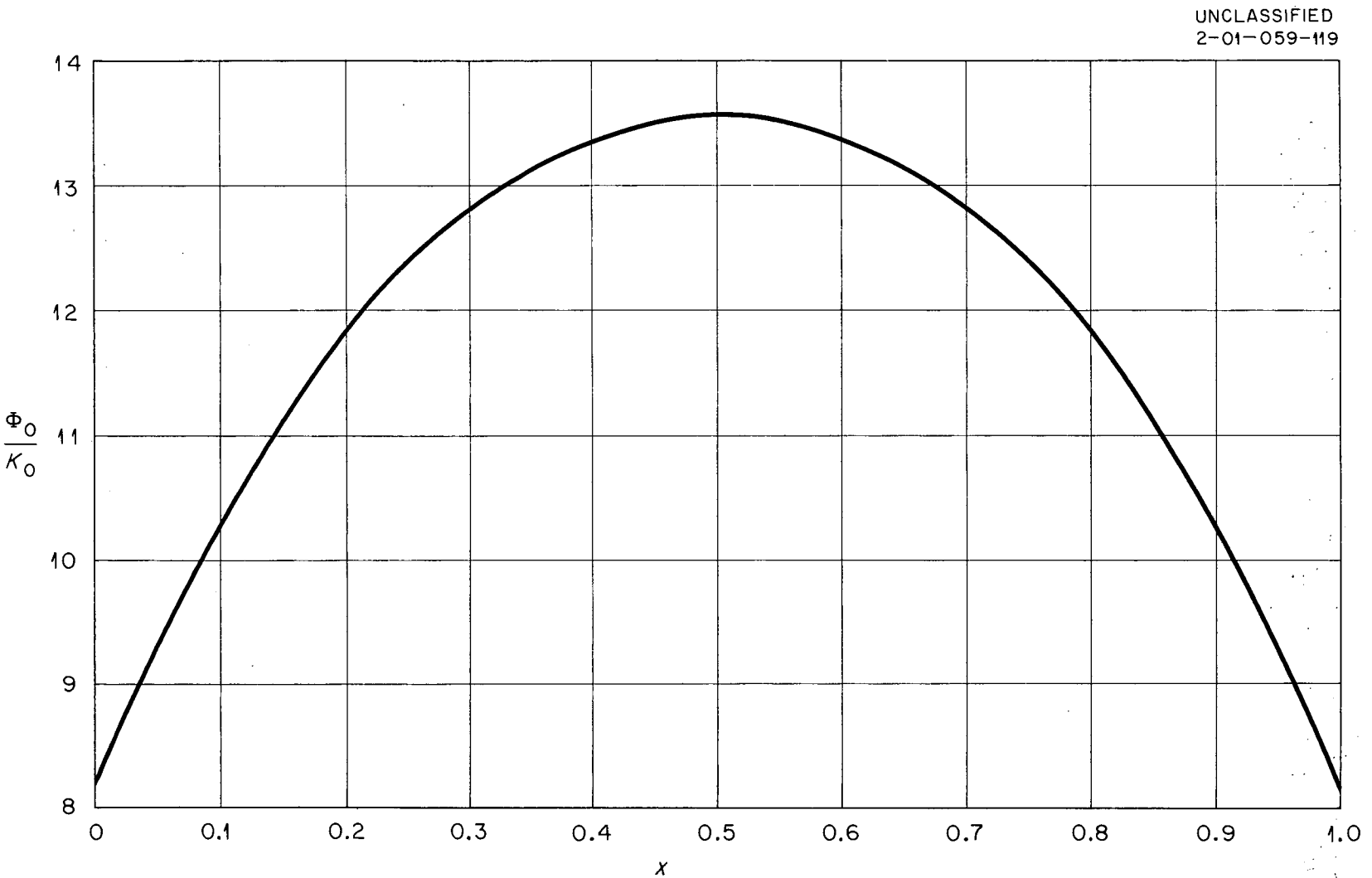

Fig. 2. Flux on the Centerline of a Cylinder of Finite Length ($j = 0$, $p = 3.701$)

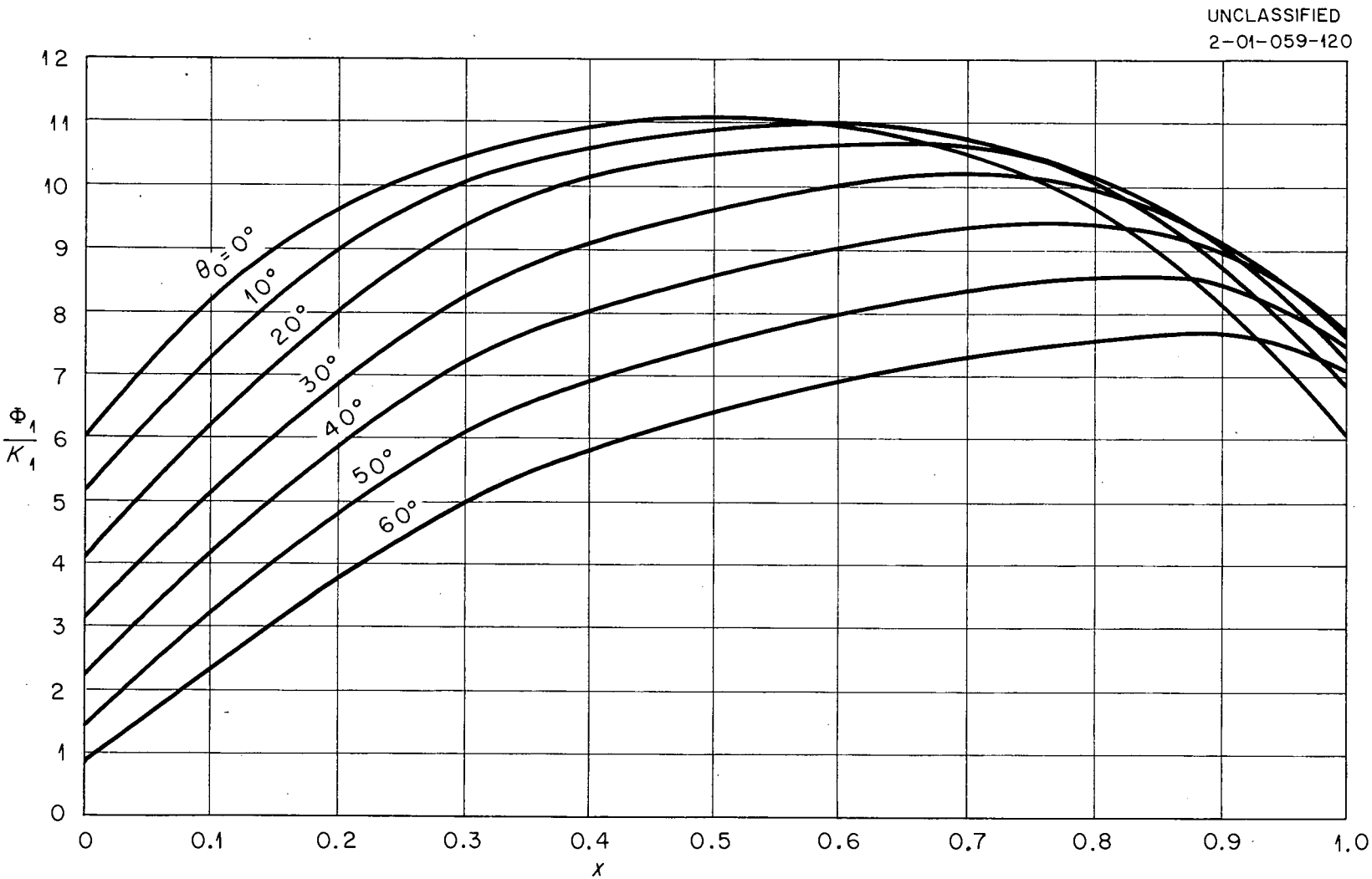

Fig. 3. Flux on the Centerline of a Cylinder of Finite Length ($j = 1$, $p = 3.701$)

