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ON KOFINK'S APPROXIMATE SOLUTION OF THE NEUTRON TRANSPORT EQUATION

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Printed in USA. Price 25 cents. Available from the

Office of Technical Services
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Washington 25, D. C.

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ORNL-2485
Physics and Mathematics

Contract No. W-7405-eng-26

MATHEMATICS PANEL

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Ky Fan and J. J. Andrews

DATE ISSUED

MAR 19 1958

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§1. Introduction

Recently W. Kofink [3], [4] has applied the spherical harmonics method for obtaining approximate solutions of the neutron transport equation. The present paper is a mathematically rigorous exposition of the main part of Kofink's work and is self-contained. Originally our interest was to investigate the question of convergence of Kofink's approximate solutions, but we have found it necessary first to reorganize his results on a more rigorous basis. Throughout this paper, we follow Kofink's treatment except when clarity or rigor requires modification or supplementary proofs. His several ingenious calculations are adopted with some occasional changes. Thus the only contribution of the present authors is to supply rigorous proofs in certain derivations and thereby make it explicit and precise under what hypothesis the results are valid.

In §2, we describe the approximate solutions f_n of the Boltzmann equation (2.1). To define these approximate solutions, several preliminaries will be needed. First, it is necessary to investigate the zeros of certain polynomials G_n. This is done in §3. Secondly, certain identities needed in §2 are derived in §4. Then the long calculation of certain coefficients α_i is carried out in §5. Only with these §§ 3-5 is the construction of the approximate solutions given in §2 complete. Finally, in the last §6, we examine the behavior of the approximate solutions.

We wish to express our thanks to Dr. A. M. Weinberg for calling our attention to Kofink's work and for his continuous advice; to Dr. A. S. Householder for his valuable discussions and for simplifying the proof of Theorem 3.1.

§2. Approximate solutions of the Boltzmann equation.

The Boltzmann equation considered in this work is

$$(2.1)\quad y\,\frac{\partial f(x,y)}{\partial x} + f(x,\,y) = \frac{1-\gamma_a}{2}\int_{-1}^{1} f(x,y)\,dy + \frac{3(1-\gamma_t)}{2}\,y\int_{-1}^{1} f(x,\,y)y\,dy\,,$$

where the unknown function f is defined for $-\infty < x < +\infty$, $-1 \leqslant y \leqslant 1$ and is required to satisfy the following boundary conditions:

$$(2.2)\qquad \lim_{x\to-\infty} e^{\frac{x}{y}} f(x,y) = 0 \quad \text{for } y > 0;$$

$$(2.3)\qquad f(0,\,y) = 0 \qquad \text{for } -1 \leqslant y \leqslant 0.$$

γ_a, γ_t are constants, but their values depend on whether $x \geqslant 0$ or $x < 0$:

$$(2.4)\qquad \begin{cases} \gamma_a = \gamma_t = 1 & \text{for } x \geqslant 0; \\ 0 < \gamma_a \leqslant 1,\ \ 0 < \gamma_t \leqslant 1 & \text{and not both equal to 1, for } x < 0. \end{cases}$$

We refer to [3], [8; Chap. IX] for the physical meaning of equation (2.1).

In the spherical harmonics method, as an approximate solution of (2.1) we shall take a linear combination of functions each of the form

$$(2.5)\qquad e^{-\frac{x}{\lambda}} \sum_{\ell=0}^{2n} (2\ell+1)\, G_\ell(\lambda)\, P_\ell(y),$$

where P_ℓ denotes, as usual, the Legendre polynomial of degree ℓ, and where λ is a parameter.

When we substitute the expression (2.5) for f(x,y) in (2.1), we obtain

$$-\frac{y}{\lambda}\sum_{\ell=0}^{2n} (2\ell+1)\, G_\ell(\lambda)\, P_\ell(y) + \sum_{\ell=0}^{2n} (2\ell+1)\, G_\ell(\lambda)\, P_\ell(y) = (1-\gamma_a)\, G_0(\lambda) + 3(1-\gamma_t)y\, G_1(\lambda).$$

Using the recurrence relation

$$(2.6) \qquad (2\ell+1)y\, P_\ell(y) = (\ell+1)\, P_{\ell+1}(y) + \ell\, P_{\ell-1}(y),$$

the above equation can be written

$$-\frac{1}{\lambda}\sum_{\ell=0}^{2n} G_\ell(\lambda)\left[(\ell+1)\, P_{\ell+1}(y) + \ell\, P_{\ell-1}(y)\right] + \sum_{\ell=0}^{2n}(2\ell+1)\, G_\ell(\lambda)\, P_\ell(y)$$

$$= (1-\gamma_a)\, G_0(\lambda)\, P_0(y) + 3(1-\gamma_t)\, G_1(\lambda)\, P_1(y),$$

or

$$(2.7) \qquad \begin{cases} G_1(\lambda) = \gamma_a\, \lambda\, G_0(\lambda), \qquad 2G_2(\lambda) + G_0(\lambda) = 3\,\gamma_t\, \lambda\, G_1(\lambda), \\ (\ell+1)\, G_{\ell+1}(\lambda) + \ell\, G_{\ell-1}(\lambda) = (2\ell+1)\, \lambda\, G_\ell(\lambda), \qquad (2 \leqslant \ell \leqslant 2n-1) \\ 2n\, G_{2n-1}(\lambda) = (4n+1)\lambda\, G_{2n}(\lambda), \qquad G_{2n}(\lambda) = 0. \end{cases}$$

Obviously (2.7) implies that $G_\ell(\lambda) \equiv 0$ for $0 \leqslant \ell \leqslant 2n$. This merely means that equation (2.1) has no solution of the form (2.5) besides the trivial solution $f(x,y) \equiv 0$.

To remedy this situation, we shall still consider functions of form (2.5), but we shall require G_ℓ to satisfy

$$(2.8) \qquad \begin{cases} G_0(\lambda) = 1, \qquad G_1(\lambda) = \gamma_a \lambda, \\ 2G_2(\lambda) + G_0(\lambda) = 3\gamma_t\, \lambda\, G_1(\lambda), \\ (\ell+1)\, G_{\ell+1}(\lambda) + \ell\, G_{\ell-1}(\lambda) = (2\ell+1)\, \lambda\, G_\ell(\lambda), \qquad (2 \leqslant \ell \leqslant 2n-1) \\ G_{2n}(\lambda) = 0 \end{cases}$$

instead of (2.7). Notice that (2.8) is obtained from (2.7) by dropping the relation $2n\, G_{2n-1}(\lambda) = (4n+1)\, \lambda\, G_{2n}(\lambda)$ and by adding $G_0(\lambda) = 1$.

Because the recurrence relation for $2 \leqslant \ell \leqslant 2n-1$ in (2.8) is the same as that for Legendre functions, we set

$$(2.9)\qquad \begin{cases} G_0(\lambda) = 1, \qquad G_1(\lambda) = \gamma_a \lambda, \\ G_\ell(\lambda) = A(\lambda)\, P_\ell(\lambda) + B(\lambda)\, W_{\ell-1}(\lambda), & (2 \leqslant \ell \leqslant 2n) \end{cases}$$

where

$$(2.10)\qquad W_{\ell-1}(\lambda) = P_\ell(\lambda)\, Q_0(\lambda) - Q_\ell(\lambda),$$

Q_ℓ being the Legendre function of the second kind (see, e.g. [2; p. 152]). If the parameter λ satisfies

$$(2.11)\qquad A(\lambda)\, P_{2n}(\lambda) + B(\lambda)\, W_{2n-1}(\lambda) = 0,$$

then the G_ℓ's defined by (2.9) will satisfy the relation $G_{2n}(\lambda) = 0$ and the recurrence relations for $3 \leqslant \ell \leqslant 2n-1$ in (2.8). We choose $A(\lambda)$, $B(\lambda)$ in such a way that the recurrence relation for $\ell = 2$, i.e.

$$3\, G_3(\lambda) + 2\, G_1(\lambda) = 5\, \lambda\, G_2(\lambda)$$

and the two remaining relations of (2.8)

$$G_1(\lambda) = \gamma_a \lambda, \qquad 2\, G_2(\lambda) + G_0(\lambda) = 3\, \gamma_t\, \lambda\, G_1(\lambda)$$

will also be satisfied. In other words, $A(\lambda)$, $B(\lambda)$ are so chosen that

$$G_2(\lambda) = A(\lambda)\, P_2(\lambda) + B(\lambda)\, W_1(\lambda) = \frac{3}{2}\gamma_a \gamma_t \lambda^2 - \frac{1}{2},$$

$$G_3(\lambda) = A(\lambda)\, P_3(\lambda) + B(\lambda)\, W_2(\lambda) = \frac{5}{2}\gamma_a \gamma_t \lambda^3 - \frac{2}{3}\gamma_a \lambda - \frac{5}{6}\lambda .$$

As

$$P_2(\lambda) = \frac{3}{2}\lambda^2 - \frac{1}{2}, \qquad P_3(\lambda) = \frac{5}{2}\lambda^3 - \frac{3}{2}\lambda,$$

$$W_1(\lambda) = \frac{3}{2}\lambda, \qquad W_2(\lambda) = \frac{5}{2}\lambda^2 - \frac{2}{3},$$

a simple calculation yields

$$A(\lambda) = 1 + 3\, \gamma_a (1-\gamma_t)\, \lambda^2, \qquad B(\lambda) = -\lambda\left[1 - \gamma_a + 3\, \gamma_a (1-\gamma_t)\, \lambda^2\right].$$

To simplify, we introduce

$$a = 3\,\gamma_a\,(1-\gamma_t), \qquad c = 1 - \gamma_a. \tag{2.12}$$

Then we can state the above result as follows:

<u>Lemma 2.1.</u> <u>Let</u>

$$G_0(\lambda) = 1,\ G_1(\lambda) = (1-c)\lambda,\ G_\ell(\lambda) = (1+a\lambda^2)\,P_\ell(\lambda) - \lambda(c+a\lambda^2)\,W_{\ell-1}(\lambda) \qquad (2 \leqslant \ell \leqslant 2n), \tag{2.13}$$

<u>where</u> a, c <u>are given by</u> (2.12). <u>Then for every zero</u> λ <u>of the polynomial</u> G_{2n}, <u>the quantities</u> $G_\ell(\lambda)$ $(0 \leqslant \ell \leqslant 2n)$ <u>satisfy all relations in</u> (2.8).

