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BOUNDARY VALUES FOR THE INNER RADIUS OF A CYLINDRICAL ANNULAR REACTOR

E. L. Zimmerman

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Office of Technical Services
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Contract No. W-7405-eng-26

Applied Nuclear Physics Division

BOUNDARY VALUES FOR THE INNER RADIUS OF A CYLINDRICAL ANNULAR REACTOR

E. L. Zimmerman*

DATE ISSUED

JUN 5 1958

OAK RIDGE NATIONAL LABORATORY
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UNION CARBIDE CORPORATION
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*Present address: Nuclear Development Corporation of America
White Plains, New York

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ABSTRACT

A boundary condition applicable to the inner radius of cylindrical annular reactors of finite length is derived. The resulting boundary condition is given as the ratio of the flux to the radial derivative of the flux at the inside radius and is expressed as a function of the geometry and the neutron mean free path in the multiplying material.

The method assumes the validity of diffusion theory and also that the flux is separable into radial and axial parts.

While the result is applicable to the more general multigroup calculation, a two-group treatment is presented by way of illustration. The latter is of practical value for such cases in which adequate two-group constants are available either from fundamental cross-section data or from a comparison to critical experiments using similar materials.

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INTRODUCTION*

The problem of a reactor in the form of a cylindrical annulus of infinite length having a central void may be solved by the application of conventional boundary conditions. The net neutron current into the inner cylindrical void is taken to be zero, which implies a zero slope of the flux in the radial direction at the inside surface. This assumption is not valid for a cylindrical annulus of finite length due to streaming losses. The problem of neutron streaming in cylindrical channels has been considered previously,[1,2] but the results do not appear to be directly applicable to the present problem.

Neutrons entering the inside cylindrical void will, on the average, traverse the void in a direction away from the region of higher flux and either re-enter the annulus at a region of lower flux or escape from the system by way of the ends of the void. This loss of neutrons will result in a net neutron current from the annulus into the void and a corresponding non-zero radial derivative at the inside radius of the annulus. As the inside radius becomes larger compared to the length of the cylinder, the loss increases and relatively fewer neutrons entering the inside void will be returned to the annulus. In the limiting case in which the ratio of length to radius of the inside cylindrical void becomes zero, the rate at which neutrons re-enter the annulus from the void approaches zero and the boundary condition approaches that for a bare external surface, i.e., the flux extrapolates to zero at a distance two thirds of a mean free path from the surface.

In the present treatment a boundary condition consisting of the ratio of the value of the flux to its radial derivative at the inside radius of the annulus is derived. The method is similar to the one used in the treatment of the problem of a gap in a reactor.[3-5] In that method it was assumed that the streaming of neutrons in the gap did not change the form of the component of the solution parallel to the gap, but did cause a depression in the flux, described by a change in the component normal to the gap. The method consists of calculating, for a particular surface element on the inside radius, the neutron current from all other such elements and equating it to the diffusion theory expression for partial neutron current. The source strength of surface elements is derived from diffusion theory. It is assumed that the flux is separable, i.e., it can be adequately described as the product of independent radial and axial functions. In the present case the axial function will be a simple cosine.

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* A part of this work was presented as an Oak Ridge National Laboratory Memo, CF-54-7-64, July, 1954. This has been revised and a section on the application to a two-group treatment has been added in the present paper.

1. T. Rubin, H. E. Stern, and F. W. Metzger, "An Axial Control Rod in Cylindrical Reactors," NEPA-1207-EAR-R14 (1949).
2. A. M. Weinberg and L. C. Noderer, "Theory of Neutron Chain Reactions, Vol. II," ORNL-CF-51-5-98 (Aug. 10, 1951).
3. M. L. Goldberger, M. G. Goldberger, and J. E. Wilkins, Jr., "The Effects of Gaps on Pile Reactivity," CP-3443 (Feb. 20, 1946).
4. S. Tamor, "The Effect of Gaps on Pile Reactivity," ORNL-1320 (1952).
5. J. Chernick and I Kaplan, "The Nuclear Reactor with a Transverse Air Gap," BNL-1996 (1954).

I. METHOD

Consider a cylindrical annular reactor having a co-axial cylindrical void of radius R and length 2L. (Fig. 1). The outside cylinder may be

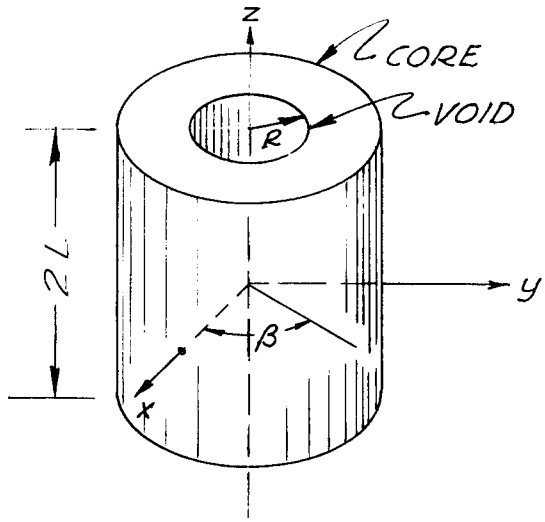

Fig. 1

bare or reflected, but this is not significant in the present problem.

Following the assumption of separability, the flux is given by

$$\phi(r,z) = \phi_r(r)\phi_z(z), \text{ and } \nabla\phi(r,z) = \phi_z(z)\,\frac{d\phi_r(r)}{dr}\,\underline{r}_o + \phi_r(r)\,\frac{d\phi_z(z)}{dz}\,\underline{z}_o \quad (1)$$

Where $\underline{r}_o$ and $\underline{z}_o$ are unit vectors in the direction of r and z. The values of the flux in the following discussion are all taken at r = R, and the abbreviated notation will be used:

$$\phi_R = \phi_r(R) \qquad \phi_z = \phi_z(z)$$

$$\phi'_R = \left.\frac{d\phi_r}{dr}\right|_{r=R}, \qquad \phi_{z_i} = \phi_z(z_i), \text{ and} \qquad \phi'_{z_i} = \left.\frac{d\phi_z}{dz}\right|_{z=z_i} \quad (2)$$

The rate at which neutrons leave a surface $\underline{n}_1\, dS_1$, at (R,z_1) directed into a solid angle $d\Omega$ about the direction $\underline{\Omega}_o$ is given by diffusion theory as

$$\frac{1}{4\pi}\left[\phi(R,z_1) - \lambda\nabla\phi(R,z_1)\cdot\underline{\Omega}_o\right] d\Omega\,\underline{\Omega}_o\cdot\underline{n}_1\, dS_1 \quad (3)$$

where λ is the mean free path of neutrons in the core material. Consider two surface elements dS_1 at z_1 and dS_2 at z_2 on the inside cylinder. The following relations are found by referring to Fig. 2.