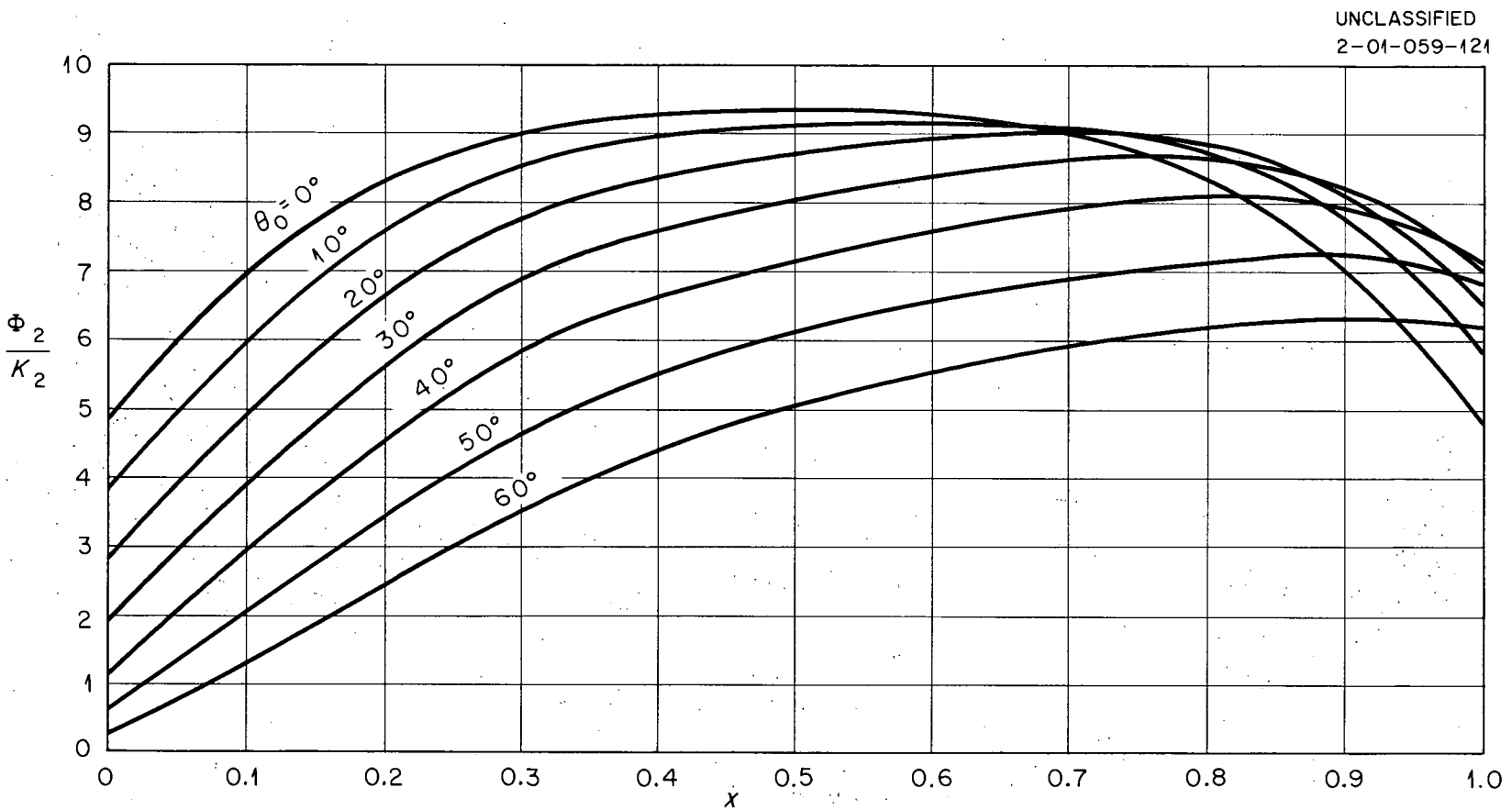

Fig. 4. Flux on the Centerline of a Cylinder of Finite Length ($j = 2$, $p = 3.701$)

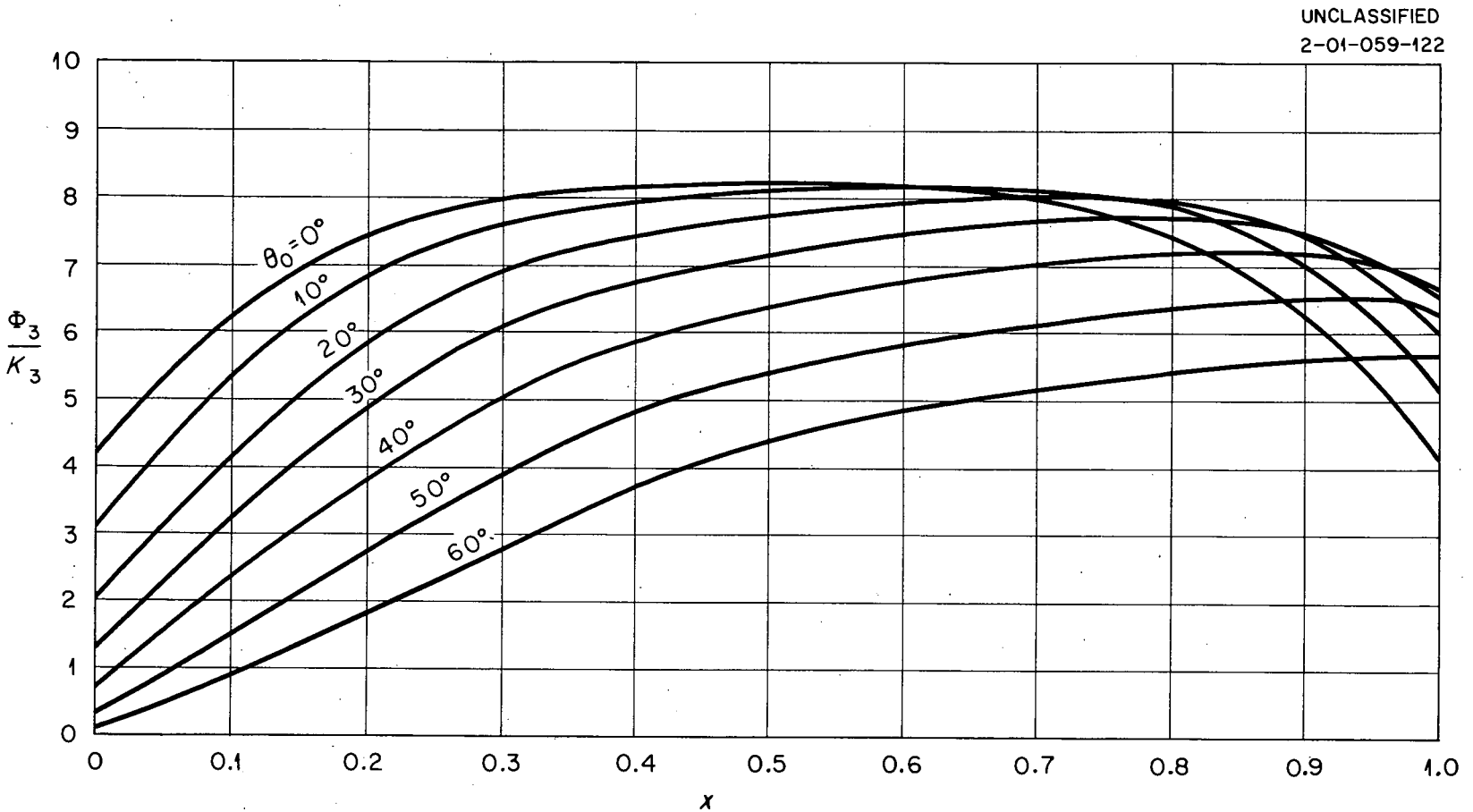

Fig. 5. Flux on the Centerline of a Cylinder of Finite Length ($j = 3$, $p = 3.701$)