In view of (2.4), the constants a, c introduced by (2.12) depend on whether $x \geqslant 0$ or $x < 0$:

$$\begin{cases} a = c = 0 \quad \text{for } x \geqslant 0; \\ 0 \leqslant c < 1,\ 0 \leqslant a < 3(1-c) \text{ and } a^2 + c^2 \neq 0 \quad \text{for } x < 0. \end{cases} \tag{2.14}$$

Consider first the case $x \geqslant 0$. In this case, $a = c = 0$, the polynomials G_ℓ defined by (2.13) are simply the Legendre polynomials P_ℓ. P_{2n} is even and has exactly n positive zeros [6; p. 43]:

$$\mu_1^{(2n)} > \mu_2^{(2n)} > \cdots > \mu_n^{(2n)}. \tag{2.15}$$

Thus we form a linear combination of n functions each of form (2.5) (ccoresponding to the n positive zeros of P_{2n}):

$$f_n(x,y) = \sum_{j=1}^{n} \beta_j\, e^{-\frac{x}{\mu_j}} \sum_{\ell=0}^{2n-1} (2\ell+1)\,P_\ell(\mu_j)\,P_\ell(y) \quad \text{for } x \geqslant 0, \tag{2.16}$$

where β_j are constant coefficients, and where we have set $\mu_j = \mu_j^{(2n)}$ for the sake of simplicity. Here we do not use the negative zeros of P_{2n} in (2.16) because we want $\lim_{x\to+\infty} f_n(x, y) = 0$ for $y > 0$. This condition did not appear among the boundary

conditions (2.2), (2.3), but the equation $y \frac{\partial f(x, y)}{\partial x} + f(x, y) = 0$ (i.e. equation (2.1) for $x \geqslant 0$) implies $f(x, y) = e^{-\frac{x}{y}} f(0, y)$ for $x \geqslant 0$, $y \neq 0$ and therefore $\lim_{x \to +\infty} f(x, y) = 0$ for $y > 0$.

For $x < 0$, we have $0 \leqslant c < 1$, $0 \leqslant a < 3(1 - c)$ and $a^2 + c^2 \neq 0$. For $n \geqslant 1$, the polynomial

$$G_{2n}(\lambda) = (1 + a\lambda^2) P_{2n}(\lambda) - \lambda(c + a\lambda^2) W_{2n-1}(\lambda)$$

is even and of degree 2n. It will be shown in §3 (Theorem 3.2) that G_{2n} has exactly n positive zeros which are all distinct. The positive zeros of G_{2n} will be denoted by

$$\lambda_1^{(2n)} > \lambda_2^{(2n)} > \dots > \lambda_n^{(2n)} . \tag{2.17}$$

The negative zero of G_{2n} of largest absolute value will be denoted by $\lambda_0^{(2n)}$, i.e.

$$\lambda_0^{(2n)} = -\lambda_1^{(2n)} . \tag{2.18}$$

Then we form a linear combination of n + 1 functions each of form (2.5) (corresponding to the zeros $-\lambda_0^{(2n)}, -\lambda_1^{(2n)}, \dots, -\lambda_n^{(2n)}$ of G_{2n}):

$$f_n(x, y) = \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} \sum_{\ell=0}^{2n-1} (2\ell+1) G_\ell(-\lambda_i) P_\ell(y) \quad \text{for } x < 0, \tag{2.19}$$

where α_i are constant coefficients and where we have set $\lambda_i = \lambda_i^{(2n)}$ $(0 \leqslant i \leqslant n)$ for the sake of simplicity. Here in (2.19), we use only the n + 1 zeros $-\lambda_0, -\lambda_1, \dots, -\lambda_n$ of G_{2n}. The reason for this lies in the boundary condition (2.2) and will become clear in the proof of Theorem 6.1.

We now have

$$f_n(x, y) = \begin{cases} \sum_{j=1}^{n} \beta_j e^{-\frac{x}{\mu_j}} \sum_{\ell=0}^{2n-1} (2\ell+1) P_\ell(\mu_j) P_\ell(y) & \text{for } x \geqslant 0, \\ \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} \sum_{\ell=0}^{2n-1} (2\ell+1) G_\ell(-\lambda_i) P_\ell(y) & \text{for } x < 0. \end{cases} \tag{2.20}$$

By use of §4, an alternative but identical expression for f_n is

$$(2.21)\qquad f_n(x, y) = \begin{cases} 2n\, P_{2n}(y) \sum_{j=1}^{n} \beta_j\, e^{-\frac{x}{\mu_j}} \dfrac{P_{2n-1}(\mu_j)}{y - \mu_j} & \text{for } x \geqslant 0, \\[2ex] -\sum_{i=0}^{n} \alpha_i \lambda_i\, e^{\frac{x}{\lambda_i}} \left\{ a\lambda_i + \dfrac{c + a\lambda_i^2}{P_{2n}(\lambda_i)} \dfrac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i} \right\} & \text{for } x < 0. \end{cases}$$

We proceed to choose the coefficients α_i, β_j so that the function f_n defined by (2.21) is continuous along the line $x = 0$.

<u>Lemma 2.2</u>. <u>The function</u> f_n <u>defined by</u> (2.21) <u>is continuous along the line</u> $x = 0$ (and therefore continuous for $-\infty < x < +\infty$, $-1 \leqslant y \leqslant 1$), <u>if and only if the coefficients</u> α_i, β_j <u>are so chosen that</u>

$$(2.22)\qquad \sum_{i=0}^{n} \alpha_i \lambda_i \left(a\lambda_i - \frac{c + a\lambda_i^2}{\lambda_i - \mu_j} \right) = 0, \qquad (1 \leqslant j \leqslant n)$$

$$(2.23)\qquad \beta_j = \frac{1}{2n\, P_{2n-1}(\mu_j)\, P'_{2n}(\mu_j)} \sum_{i=0}^{n} \alpha_i \lambda_i \left(\frac{c + a\lambda_i^2}{\lambda_i + \mu_j} - a\lambda_i \right), \qquad (1 \leqslant j \leqslant n).$$

<u>Proof</u>. f_n is continuous along the line $x = 0$, if and only if the two expressions in (2.21) are identical for $x = 0$, i.e.

$$(2.24)\qquad 2n\, P_{2n}(y) \sum_{j=1}^{n} \beta_j \frac{P_{2n-1}(\mu_j)}{y - \mu_j} =$$

$$= -\sum_{i=0}^{n} \alpha_i \lambda_i \left\{ a\lambda_i + \frac{c + a\lambda_i^2}{P_{2n}(\lambda_i)} \frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i} \right\}, \quad (-1 \leqslant y \leqslant 1).$$

Since the left-hand side vanishes for $y = -\mu_j$ $(1 \leqslant j \leqslant n)$, the α_i's must necessarily satisfy (2.22). Now

$$\varphi(y) = -\sum_{i=0}^{n} \alpha_i \lambda_i \left\{ a\lambda_i + \frac{c + a\lambda_i^2}{P_{2n}(\lambda_i)} \frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i} \right\}$$

being a polynomial of degree $\leqslant 2n - 1$, we have the partial fraction decomposition

$$\frac{\varphi(y)}{P_{2n}(y)} = \sum_{j=1}^{n} \frac{\varphi(\mu_j)}{P'_{2n}(\mu_j)} \frac{1}{y - \mu_j} + \sum_{j=1}^{n} \frac{\varphi(-\mu_j)}{P'_{2n}(-\mu_j)} \frac{1}{y + \mu_j} .$$

If the necessary condition (2.22) is satisfied, then we have $\varphi(-\mu_j) = 0$ $(1 \leqslant j \leqslant n)$ and

$$\frac{\varphi(y)}{P_{2n}(y)} = \sum_{j=1}^{n} \frac{\varphi(\mu_j)}{P'_{2n}(\mu_j)} \frac{1}{y - \mu_j} .$$

Hence, under the assumption (2.22), relation (2.24) will hold, if and only if

$$\beta_j = \frac{\varphi(\mu_j)}{2n\, P_{2n-1}(\mu_j)\, P'_{2n}(\mu_j)} \qquad (1 \leqslant j \leqslant n);$$

which is precisely (2.23).

As (2.22) is a system of n homogeneous linear equations in n + 1 unknowns α_i, we shall add a normalizing condition

$$\alpha_0 + \alpha_1 = 1. \tag{2.25}$$

In §5, we shall solve the system of n + 1 equations (2.22), (2.25) for the α_i's. It will be shown (Theorems 5.1, 5.2, 5.3) that α_i's are uniquely determined by (2.22), (2.25) in the following three cases:

(Case 1) $0 < c < 1$, $0 < a < 3(1 - c)$ and $i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1 F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right) \neq 0$; where F_+, F_- are defined by (5.7), (5.8).

(Case 2) $0 < c < 1$, $a = 0$.

(Case 3) $c = 0$, $0 < a < 3$.

In each of these cases, the function f_n defined by (2.21), with α_i, β_j uniquely determined by (2.22), (2.25), (2.23), will be called the nth <u>approximate solution</u> of equation (2.1).

§3. Zeros of the polynomials G_n.

In §2, we have encountered (see (2.13)) polynomials G_n defined by

$$G_0(\lambda) = 1, \quad G_1(\lambda) = (1 - c)\lambda, \quad G_n(\lambda) = (1 + a\lambda^2) P_n(\lambda) - \lambda(c + a\lambda^2) W_{n-1}(\lambda)$$

for $n \geqslant 2$.

