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STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY

PART II

Walter Kofink

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Erratum

STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY
PART II

Walter Kofink

Page 3. The second line of equation (1) should read:

$$= \sum_{k=\overline{0},0,1,\ldots}^{(L-1)/2} \alpha_k e^{\zeta/\lambda_k} \left\{ (\mu+\lambda_k) \left[-a\lambda_k + \frac{c+a\lambda_k^2}{\lambda_k+\mu} \left(1 - \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right) \right] - c + a\mu\lambda_k \right\} \qquad (1)$$

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STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY: II. Behavior of the solution of the Milne problem with anisotropic scattering for $L \rightarrow \infty$.

Walter Kofink

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Studies of the Spherical Harmonics Method in Neutron Transport Theory:

II. Behavior of the solution of the Milne problem with anisotropic scattering for $L \rightarrow \infty$.

Walter Kofink*

In Part I of this series it was shown that the partial solutions of the Chandrasekhar-Wick method which contain the factor $g(\mu)$ cf. Eq.[1](I-27), dependent on the direction $\mu = \cos\vartheta$, are solutions of the <u>non-truncated Boltzmann integro-differential equation</u> (I-1). An error occurs only because the Gauss quadrature method is used to evaluate approximately the integrals on the right side of (I-1). The integrands are not polynomials but are meromorphic functions[2] of μ; hence, the integral is evaluated only approximately. This error is manifested in L-approximation as an error in the characteristic equation, which is an algebraic equation instead of a transcendental. But the sequence of algebraic characteristic equations converges for $L \rightarrow \infty$ to the exact transcendental characteristic equation. Therefore one may expect that the solutions of the C-W method tend to the correct Wiener-Hopf solution for $L \rightarrow \infty$.

The solutions (I-16) of the spherical harmonics method, however, can be used for a correct integration of the integrals on the right side of Eq. (I-1). They are continuous functions in the whole interval $-1 \leq \mu \leq 1$ and their μ-dependence is of such a kind to satisfy a truncated system of differential equations (compare

* Fulbright and Deutsche Forschungsgemeinschaft grantee assigned to Oak Ridge National Laboratory. Permanent address: Institute of Technology, Karlsruhe, Germany.

1. The numeral I in the number of an equation refers to the corresponding equation in the author's first report, ORNL-2334.

2. A meromorphic function is a function which is analytic except for poles.

section 1 in I) instead of equation (I-1). The approximate characteristic equation of the S-H method coincides with that of the C-W method, but the truncation of the infinite system of differential equations (I-3) introduces a further, and more serious, error. It is shown in the following section 1 that the spherical harmonics method introduces an "error-source" in the Boltzmann equation. It turns out that this error-source is important especially at those places where the contribution of the transients to the solution is large. In the Milne problem this region is near the boundary of the medium. The error-source does not show a tendency to vanish in the limit $L \to \infty$ in this case, indicating a lack of convergence of the spherical harmonics method. In a denumerably infinite set of directions, however, the spherical harmonics solution coincides with the C-W solution in the limit $L \to \infty$ and the error-source is zero there also. Hence it is possible to devise a scheme for passing to the limit by using only the C-W part of the S-H solution. In the sections 3-6 the limiting process will be carried out for the case $a/c = 3\gamma_a(1-\gamma_t)/(1-\gamma_a) \geq 0$, but in a way, which cannot claim complete mathematical rigour.

Although the convergence of the angular distribution $f(\xi,\mu)$ given by the spherical harmonics method presents serious difficulties, it will be seen that the convergence of the total flux $f_{00}(\xi)$ shows no such difficulties. Thus for applications to reactors -- for example, in the calculation of thermal utilization -- where the spherical harmonics method is used really to compute $f_{00}(\xi)$, not $f(\xi,\mu)$, subtle questions of convergence do not arise: the spherical harmonics method, used in the usual straightforward way, does converge to the proper answer in the limit $L \to \infty$.

1. <u>The error-source in the Boltzmann equation caused by spherical harmonics solutions.</u>

If one puts the solution $f(\xi,\mu)$ (I-16) of the spherical harmonics method in P_L-approximation in the Boltzmann integro-differential equation (I-1), one obtains a remainder

$$R(\xi,\mu) = \mu \frac{\partial f(\xi,\mu)}{\partial \xi} + f(\xi,\mu) - \frac{c}{2}\int_{-1}^{+1} f(\xi,\mu')\,d\mu' - \frac{1}{2}\frac{a}{1-c}\mu\int_{-1}^{+1}\mu' f(\xi,\mu')\,d\mu'$$

$$= \sum_{k=\bar{0},0,1,\ldots}^{(L-1)/2} \alpha_k e^{\zeta/\lambda_k}\left\{(\mu+\lambda_k)\left[-a\lambda_k + \frac{c+a\lambda_k^2}{\lambda_k+\mu}\left(1-\frac{p_{L+1}(\mu)}{p_{L+1}(-\lambda_k)}\right)\right] - c + a\mu\lambda_k\right\} \qquad (1)$$

using the integrals (I-51,54). In (1) all terms which do not contain spherical harmonics cancel one another. Hence, the remainder is

$$R(\xi,\mu) = -\sum_K \alpha_K e^{\xi/\lambda_K}(c+a\lambda_K^2)\frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_K)} \qquad (2)$$

$$= -\frac{P_{L+1}(\mu)}{P_{L+1}(0)}\sum_K \alpha_K e^{\xi/\lambda_K}(c+a\lambda_K^2)\frac{P_{L+1}(0)}{P_{L+1}(-\lambda_K)}$$

$$= \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f^{II}(\xi,0). \qquad (3)$$

The last equation was found by comparison of the $\sum_K$ with equation (I-18) for $\mu = 0$. Therefore, the remainder is closely related to $f^{II}(\xi,0)$ i.e., to the difference between the spherical harmonics and Gauss quadrature method solutions for $\mu = +0$. Instead of fulfilling the Boltzmann equation (I-1) exactly,

$f(z,\mu) = f^{I}(z,\mu) + f^{II}(z,\mu)$ satisfies the equation

$$i\frac{\partial f(z,\mu)}{\partial z} + f(z,\mu) = \frac{1}{2}(1-\gamma_a)\int_{-1}^{+1} f(z,\mu')\,d\mu' + \frac{3}{2}(1-\gamma_t)\mu\int_{-1}^{+1} f(z,\mu')\mu'\,d\mu' + \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f^{II}(z,0) \tag{4}$$

with a "2^{L+1}-pole-source," the strength of which depends on the part $f^{II}(z,0)$ of the directed flux perpendicular to the z-axis in every z-plane.

This result is valid also for every partial solution contained in equation (I-16) with a corresponding partial remainder contained in equation (4). Therefore it is not a special feature of the Milne problem, but a common feature of all spherical harmonics solutions. The remainders of the different partial solutions, however, may accumulate or distroy one another. One recognizes that the true Boltzmann equation can be satisfied for arbitrary μ $(-1 \leq \mu \leq 1)$ only for problems, in which the boundary conditions are of such a kind that $f^{II}(z,0)$ tends to zero in the limit $L \to \infty$ as $\frac{1}{L^{\alpha}}$ with $\alpha > \frac{1}{2}$.

This will presumably not happen in the Milne problem, because in this case $f^{II}(z,0)$ has just the task of joining smoothly the directed fluxes of medium and vacuum side at the point of discontinuity $z=0$; hence one cannot expect that $\lim_{L\to\infty} f^{II}(z,0) = 0$. On the boundary between medium and vacuum especially $\lim_{L\to\infty} f^{II}(0,0) = -\frac{1}{2}\lim_{L\to\infty} f^{I}(0,0)$, if the spherical harmonics series tends to 1/2 of the discontinuity of the exact Wiener-Hopf solution. Normalized to unit density at the boundary the exact values are $f(0,\mu=+0)=\frac{1}{2}$, $f(0,\mu=-0)=0$ with a discontinuity of 1/2 in the case of isotropic scattering without absorption. The spherical harmonics method yields with the same normalization in

L = 3 approximation $f(0,0) = 0.3737$, in L = 5 approximation $f(0,0) = 0.3382$, in L = 15 approximation** $f(0,0) = 0.2898$. This behavior does not contradict to the expectation above, according to which one expects $\lim_{L\to\infty} f(0,0) = \frac{1}{4}$ or $\lim_{L\to\infty} f^{II}(0,0) = -\frac{1}{4}$.

For the special directions $\mu = \mu_j \left(j = 1, 2, \cdots, \frac{L+1}{2}\right)$ the error in (4) vanishes, because $P_{L+1}(\mu_j) = 0$. From equations (I-16,17,18) we know that in these directions $f^{II}(z,\mu_j) = 0$ and $f(z,\mu_j) = f^{I}(z,\mu_j)$ which is identical with the Gauss quadrature solution for $\mu = +\mu_j$ and is zero for $\mu = -\mu_j$ (this last was our boundary condition for the Milne problem in the spherical harmonics method). Therefore we may fulfill the boundary condition and the true integro-differential equation in the limit $L\to\infty$ at a denumerable infinite set of directions $\mu = \pm\mu_j$. In these directions the spherical harmonics- and Gauss quadrature-method solutions coincide, in all other intermediate directions they differ and the spherical harmonics method leads in general to a non-vanishing remainder in (4). Therefore we have to select in the limit $L\to\infty$ only the denumerably infinite set of directions $\mu = \mu_j$, in which both methods coincide, if we wish to proceed to the exact limiting function of the Wiener-Hopf method.

The following consideration shows the behavior of the remainder in equation (4) as function of μ at a <u>fixed z</u>. If $f^{II}(z,0)$ is finite and different from zero, we obtain the quotient of the remainders at the end-points of the μ-interval

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** I owe the calculation of the P_{15}-approximation to Mr. H. S. Moran, Oak Ridge National Laboratory.

$|\mu| = 1$ and the center $\mu = 0$, (L + 1 even),

$$\frac{R(\xi,1)}{R(\xi,0)} = \frac{P_{L+1}(1)}{P_{L+1}(0)} = (-1)^{\frac{L+1}{2}} \frac{2\cdot 4\cdot 6\cdot 8\cdot \ldots (L+1)}{1\cdot 3\cdot 5\cdot 7\cdot \ldots L}$$

This quotient is 2.67 for L = 3, - 3.2 for L = 5, 5.09 for L = 15. For large L it diverges as $\sqrt{L}$

$$\frac{R(\xi,1)}{R(\xi,0)} \sim (-1)^{\frac{L+1}{2}} \sqrt{\frac{\pi}{2}}\sqrt{L+1} \,. \tag{5}$$

Hence the remainder in (4) increases strongly from the center to the end-points of the μ-interval.