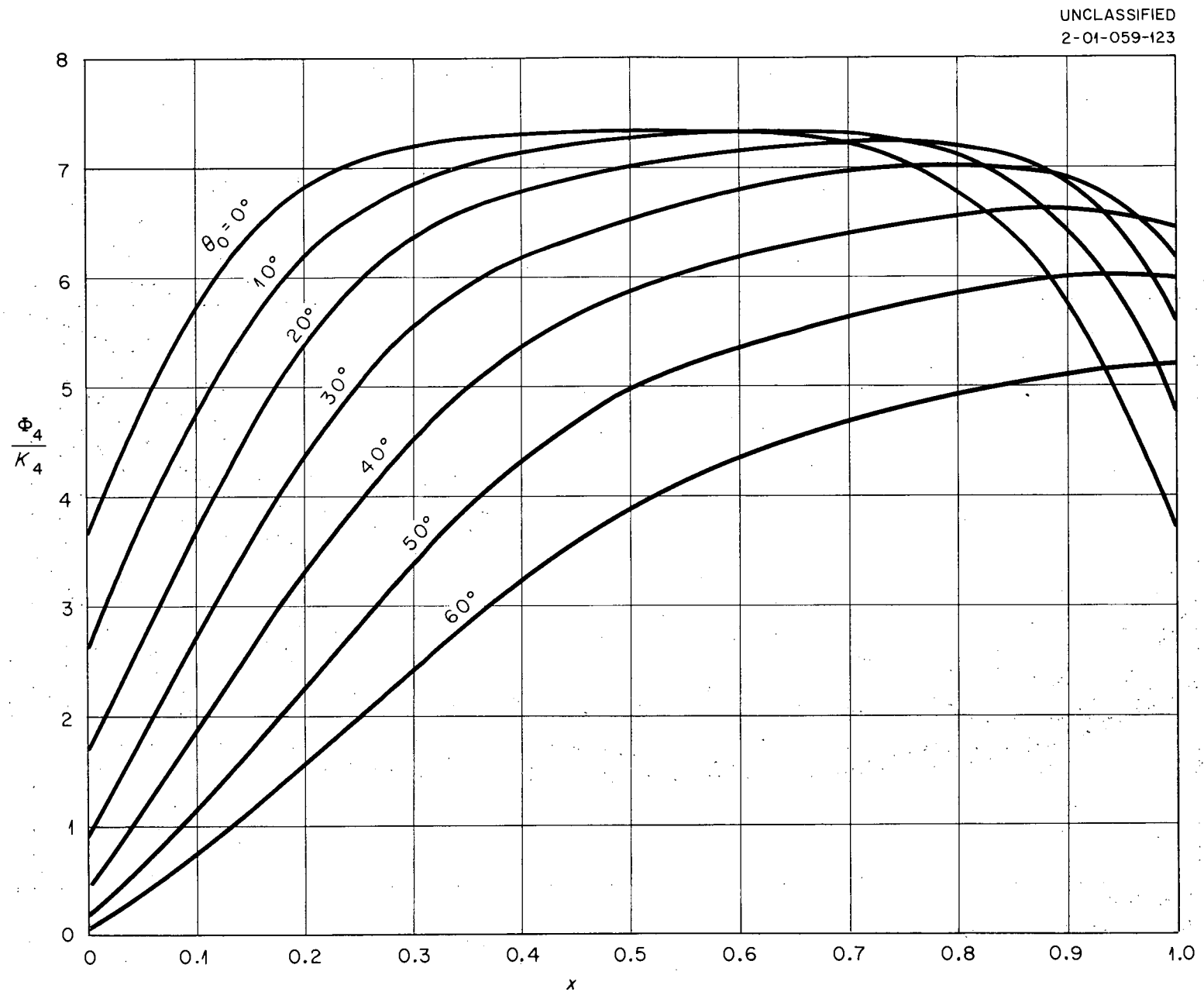

Fig. 6. Flux on the Centerline of a Cylinder of Finite Length ($j = 4$, $p = 3.701$)

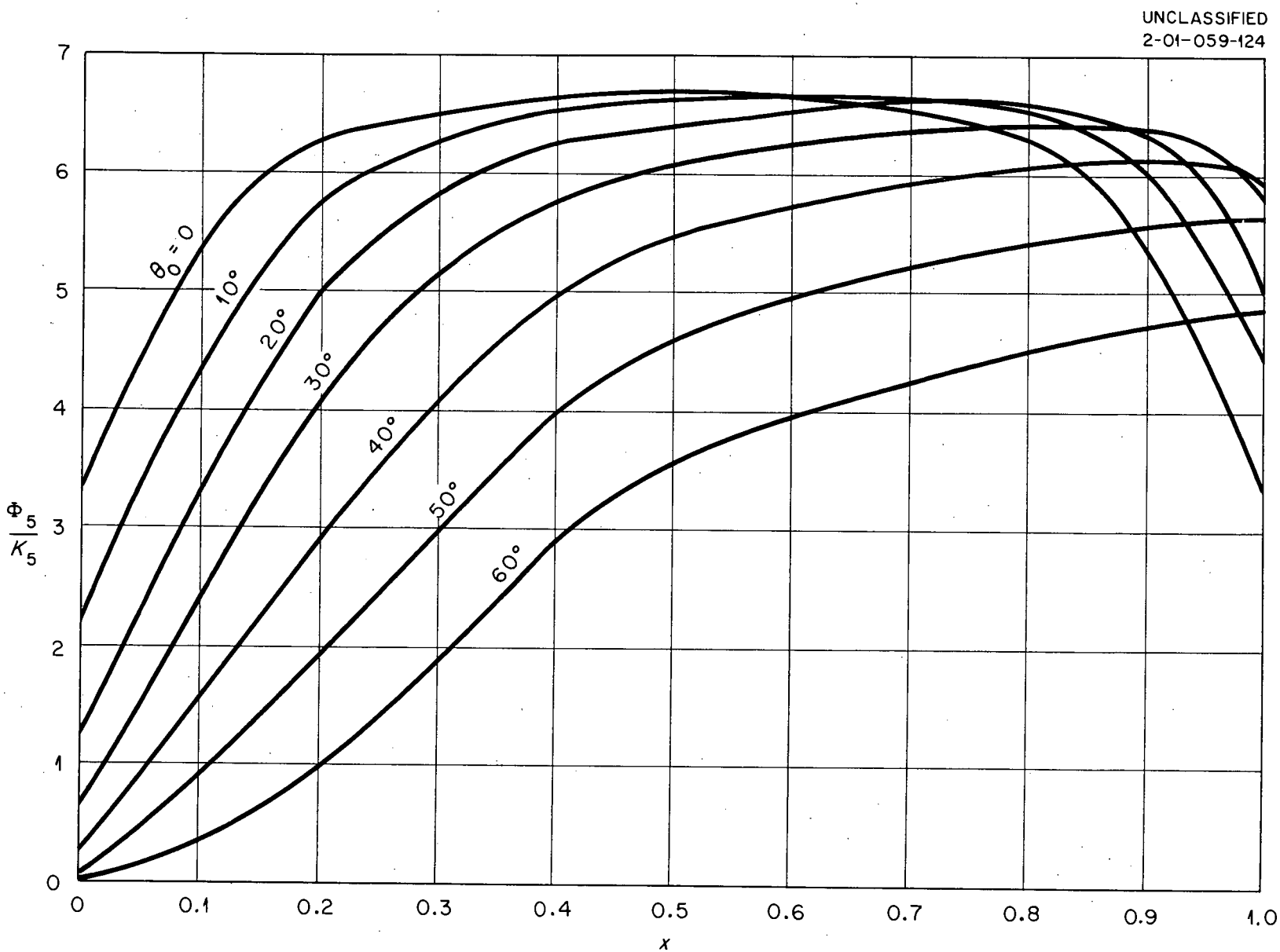

Fig. 7. Flux on the Centerline of a Cylinder of Finite Length ($j = 5$, $p = 3.701$)