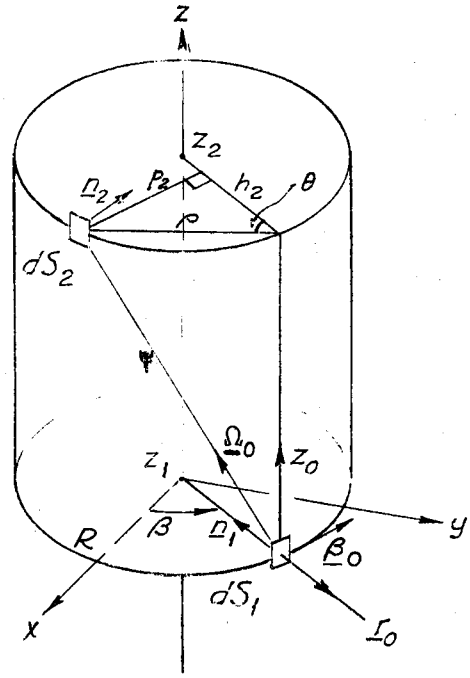

Fig. 2

$$dS_1 = R d\beta \, dz_1$$

$$dS_2 = 2R \, d\theta \, dz_2$$

$$\rho = 2R \cos\theta$$

$$h_2 = \rho \cos\theta = 2R \cos^2\theta$$

$$p_2 = \rho \sin\theta = 2R \cos\theta \sin\theta \tag{4}$$

$$\underline{n}_1 = -\underline{r}_0$$

$$\underline{n}_2 = \underline{r}_0 \cos 2\theta + \underline{\beta}_0 \sin 2\theta$$

$$\underline{\Omega}_0 = -\frac{h_2}{\psi} \cdot \underline{r}_0 - \frac{p_2}{\psi} \underline{\beta}_0 + \frac{z_2 - z_1}{\psi} \underline{z}_0$$

$$\nabla \phi(R, z_1) \cdot \underline{\Omega}_0 = -\phi_{z_1} \phi'_R \frac{h_2}{\psi} + \phi_R \phi'_{z_1} \frac{z_2 - z_1}{\psi}$$

$$\underline{\Omega}_0 \cdot \underline{n}_1 = \frac{h_2}{\psi}$$

$$d\Omega = \frac{dS_2}{\psi} \underline{n}_2 \cdot (-\underline{\Omega}_0) = \frac{dS_2}{\psi} \cdot \frac{h_2}{\psi}$$

$$\psi^2 = (z_2 - z_1)^2 + \rho^2 = 4R^2(\zeta^2 + \cos^2\theta) = \frac{4R^2}{k^2}(1 - k^2 \sin^2\theta) = \frac{4R^2 \Delta^2}{k^2}$$

Where $\zeta = \frac{(z_2 - z_1)}{2R}$, $k^2 = \frac{1}{\zeta^2 + 1}$, $\Delta^2 = (1 - k^2 \sin^2\theta)$

Substituting into (3) one has the rate at which neutrons from dS_1 reach dS_2,

$$j_{dS_1 \to dS_2} = \frac{1}{8\pi}\left[\phi_R\, \phi_{z_1} \frac{k^4\cos^4\theta}{\Delta^4} + \lambda\phi_{z_1}\, \phi_R' \frac{k^5\cos^6\theta}{\Delta^5} - \lambda\phi_R\phi_{z_1}'\, (1-k^2)^{1/2} \frac{k^4\cos^4\theta}{\Delta^4}\right] d\theta\, d\beta\, dz_1\, dz_2 \tag{5}$$

The rate at which neutrons from dS_1 reach the inside of the cylinder in dz_2 about z_2 is found by integrating $j_{dS_1 \to dS_2}$ with respect to θ. The resulting expression is integrated over β and z_1 to find the total rate at which neutrons reach the element about z_2, which is

$$j_+ (R,z_2) = \frac{1}{2\pi R dz_2} \int_{z_1=-L}^{L} \int_{\beta=0}^{2\pi} 2 \int_{\theta=0}^{\pi/2} j_{dS_1 \to dS_2} \tag{6}$$

This is the partial current at z_2 which is to be compared to the diffusion theory expression for the same quantity,

$$j_+(R,z_2) = \frac{\phi(R,z_2)}{4} - \frac{\lambda\partial\phi(r,z)}{6\,\partial r}\Bigg|_{\substack{z=z_2\\ r=R}} = \phi_{z_2}\left[\frac{\phi_R}{4} - \frac{\lambda\phi_R'}{6}\right] \tag{7}$$

By the assumption of separability the functions ϕ_R and ϕ_R' are independent of z and the ratio ϕ_R/ϕ_R' may be found by equating the expressions in (6) and (7) at any arbitrary z_2. It should be noted that the functions ϕ and ϕ' are not strictly separable and the above ratio found by equating (6) and (7) will depend on the choice of z_2. The assumption of separability will be somewhat justified however by the agreement between the results found using this assumption and the same quantities calculated by conventional methods for certain limiting cases. These limiting cases are shown in a later paragraph.

The expressions in (6) and (7) are now equated at a particular z_2 which, for convenience in performing the integration, is taken to be zero. In this case, by symmetry, the integral with respect to z_1 is taken only over the range $0 \leq z_1 \leq L$. Substituting (5) into (6), integrating with respect to β, and combining with (7), one has:

$$\left.\begin{aligned} \phi_{z_0}\left(\frac{\phi_R}{4} - \lambda\frac{\phi_R'}{6}\right) = \frac{1}{2\pi R}\Bigg[& \phi_R \int_0^L \phi_{z_1} dz_1 \int_0^{\pi/2} \frac{k^4\cos^4\theta\, d\theta}{\Delta^4} \\ & + \lambda\phi_R' \int_0^L \phi_{z_1}\, dz_1 \int_0^{\pi/2} \frac{k^5\cos^6\theta\, d\theta}{\Delta^5} \\ & - \lambda\phi_R \int_0^L - \phi_{z_1}'\, dz_1 (1-k^2)^{1/2} \int_0^{\pi/2} \frac{k^4\cos^4\theta\, d\theta}{\Delta^5}\Bigg] \end{aligned}\right\} \tag{8}$$

$$= \frac{1}{2\pi R}\left[\phi_R \int_0^L \phi_{z_1} f_1(k)\, dz_1 + \lambda\phi_R' \int_0^L \phi_{z_1} f_2(k) dz_1 - \lambda\phi_R \int_0^L -\phi_{z_1}' f_3(k) dz_1\right] \tag{8}$$

$$\left.\begin{aligned} \text{where } f_1(k) &= k^4 \int_0^{\pi/2} \frac{\cos^4\theta\, d\theta}{\Delta^4} \\ f_2(k) &= k^5 \int_0^{\pi/2} \frac{\cos^6\theta\, d\theta}{\Delta^5} \\ f_3(k) &= (1-k^2)^{1/2} k^4 \int_0^{\pi/2} \frac{\cos^4\theta\, d\theta}{\Delta^5} \end{aligned}\right\} \tag{9}$$