Since $W_0(\lambda) = 1$, we have actually

$$(3.1) \qquad G_n(\lambda) = (1 + a\lambda^2) P_n(\lambda) - \lambda(c + a\lambda^2) W_{n-1}(\lambda) \qquad \text{for } n \geqslant 1$$

and therefore the recurrence relation (same as that for P_n, Q_n)

$$(3.2) \qquad (n + 1) G_{n+1}(\lambda) - (2n+1)\lambda G_n(\lambda) + n G_{n-1}(\lambda) = 0 \qquad \text{for } n \geqslant 2.$$

This relation together with

$$(3.3) \qquad G_1(\lambda) = (1 - c)\lambda, \qquad G_2(\lambda) = \frac{3(1-c) - a}{2}\lambda^2 - \frac{1}{2}$$

determine the G_n's completely. The purpose of the present §3 is to study the distribution of the zeros of G_n $(n \geqslant 1)$. In §2, the constants a, c are either both 0, or they satisfy the restrictions

$$(3.4) \qquad 0 \leqslant c < 1, \qquad 0 \leqslant a < 3(1 - c), \qquad a^2 + c^2 \neq 0.$$

When $a = c = 0$, we have $G_n = P_n$. As the distribution of the zeros of P_n is well-known [6; pp. 118, 121], we shall assume (3.4) throughout this §3.

Theorem 3.1. For $n \geqslant 1$, all n zeros of G_n are real and distinct. Furthermore, when the zeros $\lambda_k^{(n)}$ of G_n and the zeros $\lambda_k^{(n+1)}$ of G_{n+1} are each arranged in descending order, they satisfy the inequalities

$$(3.5) \qquad \lambda_1^{(n+1)} > \lambda_1^{(n)} > \lambda_2^{(n+1)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n+1)} > \lambda_n^{(n)} > \lambda_{n+1}^{(n+1)}.$$

<u>Proof</u>. By (3.2), (3.3) and (3.4), G_n $(n \geqslant 1)$ is a polynomial of degree n and the coefficient of λ^n in G_n is positive. Therefore

$$\lim_{\lambda \to +\infty} G_n(\lambda) = +\infty, \qquad \lim_{\lambda \to -\infty} (-1)^n G_n(\lambda) = +\infty. \tag{3.6}$$

Clearly the zeros of G_1, G_2 are real and satisfy $\lambda_1^{(2)} > \lambda_1^{(1)} > \lambda_2^{(2)}$. Assume for $i \leqslant n$, the zeros of each G_i are all real and distinct and separate the zeros of G_{i-1}. We shall prove that the same is true for G_{n+1}. Consider two consecutive zeros $\lambda_{k-1}^{(n)}$, $\lambda_k^{(n)}$ of G_n. Then

$$G_{n-1}\left(\lambda_{k-1}^{(n)}\right) \; G_{n-1}\left(\lambda_k^{(n)}\right) < 0.$$

This and (3.2) imply that

$$G_{n+1}\left(\lambda_{k-1}^{(n)}\right) \; G_{n+1}\left(\lambda_k^{(n)}\right) < 0,$$

so that G_{n+1} has at least one zero, say $\lambda_k^{(n+1)}$, such that $\lambda_{k-1}^{(n)} > \lambda_k^{(n+1)} > \lambda_k^{(n)}$. Next, from $G_{n-1}\left(\lambda_1^{(n)}\right) \neq 0$ and (3.2), we have

$$G_{n+1}\left(\lambda_1^{(n)}\right) \; G_{n-1}\left(\lambda_1^{(n)}\right) < 0.$$

This and (3.6) imply that G_{n+1} must change sign in the interval $\lambda_1^{(n)} < \lambda < +\infty$. Hence G_{n+1} has at least one zero, say $\lambda_1^{(n+1)}$, in this interval. Similarly, G_{n+1} must have a zero $\lambda_{n+1}^{(n+1)} < \lambda_n^{(n)}$. This accounts for all zeros of G_{n+1}.

For the sake of simplicity, whenever only one polynomial G_n is involved, its zeros will be denoted by λ_k instead of $\lambda_k^{(n)}$.

<u>Theorem 3.2</u>. <u>For</u> $n \geqslant 1$, G_{2n} <u>has exactly</u> n <u>positive zeros</u> λ_k $(1 \leqslant k \leqslant n)$ <u>and they satisfy the inequalities</u>

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_n > \mu_n, \tag{3.7}$$

<u>where</u> μ_k $(1 \leqslant k \leqslant n)$ <u>denote the positive zeros of</u> P_{2n}[1].

<u>Proof</u>. By Theorem 3.1, all 2n zeros of G_{2n} are real and distinct. Since G_{2n} is even, it follows that G_{2n} has exactly n positive zeros.

Let

$$\mathbb{Q}_n(\lambda) = \frac{1}{2}\left\{Q_n(\lambda + i\,0) + Q_n(\lambda - i\,0)\right\}, \qquad (-1 < \lambda < 1),$$

Then

$$\mathbb{Q}_n(\lambda) = \mathbb{Q}_0(\lambda)\, P_n(\lambda) - W_{n-1}(\lambda) \qquad (-1 < \lambda < 1)$$

and therefore

$$\mathbb{Q}_{2n}(\mu_k) = -\,W_{2n-1}(\mu_k) \qquad (1 \leqslant k \leqslant n).$$

Since $\operatorname{sgn} \mathbb{Q}_{2n}(\mu_k) = (-1)^k$ [6; p. 152], we have

$$\operatorname{sgn} W_{2n-1}(\mu_k) = (-1)^{k-1} \qquad (1 \leqslant k \leqslant n).$$

By (3.1),

$$G_{2n}(\mu_k) = -\mu_k\,(c + a\,\mu_k^2)\, W_{2n-1}(\mu_k).$$

By (3.4), we have

$$\operatorname{sgn} G_{2n}(\mu_k) = -\operatorname{sgn} W_{2n-1}(\mu_k) = (-1)^k. \qquad (1 \leqslant k \leqslant n) \tag{3.8}$$

---

(1) As G_{2n}, P_{2n} are even polynomials, it suffices to consider only their positive zeros.

Hence none of μ_k is a zero of G_{2n}, and the number of zeros of G_{2n} in the interval $\mu_k < \lambda < \mu_{k+1}$ $(1 \leqslant k \leqslant n-1)$ is odd. Next, by comparing sgn $G_{2n}(\mu_n) = (-1)^n$ with sgn $G_{2n}(0) =$ sgn $P_{2n}(0) = (-1)^n$, we infer that G_{2n} has an even number of zeros in $0 < \lambda < \mu_n$. Finally, since the total number of positive zeros of G_{2n} is n, it follows that G_{2n} has exactly one zero in each of the intervals $+\infty > \lambda > \mu_1$ and $\mu_k > \lambda > \mu_{k+1}$ $(1 \leqslant k \leqslant n-1)$.

Theorem 3.3. Let $\lambda_1^{(n)}$ denote the largest positive zero of G_n $(n \geqslant 2)$. Then the increasing sequence

$$\lambda_1^{(2)} < \lambda_1^{(3)} < \cdots < \lambda_1^{(n)} < \cdots \tag{3.9}$$

converges to a finite limit

$$x_0 = \lim_{n\to\infty} \lambda_1^{(n)} > 1. \tag{3.10}$$

And $\pm x_0$ are the only zeros of the function

$$G(z) = \frac{1 + a z^2}{c + a z^2} - \frac{z}{2} \log \frac{z+1}{z-1}, \tag{3.11}$$

where we take the principal branch of $\log \frac{z+1}{z-1}$.

Proof. Let

$$R_n(z) = \frac{n\, G_n(z)}{z\, G_{n-1}(z)}. \qquad (n \geqslant 2) \tag{3.12}$$

Then the recurrence relation (3.2) can be written

$$R_n(z) = \frac{n^2}{z^2} \frac{1}{2n + 1 - R_{n+1}(z)}. \qquad (n \geqslant 2) \tag{3.13}$$

Now

$$R_2(z) = \frac{[3(1-c) - a]\, z^2 - 1}{(1-c)\, z^2}$$

or

$$\frac{c + a z^2}{1 + a z^2} = 1 - \frac{1}{z^2 \left[3 - R_2(z)\right]}.$$

By repeated application of (3.13), we get

$$\frac{c + a z^2}{1 + a z^2} = 1 - \frac{w^2}{3 - R_2(z)} = 1 - \cfrac{w^2}{3 - \cfrac{4w^2}{5 - R_3(z)}} = \cdots$$

or

$$(3.14) \qquad \frac{c + a z^2}{1 + a z^2} = 1 - \frac{w^2}{3} - \frac{4w^2}{5} - \frac{9w^2}{7} - \cdots - \frac{(n-1)^2 w^2}{(2n-1) - R_n(z)},$$

where $w = \frac{1}{z}$. Consider now the rational function

$$(3.15) \qquad S_n(z) = \frac{c + a z^2}{1 + a z^2} - \left\{ 1 - \frac{w^2}{3} - \frac{4w^2}{5} - \cdots - \frac{(n-1)^2 w^2}{2n - 1} \right\}, \quad (w = \frac{1}{z}).$$

The continued fraction

$$1 - \frac{w^2}{3} - \frac{4w^2}{5} - \cdots - \frac{(n-1)^2 w^2}{(2n - 1)} - \cdots$$

converges to

$$\left(\frac{1}{2w} \log \frac{1 + w}{1 - w} \right)^{-1}$$

uniformly in every closed region in the complex w-plane cut along the real axis from -1 to $-\infty$ and from $+1$ to $+\infty$ [1; p. 125]. Therefore $S_n(z)$ converges to

$$(3.16) \qquad S(z) = \frac{c + a z^2}{1 + a z^2} - \left(\frac{z}{2} \log \frac{z + 1}{z - 1} \right)^{-1}$$

uniformly in every closed region in the z-plane which is disjoint from the closed interval $[-1, 1]$ on the real axis.