Table 1. Flux ($\bar{\phi}_j/K_j$) on Centerline of a Cylinder of Finite Length

x \ θ_o	0°	10°	20°	30°	40°	50°	60°
			j = 0, p = 3.701				
0	8.21						
0.2	11.83						
0.4	13.35						
0.6	13.35						
0.8	11.83						
0.9	10.26						
0.95	9.28						
1.00	8.21						
			j = 1, p = 3.701				
0	6.07	5.17	4.13	3.14	2.24	1.47	0.84
0.2	9.69	9.02	8.07	6.99	5.90	4.82	3.78
0.4	10.94	10.61	10.17	9.03	8.01	6.94	5.84
0.6	10.94	10.93	10.60	9.94	9.01	7.99	6.91
0.8	9.69	10.07	10.15	9.91	9.38	8.56	7.52
0.9	8.20	8.78	9.10	9.14	8.91	8.40	7.64
0.95	7.19	7.85	8.28	8.46	8.37	8.04	7.46
1.00	6.07	6.78	7.29	7.58	7.63	7.46	7.05
			j = 2, p = 3.701				
0	4.90	3.85	2.83	1.93	1.20	0.65	0.29
0.2	8.37	7.64	6.70	5.65	4.55	3.48	2.49
0.4	9.31	8.99	8.40	7.60	6.64	5.59	4.49
0.6	9.31	9.15	8.98	8.38	7.58	6.62	5.56
0.8	8.37	8.80	8.89	8.62	8.02	7.17	6.16
0.9	7.02	7.76	8.19	8.24	7.93	7.27	6.36
0.95	6.03	6.90	7.50	7.76	7.64	7.16	6.37
1.00	4.90	5.85	6.59	7.04	7.13	6.85	6.25
			j = 3, p = 3.701				
0	4.18	3.10	2.01	1.31	0.71	0.32	0.11
0.2	7.47	6.81	5.87	4.90	3.81	2.75	1.82
0.4	8.16	7.92	7.46	6.76	5.87	4.84	3.75
0.6	8.16	8.15	7.91	7.44	6.75	5.85	4.82
0.8	7.47	7.86	7.93	7.69	7.16	6.37	5.41
0.9	6.27	7.02	7.45	7.51	7.19	6.52	5.61
0.95	5.31	6.24	6.89	7.17	7.04	6.51	5.67
1.00	4.18	5.21	6.06	6.58	6.68	6.36	5.66

Table 1. (cont)

x \ θ_o	0°	10°	20°	30°	40°	50°	60°
			j = 4, p = 3.701				
0	3.70	2.63	1.68	0.94	0.44	0.17	0.04
0.2	6.81	6.24	5.42	4.41	3.32	2.28	1.54
0.4	7.31	7.15	6.79	6.19	5.37	4.35	3.26
0.6	7.31	7.31	7.14	6.78	6.18	5.35	4.33
0.8	6.81	7.13	7.19	6.99	6.55	5.84	4.91
0.9	5.75	6.46	6.86	6.91	6.61	5.99	5.11
0.95	4.82	5.75	6.39	6.67	6.54	6.02	5.18
1.00	3.70	4.76	5.59	6.18	6.46	5.95	5.22
			j = 5, p = 3.701				
0	3.35	2.28	1.23	0.69	0.29	0.09	0.02
0.2	6.30	5.81	5.05	4.07	2.97	1.94	0.99
0.4	6.66	6.55	6.26	5.78	4.99	4.00	2.91
0.6	6.66	6.66	6.55	6.27	5.76	4.98	3.98
0.8	6.30	6.55	6.59	6.44	6.08	5.44	4.55
0.9	5.36	6.01	6.36	6.40	6.15	5.58	4.74
0.95	4.47	5.37	5.99	6.25	6.13	5.63	4.82
1.00	3.35	4.42	5.00	5.84	5.95	5.64	4.88

Appendix B

Formulas, Graphs, and Tables for the Flux on the Radius at the End of a Cylinder of Semi-infinite Length

The equations that result from Eq. 20 for values of j from zero through five are given below.

Case 1: $j = 0$

$$\Phi_o = 2\pi K_o K(k)$$

Case 2: $j = 1$

$$\Phi_1 = 4k_1 \left[\frac{\pi}{2} \cos\theta_o + K(k) \sin\theta_o \right]$$

Case 3: $j = 2$

$$\Phi_2 = \pi K_2 \left[E(k) \cos^2\theta_o + 2\sin\theta_o \cos\theta_o + K(k) \sin^2\theta_o \right]$$

Case 4: $j = 3$

$$\Phi_3 = 4K_3 \left[\frac{\pi}{6} (2-k^2) \cos^3\theta_o + E(k) \sin\theta_o \cos^2\theta_o + \frac{\pi}{2} \sin^2\theta_o \cos\theta_o + \frac{2}{3} K(k) \sin^3\theta_o \right]$$

Case 5: $j = 4$

$$\Phi_4 = \pi K_4 \left\{ \frac{1}{4} \left[2(2-k^2) E(k) - (1-k^2) K(k) \right] \cos^4\theta_o + (2-k^2) \sin\theta_o \cos^3\theta_o \right.$$

$$\left. + \frac{3}{2} E(k) \sin^2\theta_o \cos^2\theta_o + 2 \sin^3\theta_o \cos\theta_o + \frac{3}{4} K(k) \sin^4\theta_o \right\}$$

<u>Case 6</u>: $j = 5$

$$\Phi_5 = 4K_5 \left\{ \frac{2\pi}{15} \left[\frac{3(2-k^2)^2}{4} - (1-k^2) \right] \cos^5\theta_o + \frac{1}{3} \left[2(2-k^2)E(k) - (1-k^2)K(k) \right] \sin\theta_o \cos^4\theta_o \right.$$

$$+ \frac{\pi}{3} (2-k^2) \sin^2\theta_o \cos^3\theta_o + \frac{4}{3} E(k) \sin^3\theta_o \cos^2\theta_o + \frac{\pi}{2} \sin^4\theta \cos\theta_o$$