The divergence proportional $\sqrt{L}$, however, occurs only in the narrowest neighborhood of the point $\mu = 1$. One recognizes this fact if one considers the quotient of the remainder at the extrema μ^E of $P_{L+1}(\mu)$ and the remainder at the center $\mu = 0$. The abscissae μ^E of the extrema, for which $P'_{L+1}(\mu^E) = 0$ are approximatively $\mu_m^E \sim \cos \frac{m + 1/4}{L + \frac{3}{2}} \pi$ for large L, where the numeration $m = 1, 2, \cdots \frac{L+1}{2}$ begins with $m = 1$ for the largest abscissa between the zeros μ_1 and μ_2 of $P_{L+1}(\mu_j) = 0$ $[\mu_2 < \mu_1^E < \mu_1]$ and ends with $\mu^E_{\frac{L+1}{2}} = \cos\frac{\pi}{2}$ for $m = \frac{L+1}{2}$. The extrema are $P_{L+1}(\mu_m^E) \sim (-1)^m \sqrt{\frac{2}{\pi(L+1)}}\,[1 - (\mu_m^E)^2]^{-1/4}$ for large L. Hence the quotient of the remainders at fixed z for large L is

$$\frac{R(\xi,\mu_m^E)}{R(\xi,0)} \sim (-1)^{\frac{L+1}{2} - m} [1 - (\mu_m^E)^2]^{-1/4} . \tag{6}$$

The absolute value of this quotient is independent of L and increases slowly with the distance from the center. The divergence proportional to $\sqrt{L}$ occurs only in the small interval between the largest zero of $P_{L+1}(\mu)$, $\mu_1 \approx \cos\frac{3\pi/2}{2L+3}$ $\approx 1 - \frac{1}{2}\left(\frac{3\pi/2}{2L+3}\right)^2 < \mu \leq 1$; this interval in the neighborhood of $\mu = 1$ decreases as L^{-2} for large L.

Hence the spherical harmonics approximation shows two possibilities for a lack of convergence. (1) If $f^{II}(z,0)$ does not vanish in the limit $L \to \infty$, an event, which may be caused by the physical conditions of the problem; (2) If $f^{II}(z,0)$ does not vanish strongly enough to reduce the remainder to zero also for $|\mu| = 1$; this is the case, if $f^{II}(z,0)$ does not tend to zero stronger than $1/\sqrt{L}$ for $L \to \infty$.

In our case of the Milne problem one observes a quick decrease of the remainder (3) from the boundary to the interior of the medium $\xi \leq 0$. To recognize this fact we separate the $\alpha_{\bar{0}}$ and α_0-terms from the sum in the remainder (L + 1 even!)

$$R(\xi,\mu) = -\left\{(c + a\lambda_0^2)\frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_0)}\left(\alpha_{\bar{0}}\, e^{-\xi/\lambda_0} + \alpha_0\, e^{\xi/\lambda_0}\right)\right.$$

$$\left. + \sum_{s=1}^{(L-1)/2} \alpha_s\, e^{\xi/\lambda_s}(c + a\lambda_s^2)\frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_s)}\right\}. \qquad (7)$$

λ_0 lies at least outside of the real interval $(-1, 1)$ and is usually large; in this region we have for large L

$$P_{L+1}(\lambda_0) \sim \frac{\left[\lambda_0 + (\lambda_0^2 - 1)^{1/2}\right]^{L+\frac{3}{2}}}{\sqrt{2\pi(L+1)}\,(\lambda_0^2 - 1)^{1/4}} \,. \tag{8}$$

μ however lies in the interior of the real interval $(-1, 1)$ and here

$$P_{L+1}(\mu) \sim \sqrt{\frac{2}{\pi(L+1)\sqrt{1-\mu^2}}}\,\sin\varphi \quad \text{with} \quad \varphi = \frac{\pi}{4} + \left(L + \frac{3}{2}\right)\arccos\mu \tag{9}$$

for large L and $1 - \varepsilon < \mu < 1 + \varepsilon$ (ε small). Therefore the quotient

$$\frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_0)} \sim \frac{\sqrt{2}\,\sin\varphi}{(1-\mu^2)^{1/4}\,(2\lambda_0)^{L+1}}$$

tends to zero for $L \to \infty$ in this μ-interval. This remains valid at $\mu = 1$ because $P_{L+1}(\pm 1) = 1$ (L + 1 even). Hence the first term in (7) plays no role in the limit $L \to \infty$; <u>the remainder contains in the limit $L \to \infty$ only the contributions of the transients</u> (with $0 < \lambda_s < 1$)

$$\lim_{L\to\infty} R(\xi, \mu) = \lim_{L\to\infty} \left\{ -\sum_{s=1}^{(L-1)/2} \alpha_s\, e^{\xi/\lambda_s} (c + a\lambda_s^2)\, \frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_s)} \right\}. \tag{10}$$

The meaning of this fact is that the spherical harmonics method produces a remainder <u>in the angular distribution</u> just in those points, in which one wants

better information than diffusion theory can give.

Introduction of the values (I-39) of α_s in (9) leads to

$$\lim_{L\to\infty} R(\xi,\mu) = \lim_{L\to\infty} \left\{ N(c+a\lambda_0^2)\, 2\lambda_0^2\, p(\mu)\, p(-\mu) \sum_{s=1}^{(L-1)/2} \frac{e^{\xi/\lambda_s}\left[1-\frac{a}{c}\lambda_s\,\delta(\lambda_a)\right]}{\lambda_s(\lambda_0^2-\lambda_s^2)\, r'(\lambda_s)\, p(-\lambda_s)} \right\} \tag{11}$$

Because the sign of $r'(\lambda_s)$ alternates with s, the sum in (11) is the difference of two infinite series with alternating terms. This renders a general statement of the behavior of (11) difficult.

One may use equation (4) to form the moments $\int_{-1}^{+1} \mu^{\ell} f(\xi,\mu)\, d\mu$. Because the remainder is proportional to $P_{L+1}(\mu)$ and $\int_{-1}^{+1} \mu^{\ell} P_{L+1}(\mu)\, d\mu = 0$ for $0 \le \ell \le L$, the relations between the moments of $f(\xi,\mu)$, which one could derive on the basis of the correct integro-differential equation (I-1), are the same in the spherical harmonics P_L-approximation up to $\ell = L$. One of those relations is equation (I-25), it is the case $\ell = 0$. In general, if we denote the moments by $m_\ell(\xi) = \int_{-1}^{+1} \mu^{\ell} f(\xi,\mu)\, d\mu$ the correct integral equation (I-1) and equation (4) for P_L-approximation in spherical harmonics method give equally the following relations connecting the moments

$$\frac{\partial}{\partial \xi} m_{\ell+1}(\xi) + m_\ell(\xi) = \begin{cases} \dfrac{1-\gamma_a}{\ell+1}\, m_0(\xi) & \text{for } \ell \text{ even} \\ 3\,\dfrac{1-\gamma_t}{\ell+2}\, m_1(\xi) & \text{for } \ell \text{ odd} \end{cases}$$

for linearly anisotropic scattering and $0 \le \ell \le L$.

2. <u>The functions, which tend for positive and negative μ in the limit $L \to \infty$ to the exact directed flux.</u>

We put in this section an index L on the approximate P_L-spherical harmonics solutions for the directed flux $f_L(\xi,\mu) = f_L^{I}(\xi,\mu) + f_L^{II}(\xi,\mu)$ (I-16)

which satisfies the Boltzmann equation (4) with the error source term

$$\mu \frac{\partial f_L(\xi,\mu)}{\partial \xi} + f_L(\xi,\mu) = F_L(\xi,\mu) , \tag{13}$$

where

$$F_L(\xi,\mu) = \frac{1}{2}(1-\gamma_a)\int_{-1}^{+1} f_L(\xi,\mu')d\mu' + \frac{3}{2}(1-\gamma_t)\mu\int_{-1}^{+1}\mu' f_L(\xi,\mu')d\mu' + \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f_L^{II}(\xi,0). \tag{14}$$

The exact directed flux, which satisfies the Boltzmann equation (I-1), may be denoted in this section by $f(\xi,\mu)$ without index:

$$\mu \frac{\partial f(\xi,\mu)}{\partial \xi} + f(\xi,\mu) = F(\xi,\mu) , \tag{15}$$

where

$$F(\xi,\mu) = \frac{1}{2}(1-\gamma_a)\int_{-1}^{+1} f(\xi,\mu')d\mu' + \frac{3}{2}(1-\gamma_t)\mu\int_{-1}^{+1}\mu' f(\xi,\mu')d\mu'. \tag{16}$$

If one considers μ for the moment as a fixed parameter and solves equations (13)

and (15) as inhomogeneous linear differential equations, one obtains

$$f(\xi,\mu) = e^{-\xi/\mu}\left[C(\mu) + \frac{1}{\mu}\int_0^{\xi} F(\xi',\mu)\, e^{\xi'/\mu}\, d\xi'\right] \tag{17}$$

for functions with or without index L, differing only by different "constants" $C(\mu)$. The constants $C(\mu)$ are determined by the boundary conditions. For large negative ξ we have the asymptotic behavior $f(\xi,\mu) \sim e^{-\xi/\lambda_{0\infty}}$ and $f_L(\xi,\mu) \sim e^{-\xi/\lambda_0}$ respectively, with $\lambda_{0\infty}$ and $\lambda_0 > 1$ <u>for positive μ</u>; hence one may use, <u>for positive μ</u>, the boundary conditions $e^{\xi/\mu} f(\xi,\mu) \to 0$ and $e^{\xi/\mu} f_L(\xi,\mu) \to 0$ for $\xi \to -\infty$; these conditions determine the constants $C(\mu)$ and $C_L(\mu)$, because $\mu \leq 1$. This yields in the first case

$$\lim_{\xi\to-\infty} e^{\xi/\mu} f(\xi,\mu) = C(\mu) + \frac{1}{\mu}\int_0^{-\infty} F(\xi',\mu)\, e^{\xi'/\mu}\, d\xi' = 0$$

or

$$C(\mu) = \frac{1}{\mu}\int_{-\infty}^{0} F(\xi',\mu)\, e^{\xi'/\mu}\, d\xi' \qquad (0 < \mu \leq 1) \tag{18}$$

and

$$f(\xi,\mu) = \frac{e^{-\xi/\mu}}{\mu}\int_{-\infty}^{\xi} F(\xi',\mu)\, e^{\xi'/\mu}\, d\xi'. \tag{19}$$