It is easy to see that the function G defined by (3.11) has no zero in the interval $[-1, 1]$ on the real axis. Consider now any zero z_0 of G. Then z_0 is also a zero of S and lies outside the interval $[-1, 1]$. According to what we have just proved and by a theorem of Hurwitz [7; p. 119], z_0 must be a point of accumulation

of the zeros of the sequence of functions $\{S_n\}$. But, from (3.15), (3.14), and (3.12), the zeros of S_n are also zeros of G_n, hence z_0 is a point of accumulation of the zeros of the sequence of polynomials $\{G_n\}$.

Since the zeros of Legendre polynomials are all between -1 and 1, so by Theorem 3.2, $\pm \lambda_1^{(2n)}$ are the only zeros of G_{2n} $(n \geqslant 1)$ which may be outside $[-1, 1]$. By Theorem 3.1, we have the inequalities (3.9) and therefore $\pm\lambda_1^{(2n+1)}$ are the only zeros of G_{2n+1} $(n \geqslant 1)$ which may be outside $[-1, 1]$. Hence, if G has zeros, then the increasing sequence (3.9) must converge to a finite limit $x_0 = \lim_{n\to\infty} \lambda_1^{(n)}$ and $\pm x_0$ are the only zeros of G.

It remains, therefore, to show that G does have a real zero > 1. Since $\lim_{x\to 1+0} G(x) = -\infty$, we need only to prove that $G(x_1) > 0$ for some $x_1 > 1$.

From the power series of $\log(1 + y)$, we have

$$\log(1 + y) - y + \frac{y^2}{2} < \frac{y^3}{3} \qquad (-1 < y < 1).$$

Hence

$$\log\left(1 + \frac{2}{x - 1}\right) - \frac{2}{x - 1} + \frac{2}{(x - 1)^2} < \frac{8}{3(x - 1)^3} \qquad \text{for } x > 3,$$

or

$$\frac{x}{2}\log\left(1 + \frac{2}{x - 1}\right) - \frac{x}{x - 1} + \frac{x}{(x - 1)^2} - \frac{4x}{3(x - 1)^3} < 0 \qquad \text{for } x > 3. \tag{3.17}$$

Let

$$\varphi(x) = \frac{1 + a x^2}{c + a x^2} - \frac{x}{x - 1} + \frac{x}{(x - 1)^2} - \frac{4x}{3(x - 1)^3}. \tag{3.18}$$

Then (3.17) becomes

$$G(x) > \varphi(x) \qquad \text{for } x > 3. \tag{3.19}$$

We find

$$\varphi(x) = \frac{3(1 - c) - a}{3(x - 1)^3 (c + a x^2)} \psi(x), \tag{3.20}$$

where

$$(3.21)\quad \psi(x) = x^3 - \frac{3[3(1-c)+a]}{3(1-c)-a}\, x^2 + \frac{9-10c}{3(1-c)-a}\, x - \frac{3}{3(1-c)-a}\,.$$

Take

$$x_1 = \frac{36}{3(1-c)-a}$$

By (3.4), we have $x_1 > 12$ and

$$\frac{3[3(1-c)+a]}{3(1-c)-a} < \frac{x_1}{2},\quad \frac{|9-10c|}{3(1-c)-a} < \frac{x_1}{2},\quad \frac{3}{3(1-c)-a} < \frac{x_1}{2}\,.$$

Consequently,

$$\psi(x_1) > x_1^3 - \frac{x_1^3}{2} - \frac{x_1^2}{2} - \frac{x_1}{2} = \frac{x_1\,[x_1(x_1-1)-1]}{2} > 0,$$

which combined with (3.19), (3.20) yields

$$G(x_1) > 0 \qquad \text{with } x_1 = \frac{36}{3(1-c)-a} > 12.$$

§4. Two identities.

We shall now derive two identities which are needed in §2 [see (2.20), (2.21)].

Lemma 4.1. Let μ_j denote the jth positive zero of P_{2n}. Then

$$(4.1)\quad \sum_{\ell=0}^{2n-1} (2\ell+1)\, P_\ell(\mu_j)\, P_\ell(y) = 2n\, P_{2n}(y)\, \frac{P_{2n-1}(\mu_j)}{y-\mu_j}.$$

Proof. This follows immediately from Christoffel's first summation formula [2; p. 162]

$$(4.2)\quad (y-x)\sum_{\ell=0}^{m} (2\ell+1)\, P_\ell(x)\, P_\ell(y) = (m+1)\left[P_m(x)\, P_{m+1}(y) - P_m(y)\, P_{m+1}(x)\right].$$

<u>Lemma 4.2</u>. <u>Let</u>

(4.3) $G_0(\lambda) = 1, \quad G_\ell(\lambda) = (1 + a\lambda^2) P_\ell(\lambda) - \lambda(c + a\lambda^2) W_{\ell-1}(\lambda)$ <u>for</u> $\ell \geqslant 1$;

<u>where</u> c, a <u>satisfy</u> (3.4). <u>Let</u> λ_i <u>denote the</u> i^{th} <u>positive zero of</u> G_{2n}. <u>Then</u>

$$(4.4) \qquad \sum_{\ell=0}^{2n-1} (2\ell + 1) G_\ell(-\lambda_i) P_\ell(y) =$$

$$= -a\lambda_i^2 + \frac{\lambda_i (c + a\lambda_i^2)}{y + \lambda_i} \left[1 - \frac{P_{2n}(y)}{P_{2n}(-\lambda_i)}\right].$$

<u>Proof</u>. Besides (4.2), we shall also need Christoffel's second formula [2; p. 162]

$$(4.5) \qquad (x - y) \sum_{\ell=0}^{m} (2\ell + 1) P_\ell(y) Q_\ell(x) =$$

$$= 1 - (m + 1)\left[P_{m+1}(y) Q_m(x) - P_m(y) Q_{m+1}(x)\right].$$

Since $G_0(\lambda) = 1$ and

$$G_\ell(-\lambda) = (1 + a\lambda^2) P_\ell(-\lambda) + \lambda(c + a\lambda^2)\left[P_\ell(-\lambda) Q_0(-\lambda) - Q_\ell(-\lambda)\right], \quad (\ell \geqslant 1)$$

we write

$$\sum_{\ell=0}^{2n-1} (2\ell + 1) G_\ell(-\lambda) P_\ell(y) =$$

$$= -a\lambda^2 + \left\{(1 + a\lambda^2) + \lambda(c + a\lambda^2) Q_0(-\lambda)\right\} \sum_{\ell=0}^{2n-1} (2\ell + 1) P_\ell(-\lambda) P_\ell(y)$$

$$- \lambda(c + a\lambda^2) \sum_{\ell=0}^{2n-1} (2\ell + 1) P_\ell(y) Q_\ell(-\lambda).$$

Then by (4.2), (4.5),

$$\sum_{\ell=0}^{2n-1} (2\ell + 1)\, G_\ell\,(-\lambda)\, P_\ell\,(y) = -\,a\,\lambda^2 +$$

$$+ \left\{(1 + a\,\lambda^2) + \lambda(c + a\,\lambda^2)\, Q_0\,(-\lambda)\right\} \frac{2n}{y+\lambda}\left[P_{2n-1}\,(-\lambda)\, P_{2n}\,(y) - P_{2n-1}(y)\, P_{2n}(-\lambda)\right]$$

$$+ \frac{\lambda(c + a\lambda^2)}{y+\lambda}\left\{1 - 2n\left[P_{2n}\,(y)\, Q_{2n-1}\,(-\lambda) - P_{2n-1}\,(y)\, Q_{2n}\,(-\lambda)\right]\right\}$$

$$= -\,a\,\lambda^2 + \frac{\lambda(c + a\,\lambda^2)}{y+\lambda} +$$

$$+ \frac{2n\, P_{2n}\,(y)}{y+\lambda}\left\{(1 + a\,\lambda^2)\, P_{2n-1}\,(-\lambda) + \lambda(c + a\,\lambda^2)\left[P_{2n-1}\,(-\lambda)\, Q_0\,(-\lambda) - Q_{2n-1}(-\lambda)\right]\right\}$$

$$- \frac{2n\, P_{2n-1}\,(y)}{y+\lambda}\left\{(1 + a\,\lambda^2)\, P_{2n}\,(-\lambda) + \lambda(c + a\,\lambda^2)\left[P_{2n}\,(-\lambda)\, Q_0\,(-\lambda) - Q_{2n}\,(-\lambda)\right]\right\}$$

or

$$\sum_{\ell=0}^{2n-1} (2\ell + 1)\, G_\ell\,(-\lambda)\, P_\ell\,(y) = -\,a\,\lambda^2 + \frac{\lambda(c + a\,\lambda^2)}{y+\lambda} +$$

$$+ \frac{2n\, P_{2n}(y)}{y+\lambda}\left\{(1 + a\,\lambda^2)\, P_{2n-1}\,(-\lambda) + \lambda(c + a\,\lambda^2)\, W_{2n-2}\,(-\lambda)\right\}$$

$$- \frac{2n\, P_{2n-1}(y)}{y+\lambda}\, G_{2n}\,(-\lambda).$$

As λ_i is a zero of G_{2n}, we have $G_{2n}\,(-\,\lambda_i) = 0$ and

$$1 + a\,\lambda_i^2 = -\,\lambda_i\,(c + a\,\lambda_i^2)\,\frac{W_{2n-1}\,(-\lambda_i)}{P_{2n}\,(-\lambda_i)}$$

and the last equation becomes

$$(4.6)\quad \sum_{\ell=0}^{2n-1} (2\ell + 1)\, G_\ell\,(-\lambda_i)\, P_\ell\,(y) = -\,a\,\lambda_i^2 + \frac{\lambda_i\,(c + a\,\lambda_i^2)}{y+\lambda_i} +$$

$$+ 2n\,\frac{P_{2n}\,(y)}{P_{2n}\,(-\lambda_i)}\,\frac{\lambda_i\,(c + a\,\lambda_i^2)}{y+\lambda_i}\left\{P_{2n}\,(-\,\lambda_i)\, W_{2n-2}\,(-\,\lambda_i) - P_{2n-1}\,(-\,\lambda_i)\, W_{2n-1}(-\,\lambda_i)\right\}.$$