$$\left. + \frac{8}{15} K(k) \sin^5\theta_o \right\}$$

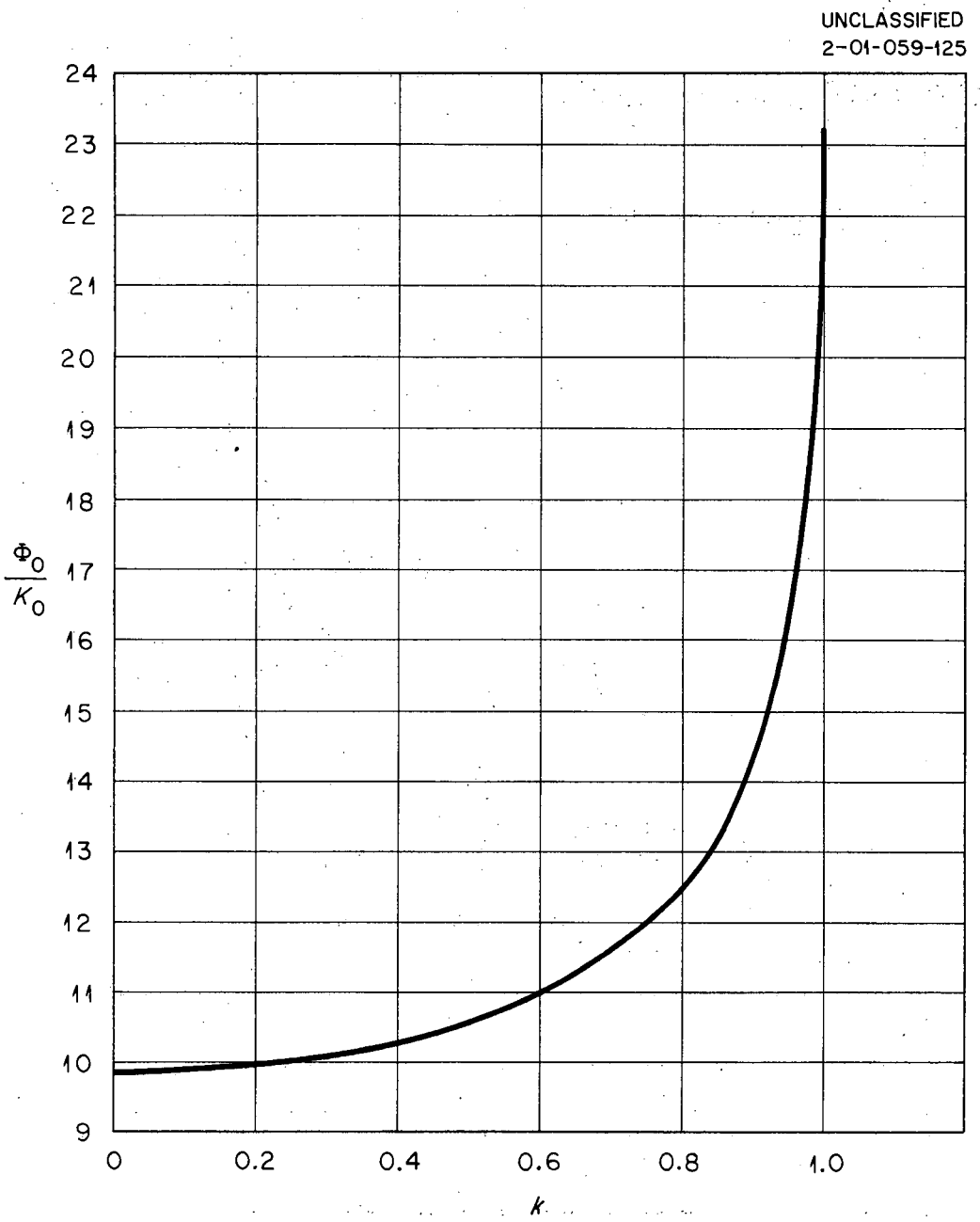

Fig. 8. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 0$)

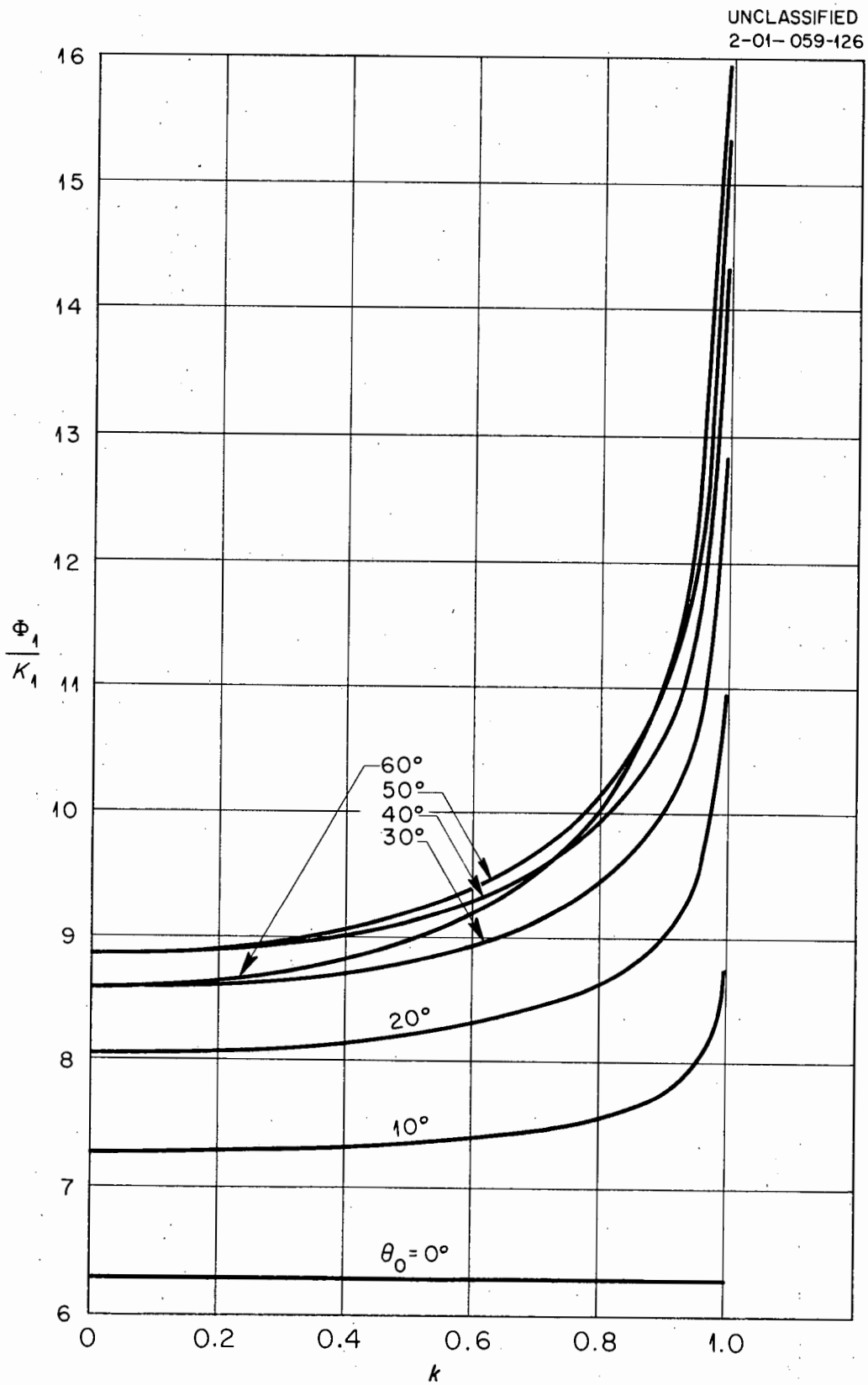

Fig. 9. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 1$)