In the second case (P_L-approximation) the source boundary condition leads to

$$C_L(\mu) = \frac{1}{\mu}\int_{-\infty}^{0} F_L(\xi',\mu)\, e^{\xi'/\mu}\, d\xi' \qquad (0 < \mu \leq 1) \tag{20}$$

and

$$f_L(\xi,\mu) = \frac{e^{-\xi/\mu}}{\mu}\int_{-\infty}^{\xi} F_L(\xi',\mu)\, e^{\xi'/\mu}\, d\xi'. \tag{21}$$

If we take the difference of the exact and the approximate directed fluxes in the limit $L \to \infty$ we obtain <u>for positive μ</u>

$$f(\xi,\mu) - \lim_{L\to\infty} f_L(\xi,\mu) = \frac{1}{\mu}\, e^{-\xi/\mu} \lim_{L\to\infty} \int_{-\infty}^{\xi} [F(\xi',\mu) - F_L(\xi',\mu)]\, e^{\xi'/\mu}\, d\xi' . \quad (22)$$

We now assume that the approximate density and current tend to the exact value for $L \to \infty$; hence the corresponding terms cancel in the integrand on the right side of (22) and the error source term in $F_L(\xi,\mu)$-- compare equations (14) and (16) -- alone remains:

$$f(\xi,\mu) - \lim_{L\to\infty} f_L(\xi,\mu) = -\frac{1}{\mu}\, e^{-\xi/\mu} \lim_{L\to\infty} \frac{P_{L+1}(\mu)}{P_{L+1}(0)} \int_{-\infty}^{\xi} f_L^{II}(\xi',0)\, e^{\xi'/\mu}\, d\xi'$$

$$= \frac{1}{\mu}\, e^{-\xi/\mu} \lim_{L\to\infty} \sum_{k=\bar{0},0,1,\ldots}^{(L-1)/2} \alpha_k (c + a\lambda_k^2) \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \int_{-\infty}^{\xi} e^{\xi'\left(\frac{1}{\mu} + \frac{1}{\lambda_k}\right)}\, d\xi'$$

$$= e^{-\xi/\mu} \lim_{L\to\infty} \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k}\, e^{\xi\left(\frac{1}{\mu} + \frac{1}{\lambda_k}\right)} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)}$$

$$= \lim_{L\to\infty} \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k}\, e^{\xi/\lambda_k} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)}$$

$$= -\lim_{L\to\infty} f_L^{II}(\xi,\mu). \qquad (0 < \mu \leq 1) \qquad (23)$$

Evidently one may include the direction $\mu = 0$ for $\xi \leq 0$ in formula (23). Therefore one finds again that <u>the Gauss quadrature part $f_L^{I}(\xi,\mu)$ tends for positive μ</u> to the exact directed flux

$$f(\xi,\mu) = \lim_{L\to\infty}\left[f_L(\xi,\mu) - f_L^{II}(\xi,\mu)\right] = \lim_{L\to\infty} f_L^{I}(\xi,\mu). \qquad (24)$$
$$(0 \leq \mu \leq 1).$$

<u>For negative μ</u> we have different boundary conditions for the exact and the approximate functions, which determine the constants $C(\mu)$ and $C_L(\mu)$. In the first case we may use the condition of no flux from the vacuum to the medium at the boundary $\xi = 0$ i.e. $f(0,\mu) = 0$ for negative μ. Equation (17) yields in this case

$$f(0,\mu) = C(\mu) = 0 \qquad (-1 \leq \mu < 0) \qquad (25)$$

and we have simply

$$f(\xi,\mu) = \frac{e^{-\xi/\mu}}{\mu}\int_0^{\xi} F(\xi',\mu)\, e^{\xi'/\mu}\, d\xi'. \quad (-1 \leq \mu < 0) \qquad (26)$$

For the approximate directed flux, however, this boundary condition holds only for the discrete set of points $\mu = -\mu_j$ $(j = 1, 2, \ldots, \frac{L+1}{2})$, where $f_L(0,-\mu_j) = 0$. In general we have a directed flux $f_L(0,\mu)$ backward from the vacuum to the medium at the boundary $\xi = 0$. Hence equation (17) gives

$$C_L(\mu) = f_L(0,\mu) \qquad (-1 \leq \mu < 0) \qquad (27)$$

and the directed flux in L-approximation is

$$f_L(\xi,\mu) = e^{-\xi/\mu}\left[f_L(0,\mu) + \frac{1}{\mu}\int_0^{\xi} F_L(\xi',\mu)\, e^{\xi'/\mu}\, d\xi'\right]. \qquad (28)$$

$$(-1 \leq \mu < 0)$$

Taking again the difference between the exact and the approximate directed fluxes in the limit $L \to \infty$ we find

$$f(\xi,\mu) - \lim_{L\to\infty} f_L(\xi,\mu) = -e^{-\xi/\mu} \lim_{L\to\infty}\left\{f_L(0,\mu) + \frac{1}{\mu}\int_0^{\xi}\left[F_L(\xi',\mu) - F(\xi',\mu)\right] e^{\xi'/\mu}\, d\xi'\right\}$$

$$= -e^{-\xi/\mu} \lim_{L\to\infty}\left\{f_L(0,\mu) + \frac{1}{\mu}\,\frac{P_{L+1}(\mu)}{P_{L+1}(0)}\int_0^{\xi} f_L^{II}(\xi',0)\, e^{\xi'/\mu}\, d\xi'\right\} \qquad (29)$$

The last integral is

$$\frac{P_{L+1}(\mu)}{P_{L+1}(0)}\int_0^{\xi} f_L^{II}(\xi',0)\, e^{\xi'/\mu}\, d\xi' = -\sum_k \alpha_k (c + a\lambda_k^2)\frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)}\int_0^{\xi} e^{\xi'\left(\frac{1}{\mu}+\frac{1}{\lambda_k}\right)}\, d\xi'$$

$$= -\mu \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k}\,\frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)}\left[e^{\xi\left(\frac{1}{\mu}+\frac{1}{\lambda_k}\right)} - 1\right].$$

By introducing this in equation (29) we get

$$f(\xi,\mu) - \lim_{L\to\infty} f_L(\xi,\mu)$$

$$= \lim_{L\to\infty}\left\{ - f_L(0,\mu)\, e^{-\xi/\mu} + \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \left[e^{\xi/\lambda_k} - e^{-\xi/\mu} \right] \right\}$$

$$= \lim_{L\to\infty}\left\{ - f_L(0,\mu)\, e^{-\xi/\mu} - f_L^{II}(\xi,\mu) + f_L^{II}(0,\mu)\, e^{-\xi/\mu} \right\}$$

$$= - \lim_{L\to\infty}\left[f_L^{I}(0,\mu)\, e^{-\xi/\mu} + f_L^{II}(\xi,\mu) \right]. \qquad (-1 \le \mu < 0)$$

remembering equation (I-18) and $f_L(0,\mu) = f_L^{I}(0,\mu) + f_L^{II}(0,\mu)$.
Hence, we see that the exact directed flux $f(\xi,\mu)$ will be, in the limit $L\to\infty$, the sum of two terms, both containing only the first part f_L^{I} (the Gauss quadrature part) of the spherical harmonics solution f_L

$$f(\xi,\mu) = \lim_{L\to\infty}\left[f_L^{I}(\xi,\mu) - f_L^{I}(0,\mu)\, e^{-\xi/\mu} \right]. \qquad (-1 \le \mu < 0) \qquad (30$$

The second term removes the poles of $f_L^{I}(\xi,\mu)$ at $\mu = -\lambda_k$; this can be seen quickly by remembering the expression (I-17) of $f_L^{I}(\xi,\mu)$. We therefore have the

following representation of equation (30) <u>for negative μ</u>

$$f(\xi,\mu) = \lim_{L\to\infty} \sum_{k=\bar{0},0,1,\ldots}^{(L-1)/2} \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \right\} \left[e^{\xi/\lambda_k} - e^{-\xi/\mu} \right]. \qquad (31)$$

In equation (31) both ξ and μ are negative, their quotient ξ/μ is always positive. In the interior of the medium -- excluding the boundary $\xi = 0$ -- the term $e^{-\xi/\mu}$ vanishes with $\mu \to 0$. Hence, we have a continuous transition to the expression (24) for positive μ. On the boundary $\xi = 0$ however, we have to exclude the direction $\mu = 0$ perpendicular to the z-axis from equation (31) and we obtain $f(0,\mu) = 0$ for all $\mu < 0$. In this way the well-known discontinuity arises between the expressions (31) and (24) for $\mu = 0$ on the boundary of the medium $\xi = 0$.