On the other hand, we have

$$P_{2n} W_{2n-2} - P_{2n-1} W_{2n-1} =$$

$$= P_{2n} (P_{2n-1} Q_0 - Q_{2n-1}) - P_{2n-1} (P_{2n} Q_0 - Q_{2n})$$

$$= P_{2n-1} Q_{2n} - P_{2n} Q_{2n-1},$$

and, by setting $x = y = -\lambda_i$, $m = 2n - 1$ in (4.5):

$$0 = 1 - 2n \left[P_{2n} (-\lambda_i) Q_{2n-1} (-\lambda_i) - P_{2n-1} (-\lambda_i) Q_{2n} (-\lambda_i) \right].$$

Hence

$$P_{2n} (-\lambda_i) W_{2n-2} (-\lambda_i) - P_{2n-1} (-\lambda_i) W_{2n-1} (-\lambda_i) = -\frac{1}{2n}$$

and (4.6) reduces to (4.4).

§5. Calculation of the coefficients α_k.

This section is devoted to the solution of the system of n + 1 linear equations

$$\sum_{k=0}^{n} \alpha_k \lambda_k \left(a \lambda_k - \frac{c + a \lambda_k^2}{\lambda_k - \mu_j} \right) = 0, \qquad (1 \leqslant j \leqslant n) \tag{5.1}$$

and

$$\alpha_0 + \alpha_1 = 1 ; \tag{5.2}$$

which has arisen in §2. We recall that c, a here are subject to restrictions

$$0 \leqslant c < 1, \qquad 0 \leqslant a < 3(1 - c), \qquad a^2 + c^2 \neq 0 ; \tag{5.3}$$

$\mu_1 > \mu_2 > \cdots > \mu_n$ are the positive zeros of P_{2n}; $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ are the positive zeros of the polynomial $G_{2n} (\lambda) = (1 + a \lambda^2) P_{2n} (\lambda) - \lambda(c + a \lambda^2) W_{2n-1} (\lambda)$; and

$\lambda_0 = -\lambda_1$. In Theorem 3.2, we have seen that

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_n > \mu_n. \tag{5.4}$$

In the calculation below, we shall use the following notation:

$$p(\lambda) = \prod_{j=1}^{n} (\lambda - \mu_j), \tag{5.5}$$

$$r(\lambda) = \prod_{j=2}^{n} (\lambda - \lambda_j), \tag{5.6}$$

$$F_+(\lambda) = r(\lambda)\, p(-\lambda) + r(-\lambda)\, p(\lambda), \tag{5.7}$$

$$F_-(\lambda) = r(\lambda)\, p(-\lambda) - r(-\lambda)\, p(\lambda). \tag{5.8}$$

<u>Lemma 5.1</u>. <u>The system of</u> n <u>equations</u> (5.1) <u>is equivalent to the system of equations</u>

$$n\, a \sum_{k=0}^{n} \alpha_k \lambda_k^2 - \sum_{k=0}^{n} \left[\alpha_k \lambda_k (c + a\lambda_k^2) \sum_{j=1}^{n} \frac{1}{\lambda_k - \mu_j} \right] = 0 \tag{5.9}$$

<u>and</u>

$$\alpha_k = \frac{\lambda_1 (c + a\lambda_1^2)}{\lambda_k (c + a\lambda_k^2)} \frac{p(\lambda_k)}{r'(\lambda_k)} \left[\frac{\alpha_1 r(\lambda_1)}{(\lambda_k - \lambda_1) p(\lambda_1)} - \frac{\alpha_0 r(-\lambda_1)}{(\lambda_k + \lambda_1)\, p(-\lambda_1)} \right], \qquad (2 \leqslant k \leqslant n). \tag{5.10}$$

<u>Proof</u>. Equation (5.9) is obtained by summing the n equations of (5.1). If we subtract the jth equation of (5.1) from the first equation, we get

$$\sum_{k=0}^{n} \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{(\lambda_k - \mu_1)(\lambda_k - \mu_j)} = 0, \qquad (2 \leqslant j \leqslant n). \tag{5.11}$$

Thus (5.1) is equivalent to (5.9) and (5.11) together.

Let

$$b_k = \frac{\alpha_k \lambda_k (c + a \lambda_k^2)}{p(\lambda_k)}, \qquad (0 \leq k \leq n). \tag{5.12}$$

Then (5.11) can be written

$$\sum_{k=2}^{n} b_k \frac{p(\lambda_k)}{(\lambda_k-\mu_1)(\lambda_k-\mu_j)} = - \frac{b_0\, p(\lambda_0)}{(\lambda_0-\mu_1)(\lambda_0-\mu_j)} - \frac{b_1\, p(\lambda_1)}{(\lambda_1-\mu_1)(\lambda_1-\mu_j)}, \qquad (2 \quad j \quad n). \tag{5.13}$$

This system (5.13) can be solved by Cramer's rule:

$$b_k = \frac{\triangle_k}{\triangle}, \qquad (2 \leq k \leq n).$$

We have

$$\triangle = \frac{\prod_{i=2}^{n} p(\lambda_i)}{\prod_{i=2}^{n} (\lambda_i-\mu_1)} \begin{vmatrix} \frac{1}{\lambda_2-\mu_2} & \frac{1}{\lambda_3-\mu_2} & \cdots & \frac{1}{\lambda_n-\mu_2} \\ \frac{1}{\lambda_2-\mu_3} & \frac{1}{\lambda_3-\mu_3} & \cdots & \frac{1}{\lambda_n-\mu_3} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{\lambda_2-\mu_n} & \frac{1}{\lambda_3-\mu_n} & \cdots & \frac{1}{\lambda_n-\mu_n} \end{vmatrix}$$

$$= \frac{\prod_{2 \leq i \leq n} p(\lambda_i)}{\prod_{2 \leq i \leq n} (\lambda_i-\mu_1)} \; \frac{\prod_{2 \leq i < j \leq n} (\lambda_i-\lambda_j) \cdot \prod_{2 \leq i < j \leq n} (\mu_j-\mu_i)}{\prod_{2 \leq i \leq n} \prod_{2 \leq j \leq n} (\lambda_i-\mu_j)}$$

$$= \prod_{2 \leq i < j \leq n} (\lambda_i-\lambda_j) \cdot \prod_{2 \leq i < j \leq n} (\mu_j-\mu_i)$$

[5; p. 98], and similarly

$$\triangle_k = - b_0 (-1)^k \prod_{\substack{0 \leq i < j \leq n \\ i,j \neq 1,k}} (\lambda_i - \lambda_j) \cdot \prod_{2 \leq i < j \leq n} (\mu_j - \mu_i)$$

$$- b_1 (-1)^k \prod_{\substack{1 \leq i < j \leq n \\ i,j \neq k}} (\lambda_i - \lambda_j) \cdot \prod_{2 \leq i < j \leq n} (\mu_j - \mu_i), \qquad (2 \leq k \leq n).$$

Thus

$$(5.14) \qquad b_k = \frac{\triangle_k}{\triangle} = \frac{-b_0 \prod_{\substack{2 \leq j \leq n \\ j \neq k}} (\lambda_0 - \lambda_j) - b_1 \prod_{\substack{2 \leq j \leq n \\ j \neq k}} (\lambda_1 - \lambda_j)}{\prod_{\substack{2 \leq j \leq n \\ j \neq k}} (\lambda_k - \lambda_j)}, \qquad (2 \leq k \leq n).$$

By (5.6) we have

$$(5.15) \qquad r'(\lambda_k) = \prod_{\substack{2 \leq j \leq n \\ j \neq k}} (\lambda_k - \lambda_j), \qquad (2 \leq k \leq n).$$

Hence (5.14) becomes

$$b_k = \frac{1}{r'(\lambda_k)} \left[\frac{b_0 \, r(\lambda_0)}{\lambda_k - \lambda_0} + \frac{b_1 \, r(\lambda_1)}{\lambda_k - \lambda_1} \right] \qquad (2 \leq k \leq n).$$

Now, (5.3) implies $c + a \lambda_k^2 \neq 0$. Therefore by (5.12) and $\lambda_0 = - \lambda_1$, we have

$$\alpha_k = \frac{p(\lambda_k)}{r'(\lambda_k)} \frac{1}{\lambda_k (c + a \lambda_k^2)} \left[\frac{b_0 \, r(\lambda_0)}{\lambda_k - \lambda_0} + \frac{b_1 \, r(\lambda_1)}{\lambda_k - \lambda_1} \right], \qquad (2 \leq k \leq n)$$

and

$$\frac{b_0 \, r(\lambda_0)}{\lambda_k - \lambda_0} + \frac{b_1 \, r(\lambda_1)}{\lambda_k - \lambda_1} = \lambda_1 (c + a \lambda_1^2) \left[\frac{\alpha_1 \, r(\lambda_1)}{p(\lambda_1)(\lambda_k - \lambda_1)} - \frac{\alpha_0 \, r(-\lambda_1)}{p(-\lambda_1)(\lambda_k + \lambda_1)} \right].$$

This yields (5.10) which solves (5.11) for α_k $(2 \leq k \leq n)$ in terms of α_0, α_1. Since (5.1) is equivalent to (5.9), (5.11) together, so it is also equivalent to (5.9) combined with (5.10).