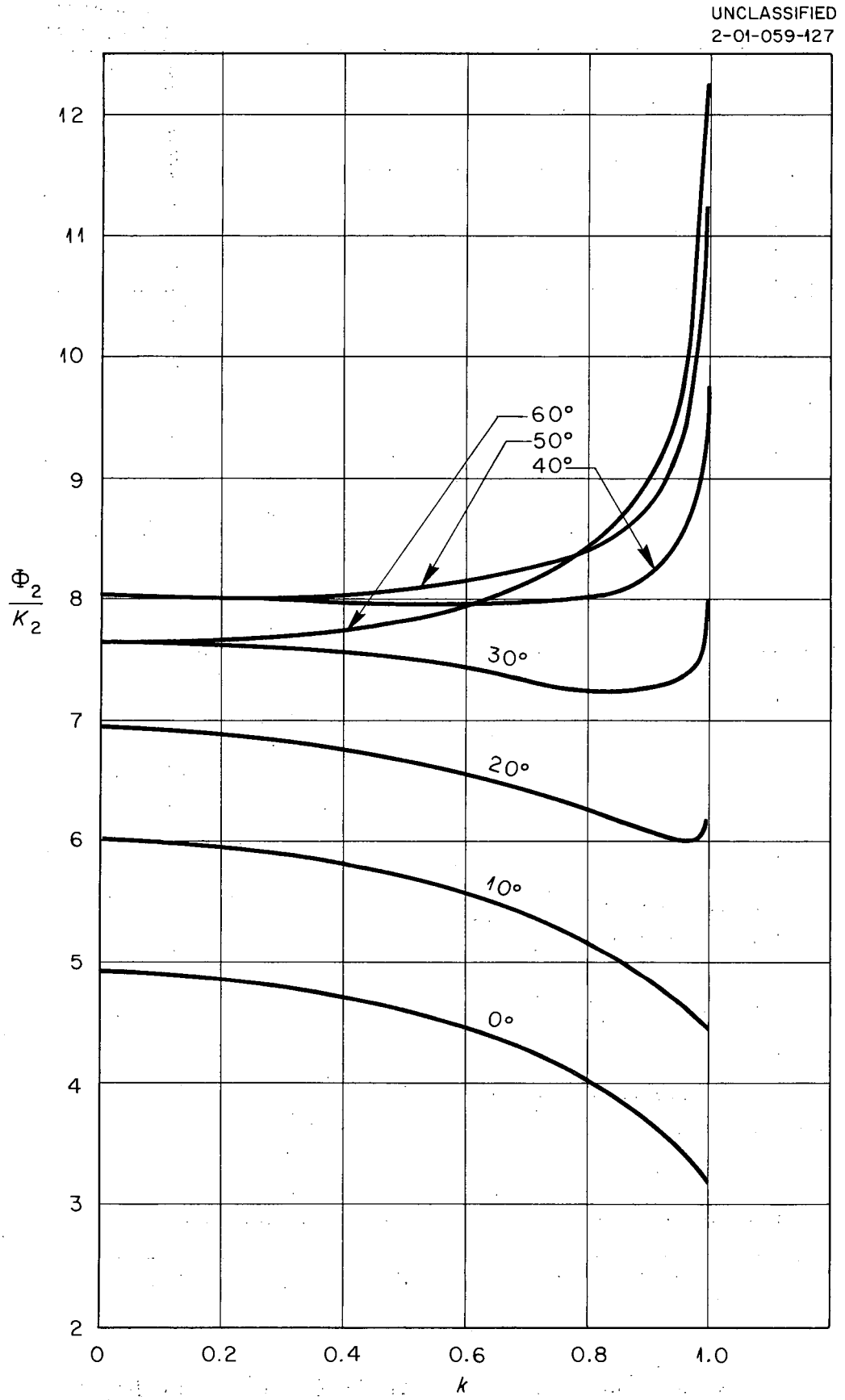

Fig. 10. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 2$)

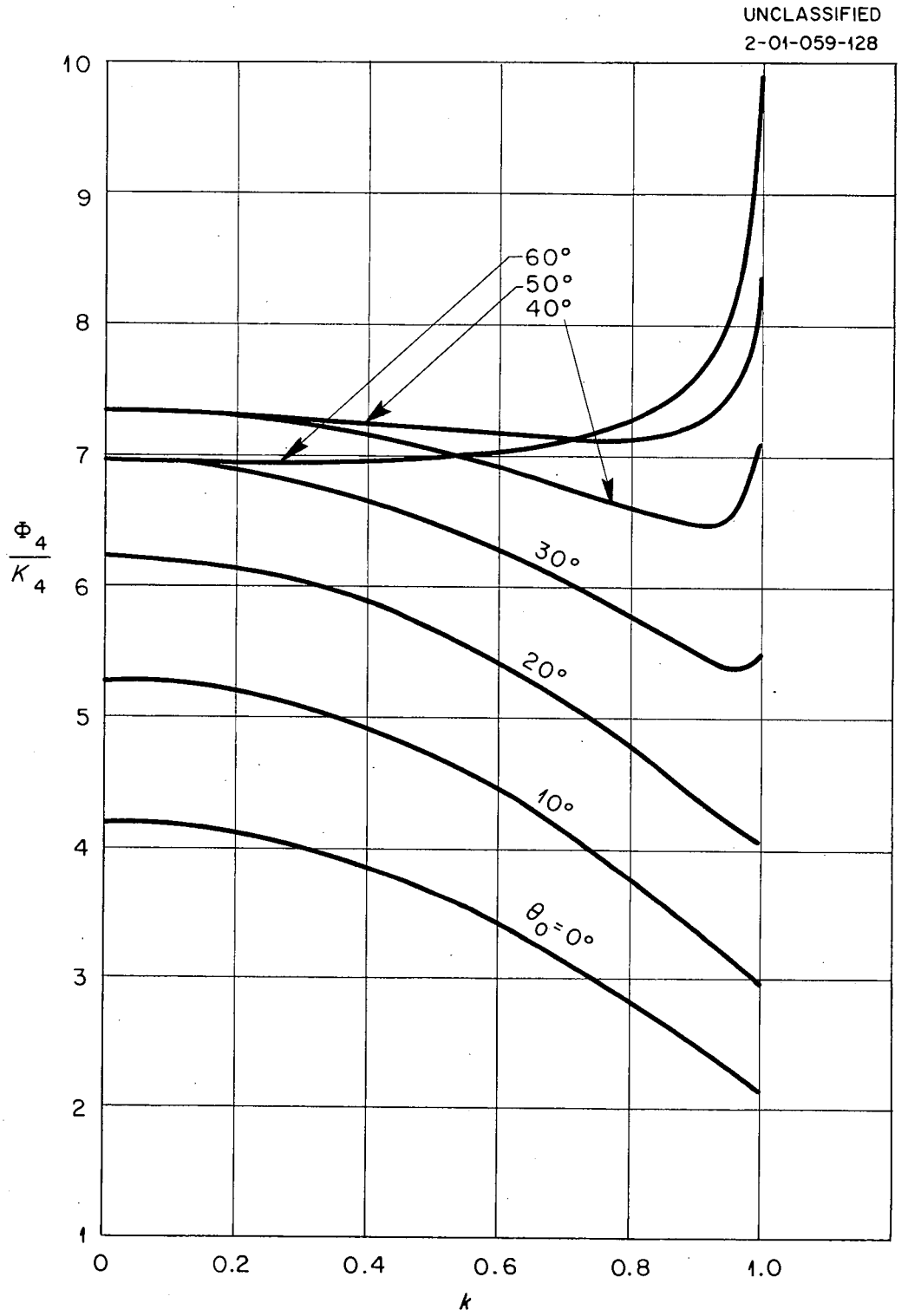

Fig. 11. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 3$)