Rearranging terms in (8), one has the final result which may be expressed in the form

$$\lambda\alpha \equiv \lambda\frac{\phi_R'}{\phi_R} = F_1\left(\frac{L}{R}\right) + \frac{\lambda}{L} F_2\left(\frac{L}{R}\right), \tag{10}$$

$$\left.\begin{aligned} \text{where } F_1\left(\frac{L}{R}\right) &= \frac{1 - \frac{2}{\pi R}\int_0^L \frac{\phi_{z_1}}{\phi_{z_o}} f_1(k) dz_1}{\frac{2}{3}\left(1 + \frac{3}{\pi R}\int_0^L \frac{\phi_{z_1}}{\phi_{z_o}} f_2(k)\, dz_1\right)} \\ F_2\left(\frac{L}{R}\right) &= \frac{\frac{2L}{\pi R}\int_0^L -\frac{\phi_{z_1}'}{\phi_{z_o}} f_3(k) dz_1}{\frac{2}{3}\left(1 + \frac{3}{\pi R}\int_0^L \frac{\phi_{z_1}}{\phi_{z_o}} f_2(k) dz_1\right)} \end{aligned}\right\} \tag{11}$$

Using the result of (10) one can immediately find an expression for the rate of streaming out of one end of the inner cylinder. This is equivalent to the net current into the cylinder integrated over half the length.

The net rate at which neutrons enter the void through a surface element $2\pi R dz$ at z is $2\pi R(\lambda/3)\phi_z\, \phi_r'\, dz$, and the total rate in half the cylinder is therefore

$$\left.\begin{aligned} S &= \frac{2\pi R\lambda}{3}\phi_R' \int_0^L \phi_z dz = \frac{4}{3}\phi_{z_o}\phi_r'\lambda RL \\ &= \frac{4}{3} RL\, \phi(R,0)\, \lambda\frac{\phi_R'}{\phi_R} \\ &= \frac{4}{3} RL\, \phi(R,0)\left[F_1\left(\frac{L}{R}\right) + \frac{\lambda}{L} F_2\left(\frac{L}{R}\right)\right] \end{aligned}\right\} \tag{12}$$

II. EVALUATION OF INTEGRALS

The integral in $f_1(k)$ may be reduced to an algebraic expression, while those in $f_2(k)$ and $f_3(k)$ are solvable in terms of complete elliptic integrals of the first and second kinds. Consider an integral of the form

$$k^n \int \frac{\cos^n\theta \, d\theta}{\Delta^m}, \text{ where } \Delta = (1-k^2\sin^2\theta)^{1/2}.$$

This may be written

$$k^{n-2} \int \frac{\cos^{n-2}\theta \left[k^2-k^2\sin^2\theta\right]}{\Delta^m} d\theta .$$

Adding and substracting Δ^2 inside the bracket, this becomes

$$+ k^{n-2} \int \frac{\cos^{n-2}\theta \left[-(1-k^2) + \Delta^2\right] d\theta}{\Delta^m}, \quad \text{and finally}$$

$$k^n \int \frac{\cos^n\theta \, d\theta}{\Delta^m} = k^{n-2} \int \frac{\cos^{n-2}\theta \, d\theta}{\Delta^{m-2}} - (1-k^2)k^{n-2} \int \frac{\cos^{n-2}\theta \, d\theta}{\Delta^m}. \tag{13}$$

Also note that

$$d\left(\frac{\cos\theta \sin\theta}{\Delta^m}\right) = \frac{1-(2-(m-1)k^2)\sin^2\theta - (m-2)k^2\sin^4\theta}{\Delta^{m+2}} d\theta$$

and

$$\int_{\theta=0}^{\pi/2} \frac{d}{d\theta}\left(\frac{\cos\theta\sin\theta}{\Delta^m}\right) d\theta = \left.\frac{\cos\theta\sin\theta}{\Delta^m}\right|_0^{\pi/2} = 0 \tag{14}$$

By application of 13) and 14), the integrals in 9) are reduced to simple forms.

$$\left.\begin{aligned} f_1(k) &= \frac{\pi}{4}\left[2-(2+k^2)(1-k^2)^{1/2}\right] \\ f_2(k) &= \frac{1}{3k}\left[(2(1-k^4) + 3(2-k^2))E-(1-k^2)(8-k^2)K\right] \\ f_3(k) &= (1-k^2)^{1/2}\left[(2+k^2)K- 2(1+k^2)E\right] \\ \text{where } K &= \int_0^{\pi/2} \frac{d\theta}{\Delta}, \quad \text{and } E = \int_0^{\pi/2} \Delta \, d\theta. \end{aligned}\right\} \tag{15}$$

The axial solution for the flux is $\phi_z = \phi_{z_o} \cos\frac{\pi z}{2L}$ (16)

and $\phi_z' = -\phi_{z_o} \frac{\pi}{2L} \sin\frac{\pi z}{2L}$

The functions in (11) are

$$F_1\left(\frac{L}{R}\right) = \frac{1 - \frac{1}{2R}\int_0^L \left[2-(2+k^2)\,(1-k^2)^{1/2}\right] \cos\frac{\pi z}{2L}\, dz}{\frac{2}{3}\left\{1+\frac{1}{\pi R}\int_0^L \frac{1}{k}\left[\left(2(1-k^4)+3(2-k^2)\right)E-(1-k^2)(8+k^2)K\right]\cos\frac{\pi z}{2L}\, dz\right\}}$$

$$F_2\left(\frac{L}{R}\right) = \frac{\frac{1}{2R}\int_0^L (1-k^2)^{1/2}\left[(2+k^2)K-2(1+k^2)E\right]\sin\frac{\pi z}{2L}\, dz}{\frac{2}{3}\left\{1+\frac{1}{\pi R}\int_0^L \frac{1}{k}\left[\left(2(1-k^4)+3(2-k^2)\right)E-(1-k^2)(8+k^2)K\right]\cos\frac{\pi z}{2L}\, dz\right\}} \qquad (17)$$

The functions in (17) have been integrated numerically and the results over a range of L/R values are shown in Figs. 3 and 4.

Equation (10) and the values of the functions from the curves give a convenient boundary condition for the inner radius of an annular reactor. This condition is valid within the limitations of diffusion theory and the separability assumption even when the radius becomes large compared to the cylinder length. The values of F_1 and F_2 for the limiting cases, i.e., where L/R becomes infinite or zero, are indicated on the curves, and may be obtained directly from (17). These cases are discussed in Section IV, page 18. The derivation is for neutrons of a single energy, but it may be used in two-group or multi-group analyses by choosing an appropriate λ for each energy group.

It is of interest to note that a similar expression for this inner radius boundary condition may be derived by first calculating the streaming of neutrons out of an end of the inner cylinder and equating this to the net rate at which neutrons enter the inner cylinder from the sides. The converse of this procedure was used in arriving at equation (12).