3. The function $p(\lambda)/r(\lambda)$ in the limit $L \to \infty$.

To show how the extrapolation lengths δ and Δ and the first part of the directed flux at the boundary $f^{I}(0,\mu)$ behave in the limit of infinite high degree of approximation $L \to \infty$, one has to investigate the behavior of the function

$$-\frac{p(-\lambda)}{r(-\lambda)} = \left(\lambda + \mu_{\frac{L+1}{2}}\right) \prod_{s=1}^{(L-1)/2} \frac{\lambda + \mu_s}{\lambda + \lambda_s} \tag{32}$$

in the limit $L \to \infty$. If one keeps λ out of the real interval $-\lambda_{\frac{L-1}{2}} > \lambda > -2\mu_1 + \lambda_1$ or $0 > \lambda > -1$, where the poles of the product in equation (32) lie, one may find a convenient representation of the limiting function by application of Euler's sum formula to

$$\log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right] = \log\left(\lambda + \mu_{\frac{L+1}{2}}\right) + \sum_{s=1}^{(L-1)/2} \log \frac{\lambda + \mu_s}{\lambda + \lambda_s} . \tag{33}$$

The validity of the development for $\lambda = -1$ is somewhat dubious for the present in the general case of an arbitrary c and a, because one should prove generally that $\lim_{L\to\infty} -2\mu_1 + \lambda_1 > -1$. In the case of c = 1, a = 0 and L = 15, one obtains $-2\mu_1 + \lambda_1 = -0.9984\ 5942$. We develop in the last sum every term in powers of $(\mu_s - \lambda_s)/(\mu_s + \lambda)$

$$\log \frac{\lambda + \mu_s}{\lambda + \lambda_s} = -\log \frac{\lambda + \lambda_s}{\lambda + \mu_s} = -\log\left(1 - \frac{\mu_s - \lambda_s}{\mu_s + \lambda}\right)$$

$$= \frac{\mu_s - \lambda_s}{\mu_s + \lambda} + \frac{1}{2}\left(\frac{\mu_s - \lambda_s}{\mu_s + \lambda}\right)^2 + \cdots , \tag{34}$$

which is convergent in the <u>exterior</u> of the circle

$$\lambda = -\mu_s + (\mu_s - \lambda_s)\, e^{i\varphi} \qquad \text{around} \quad \lambda = -\mu_s \,. \tag{35}$$

An estimate of $\mu_s - \lambda_s$ shows that this development is tantamount to a development in powers of $\frac{1}{L}$. Hence in the limit $L \to \infty$ only the first term of the series (34) will be essential.

To find the difference of the roots $\mu_s - \lambda_s$ we remember that μ_s and λ_s are the zeros of the characteristic equations corresponding to the vacuum- and the medium-case:

$$\begin{aligned} &a)\ P_{L+1}(\mu_s) = 0\,, \\ &b)\ (1 + a\lambda_s^2)\, P_{L+1}(\lambda_s) - \lambda_s (c + a\lambda_s^2)\, W_L(\lambda_s) = 0\,. \end{aligned} \tag{36}$$

Because the λ_s are those zeros of (b) which lie between 0 and 1, we may choose $Q_0(\lambda)$ in $W_L(\lambda) = P_{L+1}(\lambda)\, Q_0(\lambda) - Q_{L+1}(\lambda)$ as $Q_0(\lambda) = \tanh^{-1}\lambda$ and we may write (b) also in the following fashion

$$[1 + a\lambda_s^2 - \lambda_s (c + a\lambda_s^2) \tanh^{-1}\lambda_s]\, P_{L+1}(\lambda_s) + \lambda_s (c + a\lambda_s^2)\, Q_{L+1}(\lambda_s) = 0\,.$$

According to E. Jahnke and F. Emde, Tables of Functions, p. 117, the asymptotic behavior of the Legendre functions P_{L+1} and Q_{L+1} for large L and angles ϑ between $\varepsilon < \vartheta < \pi - \varepsilon$ $\left(0 < \varepsilon \ll \frac{\pi}{6}\right)$ is

$$\begin{aligned} P_{L+1}(\mu) &\sim \sqrt{\frac{2}{\pi(L+1)\sin\vartheta}}\, \left\{ \sin\varphi - \frac{1}{4(L+1)} \left[\sin\varphi + \frac{1}{2} \operatorname{cotan}\vartheta \cos\varphi \right] \right\} \\ Q_{L+1}(\mu) &\sim \sqrt{\frac{\pi}{2(L+1)\sin\vartheta}}\, \left\{ \cos\varphi - \frac{1}{4(L+1)} \left[\cos\varphi - \frac{1}{2} \operatorname{cotan}\vartheta \sin\varphi \right] \right\} \end{aligned} \tag{37}$$

where $\mu = \cos\vartheta$ and $\varphi = (L+\frac{3}{2})\vartheta + \frac{\pi}{4}$.

(a) This yields in first approximation (upper index 1 is related to the zeros μ_s):

$$\sin\varphi_s^{(1)} = \sin\left[(L+\tfrac{3}{2})\vartheta_s^{(1)} + \tfrac{\pi}{4}\right] = 0 \tag{38}$$

$$\varphi_s^{(1)} = s\pi \qquad \left(s = 1, 2, \ldots \tfrac{L+1}{2}\right). \tag{39}$$

$$\vartheta_s^{(1)} = \frac{(2s-\frac{1}{2})\pi}{2L+3}, \quad \mu_s = \cos\vartheta_s^{(1)} = \cos\frac{(2s-\frac{1}{2})\pi}{2L+3}. \tag{40}$$

We remark for later use

$$d\mu_s = -\frac{2\pi}{2L+3}\sin\frac{(2s-\frac{1}{2})\pi}{2L+3}\,ds = -\frac{2\pi}{2L+3}\sqrt{1-\mu_s^2}\,ds \tag{41}$$

$$\begin{aligned}
\lim_{L\to\infty}\mu_1 &= \lim_{L\to\infty}\cos\frac{3\pi/2}{2L+3} \to \cos 0 = 1\\
\lim_{L\to\infty}\mu_{\frac{L-1}{2}} &= \lim_{L\to\infty}\cos\frac{(L-3/2)\pi}{2L+3} \to \cos\frac{\pi}{2} = 0\\
\lim_{L\to\infty}\mu_{\frac{L+1}{2}} &= \lim_{L\to\infty}\cos\frac{(L+\frac{1}{2})\pi}{2L+3} \to \cos\frac{\pi}{2} = 0
\end{aligned} \tag{42}$$

As long as L is finite, however, μ_1 belongs to an angle $\vartheta_1^{(1)} = \frac{(3\pi)/2}{2L+3} > 0$, for which presumably the asymptotic expression (37) is valid, because its violation becomes sensible only in the narrowest neighborhood of $\vartheta = 0$ in the interval $0 \le \vartheta \le \vartheta_1^{(1)}$.

(b) In the same order of approximation we obtain an equation for the

zeros λ_s (upper index 2 is related to the zeros λ_s):

$$[1+a\lambda_s^2-\lambda_s(c+a\lambda_s^2)\tanh^{-1}\lambda_s]\sin\varphi_s^{(2)}+\frac{\pi}{2}\lambda_s(c+a\lambda_s^2)\cos\varphi_s^{(2)}=0 \quad (43)$$

with $\varphi_s^{(2)}=(L+\frac{3}{2})\vartheta_s^{(2)}+\frac{\pi}{4}$

and $\lambda_s=\cos\vartheta_s^{(2)}$. (44)

We introduce the angular difference $\varepsilon_s=\vartheta_s^{(2)}-\vartheta_s^{(1)}$ of the zeros. Then we have

$$\lambda_s=\cos\vartheta_s^{(2)}=\cos(\vartheta_s^{(1)}+\varepsilon_s)=\cos\vartheta_s^{(1)}\cos\varepsilon_s-\sin\vartheta_s^{(1)}\sin\varepsilon_s\approx\mu_s-\varepsilon_s\sqrt{1-\mu_s^2}$$

neglecting higher powers of ε_s than the first, and we get

$$\varepsilon_s=\vartheta_s^{(2)}-\vartheta_s^{(1)}\sim\frac{\mu_s-\lambda_s}{\sqrt{1-\mu_s^2}}\,. \quad (45)$$

Using equations (44, 45, 38) one obtains for $\sin\varphi_s^{(2)}$ and $\cos\varphi_s^{(2)}$ asymptotically for large L

$$\sin\varphi_s^{(2)}=(-1)^s\sin(L+\tfrac{3}{2})\varepsilon_s$$
$$\cos\varphi_s^{(2)}=(-1)^s\cos(L+\tfrac{3}{2})\varepsilon_s \quad (46)$$

Putting these expressions in the characteristic equation (43) one finds an approximate equation for ε_s

$$[1+a\mu_s^2-\mu_s(c+a\mu_s^2)\tanh^{-1}\mu_s]\sin(L+\tfrac{3}{2})\varepsilon_s+\frac{\pi}{2}\mu_s(c+a\mu_s^2)\cos(L+\tfrac{3}{2})\varepsilon_s\approx 0$$

and its solution

$$\lambda_s-\mu_s=\frac{\sqrt{1-\mu_s^2}}{L+\frac{3}{2}}\tan^{-1}\left[\frac{\frac{\pi}{2}\mu_s(c+a\mu_s^2)}{1+a\mu_s^2-\mu_s(c+a\mu_s^2)\tanh^{-1}\mu_s}\right]+O\left(\frac{1}{L^2}\right) \quad (47)$$

Because our earlier exemption of λ_0 from the sequence of the roots of the characteristic equation (36b), s runs from 1 to $\frac{L-1}{2}$, and the last root $\mu_{\frac{L+1}{2}}$ of the other characteristic equation (36a) is left without a λ-partner. Hence, in the limit $c \to 0$, $a \to 0$ the λ_s go to μ_{s+1} because $\lambda_0 \to \mu_1$, $\lambda_2 \to \mu_2, \cdots\cdots \lambda_{\frac{L-1}{2}} \to \mu_{\frac{L+1}{2}}$ and we have to take that branch of $\tan^{-1}$ in formula (47), which is $-\pi$ for zero-argument. This situation remains unchanged in the case of isotropic scattering without absorption $c = 1$, $a = 0$ where the differences of the roots $\lambda_s - \mu_s$ are always negative. For illustration[7] we give a short table of these differences in the L = 15 approximation, calculated exactly and with the asymptotic formula (47) for

s	λ_s	μ_s	$\lambda_s - \mu_s$ (exact)	$\tan^{-1} \frac{\frac{\pi}{2}\mu_s}{1-\mu_s \tanh^{-1}\mu_s}$	$\lambda_s - \mu_s$ (equ. 47)
0	∞	--	--	--	--
1	0.9803 4244	0.9894 0093	- 0.0090 5849	- 0.7741 217	- 0.0066 691
2	0.9164 7056	0.9445 7502	- 0.0281 0446	- 1.1408 972	- 0.0227 001
3	0.8119 9386	0.8656 3120	- 0.0536 3734	- 1.4691 894	- 0.0445 816
4	0.6717 7145	0.7554 0441	- 0.0836 3296	- 1.7829 688	- 0.0708 064
5	0.5020 5099	0.6178 7624	- 0.1158 2525	- 2.0895 982	- 0.0995 757
6	0.3103 1156	0.4580 1678	- 0.1477 0522	- 2.3922 868	- 0.1288 853
7	0.1049 7050	0.2816 0355	- 0.1766 3305	- 2.6927 734	- 0.1565 939
8	--	0.0950 1251	--	--	--