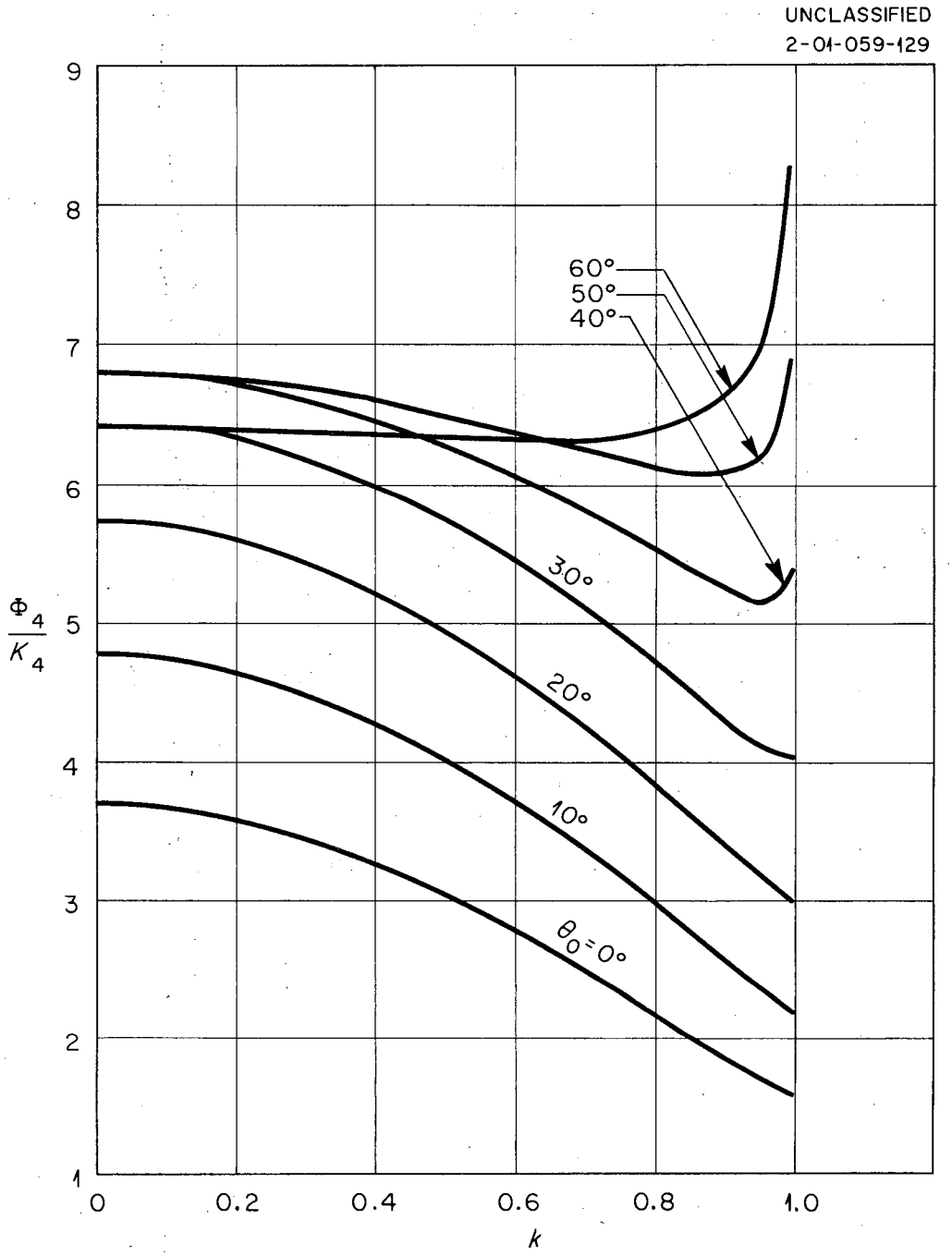

Fig. 12. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 4$)

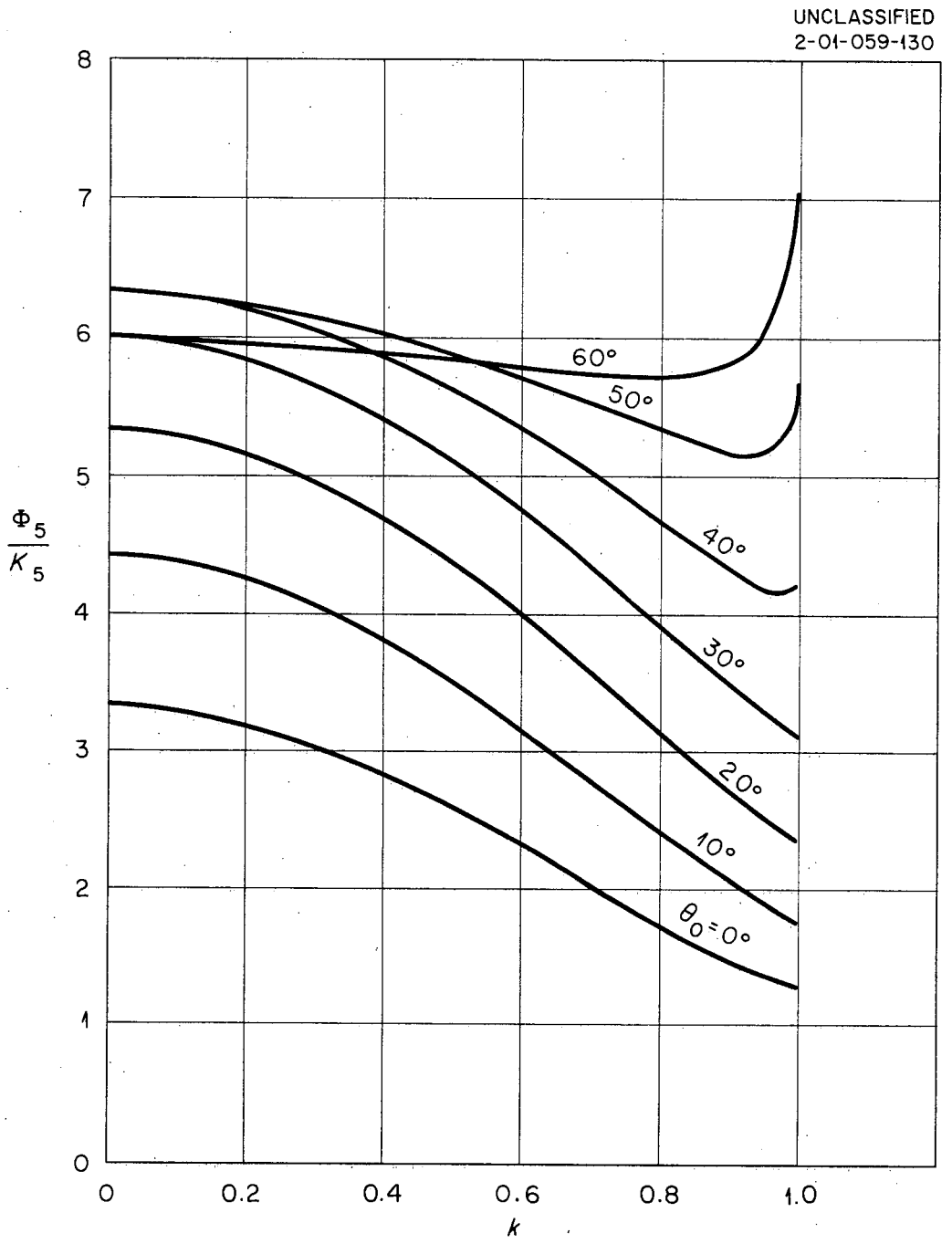

Fig. 13. Flux on the Radius at the End of a Cylinder of Semi-infinite Length ($j = 5$)