III. APPLICATION TO THE TWO-GROUP METHOD

A. General Solution for a Cylinder of Finite Length.

The neutrons in a system are assumed to be divided into fast and slow energy groups. The behavior of each of the two groups of neutrons is described by the two diffusion equations:[6]

$$\nabla^2\phi_1 - \frac{\Sigma_1}{D_1}\phi_1 + \frac{k\,\Sigma_2\,\phi_2}{D_1} = 0$$

$$\text{and } \nabla^2\phi_2 - \frac{\Sigma_2}{D_2}\phi_2 + \frac{\Sigma_1\,\phi_1}{D_2} = 0 \qquad (18)$$

6. S. Glasstone and M. C. Edlund, The Elements of Nuclear Reactor Theory, pp 238-243, D. Van Nostrand Company, New York (1952).

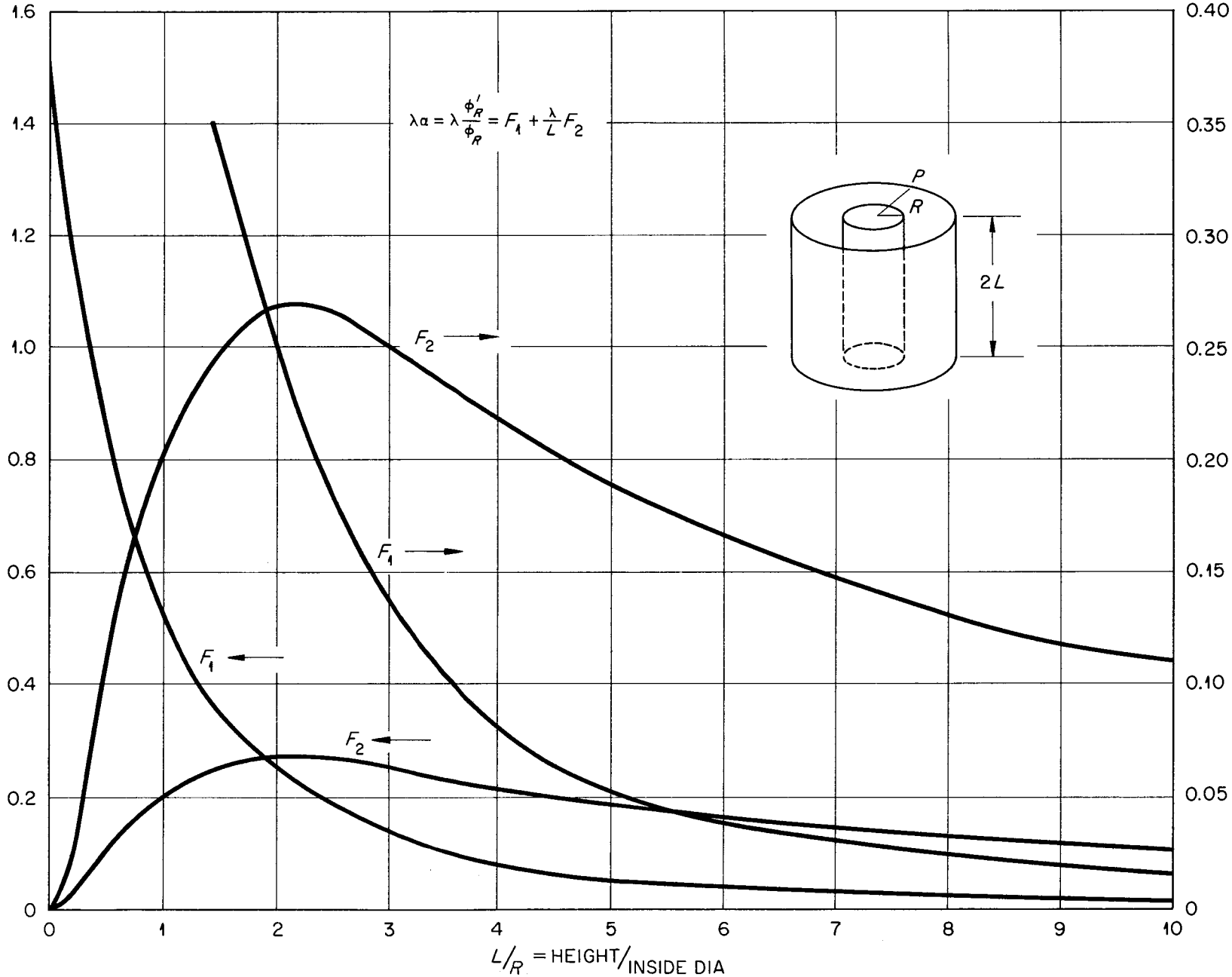
$\lambda\alpha = \lambda \frac{\phi_R'}{\phi_R} = F_1 + \frac{\lambda}{L} F_2$
F_2
F_1
F_1
F_2
P
R
2L
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
0
1
2
3
4
5
6
7
8
9
10
L/R = HEIGHT/INSIDE DIA

Fig. 3

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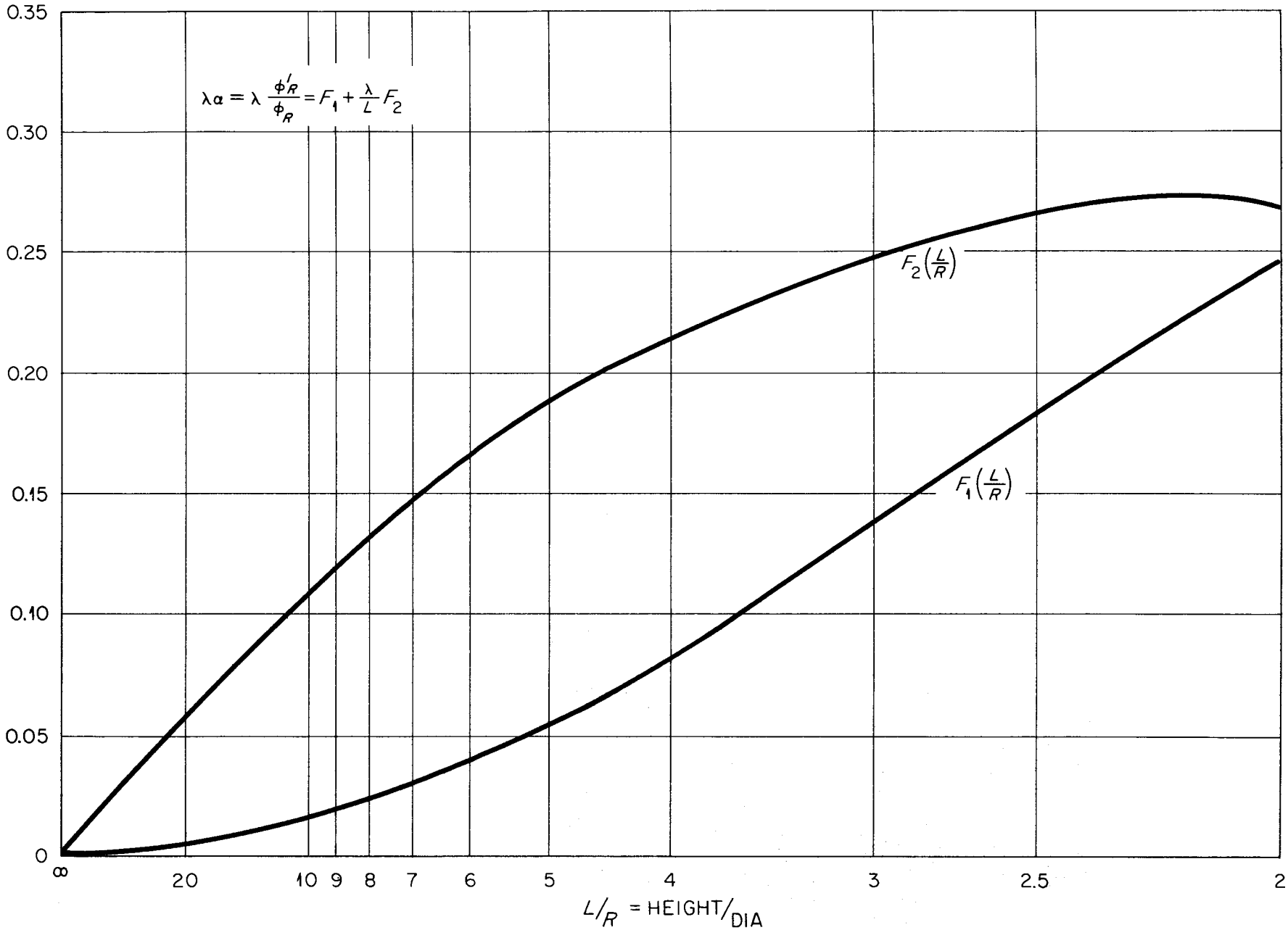