The table shows that in L = 15 approximation the $\lambda_s - \mu_s$ of equation (47) are

[7] I owe the calculation of the table to Mr. H. S. Moran, Oak Ridge National Laboratory; the μ's are taken from A. N. Lowan, N. Davids, and A. Levenson, Tables for Gauss mechanical quadrature formula, 1941.

still far from the exact values, but it shows also the branch of $\tan^{-1}$, which we use. This is in accord with

$$\lambda_s - \mu_s \Big|_{\substack{c=0\\a=0}} \to \mu_{s+1} - \mu_s = \cos\frac{(2s+\frac{3}{2})\pi}{2L+3} - \cos\frac{(2s-\frac{1}{2})\pi}{2L+3}$$

$$\to -\sin\frac{(2s-\frac{1}{2})\pi}{2L+3}\sin\frac{2\pi}{2L+3} \qquad \text{for large } L$$

$$\to \frac{\sqrt{1-\mu_s^2}}{L+\frac{3}{2}}(-\pi).$$

With the asymptotic expression (47) for $\lambda_s - \mu_s$ and neglecting higher powers of $\frac{\mu_s - \lambda_s}{\mu_s + \lambda}$ in (34) than the first, because they are small of higher order in $1/L$, one obtains for large L and λ not on the real axis between - 1 and 0

$$\lim_{L\to\infty} \log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right]$$

$$= \lim_{L\to\infty}\left\{\log\left(\lambda+\mu_{\frac{L+1}{2}}\right) - \frac{1}{L+\frac{3}{2}}\sum_{s=1}^{(L-1)/2}\frac{\sqrt{1-\mu_s^2}}{\mu_s+\lambda}\tan^{-1}\left[\frac{\frac{\pi}{2}\mu_s(c+a\mu_s^2)}{1+a\mu_s^2-\mu_s(c+a\mu_s^2)\tanh^{-1}\mu_s}\right]\right\}. \tag{48}$$

$\mu_{\frac{L+1}{2}}$ in the first term tends to zero for $L\to\infty$. For the evaluation of the sum over S we use the first term of the Euler sum formula, -- the other terms are smaller of higher order in $\frac{1}{L}$ --, to represent the sum in the limit $L\to\infty$ as an integral

$$\lim_{L\to\infty}\log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right] = \log\lambda - \lim_{L\to\infty}\int_1^{(L-1)/2}\frac{\lambda_s-\mu_s}{\lambda+\mu_s}\,ds$$

$$= \log\lambda + \frac{1}{\pi}\lim_{L\to\infty}\int_{\cos\frac{3\pi/2}{2L+3}}^{\cos\frac{(L-\frac{3}{2})\pi}{2L+3}} \frac{d\mu_s}{\lambda_s+\mu_s}\tan^{-1}\left[\frac{\frac{\pi}{2}\mu_s(c+a\mu_s^2)}{1+a\mu_s^2-\mu_s(c+a\mu_s^2)\tanh^{-1}\mu_s}\right]$$

$$= \log\lambda - \frac{1}{\pi}\int_0^1 \frac{du}{\lambda+u}\tan^{-1}\left[\frac{\frac{\pi}{2}u(c+au^2)}{1+au^2-u(c+au^2)\tanh^{-1}u}\right]. \qquad (49)$$

In (49) equations (41, 42) were used to change the variables of integration from s to $\mu_s = u$ in the limit $L \to \infty$.

Application of partial integration to the integral yields

$$\lim_{L\to\infty}\log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right] = \log\lambda - \frac{1}{\pi}\left[\log(\lambda+u)\cdot\tan^{-1}\left\{\frac{\frac{\pi}{2}u(c+au^2)}{1+au^2-u(c+au^2)\tanh^{-1}u}\right\}\right]_0^1$$

$$+\frac{1}{2}\int_0^1 \log(\lambda+u)\,\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (50)$$

with

$$h \equiv h(c,a;u) = c + [c^2 - c + a(3-c)]u^2 + a[a+3(c-1)]u^4 \qquad (51)$$

and

$$g \equiv g(c,a;u) = \frac{1}{[1+au^2-u(c+au^2)\tanh^{-1}u]^2+[\frac{\pi}{2}u(c+au^2)]^2}. \qquad (52)$$

From our consideration of the dependence of $\lambda_s - \mu_s$ on μ_s we may recognize that the values of our $\tan^{-1}\{\cdots\}$ are always negative; therefore $\tan^{-1}\{\cdots\}$ $= 0$ for $u = 1$ and $-\pi$ for $u = 0$. Therefore the second term cancels just the first in (50) and one obtains finally

$$\lim_{L\to\infty} \log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right] = \frac{1}{2}\int_0^1 \log(\lambda+u)\,\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (53)$$

Presumably it is not forbidden to put $\lambda = 0$ in this formula, because the upper end $-\lambda_{\frac{L-1}{2}}$ of the excluded λ-interval is lower than 0 for every finite L. Then we would have

$$\lim_{L\to\infty} \log\left[-\frac{p(0)}{r(0)}\right] = \frac{1}{2}\int_0^1 \log u\,\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (54)$$

For later use one wants to reverse the sign of λ in (53). We write

$$\lim_{L\to\infty} \log\left[-\frac{p(-\lambda)}{r(-\lambda)}\right] = (\log\lambda)\cdot\frac{1}{2}\int_0^1 \frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du$$

$$+\frac{1}{2}\int_0^1 \log\left(1+\frac{u}{\lambda}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du\,.$$

The first integral on the right side is

$$\int_0^1 \frac{hg}{1-u^2}\,du = \frac{2}{\pi}\int_0^1 \frac{d}{du}\left[\tan^{-1}\frac{\frac{\pi}{2}u(c+au^2)}{1+au^2-u(c+au^2)\tanh^{-1}u}\right]du$$

$$= \frac{2}{\pi}\cdot\pi = 2$$

and we may join this term with the left side

$$\lim_{L\to\infty}\log\frac{p(-\lambda)}{(-\lambda)\,r(-\lambda)} = \frac{1}{2}\int_0^1 \log\left(1+\frac{u}{\lambda}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (55$$

Now we reverse the sign of λ and we obtain, with the restriction λ not on the real axis between 0 and 1,

$$\lim_{L\to\infty}\log\frac{p(\lambda)}{\lambda\,r(\lambda)} = \frac{1}{2}\int_0^1 \log\left(1-\frac{u}{\lambda}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (56)$$

and finally with the restriction λ <u>outside</u> of the real interval -1 to + 1

$$\lim_{L\to\infty}\log\left[-\frac{p(-\lambda)}{r(-\lambda)}\Big/\frac{p(\lambda)}{r(\lambda)}\right] = \frac{1}{2}\int_0^1 \log\left(\frac{1+\frac{u}{\lambda}}{1-\frac{u}{\lambda}}\right)\frac{hg}{1-u^2}\,du$$

$$= \int_0^1 \left(\tanh^{-1}\frac{u}{\lambda}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du\,. \qquad (57)$$

In the limit $L \to \infty$ the radii $\mu_s - \lambda_s$ of the circles of non-convergence (35) vanish and only the real interval $-1 < \lambda < +1$ will be excluded from an application of the last formula.

4. Behavior of λ_0 and $p(0)/r(0)$ in the limit $L \to \infty$.

(a) λ_0 lies outside the real interval -1 to +1. Therefore we have to use asymptotic formula, which are adapted to this case. We follow a procedure of B. Carlson[3], according to whom (see Szegö, Orthogonal polynomials, p. 188, 219).

$$P_{L+1}(\lambda) \sim \frac{1}{\sqrt{2\pi(L+1)}} \frac{1}{(\lambda^2-1)^{1/4}} \left[\lambda + (\lambda^2-1)^{1/2}\right]^{L+\frac{3}{2}}$$

$$Q_{L+1}(\lambda) \sim \frac{1}{\sqrt{L+1}} \left[\lambda - (\lambda^2-1)^{1/2}\right]^{L+2} \phi(\lambda)$$

$$= \frac{1}{\sqrt{L+1}} \frac{\phi(\lambda)}{\left[\lambda + (\lambda^2-1)^{1/2}\right]^{L+2}}$$

for large L and λ outside the cut $-1 \leq \lambda \leq 1$. $\phi(\lambda)$ is a regular function, independent of L and different from zero outside the cut. Hence the quotient

$$\frac{Q_{L+1}(\lambda_0)}{P_{L+1}(\lambda_0)} \sim \frac{\sqrt{2\pi}\, \phi(\lambda_0)(\lambda_0^2-1)^{1/4}}{\left[\lambda_0 + (\lambda_0^2-1)^{1/2}\right]^{2L+7/2}} \to 0 \quad \text{for } L \to \infty ,$$

3 B. Carlson, Neutron Diffusion Spherical Harmonics Theory, MDDC 236, p. 21.

If we write the characteristic equation (I-12) in the case of λ_0 in the form $(1+a\lambda_0^2)\,P_{L+1}(\lambda_0) - (c+a\lambda_0^2)\,\lambda_0\,W_L(\lambda_0)$

$$= P_{L+1}(\lambda_0)\left\{1+a\lambda_0^2 - \lambda_0(c+a\lambda_0^2)\,\mathrm{cotanh}^{-1}\lambda_0 + \lambda_0(c+a\lambda_0^2)\,\frac{Q_{L+1}(\lambda_0)}{P_{L+1}(\lambda_0)}\right\} = 0$$

using outside the cut the formula

$$W_L(\lambda_0) = P_{L+1}(\lambda_0)\,\mathrm{cotanh}^{-1}\lambda_0 - Q_{L+1}(\lambda_0)$$

we recognize that the characteristic equation in the limit $L \to \infty$ will be

$$1 + a\lambda_{0\infty}^2 - \frac{1}{2}\lambda_{0\infty}(c+a\lambda_{0\infty}^2)\log\frac{\lambda_{0\infty}+1}{\lambda_{0\infty}-1} = 0 \qquad (58.$$

with

$$\lambda_{0\infty} = \lim_{L\to\infty}\lambda_0 .$$

An estimate of $\lambda_{0\infty}$ is given by formula (I-58).