<u>Lemma 5.2</u>. <u>When the</u> α_k's <u>satisfy</u> (5.10), <u>we have</u>

$$(5.16)\quad \sum_{k=0}^{n}\left[\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\sum_{j=1}^{n}\frac{1}{\lambda_k - \mu_j}\right] =$$

$$= n\,\lambda_1\,(c + a\,\lambda_1^2)\left[\alpha_1\,\frac{r(\lambda_1)}{p\,(\lambda_1)} - \alpha_0\,\frac{r(-\lambda_1)}{p(-\lambda_1)}\right].$$

<u>Proof</u>. First, since $\lambda_0 = -\lambda_1$, we write

$$(5.17)\quad \sum_{k=0}^{n}\left[\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\sum_{j=1}^{n}\frac{1}{\lambda_k - \mu_j}\right] = \sum_{k=0}^{n}\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\,\frac{p'(\lambda_k)}{p(\lambda_k)}$$

$$= \lambda_1\,(c + a\,\lambda_1^2)\left[\alpha_1\,\frac{p'(\lambda_1)}{p(\lambda_1)} - \alpha_0\,\frac{p'(-\lambda_1)}{p(-\lambda_1)}\right] + \sum_{k=2}^{n}\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\,\frac{p'(\lambda_k)}{p(\lambda_k)}.$$

By (5.10),

$$\sum_{k=2}^{n}\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\,\frac{p'(\lambda_k)}{p\,(\lambda_k)} =$$

$$= \sum_{k=2}^{n}\lambda_1\,(c + a\,\lambda_1^2)\,\frac{p'(\lambda_k)}{r'(\lambda_k)}\left[\frac{\alpha_1\,r(\lambda_1)}{(\lambda_k-\lambda_1)\,p(\lambda_1)} - \frac{\alpha_0\,r(-\lambda_1)}{(\lambda_k + \lambda_1)\,p(-\lambda_1)}\right]$$

or

$$(5.18)\quad \sum_{k=2}^{n}\alpha_k\,\lambda_k\,(c + a\,\lambda_k^2)\,\frac{p'(\lambda_k)}{p(\lambda_k)} =$$

$$= \frac{\lambda_1\,(c + a\,\lambda_1^2)}{p(\lambda_1)\,p(-\lambda_1)}\sum_{k=2}^{n}\frac{p'(\lambda_k)}{r'(\lambda_k)}\,\frac{\alpha_1\,r(\lambda_1)\,p(-\lambda_1)(\lambda_k + \lambda_1) - \alpha_0\,r(-\lambda_1)\,p(\lambda_1)(\lambda_k - \lambda_1)}{\lambda_k^2 - \lambda_1^2}.$$

To compute the Σ-sum on the right-hand side of (5.18), we introduce

$$\varphi(\lambda) = p'(\lambda)\,[\alpha_1\, r(\lambda_1)\, p(-\lambda_1)(\lambda + \lambda_1) - \alpha_0\, r(-\lambda_1)\, p(\lambda_1)(\lambda-\lambda_1)],$$

$$\psi(\lambda) = \prod_{j=1}^{n} (\lambda-\lambda_j) = (\lambda-\lambda_1)\, r(\lambda).$$

φ is a polynomial in λ of degree $\leqslant n$. The coefficient of λ^n in $\varphi(\lambda)$ is $n\,\gamma_0$, where

$$\gamma_0 = \alpha_1\, r(\lambda_1)\, p(-\lambda_1) - \alpha_0\, r(-\lambda_1)\, p(\lambda_1). \tag{5.19}$$

ψ is a polynomial in λ of degree n, the coefficient of λ^n in $\psi(\lambda)$ is 1; and the zeros λ_j $(1 \leqslant j \leqslant n)$ of ψ are all distinct. Therefore we have the partial fraction decomposition

$$\frac{\varphi(\lambda)}{\psi(\lambda)} = n\,\gamma_0 + \sum_{j=1}^{n} \frac{\varphi(\lambda_j)}{\psi'(\lambda_j)}\,\frac{1}{\lambda - \lambda_j}\,.$$

Then we have

$$\sum_{j=2}^{n} \frac{\varphi(\lambda_j)}{\psi'(\lambda_j)}\,\frac{1}{\lambda + \lambda_j} = n\,\gamma_0 - \frac{\varphi(-\lambda)}{\psi(-\lambda)} - \frac{\varphi(\lambda_1)}{\psi'(\lambda_1)(\lambda + \lambda_1)}\,.$$

Now the Σ-sum on the right-hand side of (5.18) is

$$\sum_{k=2}^{n} \frac{\varphi(\lambda_k)}{\psi'(\lambda_k)(\lambda_1 + \lambda_k)} = n\,\gamma_0 - \frac{\varphi(-\lambda_1)}{\psi(-\lambda_1)} - \frac{\varphi(\lambda_1)}{2\lambda_1\,\psi'(\lambda_1)}$$

$$= n\,\gamma_0 + \alpha_0\, p(\lambda_1)\, p'(-\lambda_1) - \alpha_1\, p(-\lambda_1)\, p'(\lambda_1).$$

Thus (5.18) becomes

$$\sum_{k=2}^{n} \alpha_k\,\lambda_k\,(c + a\lambda_k^2)\,\frac{p'(\lambda_k)}{p(\lambda_k)} =$$

$$= \frac{\lambda_1\,(c + a\,\lambda_1^2)}{p(\lambda_1)\,p(-\lambda_1)} \Big\{ n\big[\alpha_1\, r(\lambda_1)\, p(-\lambda_1) - \alpha_0\, r(-\lambda_1)\, p(\lambda_1)\big] +$$

$$+ \alpha_0\, p(\lambda_1)\, p'(-\lambda_1) - \alpha_1\, p(-\lambda_1)\, p'(\lambda_1)\Big\}\,,$$

which combined with (5.17) yields the desired relation (5.16).

Lemma 5.3. If, in addition to (5.3), we assume $c > 0$, $a > 0$, and if the α_k's satisfy (5.10), then

$$(5.20)\quad a\sum_{k=0}^{n} \alpha_k \lambda_k^2 = \frac{\lambda_1}{p(\lambda_1)\, p(-\lambda_1)} \left\{ (c + a\lambda_1^2)\left[\alpha_1 r(\lambda_1)\, p(-\lambda_1) - \alpha_0 r(-\lambda_1)\, p(\lambda_1)\right] \right.$$

$$\left. + a \operatorname{Re} \frac{p\left(i\sqrt{\frac{c}{a}}\right)}{r\left(i\sqrt{\frac{c}{a}}\right)} \left[\alpha_1 r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right) + \alpha_0 r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right)\right]\right\}.$$

Proof. By (5.10),

$$a\sum_{k=0}^{n} \alpha_k \lambda_k^2 = a(\alpha_0 + \alpha_1)\lambda_1^2 + a\sum_{k=2}^{n} \alpha_k \lambda_k^2 =$$

$$= a(\alpha_0 + \alpha_1)\lambda_1^2 + a\lambda_1 (c + a\lambda_1^2)\sum_{k=2}^{n} \frac{\lambda_k}{c + a\lambda_k^2} \frac{p(\lambda_k)}{r'(\lambda_k)} \left[\frac{\alpha_1 r(\lambda_1)}{(\lambda_k - \lambda_1)p(\lambda_1)} - \frac{\alpha_0 r(-\lambda_1)}{(\lambda_k + \lambda_1)p(-\lambda_1)}\right],$$

or

$$(5.21)\quad a\sum_{k=0}^{n} \alpha_k \lambda_k^2 = a(\alpha_0 + \alpha_1)\lambda_1^2 + \frac{\lambda_1(c + a\lambda_1^2)}{p(\lambda_1)\, p(-\lambda_1)} \sum_{k=2}^{n} \frac{f(\lambda_k)}{r'(\lambda_k)(\lambda_k^2 - \lambda_1^2)(\lambda_k^2 + \frac{c}{a})},$$

where

$$(5.22)\quad f(\lambda) = \lambda\, p(\lambda)\left[\alpha_1 r(\lambda_1)\, p(-\lambda_1)(\lambda + \lambda_1) - \alpha_0 r(-\lambda_1)\, p(\lambda_1)(\lambda - \lambda_1)\right].$$

f is a polynomial in λ of degree $\leq n + 2$. The coefficient of λ^{n+2} in $f(\lambda)$ is γ_0 given by (5.19). Consider the polynomial of degree $n + 2$

$$(5.23)\qquad g(\lambda) = (\lambda^2 + \frac{c}{a}) \prod_{j=1}^{n} (\lambda - \lambda_j) = (\lambda^2 + \frac{c}{a})(\lambda - \lambda_1)\, r(\lambda).$$

Since $c > 0$, $a > 0$, the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n, i\sqrt{\frac{c}{a}}, -i\sqrt{\frac{c}{a}}$ of g are all distinct. Hence we have the partial fraction decomposition

$$(5.24) \qquad \frac{f(\lambda)}{g(\lambda)} = \gamma_0 + \sum_{j=1}^{n} \frac{f(\lambda_j)}{g'(\lambda_j)} \frac{1}{\lambda - \lambda_j} + 2\,\mathrm{Re}\,\frac{f\left(i\sqrt{\frac{c}{a}}\right)}{g'\left(i\sqrt{\frac{c}{a}}\right)\left(\lambda - i\sqrt{\frac{c}{a}}\right)} .$$

Then the Σ-sum on the right-hand side of (5.21) is

$$(5.25) \qquad \sum_{k=2}^{n} \frac{f(\lambda_k)}{r'(\lambda_k)(\lambda_k^2 - \lambda_1^2)(\lambda_k^2 + \frac{c}{a})} = \sum_{k=2}^{n} \frac{f(\lambda_k)}{g'(\lambda_k)(\lambda_1 + \lambda_k)}$$

$$= \gamma_0 - \frac{f(-\lambda_1)}{g(-\lambda_1)} - \frac{f(\lambda_1)}{2\lambda_1\, g'(\lambda_1)} - 2\,\mathrm{Re}\,\frac{f\left(i\sqrt{\frac{c}{a}}\right)}{g'\left(i\sqrt{\frac{c}{a}}\right)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right)} .$$