Fig. 4

Where the terms ϕ, Σ and D are flux, removal cross section and diffusion constants for the fast and slow groups indicated by subscripts 1 and 2, respectively. The neutron multiplication in an infinite system containing the same materials is given by k. Solutions satisfying

$$\left.\begin{aligned} &\nabla^2\phi_1 + B^2\phi_1 = 0 \\ \text{and}\quad &\nabla^2\phi_2 + B^2\phi_2 = 0 \end{aligned}\right\} \qquad (19)$$

are assumed from which B^2 is found by direct substitution into (18) to have the possible values

$$\left.\begin{aligned} B^2 = \mu^2 &= \frac{1}{2}\left[-\left(\frac{1}{\tau}+\frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{\tau}+\frac{1}{L^2}\right)^2 + \frac{4(k-1)}{\tau L^2}}\right] \\ \text{or}\quad B^2 = \nu^2 &= \frac{1}{2}\left[-\left(\frac{1}{\tau}+\frac{1}{L^2}\right) - \sqrt{\left(\frac{1}{\tau}+\frac{1}{L^2}\right)^2 + \frac{4(k-1)}{\tau L^2}}\right] \end{aligned}\right\} \qquad (20)$$

where $\tau \equiv \frac{D_1}{\Sigma_1}$, and $L^2 \equiv \frac{D_2}{\Sigma_2}$.

For an infinite cylinder the general solutions of the equations in (18) are

$$\left.\begin{aligned} \phi_1(r) &= A_1 J_o(\mu r) + A_2 Y_o(\mu r) + C_1\, I_o(\nu r) + C_2\, K_o(\nu r) \\ \phi_2(r) &= \left[A_1 J_o(\mu r) + A_2 Y_o(\mu r)\right] S_1 + \left[C_1 I_o(\nu r) + C_2 K_o(\nu r)\right] S_2 \end{aligned}\right\} \qquad (21)$$

Where J_o, Y_o, I_o, and K_o are zero-order Bessel functions, A_1, A_2, C_1, and C_2 are arbitrary constants, and the coupling constants, S_1 and S_2 are given by

$$\left.\begin{aligned} S_1 &\equiv \frac{D_1}{\tau D_2}\;\frac{1}{\frac{1}{L^2}+\mu^2} \\ \text{and}\quad S_2 &\equiv \frac{D_1}{\tau D_2}\;\frac{1}{\frac{1}{L^2}-\nu^2} \end{aligned}\right\} \qquad (22)$$

Consider a cylinder of finite length h. (This is taken to be the augmented length which is the actual length plus appropriate extrapolation distances). One now assumes the solutions for ϕ_1 and ϕ_2 to be separable into axial and radial dependent parts and further that the fast and slow axial solutions are related by a

constant factor G. These assumptions may be written as equations

$$\phi_1(r,z) = \phi_{1r}(r)\ \phi_{1z}(z),$$
$$\phi_2(r,z) = \phi_{2r}(r)\ \phi_{2z}(z), \tag{23}$$
$$\phi_{2z}(z) = G\ \phi_{1z}(z)$$

Direct substitution of (23) into (18) gives the equations in r,

$$\nabla^2_r\ \phi_{1r}(r) - \left(\frac{\Sigma_1}{D_1}\right)^* \phi_{1r}(r) + k^* \left(\frac{\Sigma_1}{D_1}\right)^* \phi_{2r}(r) = 0$$
$$\text{and}\quad \nabla^2_r\ \phi_{2r}(r) - \left(\frac{\Sigma_2}{D_2}\right)^* \phi_{2r}(r) + \left(\frac{\Sigma_1}{D_2}\right)^* \phi_{1r}(r) = 0 \tag{24}$$

which have the same form as (18), with the substitutions

$$\left(\frac{\Sigma_1}{D_1}\right)^* = \left(B_z^2 + \frac{\Sigma_1}{D_1}\right) \text{ or } \tau^* = \frac{\tau}{1+\tau B_z^2},$$
$$\left(\frac{\Sigma_2}{D_2}\right)^* = \left(B_z^2 + \frac{\Sigma_2}{D_2}\right) \text{ or } L^{2*} = \frac{L^2}{1+L^2B_z^2} \tag{25}$$

where

$$B_z^2 = -\frac{\nabla^2_z\ \phi_{1z}(z)}{\phi_{1z}(z)} = -\frac{\nabla^2_z\ \phi_{2z}(z)}{\phi_{2z}(z)} \tag{26}$$

is the axial buckling. Since (24) and (18) have the same form, the finite cylinder may be treated as an infinite cylinder having the adjusted constants given in (25). The general solutions for the radial flux is given by (21) where constants ν and μ have been adjusted according to (24) and their definitions (20). It may be easily verified that the ratio of the coupling constants in (22) is unchanged by (25), i.e.,

$$\frac{S_1^*}{S_2^*} = \frac{S_1}{S_2}$$

B. <u>Boundary Condition for the Cylindrical Annulus.</u>

In the following discussion it may be assumed that the adjustments of (25) have been made and the star and subscript will be dropped for simplicity in notation. Also the cylindrical annulus will be considered as unreflected, or to have an appropriate reflector savings which will reduce the problem to the unreflected case. The boundary conditions are

$$\phi_1(P) = 0$$

$$\phi_2(P) = 0$$

$$\frac{\phi_1'(R)}{\phi_1(R)} = \alpha_1 \qquad (27)$$

$$\frac{\phi_2'(R)}{\phi_2(R)} = \alpha_2$$

where P and R are the outside and inside radii of the annulus respectively.* The values of the α's are taken from the curves in Figs. 3 and 4. The boundary conditions in (27) are substituted into (21) and the condition for criticality is that the determinant of the coefficients vanish, i.e.