(b) We know $\frac{p(0)}{r(0)}$ for finite L: If we write the two polynomials (c_{L+1}, p_2, c_0, w_2 are some numbers).

$$P_{L+1}(\lambda) = c_{L+1}\left[\lambda^{L+1} - \frac{(L+1)L}{2(2L+1)}\lambda^{L-1} + \frac{(L+1)L(L-1)(L-2)}{2\cdot 4\cdot(2L+1)(2L-1)}\lambda^{L-3} - + \cdots + p_2\lambda^2\right] + c_0$$

and

$$\lambda W_L(\lambda) = c_{L+1}\left[\lambda^{L+1} + \left(\frac{1}{3} - \frac{(L+1)L}{2(2L+1)}\right)\lambda^{L-1} + \left(\frac{1}{5} - \frac{1}{3}\frac{(L+1)L}{2(2L+1)} + \frac{(L+1)L(L-1)(L-2)}{2\cdot 4\cdot(2L+1)(2L-1)}\right)\lambda^{L-}\right.$$
$$\left. + \cdots + w_2\lambda^2\right],$$

we see that $P_{L+1}(\lambda)$ and $\lambda W_L(\lambda)$ have the same coefficient of the highest power of λ and that $\lambda W_L(\lambda)$ has no constant term. As was pointed out by S. Chandrasekhar, Radiative Transfer, this fact enables us to find a relation between the roots μ_j of $P_{L+1}(\mu_j) = 0$ and the λ_k's of the more complicated characteristic equation

$$(1+a\lambda^2)\, P_{L+1}(\lambda) - (c + a\lambda^2)\, \lambda W_L(\lambda)$$

$$= c_{L+1}\left(1 - c - \frac{a}{3}\right)\lambda^{L+1} - \left[\frac{c}{3} + \frac{a}{5} + \left(1 - c - \frac{a}{3}\right)\frac{(L+1)L}{2(2L+1)}\right] c_{L+1}\lambda^{L-1} + \cdots + c_0 = 0$$

In the first case $P_{L+1}(\mu_j) = 0$ the product of the squares of the roots is

$$\mu_1^2 \mu_2^2 \cdots\cdots \mu_{\frac{L+1}{2}}^2 = (-1)^{(L+1)/2} \frac{c_0}{c_{L+1}} \tag{59}$$

and in the second case

$$\lambda_0^2 \lambda_1^2 \cdots\cdots \lambda_{\frac{L-1}{2}}^2 = (-1)^{(L+1)/2} \frac{c_0}{\left(1 - c - \frac{a}{3}\right) c_{L+1}} \,. \tag{60}$$

Hence the quotient is

$$\frac{\mu_1 \mu_2 \cdots\cdots \mu_{\frac{L+1}{2}}}{\lambda_0 \lambda_1 \cdots\cdots \lambda_{\frac{L-1}{2}}} = \sqrt{1 - c - \frac{a}{3}} = \sqrt{\gamma_a \gamma_t} \,. \tag{61}$$

Now we have from equations (I-33)

$$\frac{p(0)}{r(0)} \equiv - \frac{\mu_1 \mu_2 \cdots\cdots \mu_{(L+1)/2}}{\lambda_1 \lambda_2 \cdots\cdots \lambda_{(L-1)/2}}$$

$$= -\lambda_0 \sqrt{1 - c - \frac{a}{3}} = -\lambda_0 \sqrt{\gamma_a \gamma_t} \,. \tag{62}$$

By the way we see here that the left side as a product of real numbers is always real and that for a negative γ_a (multiplication) λ_0 must be pure imaginary, if γ_t is positive.

Equation (62) is an exact equation, which relates $p(0)/r(0)$ with λ_0 for every finite L and remains valid in the limit $L \to \infty$, when $\lambda_0 \to \lambda_{0\infty}$ is the solution of (58). Hence we have

$$\lim_{L\to\infty} \frac{p(0)}{r(0)} = -\lambda_{0\infty}\sqrt{\gamma_a \gamma_t}\,. \tag{63}$$

This formula may be used for a completion of the expression for $p(-\lambda)/r(-\lambda)$. Taking the difference of the equations (53) and (54) we obtain <u>for λ outside of $-1 < \lambda < 0$</u>

$$\lim_{L\to\infty} \log\left[\frac{p(-\lambda)}{r(-\lambda)} \Big/ \left(\frac{p(0)}{r(0)}\right)\right] = \frac{1}{2}\int_0^1 \log\left(1+\frac{\lambda}{u}\right)\frac{hg}{1-u^2}\,du \tag{64}$$

or with equation (63)

$$\lim_{L\to\infty} \frac{p(-\lambda)}{r(-\lambda)} = -\lambda_{0\infty}\sqrt{\gamma_a \gamma_t}\; e^{\frac{1}{2}\int_0^1 \log\left(1+\frac{\lambda}{u}\right)\frac{hg}{1-u^2}\,du} \tag{65}$$

5. Application of the limit-function to the extrapolation-distances.

We shall indicate how the previous results can be used to compute rigorously the extrapolation distances in the case of linearly anisotropic scattering. The limiting process for isotropic scattering has been considered

by Carlson;[4] these considerations therefore represent an extension of Carlson's work. The quotient (57) occurs in

$$\tanh^{-1}\frac{\delta(\lambda)}{\lambda} = \frac{1}{2}\log\left[-\frac{p(-\lambda)}{r(-\lambda)}\Big/\left(\frac{p(\lambda)}{r(\lambda)}\right)\right],$$

where $\delta(\lambda)$ is the function (I-34) introduced as an abbreviation; hence one obtains in the limit and for λ's outside $-1 < \lambda < +1$

$$\lim_{L\to\infty}\tanh^{-1}\frac{\delta(\lambda)}{\lambda} = \frac{1}{2}\int_0^1\left(\tanh^{-1}\frac{u}{\lambda}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}\,du \qquad (66)$$

and

$$\lim_{L\to\infty}\delta(\lambda) = \lambda\tanh\left[\frac{1}{2}\int_0^1\left(\tanh^{-1}\frac{u}{\lambda}\right)\frac{hg}{1-u^2}\,du\right]. \qquad (67)$$

Formulae (I-40) show the connection of the linear extrapolation distance δ and of the distance of the extrapolated endpoint Δ with this function. We consider

(1) <u>the distance of the extrapolated endpoint Δ</u>.

For $c/a > 0$, $\lambda_a = i\sqrt{\frac{c}{a}}$, we have $\frac{u}{\lambda_a} = -i\sqrt{\frac{a}{c}}\,u$,

$$\tanh^{-1}\left(\frac{u}{\lambda_a}\right) = -i\tan^{-1}\left(\sqrt{\frac{a}{c}}\,u\right),$$

$$\lambda_a\tanh\left[\frac{1}{2}\tanh^{-1}\left(\frac{u}{\lambda_a}\right)\right] = \lambda_a\tanh\left[-\frac{i}{2}\tan^{-1}\sqrt{\frac{a}{c}}\,u\right]$$

$$= \sqrt{\frac{c}{a}}\tan\left(\frac{1}{2}\tan^{-1}\sqrt{\frac{a}{c}}\,u\right) \qquad (68)$$

4 B. Carlson, Neutron Diffusion-Spherical Harmonics Theory, MDDC 236.

Using this in equation (48), one gets

$$\lim_{L\to\infty} \delta(\lambda_a) = \sqrt{\frac{c}{a}} \tan\left[\frac{1}{2}\int_0^1 \tan^{-1}\left(\sqrt{\frac{a}{c}}\, u\right) \frac{hg}{1-u^2}\, du\right]. \tag{69}$$

The distance Δ of the extrapolated endpoint becomes with equations (67) and (69) in the limit

$$\lim_{L\to\infty} \Delta \equiv \lim_{L\to\infty} \lambda_0 \left[\tanh^{-1} \frac{\delta(\lambda_0)}{\lambda_0} + \tanh^{-1} \frac{a}{c} \lambda_0\, \delta(\lambda_a)\right]$$

$$= \lambda_{0\infty} \left\{ \frac{1}{2}\int_0^1 \left(\tanh^{-1} \frac{u}{\lambda_{0\infty}}\right) \frac{hg}{1-u^2}\, du \right. \tag{70}$$

$$\left. + \tanh^{-1}\left[\lambda_{0\infty}\sqrt{\frac{a}{c}} \tan\left(\frac{1}{2}\int_0^1 \left(\tan^{-1}\sqrt{\frac{a}{c}}\, u\right) \frac{hg}{1-u^2}\, du\right)\right]\right\}$$

$\lambda_{0\infty}$ is the limit of λ_0 in the case $L \to \infty$, it is the corresponding positive root of (58).

For the special case of <u>isotropic scattering with absorption</u> ($\gamma_t = 1$, $a = 0$) we have the following simplifications

$$h \to h(c, 0; u) = c + (c^2 - c)u^2$$

$$h/(1-u^2) = c\left(1 + \frac{cu^2}{1-u^2}\right)$$

$$g \to g(c, 0; u) = \frac{1}{[1 - cu \tanh^{-1} u]^2 + [\frac{\pi}{2} cu]^2} \tag{71}$$

and, because $a/c = 0$, $\tan^{-1}\left(\sqrt{\frac{a}{c}}\,u\right) = 0$,

$$\lim_{L\to\infty} \Delta\Big|_{a=0} = \frac{c\lambda_{0\infty}}{2}\int_0^1 \left(1+\frac{cu^2}{1-u^2}\right)\frac{\tanh^{-1}(u/\lambda_{0\infty})\,du}{[1-cu\tanh^{-1}u]^2+[\frac{\pi}{2}cu]^2} \tag{72}$$

This result agrees with the result which K. M. Case, F. de Hoffmann, and G. Placzek[5] have found by the Wiener-Hopf method, if one remembers that their $K_0 = 1/\lambda_{0\infty}$, $z_0(c) = \lim_{L\to\infty} \Delta\big|_{a=0}$.