Table 2. Flux (Φ_j/K_j) on the Radius at the End of a Cylinder of Semi-infinite Length

k \ θ_o	0°	10°	20°	30°	40°	50°	60°
				j = 0			
0	9.87						
0.1	9.89						
0.2	9.97						
0.3	10.10						
0.4	10.30						
0.5	10.59						
0.6	11.00						
0.7	11.60						
0.8	12.54						
0.9	14.33						
0.95	16.51						
0.995	23.22						
				j = 1			
0	6.28	7.28	8.05	8.58	8.85	8.85	8.58
0.1	6.28	7.28	8.06	8.59	8.86	8.86	8.60
0.2	6.28	7.29	8.07	8.61	8.89	8.90	8.64
0.3	6.28	7.30	8.10	8.66	8.95	8.97	8.71
0.4	5.28	7.33	8.15	8.72	9.03	9.06	8.82
0.5	6.28	7.36	8.21	8.81	9.15	9.20	8.98
0.6	6.28	7.40	8.30	8.94	9.31	9.40	9.21
0.7	6.28	7.47	8.43	9.13	9.56	9.69	9.53
0.8	6.28	7.57	8.63	9.43	9.94	10.15	10.05
0.9	6.28	7.77	9.02	10.00	10.68	11.03	11.04
0.95	6.28	8.01	9.50	10.70	11.57	12.09	12.24
0.995	6.28	8.75	10.96	12.83	14.31	15.36	15.94
				j = 2			
0	4.93	6.01	6.95	7.66	8.03	8.03	7.66
0.1	4.92	6.00	6.94	7.65	8.03	8.03	7.66
0.2	4.89	5.96	6.92	7.63	8.02	8.04	7.68
0.3	4.82	5.90	6.87	7.60	8.01	8.05	7.71
0.4	4.73	5.82	6.80	7.56	8.00	8.07	7.77
0.5	4.61	5.71	6.71	7.50	7.99	8.11	7.85
0.6	4.46	5.56	6.60	7.44	7.98	8.16	7.96
0.7	4.26	5.38	6.46	7.36	7.99	8.26	8.13
0.8	4.01	5.15	6.29	7.29	8.04	8.43	8.42
0.9	3.68	4.86	6.11	7.27	8.21	8.82	9.01
0.95	3.44	4.66	6.03	7.37	8.53	9.36	9.77
0.995	3.19	4.52	6.19	8.02	9.76	11.23	12.23

Table 2. Cont.

k \ θ_o	0°	10°	20°	30°	40°	50°	60°
				j = 3			
0	4.19	5.27	6.23	6.96	7.35	7.35	6.96
0.1	4.17	5.24	6.21	6.94	7.34	7.35	6.96
0.2	4.10	5.18	6.14	6.89	7.30	7.33	6.96
0.3	4.00	5.06	6.04	6.80	7.24	7.30	6.97
0.4	3.85	4.90	5.88	6.67	7.15	7.27	6.98
0.5	3.67	4.70	5.68	6.50	7.04	7.22	7.00
0.6	3.43	4.45	5.44	6.30	6.91	7.18	7.05
0.7	3.16	4.15	5.15	6.06	6.76	7.14	7.12
0.8	2.85	3.79	4.81	5.79	6.61	7.13	7.27
0.9	2.49	3.39	4.42	5.50	6.49	7.25	7.63
0.95	2.28	3.14	4.19	5.36	6.53	7.51	8.14
0.995	2.12	2.94	4.07	5.49	7.09	8.65	9.90
				j = 4			
0	3.70	4.78	5.72	6.42	6.80	6.80	6.42
0.1	3.67	4.74	5.69	6.39	6.77	6.78	6.42
0.2	3.59	4.64	5.59	6.31	6.71	6.75	6.41
0.3	3.46	4.49	5.43	6.17	6.60	6.69	6.39
0.4	3.27	4.28	5.21	5.97	6.46	6.60	6.37
0.5	3.04	4.01	4.94	5.72	6.27	6.50	6.36
0.6	2.77	3.70	4.61	5.42	6.05	6.38	6.34
0.7	2.48	3.34	4.23	5.08	5.79	6.25	6.35
0.8	2.16	2.96	3.82	4.71	5.52	6.13	6.41
0.9	1.85	2.56	3.38	4.31	5.25	6.08	6.62
0.95	1.69	2.35	3.15	4.10	5.16	6.19	6.97
0.995	1.58	2.20	2.98	4.04	5.39	6.89	8.28
				j = 5			
0	3.35	4.42	5.34	5.99	6.33	6.33	5.99
0.1	3.32	4.38	5.30	5.95	6.30	6.31	5.98
0.2	3.22	4.26	5.17	5.84	6.21	6.25	5.96
0.3	3.06	4.07	4.97	5.65	6.07	6.17	5.93
0.4	2.85	3.81	4.70	5.40	5.87	6.05	5.89
0.5	2.59	3.50	4.36	5.09	5.62	5.90	5.84
0.6	2.31	3.15	3.98	4.73	5.34	5.72	5.79
0.7	2.01	2.77	3.56	4.33	5.02	5.53	5.74
0.8	1.72	2.40	3.13	3.91	4.67	5.33	5.73
0.9	1.46	2.05	2.72	3.48	4.34	5.18	5.84
0.95	1.34	1.88	2.51	3.27	4.19	5.19	6.07
0.995	1.27	1.77	2.36	3.15	4.24	5.63	7.07

Appendix C

Development of the Recursion Relationship for the Function $I_n(k)$

In Eq. 21 the function $I_n(k)$ was defined as

$$I_n(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\alpha)^n \, d\alpha. \tag{C.1}$$

In what follows n will be interpreted as a number having any value in the interval $(-\infty, +\infty)$ and not necessarily integral or half-odd integral values, however, the general use of the function will be for the latter case.

A recursion relationship for the function can be obtained by establishing two simple relationships. First make the transformation $\cos\alpha = \frac{\sqrt{1 - k^2}}{k} \tan\delta$ in Eq. C.1, then make the transformation $\sin\delta = k \sin\beta$ to obtain

$$\int_0^{\pi/2} (1-k^2 \sin^2\alpha)^n \, d\alpha = (1-k^2)^{n+1/2} \int_0^{\pi/2} (1-k^2 \sin^2\beta)^{-(n+1)} \, d\beta, \tag{c.2}$$

or in terms of the function $I_n(k)$

$$I_n(k) = (1-k^2)^{n+1/2} \, I_{-(n+1)}(k) \tag{C.3}$$

Equation C.3 is one of the required relations, the other is obtained by noting that

$$\int_0^{\pi/2} (1-k^2 \sin^2\alpha)^n \, d\alpha = \int_0^{\pi/2} (1-k^2 \sin^2\alpha)^{n-1} \, d\alpha - \int_0^{\pi/2} k^2 \sin^2\alpha \, (1-k^2 \sin^2\alpha)^{n-1} \, d\alpha, \quad (C.4)$$

and

$$\frac{\partial I_n(k)}{\partial k} = - \frac{2n}{k} \int_0^{\pi/2} k^2 \sin^2\alpha \, (1-k^2 \sin^2\alpha)^{n-1} \, d\alpha, \quad (C.5)$$

so that one finally obtains from Eq. C.1

$$I_n(k) = I_{n-1}(k) + \frac{k}{2n} \frac{\partial I_n(k)}{\partial k} , \quad (C.6)$$

Since both Eqs. C.3 and C.6 apply for n in the interval $(-\infty,\infty)$, Eq. C.6 can be rewritten as

$$I_{-(n+1)}(k) = I_{-n}(k) + \frac{k}{2n} \frac{\partial I_{-n}(k)}{\partial k} \quad (C.7)$$

and Eq. C.3 can be substituted into Eq. C.7 to give

$$\frac{I_n}{(1-k^2)^{n+1/2}} = \frac{I_{n-1}}{(1-k^2)^{n-1/2}} + \frac{k}{2n} \frac{\partial}{\partial k} \left[(1-k^2)^{-n+1/2} I_{n-1}(k)\right] \quad (C.8)$$

or after some reduction

$$I_n(k) = \frac{2n-k^2}{2n} I_{n-1}(k) + \frac{k(1-k^2)}{n} \frac{\partial I_{n-1}(k)}{\partial k} . \quad (C.9)$$

Finally, use Eq. C.6 to substitute for $\frac{\partial I_{n-1}(k)}{\partial k}$ and arrive at

$$I_n(k) = \frac{(2n-1)(2-k^2)}{n} I_{n-1}(k) - \frac{(1-k^2)(n-1)}{n} I_{n-2}(k), \; n \neq 0 \qquad (C.10)$$

which is the desired recursion relation.