$$\left.\begin{vmatrix} J_o(\mu P) & Y_o(\mu P) & I_o(\nu P) & K_o(\nu P) \\ S_1 J_o(\mu P) & S_1 Y_o(\mu P) & S_2 I_o(\nu P) & S_2 K_o(\nu P) \\ \alpha_1 J_o(\mu R)+\mu J_1(\mu R) & \alpha_1 Y_o(\mu R)+\mu Y_1(\mu R) & \alpha_1 I_o(\nu R)-\nu I_1(\nu R) & \alpha_1 K_o(\nu R)+\nu K_1(\nu R) \\ S_1\left[\alpha_2 J_o(\mu R)+\mu J_1(\mu R)\right] & S_1\left[\alpha_2 Y_o(\mu R)+\mu Y_1(\mu R)\right] & S_2\left[\alpha_2 I_o(\nu R)-\nu I_1(\nu R)\right] & S_2\left[\alpha_2 K_o(\nu R)+\nu K_1(\nu R)\right] \end{vmatrix} = 0 \right\} \qquad (28)$$

This may be reduced to the second order determinant

$$\begin{vmatrix} \left(\alpha_1+\mu\frac{f_2}{f_1}\right) & \left(\alpha_1-\nu\frac{f_4}{f_3}\right) \\ \frac{S_1}{S_2}\left(\alpha_2+\mu\frac{f_2}{f_1}\right) & \left(\alpha_2-\nu\frac{f_4}{f_3}\right) \end{vmatrix} = 0 \qquad (29)$$

where $f_1 = \left[\frac{J_o(\mu R)}{Y_1(\mu R)} - \frac{Y_o(\mu R)}{Y_1(\mu R)}\,\frac{J_o(\mu P)}{Y_o(\mu P)}\right]$

$$f_2 = \left[\frac{J_1(\mu R)}{Y_1(\mu R)} - \frac{J_o(\mu P)}{Y_o(\mu P)}\right]$$

* P includes an appropriate extrapolation distance or reflector savings.

$$\left.\begin{aligned} f_3 &= \left[\frac{I_o(\nu R)}{I_1(\nu R)}\,\frac{K_o(\nu P)}{I_o(\nu P)} - \frac{K_o(\nu R)}{I_1(\nu R)}\right] \\ f_4 &= \left[\frac{K_o(\nu P)}{I_o(\nu P)} + \frac{K_1(\nu R)}{I_1(\nu R)}\right] \end{aligned}\right\} \quad (30)$$

In general the critical size of a cylindrical annulus may be found from (29), using fundamental cross section data. However, if information on the critical size of solid cylinders of similar material is available the problem may be greatly simplified. If the material buckling is known to be B^2, one has to a good approximation

$$\mu^2 \doteq B^2 - B_z^2 \quad (31)$$

and from (20)

$$\nu^2 = \mu^2 + \left(\frac{1}{\tau} + \frac{1}{L^2}\right) \doteq B^2 - B_z^2 + \left(\frac{1}{\tau} + \frac{1}{L^2}\right) \quad (32)$$

In certain cases where $\nu P > \nu R > 1$, the $\nu \frac{f_4}{f_3}$ in (29) may be approximated quite simply. For large arguments, one has the expansions

$$\left.\begin{aligned} I_o(x) &= \frac{e^x}{(2\pi x)^{1/2}}\left[1 + \frac{1}{8x} + \frac{3^2}{2!(8x)^2} + \frac{3^2 \cdot 5^2}{3!(8x)^3} + \cdots\right] \\ I_1(x) &= \frac{e^x}{(2\pi x)^{1/2}}\left[1 - \frac{3}{8x} - \frac{3 \cdot 5}{2!(8x)^2} - \frac{3^2 \cdot 5 \cdot 7}{3!(8x)^3} + \cdots\right] \\ K_o(x) &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 - \frac{1}{8x} + \frac{3^2}{2!(8x)^2} - \frac{3^2 \cdot 5^2}{3!(8x)^3} + \cdots\right] \\ K_1(x) &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 + \frac{3}{8x} - \frac{3 \cdot 5}{2!(8x)^2} + \frac{3^2 \cdot 5 \cdot 7}{3!(8x)^3} + \cdots\right] \end{aligned}\right\} \quad (33)$$

and the ratio $\frac{K_o(x)}{I_o(x)} \cong \frac{K_1(x)}{I_1(x)} \cong \pi e^{-2x}$ (34)

Since $\nu P > \nu R$, the term in the ratio $\frac{f_4}{f_3}$ involving P will be small compared to those involving R.

For this case one has

$$\left.\begin{aligned} \nu \frac{f_4}{f_3} &\doteq \frac{\nu \dfrac{K_1(\nu R)}{I_1(\nu R)}}{-\dfrac{K_o(\nu R)}{I_1(\nu R)}} = -\nu \frac{K_1(\nu R)}{K_o(\nu R)} \\ &\doteq -\nu\left[1 + \frac{1}{2\nu R} - \frac{1}{8(\nu R)^2} + \frac{5}{32(\nu R)^3} + \cdots\right]. \end{aligned}\right\} \quad (35)$$

The only term in (29) which depends on P, the outer radius, is $\mu \frac{f_2}{f_1}$.

The critical equation (29) may now be written:

$$\mu \frac{f_2}{f_1} = \frac{\alpha_1 \alpha_2 \left(1 - \frac{S_1}{S_2}\right) - \left(\alpha_1 - \frac{S_1}{S_2}\alpha_2\right) \nu \frac{f_4}{f_3}}{\left(1 - \frac{S_1}{S_2}\right) \nu \frac{f_4}{f_3} - \left(\alpha_2 - \frac{S_1}{S_2}\alpha_1\right)} \tag{36}$$

in which the right hand side of the equation is independent of P.