(2) <u>The linear extrapolation distance δ</u> is represented by a more lengthy formula than Δ in the general case $c \neq 0$, $a \neq 0$:

$$\lim_{L\to\infty} \delta \equiv \lim_{L\to\infty} \frac{\delta(\lambda_0) + \frac{a}{c}\lambda_0^2\,\delta(\lambda_a)}{1+\frac{a}{c}\,\delta(\lambda_0)\,\delta(\lambda_a)}$$

$$\lim_{L\to\infty} = \frac{\tanh\left[\frac{1}{2}\int_0^1\left(\tanh^{-1}\frac{u}{\lambda_{0\infty}}\right)\frac{dg}{1-u^2}\,du\right] + \lambda_{0\infty}\sqrt{\frac{a}{c}}\tan\left[\frac{1}{2}\int_0^1 \tan^{-1}\left(\sqrt{\frac{a}{c}}\,u\right)\frac{dg}{1-u^2}\,du\right]}{1+\lambda_{0\infty}\sqrt{\frac{a}{c}}\tanh\left[\frac{1}{2}\int_0^1\left(\tanh^{-1}\frac{u}{\lambda_{0\infty}}\right)\frac{dg}{1-u^2}\,du\right]\cdot\tan\left[\frac{1}{2}\int_0^1 \tan^{-1}\left(\sqrt{\frac{a}{c}}\,u\right)\frac{dg}{1-u^2}\,du\right]} \tag{73}$$

It is related to Δ by the formula (I-20) or

$$\lim_{L\to\infty} \delta = \lambda_{0\infty}\tanh\left(\frac{1}{\lambda_{0\infty}}\lim_{L\to\infty}\Delta\right). \tag{74}$$

5 K. M. Case, F. de Hoffmann, G. Placzek, Introduction to the Theory of Neutron Diffusion, p. 134, Eq. (21).

For the special case of isotropic scattering the second terms in numerator and denominator are zero and the formula agrees with Case, de Hoffmann, and Placzek's equation (21a), p. 137.

6. Application of the limit-function on the directed flux at the boundary.

To find the directed flux at the boundary in the limit $L \to \infty$, one may normalize it first to unit total current (I-55)

$$\int_{-1}^{+1} \mu\, f(0,\mu)\, d\mu = \frac{2}{3} f_{10}(0) = - \frac{8 N \gamma_a \lambda_0^2\, p(\lambda_a)\, p(-\lambda_a)}{\lambda_a \left[r(\lambda_a)\, p(-\lambda_a) - r(-\lambda_a)\, p(\lambda_a) \right]}$$

without integration over the azimuthal angle. To find the limit-function for the directed flux at the boundary, we choose according to the considerations of section 1 the Gauss quadrature part $f^{I}(0,\mu)$ - see equation (I-41) - , which coincides with the spherical harmonics solution $f(0,\mu)$ in the limit $L \to \infty$ in a denumerable infinite number of points $\mu = \mu_j$. Hence with the normalization above we seek the limit function

$$\lim_{L \to \infty} \frac{f^{I}(0,\mu)}{\int_{-1}^{+1} \mu\, f(0,\mu)\, d\mu}$$

$$= \lim_{L \to \infty} \frac{c + a\lambda_0^2}{4\gamma_a} \lambda_a \left[\frac{r(\lambda_a)}{p(\lambda_a)} - \frac{r(-\lambda_a)}{p(-\lambda_a)} \right] \frac{1 + \frac{a}{c}\, \delta(\lambda_a)\mu}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{-r(-\mu)} .$$

(75

Using the equations (55, 56) we get

$$\lim_{L\to\infty} \frac{\lambda_a\, r(\lambda_a)}{p(\lambda_a)} = e^{-\frac{1}{2}\int_0^1 \log\left(1-\frac{u}{\lambda_a}\right)\frac{hg}{1-u^2}\,du}$$

$$\lim_{L\to\infty} \frac{-\lambda_a\, r(-\lambda_a)}{p(-\lambda_a)} = e^{-\frac{1}{2}\int_0^1 \log\left(1+\frac{u}{\lambda_a}\right)\frac{hg}{1-u^2}\,du} \tag{76}$$

and

$$\lim_{L\to\infty} \lambda_a\left[\frac{r(\lambda_a)}{p(\lambda_a)} - \frac{r(-\lambda_a)}{p(-\lambda_a)}\right]$$

$$= e^{-\frac{1}{2}\int_0^1 \log\left(1-\frac{u}{\lambda_a}\right)\frac{hg}{1-u^2}\,du} + e^{-\frac{1}{2}\int_0^1 \log\left(1+\frac{u}{\lambda_a}\right)\frac{hg}{1-u^2}\,du} \tag{77}$$

With $\lambda_a = i\sqrt{\frac{c}{a}}$, $\log\left(1 \mp \frac{u}{\lambda_a}\right) = \log\left(1 \pm i\sqrt{\frac{a}{c}}\,u\right)$

$$= \log\sqrt{1+\frac{a}{c}u^2} \pm i\tan^{-1}\sqrt{\frac{a}{c}}\,u$$

$$= \frac{1}{2}\log\left(1+\frac{a}{c}u^2\right) \pm i\tan^{-1}\sqrt{\frac{a}{c}}\,u$$

equation (77) becomes

$$\lim_{L\to\infty} \lambda_a\left[\frac{r(\lambda_a)}{p(\lambda_a)} - \frac{r(-\lambda_a)}{p(-\lambda_a)}\right]$$

$$= 2\, e^{-\frac{1}{4}\int_0^1 \log\left(1+\frac{a}{c}u^2\right)\frac{hg}{1-u^2}\,du} \cdot \cos\left[\frac{1}{2}\int_0^1 \left(\tan^{-1}\sqrt{\frac{a}{c}}\,u\right)\frac{hg}{1-u^2}\,du\right]. \tag{78}$$

Using equation (69) for $\lim_{L\to\infty} \delta(\lambda_a)$, and equation (65) for $\lim_{L\to\infty} \frac{p(-\mu)}{r(-\mu)}$

we obtain the directed flux at the boundary for $0 \leq \mu \leq 1$

$$\lim_{L\to\infty} \frac{f^I(0,\mu)}{\int_{-1}^{+1} \mu\, f(0,\mu)\, d\mu} = \frac{\lambda_{0\infty}(c + a\lambda_{0\infty}^2)\sqrt{\gamma_t}}{2\sqrt{\gamma_a}\,(\lambda_{0\infty}^2 - \mu^2)} \cdot e^{1/2 \int_0^1 \log\left(1+\frac{\mu}{u}\right)\frac{hg}{1-u^2}\,du} \cdot$$

$$\cdot\, e^{-1/4 \int_0^1 \log\left(1+\frac{a}{c}u^2\right)\frac{hg}{1-u^2}\,du} \cdot \left\{\cos\omega + \mu\sqrt{\frac{a}{c}}\sin\omega\right\} \tag{79}$$

with ω dependent only on a/c

$$\omega = \frac{1}{2}\int_0^1 \left(\tan^{-1}\sqrt{\frac{a}{c}}\,u\right)\frac{hg}{1-u^2}\,du \tag{80}$$

<u>under the supposition that</u> $\frac{a}{c} = \frac{3\gamma_a(1-\gamma_t)}{1-\gamma_a}$ <u>is positive</u>. If multiplication prevails, γ_a will be negative and for positive $1-\gamma_t$ the quantity a/c will be negative. Then λ_a is real and λ_0 pure imaginary. In some respect, the roles of λ_0 and λ_a are exchanged by going through $\gamma_a = 0$. In the special case of isotropic scattering $(\gamma_t = 1,\ a = 0)$ we have a simpler expression again <u>for $0 \leq \mu \leq 1$</u>

$$\lim_{L\to\infty} \frac{f^I(0,\mu)}{\int_{-1}^{+1} \mu\, f(0,\mu)\, d\mu}\Bigg|_{a=0}$$

$$= \frac{c}{2}\,\frac{\lambda_{0\infty}}{\sqrt{\gamma_a}\,(\lambda_{0\infty}^2 - \mu^2)}\, \exp\left\{\frac{c}{2}\int_0^1 \left(1 + \frac{cu^2}{1-u^2}\right)\frac{\log\left(1+\frac{\mu}{u}\right)du}{\left[1 - cu\tanh^{-1}u\right]^2 + \left[\frac{\pi}{2}cu\right]^2}\right\} \tag{81}$$

This expression is identical with the result of Case, de Hoffmann, Placzek equation (13), p. 133, if one remembers that their $\psi(0,-\mu) = \frac{f(-\mu)}{-\mu}$ corresponds to our expression above and that their $\kappa_0 = 1/\lambda_{0\infty}$, $\gamma_a = 1-c$, according to which their $\sqrt{\frac{\kappa_0^2}{1-c}}\,\frac{1}{1-\kappa_0^2\mu^2}$ corresponds to $\frac{\lambda_{0\infty}}{\sqrt{\gamma_a}}\,\frac{1}{\lambda_{0\infty}^2-\mu^2}$ here.

7. The density in the limit $L \to \infty$.

For the density $\rho(\xi) = 4\pi f_{00}(\xi)$ the expression (I-21) holds

$$f_{00}(\xi) = \sum_{k=\bar{0},0,1,\cdots}^{(L-1)/2} \alpha_k e^{\xi/\lambda_k} = \frac{\sinh(\Delta-\xi)/\lambda_0}{\sinh \Delta/\lambda_0} + \sum_{s=1,2,}^{(L-1)/2} \alpha_s e^{\xi/\lambda_s}$$

under the normalization condition $\alpha_{\bar{0}} + \alpha_0 = 1$. The first term on the right side is the asymptotic part and its limit for $L \to \infty$ is quickly found by taking the corresponding values (70) for $\lim_{L\to\infty} \Delta$ and $\lambda_{0\infty}$ instead of λ_0. To calculate the transient part of the density

$$\sum_{s=1}^{(L-1)/2} \alpha_s e^{\xi/\lambda_s}$$

we define a function

$$\chi(\mu) = \sum_{s=1}^{(L-1)/2} \frac{\alpha_s \lambda_s}{\mu-\lambda_s} \equiv 2\lambda_0^2 N(c+a\lambda_0^2) \sum_{s=1}^{(L-1)/2} \frac{\left[1-\frac{a}{c}\lambda_s \delta(\lambda_a)\right] p(\lambda_s)}{(\mu-\lambda_s)(c+a\lambda_s^2)(\lambda_s^2-\lambda_0^2)\, r'(\lambda_s)} \qquad (82)$$

which is by application of the Lagrange interpolation formula

$$\chi(\mu) = 2\lambda_0^2 N(c+a\lambda_0^2)\Big\{ \frac{[1-\frac{a}{c}\delta(\lambda_a)\mu]\, p(\mu)}{(c+a\mu^2)(\mu^2-\lambda_0^2)\, r(\mu)}$$

$$-\frac{[1+\frac{a}{c}\lambda_a\delta(\lambda_a)]\, p(-\lambda_a)}{(\mu+\lambda_a)\,2\lambda_a(c+a\lambda_0^2)\, r(-\lambda_a)} + \frac{[1-\frac{a}{c}\lambda_a\,\delta(\lambda_a)]\, p(\lambda_a)}{(\mu-\lambda_a)\,2\lambda_a(c+a\lambda_0^2)\, r(\lambda_a)}$$

$$+\frac{[1+\frac{a}{c}\lambda_0\delta(\lambda_a)]\, p(-\lambda_0)}{(\mu+\lambda_0)\,2\lambda_0(c+a\lambda_0^2)\, r(-\lambda_0)} - \frac{[1-\frac{a}{c}\lambda_0\,\delta(\lambda_a)]\, p(\lambda_0)}{(\mu-\lambda_0)\,2\lambda_0(c+a\lambda_0^2)\, r(\lambda_0)} \tag{83}$$