One verifies easily

$$(5.26) \quad \begin{cases} \dfrac{f(-\lambda_1)}{g(-\lambda_1)} = \dfrac{\alpha_0\, \lambda_1\, p(\lambda_1)\, p(-\lambda_1)}{\lambda_1^2 + \frac{c}{a}} , \\[2ex] \dfrac{f(\lambda_1)}{2\lambda_1\, g'(\lambda_1)} = \dfrac{\alpha_1\, \lambda_1\, p(\lambda_1)\, p(-\lambda_1)}{\lambda_1^2 + \frac{c}{a}} , \\[2ex] \dfrac{f\left(i\sqrt{\frac{c}{a}}\right)}{g'\left(i\sqrt{\frac{c}{a}}\right)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right)} = \dfrac{p\left(i\sqrt{\frac{c}{a}}\right)}{2\, r\left(i\sqrt{\frac{c}{a}}\right)} \left[\dfrac{\alpha_1\, r(\lambda_1)\, p(-\lambda_1)}{i\sqrt{\frac{c}{a}} - \lambda_1} - \dfrac{\alpha_0\, r(-\lambda_1)\, p(\lambda_1)}{i\sqrt{\frac{c}{a}} + \lambda_1} \right] . \end{cases}$$

From (5.21), (5.25) and (5.26), we obtain

$$a \sum_{k=0}^{n} \alpha_k \lambda_k^2 = a(\alpha_0 + \alpha_1)\lambda_1^2 + \frac{\lambda_1 (c + a\lambda_1^2)}{p(\lambda_1)\, p(-\lambda_1)} \left\{ \alpha_1\, r(\lambda_1)\, p(-\lambda_1) - \alpha_0\, r(-\lambda_1)\, p(\lambda_1) - \right.$$

$$\left. - (\alpha_0 + \alpha_1) \frac{\lambda_1\, p(\lambda_1)\, p(-\lambda_1)}{\lambda_1^2 + \frac{c}{a}} - \mathrm{Re}\, \frac{p\left(i\sqrt{\frac{c}{a}}\right)}{r\left(i\sqrt{\frac{c}{a}}\right)} \left[\frac{\alpha_1\, r(\lambda_1)\, p(-\lambda_1)}{i\sqrt{\frac{c}{a}} - \lambda_1} - \frac{\alpha_0\, r(-\lambda_1)\, p(\lambda_1)}{i\sqrt{\frac{c}{a}} + \lambda_1} \right] \right\} ,$$

which is precisely (5.20).

Lemma 5.4. If, in addition to (5.3), we assume $c = 0$, and if the α_k's satisfy (5.10), then

$$(5.27)\qquad \sum_{k=0}^{n} \alpha_k \lambda_k^2 = \lambda_1^3 \left[\alpha_1 \frac{r(\lambda_1)}{p(\lambda_1)} - \alpha_0 \frac{r(-\lambda_1)}{p(-\lambda_1)} \right] + \lambda_1^2 \frac{p(0)}{r(0)} \left[\alpha_1 \frac{r(\lambda_1)}{p(\lambda_1)} + \alpha_0 \frac{r(-\lambda_1)}{p(-\lambda_1)} \right].$$

Proof. By (5.10) and $c = 0$, we have

$$\sum_{k=0}^{n} \alpha_k \lambda_k^2 = (\alpha_0 + \alpha_1) \lambda_1^2 + \lambda_1^3 \sum_{k=2}^{n} \frac{1}{\lambda_k} \frac{p(\lambda_k)}{r'(\lambda_k)} \left[\frac{\alpha_1 r(\lambda_1)}{(\lambda_k - \lambda_1) p(\lambda_1)} - \frac{\alpha_0 r(-\lambda_1)}{(\lambda_k + \lambda_1) p(-\lambda_1)} \right]$$

or

$$(5.28)\qquad \sum_{k=0}^{n} \alpha_k \lambda_k^2 = (\alpha_0 + \alpha_1) \lambda_1^2 + \frac{\lambda_1^3}{p(\lambda_1)\, p(-\lambda_1)} \sum_{k=2}^{n} \frac{\Phi(\lambda_k)}{\lambda_k\, r'(\lambda_k)(\lambda_k^2 - \lambda_1^2)}$$

with

$$(5.29)\qquad \Phi(\lambda) = p(\lambda) \left[\alpha_1 r(\lambda_1)\, p(-\lambda_1)\, (\lambda + \lambda_1) - \alpha_0 r(-\lambda_1)\, p(\lambda_1)\, (\lambda - \lambda_1) \right].$$

If we set

$$(5.30)\qquad \Psi(\lambda) = \lambda \prod_{j=1}^{n} (\lambda - \lambda_j) = \lambda(\lambda - \lambda_1)\, r(\lambda),$$

then we have the partial fraction decomposition

$$(5.31)\qquad \frac{\Phi(\lambda)}{\Psi(\lambda)} = \gamma_0 + \frac{\Phi(0)}{\Psi'(0)} \frac{1}{\lambda} + \sum_{j=1}^{n} \frac{\Phi(\lambda_j)}{\Psi'(\lambda_j)} \frac{1}{\lambda - \lambda_j},$$

where γ_0 is given by (5.19). Now,

$$\sum_{k=2}^{n} \frac{\Phi(\lambda_k)}{\lambda_k\, r'(\lambda_k)(\lambda_k^2 - \lambda_1^2)} = \sum_{k=2}^{n} \frac{\Phi(\lambda_k)}{\Psi'(\lambda_k)(\lambda_k + \lambda_1)} =$$

$$= \gamma_0 - \frac{\Phi(-\lambda_1)}{\Psi(-\lambda_1)} - \frac{\Phi(0)}{\Psi'(0)} \frac{1}{\lambda_1} - \frac{\Phi(\lambda_1)}{2\lambda_1 \Psi'(\lambda_1)}$$

$$= \gamma_0 - \frac{\alpha_0\, p(\lambda_1)\, p(-\lambda_1)}{\lambda_1} + \frac{p(0)}{r(0)} \frac{\alpha_1\, r(\lambda_1)\, p(-\lambda_1) + \alpha_0\, r(-\lambda_1)\, p(\lambda_1)}{\lambda_1} - \frac{\alpha_1\, p(\lambda_1)\, p(-\lambda_1)}{\lambda_1}$$

$$= \gamma_0 - (\alpha_0 + \alpha_1) \frac{p(\lambda_1)\, p(-\lambda_1)}{\lambda_1} + \frac{p(0)}{r(0)} \frac{\alpha_1\, r(\lambda_1)\, p(-\lambda_1) + \alpha_0\, r(-\lambda_1)\, p(\lambda_1)}{\lambda_1},$$

so (5.28) becomes (5.27).

We are now ready to solve the system of equations (5.1), (5.2).

<u>Theorem 5.1</u>. <u>If</u> $0 < c < 1$, $0 < a < 3(1 - c)$ <u>and if</u>

$$(5.32) \qquad i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1 F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right) \neq 0,$$

<u>then the system of equations</u> (5.1), (5.2) <u>has a unique solution given by</u> (5.10) <u>and</u>

$$(5.33) \qquad \alpha_1 = \frac{p(\lambda_1)\, r(-\lambda_1) \left[i\sqrt{\frac{c}{a}}\, F_-\left(i\sqrt{\frac{c}{a}}\right) + \lambda_1 F_+\left(i\sqrt{\frac{c}{a}}\right) \right]}{i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1 F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right)},$$

$$(5.34) \qquad \alpha_0 = \frac{p(-\lambda_1)\, r(\lambda_1) \left[i\sqrt{\frac{c}{a}}\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1 F_+\left(i\sqrt{\frac{c}{a}}\right) \right]}{i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1 F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right)}.$$

From the definition (5.7), (5.8) of F_+, F_-, it is clear that $F_+\left(i\sqrt{\frac{c}{a}}\right)$ is real and $F_-\left(i\sqrt{\frac{c}{a}}\right)$ is purely imaginary. Thus in (5.33), (5.34), each of α_0, α_1 is expressed as the quotient of two real numbers.

Proof. According to Lemma 5.1, the system of equations (5.1), (5.2) is equivalent to the system (5.9), (5.10), (5.2). By Lemmas 5.2 and 5.3, equation (5.9) can be written

$$\mathrm{Re}\,\frac{p\left(i\sqrt{\frac{c}{a}}\right)}{r\left(i\sqrt{\frac{c}{a}}\right)}\left[\alpha_1\, r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right) + \alpha_0\, r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right)\right] = 0,$$

or, what is the same,

$$p\left(i\sqrt{\frac{c}{a}}\right) r\left(-i\sqrt{\frac{c}{a}}\right)\left\{\alpha_1\, r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right) + \alpha_0\, r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right)\right\}$$

$$+\, p\left(-i\sqrt{\frac{c}{a}}\right) r\left(i\sqrt{\frac{c}{a}}\right)\left\{\alpha_1\, r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right) + \alpha_0\, r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right)\right\} = 0.$$