C. Special Cases

1. Center of annulus is a void.

   In this case α_1 and α_2 are found from the curves Fig. 3 or 4 by

$$\alpha_1 = \frac{F_1}{\lambda_1} + \frac{F_2}{L}, \qquad \alpha_2 = \frac{F_1}{2} + \frac{F_2}{L} \tag{37}$$

2. Center of annulus is lined with a poison, black to slow neutrons, but transparent to fast neutrons. Otherwise the center is void.

$$\begin{aligned} \alpha_1 &= \frac{F_1}{\lambda_1} + \frac{F_2}{L} \\ \alpha_2 &= \frac{1.5}{\lambda_2} \end{aligned} \tag{38}$$

3. Center is lined with thin poison sheet, black to slow neutrons but transparent to fast neutrons. In addition the central region is filled with a non-multiplying material having fast group constants D_1^h and Σ_1^h.

   The fast flux will satisfy the equation

$$\nabla^2 \phi_1^h - \frac{\Sigma_1^h}{D_1^h} \phi_1^h = 0, \text{ which has the solution}$$

$$\phi_1^h = E\, I_o\, (\varkappa_1^h r), \text{ where}$$

$$(\varkappa_1^h)^2 = \frac{\Sigma_1^h}{D_1^h} = \frac{1}{\tau^h}.$$

At the inside boundary, $\dfrac{\phi_1^{h'}}{\phi_1^{h}} = \dfrac{\mathcal{H}_1^{h}\, I_1(\mathcal{H}_1^{h}R)}{I_o(\mathcal{H}_1^{h}R)}$.

By the continuity of flux and current, one finds

$$\left.\begin{aligned} \alpha_1 &= \frac{D_1^{h}}{D_1} \cdot \frac{\phi_1^{h'}}{\phi_1^{h}} = \frac{D_1^{h}}{D_1} \cdot \frac{\mathcal{H}_1^{h}\, I_1(\mathcal{H}_1^{h}R)}{I_o(\mathcal{H}_1^{h}R)} \\ &\text{As in case 2 above,} \\ \alpha_2 &= \frac{1.5}{\lambda_2} \end{aligned}\right\} \qquad (39)$$

4. This method is not applicable to the case in which the inner region is filled with a non-multiplying material without a poison shell.

5. Limiting cases.

(a) Infinite length.

In this limit $\alpha_1 = \alpha_2 = 0$ and the critical determinant (29) reduces to $f_2 = 0$, or

$$\frac{J_1(\mu R)}{Y_1(\mu R)} = \frac{J_o(\mu P)}{Y_o(\mu P)} \qquad (40)$$

Let P_o be the outside critical radius of a solid cylinder of infinite length and define

$$\begin{aligned} P_2 &\equiv \frac{P}{P_o} \\ P_1 &\equiv \frac{R}{P_o} \end{aligned} \qquad (41)$$

The difference $P_2 - P_1 = \frac{P-R}{P_o}$, or the annular thickness in terms of the critical solid cylinder radius is plotted in Fig. 5 as a function of P_1. For large P_1 the roots of the Bessel functions are separated by π, and the ratio approaches the value

$$P_2' - P_1 \rightarrow \frac{\pi}{2(2.405)} = 0.65314 \qquad (42)$$

It is of interest to note that the critical volume for unit length of annulus (ignoring the extrapolation distance) is

$$V(P_1) = \pi(P^2-R^2) = \pi P_o^{\,2}(P_2^{\,2} - P_1^2) = V_o\left[(P_2-P_1)^2 + 2P_1(P_2-P_1)\right]$$

where V_o is the volume per unit length of the critical solid cylinder. For large P_1, $P_2 - P_1$ approaches a constant and the volume increases linearly with P_1, (Fig. 6).

IV. VALIDITY OF THE METHOD

The practical utility of the method lies in the fact that the effects of neutron streaming in the void of a cylindrical annulus reactor are taken into account, to at least a first approximation, by the simple application of a boundary condition. Once the functions $F_1(L/R)$ and $F_2(L/R)$ have been calculated, values for the boundary conditions may be read directly from the curves of these functions. The boundary conditions are valid even for systems in which the inside radius, i.e., the radius of the central void, is large compared to the length of the cylinder.

As has been indicated in an earlier section, the assumption of separability of variables is not strictly valid for this problem. The neutron current found by a detailed summing of incremental contributions will, in general, not equal the neutron current found by a diffusion expression which is itself only a first degree, or P_1, approximation. Further, equating neutron currents at the lateral midplane of the annulus, i.e., at $z = 0$, is an arbitrary choice and, strictly speaking, neutron currents will not be exactly matched at other values of z. However, this choice does lead to a useful solution. Despite these obvious difficulties, the method results in a good approximation as is indicated by the agreement of the two limiting cases which may be found by other considerations.

In the limiting case of an annulus of infinite length, there will be no loss of neutrons to the inside void due to streaming. This implies a zero current and zero radial derivative of the flux at the inside radius of the annulus. This corresponds to the limit found by the present method. (see equations (10) and (11)),

$$\lim_{L/R\to\infty} (\lambda\phi_R'/\phi_R) = \lim_{L/R\to\infty} F_1(L/R) + \lim_{L/R\to\infty} (\lambda/L)\, F_2(L/R) = 0$$

where $L \neq 0$.

It may be shown that the denominators in the expressions for F_1 and F_2 both become infinite in this limit.

In the opposite extreme case, i.e., one in which the void radius becomes very large compared to the length of the cylinder, the return current will approach zero. This is similar to the case of a bare slab for which a boundary condition is

$$j+ = \phi/4 - \lambda\phi'/6 = 0,$$

$$\text{or} \quad \lambda\phi'/\phi = 3/2$$

corresponding to the limit, in the present method

$$\lim_{L/R \to 0} (\lambda\phi_R'/\phi_R) = \lim_{L/R \to 0} F_1(L/R) + \lim_{L/R \to 0} (\lambda/L)\, F_2(L/R)$$