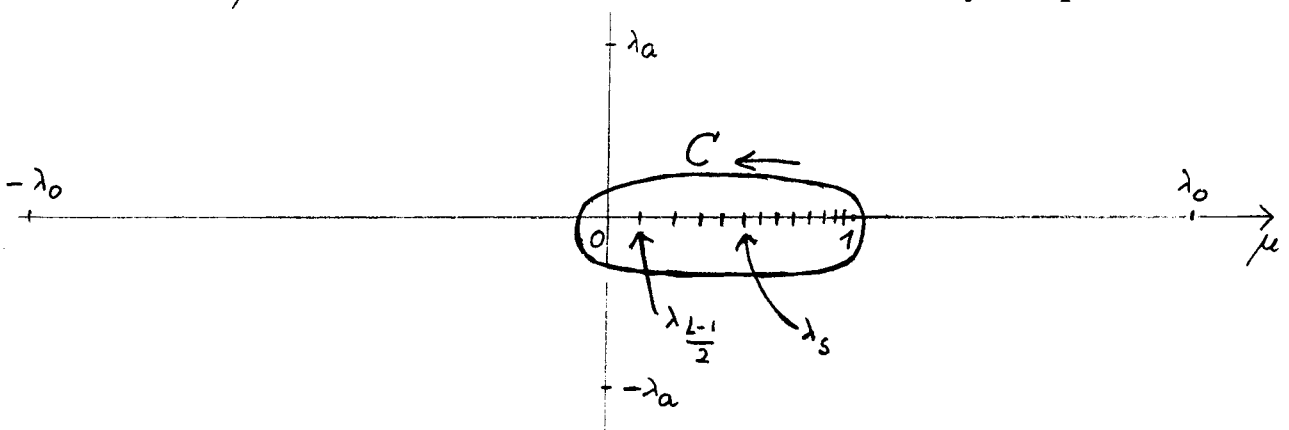
We now use a method for the calculation of the transient part of the current similar to the method which was applied first by Kourganoff and Busbridge[6] in their calculation of the density in the case of isotropic scattering without absorption. Equation (I-54) shows that the transient part of the current is proportional to $\sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s e^{\xi/\lambda_s}$. This sum may be represented as a contour integral along a path C in the complex μ-plane enclosing all poles $\mu = \lambda_s$ but excluding the points $\mu = \pm\lambda_a$, $\mu = \pm\lambda_0$

$$\sum_s \alpha_s \lambda_s e^{\xi/\lambda_s} = \frac{1}{2\pi i}\oint_C \sum_s \frac{\alpha_s \lambda_s e^{\xi/\mu}}{\mu-\lambda_s}\, d\mu = \frac{1}{2\pi i}\oint_C \chi(\mu)\, e^{\xi/\mu}\, d\mu .$$

By calculating this sum instead of $\sum_s \alpha_s e^{\xi/\lambda_s}$ one avoids jamming the path into the region $0<\mu<\lambda_{\frac{L-1}{2}}$; this region vanishes in the limit $L\to\infty$ and may in-

6 V. Kourganoff, I. W. Busbridge, Basic Methods in Transfer Problems, Oxford 1952, 27.3.

clude the point $\mu = 0$ in the interior of the area enclosed by the path C.



To calculate the density we remember equation (I-25), which represents the density as a differential quotient of the current.

Only the first term in equation (83) for $\chi(\mu)$ contributes to the contour integral, because the poles of the other four terms lie outside of C. Hence we obtain

$$\sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s e^{t/\lambda_s} = 2\lambda_0^2 (c + a\lambda_0^2) N \cdot \frac{1}{2\pi i} \oint_C \frac{\left[1 - \frac{a}{c}\mu\,\delta(\lambda_a)\right] p(\mu)\, e^{t/\mu}}{(c + a\mu^2)(\mu^2 - \lambda_0^2)\, r(\mu)}\, d\mu .$$

Normalizing to unit density at the boundary, i. e. multiplying with the normalization factor (see equation (I-53) and (63))

$$\frac{1}{\sum\limits_{k=\bar{0},0,1,\ldots}^{(L-1)/2} \alpha_k} = \frac{c\, r(0)}{2N(c + a\lambda_0^2)\, p(0)} = -\frac{c}{2N\lambda_0 (c + a\lambda_0^2)\sqrt{\gamma_a \gamma_t}}$$

we obtain two different representations of

$$\frac{\sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s e^{\xi/\lambda_s}}{\sum_{k=\bar{0},0,1,\dots}^{(L-1)/2} \alpha_k} = -\lambda_0^2 \frac{1}{2\pi i} \oint_C \frac{1-\frac{a}{c}\mu\,\delta(\lambda_a)}{(1-\frac{a}{c}\mu^2)(\lambda_0^2-\mu^2)} \frac{p(\mu)\,r(0)}{r(\mu)\,p(0)} e^{\xi/\mu}\, d\mu$$

$$\text{or} \quad = \frac{\lambda_0}{\sqrt{\gamma_a \gamma_t}} \frac{1}{2\pi i} \oint_C \frac{1-\frac{a}{c}\mu\,\delta(\lambda_a)}{(1-\frac{a}{c}\mu^2)(\lambda_0^2-\mu^2)} \frac{p(\mu)}{r(\mu)} e^{\xi/\mu}\, d\mu .$$

In the limit $L \to \infty$ we may use formula (64) for the first representation and formula (56) for the second representation; this yields

$$\lim_{L\to\infty} \frac{\sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s e^{\xi/\lambda_s}}{\sum_{k=\bar{0},0,1,\dots}^{(L-1)/2}} = -\frac{\lambda_{0\infty}^2}{2\pi i} \oint_C \Big[\frac{1-\frac{a}{c}\mu \lim_{L\to\infty} \delta(\lambda_a)}{(1+\frac{a}{c}\mu^2)(\lambda_{0\infty}^2-\mu^2)} \cdot$$

$$\cdot \exp\Big\{ \frac{\xi}{\mu} + \frac{1}{2}\int_0^1 \log\Big(1-\frac{\mu}{u}\Big) \frac{h_g}{1-u^2}\, du \Big\} \Big]\, d\mu$$

or

$$= \frac{\lambda_{0\infty}}{2\pi i \sqrt{\gamma_a \gamma_t}} \oint_C \Big[\frac{1-\frac{a}{c}\mu \lim_{L\to\infty} \delta(\lambda_a)}{(1+\frac{a}{c}\mu^2)(\lambda_{0\infty}^2-\mu^2)} \mu \exp\Big\{ \frac{\xi}{\mu} + \frac{1}{2}\int_0^1 \log\Big(1-\frac{u}{\mu}\Big) \frac{h_g}{1-u^2}\, du \Big\} \Big]\, d\mu .$$

This procedure is allowed because the integration path C does not contain points of the real interval between 0 and 1. The corresponding expressions for

the transient part of the density are

$$\lim_{L\to\infty} \frac{\sum_{s=1}^{(L-1)/2} \alpha_s e^{\xi/\lambda_s}}{\sum_{k=\bar{0},0,1,\ldots}^{(L-1)/2} \alpha_k}$$

$$= \frac{i}{2\pi}\frac{\partial}{\partial\xi}\oint_C \frac{d\mu\left[1-\frac{a}{c}\mu \lim_{L\to\infty}\delta(\lambda_a)\right]}{\left(1+\frac{a}{c}\mu^2\right)\left(1-\mu^2/\lambda_{0\infty}^2\right)} \exp\left\{\frac{\xi}{\mu}+\frac{1}{2}\int_0^1 \log\left(1-\frac{u}{\mu}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}du\right\}$$

$$\frac{1}{\pi i\,\lambda_{0\infty}\sqrt{\gamma_a\gamma_t}}\frac{\partial}{\partial\xi}\oint_C \frac{\mu\,d\mu\left[1-\frac{a}{c}\mu \lim_{L\to\infty}\delta(\lambda_a)\right]}{\left(1+\frac{a}{c}\mu^2\right)\left(1-\mu^2/\lambda_{0\infty}^2\right)} \exp\left\{\frac{\xi}{\mu}+\frac{1}{2}\int_0^1 \log\left(1-\frac{u}{\mu}\right)\frac{h(c,a;u)\,g(c,a;u)}{1-u^2}du\right\}.$$

$\lim_{L\to\infty}\delta(\lambda_a)$ in these formulas is given by equation (69).

The transformation of the contour integrals into real integrals, which are more convenient for calculations, is left here for later study. Also, no attempt has been made until now to compare these results with those obtained by B. Davison in his report CTR-358 by the Wiener-Hopf method.

Acknowledgments

I wish to express my deepest gratitude to Dr. Alvin M. Weinberg; this work originated from discussions with him and had his continuous advice during its performance. Further, it is a pleasure to acknowledge the hospitality of the Oak Ridge National Laboratory, in particular of the High Voltage Laboratory staff.

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ERRATA FOR ORNL-2334, "Studies of the Spherical Harmonics Method in Neutron Transport Theory," by Walter Kofink.

On page 2, ref. 1, and on page 14, ref. 5, change "T. C. Mark" to read "J. C. Mark."

On page 27, equation (53), change "$4\pi \Sigma \alpha_k$" to read "$4\pi \sum_k \alpha_k$".

On page 47, change last word in line 7 from bottom of page to read "special" instead of "spacial."

On page 51, after word "with" in 8th line from bottom of page, add "$\lambda = -\mu$".