$$= 3/2 + 0 = 3/2, \text{ where } L \neq 0.$$

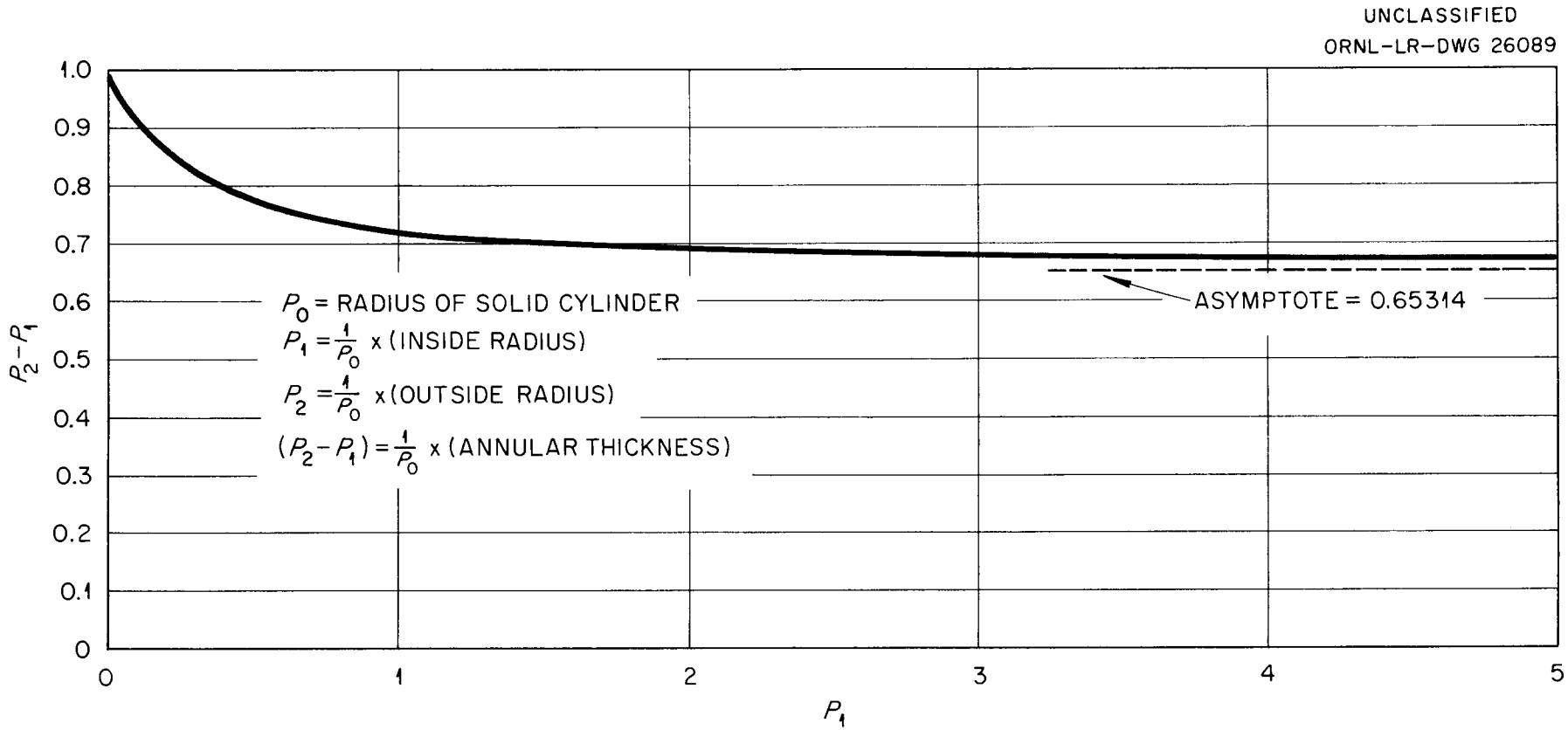

Fig. 5. Critical Thickness of Annulus as a Function of Inside Radius.—(Limiting Case $\alpha_1 = \alpha_2 = 0$)

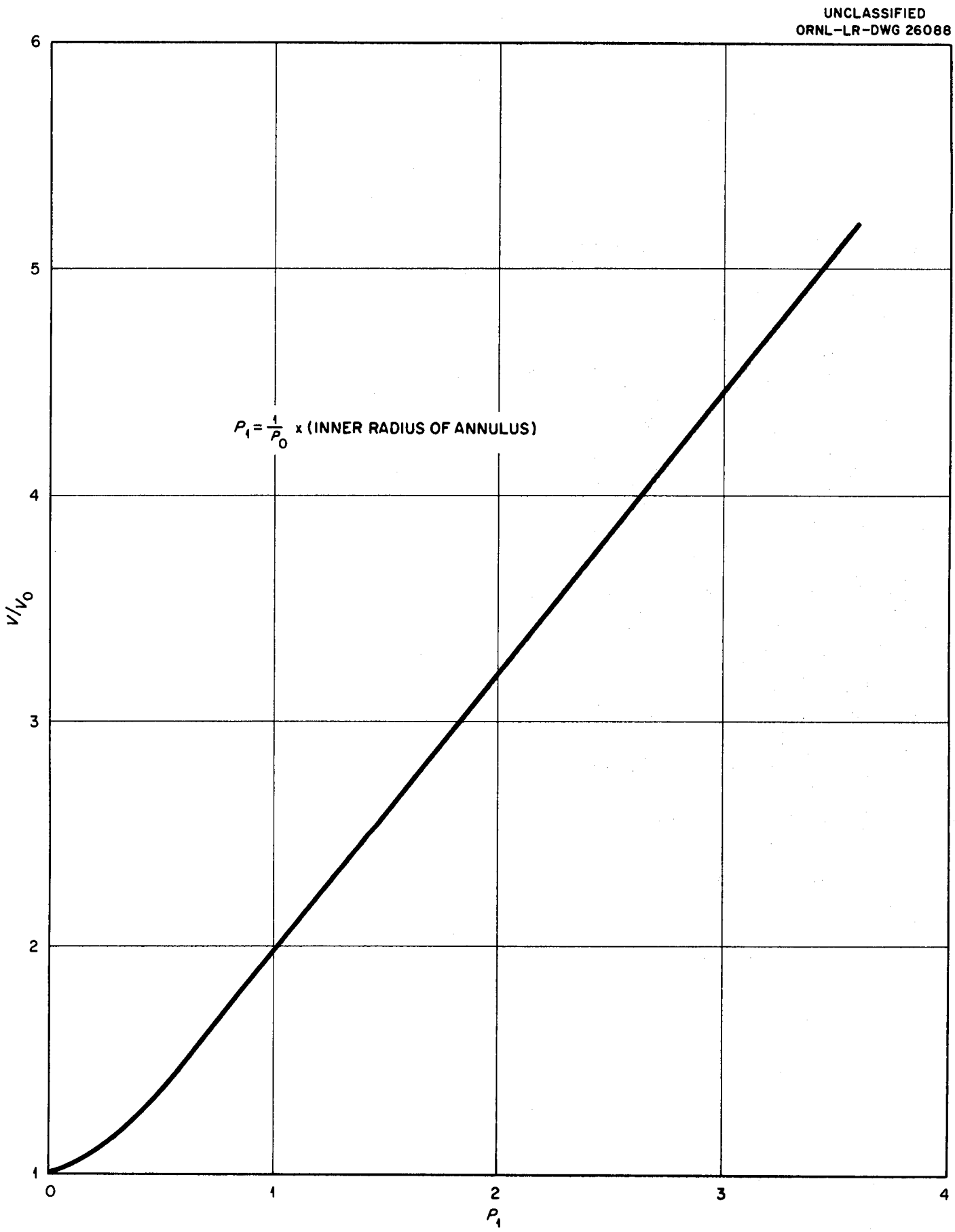

Fig. 6. Critical Volume Per Unit Length of Annulus Compared to Volume Per Unit Length of a Solid Cylinder.—(Limiting Case $\alpha_1 = \alpha_2 = 0$)

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. A. D. Callihan for his help throughout this work, to Professor Richard C. Keen for assistance in comparing a number of the results obtained by the two-group treatment to experimental results, to Dr. A. Simon for many suggestions and to Mrs. M. L. Rueff for typing the manuscript.

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