By (5.2), this may be written

$$\alpha_1\, p\left(i\sqrt{\frac{c}{a}}\right) r\left(-i\sqrt{\frac{c}{a}}\right)\left\{r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right) - r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right)\right\}$$

$$+\, \alpha_1\, p\left(-i\sqrt{\frac{c}{a}}\right) r\left(i\sqrt{\frac{c}{a}}\right)\left\{r(\lambda_1)\, p(-\lambda_1)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right) - r(-\lambda_1)\, p(\lambda_1)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right)\right\}$$

$$= -\, r(-\lambda_1)\, p(\lambda_1)\left\{p\left(i\sqrt{\frac{c}{a}}\right) r\left(-i\sqrt{\frac{c}{a}}\right)\left(\lambda_1 - i\sqrt{\frac{c}{a}}\right) + p\left(-i\sqrt{\frac{c}{a}}\right) r\left(i\sqrt{\frac{c}{a}}\right)\left(\lambda_1 + i\sqrt{\frac{c}{a}}\right)\right\},$$

or, according to (5.7), (5.8),

$$\text{(5.35)}\qquad \alpha_1\left\{i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1\, F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right)\right\}$$

$$= p(\lambda_1)\, r(-\lambda_1)\left\{i\sqrt{\frac{c}{a}}\, F_-\left(i\sqrt{\frac{c}{a}}\right) + \lambda_1\, F_+\left(i\sqrt{\frac{c}{a}}\right)\right\}.$$

From (5.35) and (5.2), we derive

$$\text{(5.36)}\qquad \alpha_0\left\{i\sqrt{\frac{c}{a}}\, F_+(\lambda_1)\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1\, F_-(\lambda_1)\, F_+\left(i\sqrt{\frac{c}{a}}\right)\right\}$$

$$= p(-\lambda_1)\, r(\lambda_1)\left\{i\sqrt{\frac{c}{a}}\, F_-\left(i\sqrt{\frac{c}{a}}\right) - \lambda_1\, F_+\left(i\sqrt{\frac{c}{a}}\right)\right\}.$$

Hence, if (5.32) holds, (5.35), (5.36) become (5.33), (5.34).

Theorem 5.2. If $0 < c < 1$, $a = 0$, then the system of equations (5.1), (5.2) has a unique solution given by (5.10) and

$$\alpha_1 = \frac{p(\lambda_1)\ r(-\lambda_1)}{F_+\ (\lambda_1)}\ , \tag{5.37}$$

$$\alpha_0 = \frac{p(-\lambda_1)\ r(\lambda_1)}{F_+\ (\lambda_1)}\ . \tag{5.38}$$

Proof. Again, the system of equations (5.1), (5.2) is equivalent to the system (5.9), (5.10), (5.2). By Lemma 5.2 and since $a = 0$, equation (5.9) can be written

$$\alpha_1 \frac{r(\lambda_1)}{p(\lambda_1)} - \alpha_0 \frac{r(-\lambda_1)}{p(-\lambda_1)} = 0.$$

This combined with (5.2) yields (5.37), (5.38). That $F_+(\lambda_1) \neq 0$ can be seen from

$$\begin{aligned} F_+\ (\lambda_1) &= r(\lambda_1)\ p(-\lambda_1) + r(-\lambda_1)\ p(\lambda_1) \\ &= (-1)^n \left\{ \prod_{k=2}^{n} (\lambda_1 - \lambda_k) \cdot \prod_{j=1}^{n} (\lambda_1 + \mu_j) - \prod_{k=1}^{n} (\lambda_1 - \mu_k) \cdot \prod_{j=2}^{n} (\lambda_1 + \lambda_j) \right\}. \end{aligned}$$

Indeed, by (5.4) we have

$$\lambda_1 - \lambda_k > \lambda_1 - \mu_{k-1} > 0, \qquad (2 \leqslant k \leqslant n),$$

$$\lambda_1 + \mu_j > \lambda_1 + \lambda_{j+1} > 0, \qquad (1 \leqslant j \leqslant n - 1),$$

$$\lambda_1 + \mu_n > \lambda_1 - \mu_n > 0;$$

and therefore

$$(-1)^n\ F_+\ (\lambda_1) > 0. \tag{5.39}$$

Theorem 5.3. If $c = 0$, $0 < a < 3$, then the system of equations (5.1), (5.2) has a unique solution given by (5.10) and

$$\alpha_1 = -\frac{p(\lambda_1)\, r(-\lambda_1)}{F_-(\lambda_1)}, \tag{5.40}$$

$$\alpha_0 = \frac{p(-\lambda_1)\, r(\lambda_1)}{F_-(\lambda_1)}. \tag{5.41}$$

Proof. As before, the system of equations (5.1), (5.2) is equivalent to the system (5.9), (5.10), (5.2). By Lemmas 5.2, 5.4 and since $c = 0$, (5.9) is reduced to

$$\alpha_1 \frac{r(\lambda_1)}{p(\lambda_1)} + \alpha_0 \frac{r(-\lambda_1)}{p(-\lambda_1)} = 0.$$

This together with (5.2) yields (5.40), (5.41). We note that $F_-(\lambda_1) \neq 0$. Indeed,

$$F_-(\lambda_1) = r(\lambda_1)\, p(-\lambda_1) - r(-\lambda_1)\, p(\lambda_1)$$

$$= (-1)^n \left\{ \prod_{k=2}^{n} (\lambda_1 - \lambda_k) \cdot \prod_{j=1}^{n} (\lambda_1 + \mu_j) + \prod_{k=2}^{n} (\lambda_1 + \lambda_k) \cdot \prod_{j=1}^{n} (\lambda_1 - \mu_j) \right\},$$

so by (5.4) we have

$$(-1)^n F_-(\lambda_1) > 0. \tag{5.42}$$

§6. Behavior of the approximate solutions

The nth approximate solution f_n of equation (2.1) has been defined by (2.21), where the coefficients α_i, β_j are determined by (2.22), (2.25) and (2.23). This §6 deals with the behavior of the approximate solutions. We shall use the following notation:

$$(6.1) \qquad f_n^{I}(x, y) = \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}} \left(- a \lambda_i + \frac{c + a \lambda_i^2}{y + \lambda_i} \right), \qquad (x \leqslant 0)$$

$$(6.2) \qquad f_n^{II}(x, y) = - \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}} \frac{c + a \lambda_i^2}{y + \lambda_i} \frac{P_{2n}(y)}{P_{2n}(\lambda_i)} ; \qquad (x \leqslant 0)$$

so that

$$(6.3) \qquad f_n(x, y) = f_n^{I}(x, y) + f_n^{II}(x, y). \qquad (x \leqslant 0)$$

Lemma 6.1. For $x \leqslant 0$, we have

$$(6.4) \qquad \int_{-1}^{1} f_n(x, y)\, dy = 2 \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} ,$$

$$(6.5) \qquad \int_{-1}^{1} f_n(x, y)\, y\, dy = 2(c - 1) \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}} .$$

Proof. As $G_{2n}(\lambda_i) = 0$ and $P_{2n}(\lambda_i) \neq 0$ (Theorem 3.2), we have

$$\frac{\lambda_i (c + a \lambda_i^2)}{P_{2n}(\lambda_i)} = \frac{1 + a \lambda_i^2}{W_{2n-1}(\lambda_i)} .$$

Thus (2.21) becomes

$$(6.6) \quad f_n(x, y) = - a \sum_{i=0}^{n} \alpha_i \lambda_i^2 e^{\frac{x}{\lambda_i}} - \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} \frac{1 + a \lambda_i^2}{W_{2n-1}(\lambda_i)} \frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i}, \quad (x \leqslant 0).$$

By a classical formula [2; p. 154]

$$\int_{-1}^{1} \frac{P_n(y) - P_n(z)}{y - z}\, dy = 2\, W_{n-1}(z),$$

we have

$$\int_{-1}^{1} \frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i}\, dy = 2\, W_{2n-1}(-\lambda_i) = -\, 2\, W_{2n-1}(\lambda_i).$$

Hence (6.6) implies

$$\int_{-1}^{1} f_n(x, y)\, dy = -\, 2a \sum_{i=0}^{n} \alpha_i \lambda_i^2 e^{\frac{x}{\lambda_i}} + 2 \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} (1 + a\lambda_i^2), \qquad (x \leqslant 0)$$

and therefore (6.4).

Next, from (6.6), we have

$$\int_{-1}^{1} f_n(x, y)\, y\, dy = \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} \frac{1 + a\lambda_i^2}{W_{2n-1}(\lambda_i)} \int_{-1}^{1} y \cdot \frac{P_{2n}(-\lambda_i) - P_{2n}(y)}{y + \lambda_i}\, dy, \qquad (x \leqslant 0).$$

Since

$$\int_{-1}^{1} y\, \frac{P_{2n}(-\lambda_i) - P_{2n}(y)}{y + \lambda_i}\, dy = \int_{-1}^{1} \left[P_{2n}(-\lambda_i) - P_{2n}(y) + \lambda_i \frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y + \lambda_i} \right] dy$$

$$= 2\, P_{2n}(\lambda_i) - 2\lambda_i\, W_{2n-1}(\lambda_i),$$

it follows that for $x \leqslant 0$:

$$\int_{-1}^{1} f_n(x, y)\, y\, dy = 2 \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}} (1 + a\lambda_i^2) \left[\frac{P_{2n}(\lambda_i)}{\lambda_i W_{2n-1}(\lambda_i)} - 1 \right]$$

$$= 2 \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}} (1 + a\lambda_i^2) \left[\frac{c + a\lambda_i^2}{1 + a\lambda_i^2} - 1 \right]$$

$$= 2\,(c - 1) \sum_{i=0}^{n} \alpha_i \lambda_i e^{\frac{x}{\lambda_i}}.$$

The following result describes to what extent the approximate solution f_n satisfies equation (2.1) and the boundary conditions (2.2), (2.3).

Theorem 6.1. The approximate solution f_n satisfies the following relations:

$$(6.7)\qquad y\,\frac{\partial f_n(x,y)}{\partial x} + f_n(x,y) = \frac{P_{2n}(y)}{P_{2n}(0)}\, f_n(x,0) \qquad \text{for } x \geqslant 0.$$

$$(6.8)\qquad y\,\frac{\partial f_n(x,y)}{\partial x} + f_n(x,y) = \frac{c}{2}\int_{-1}^{1} f_n(x,y)\,dy$$

$$+\frac{1}{2}\,\frac{a}{1-c}\,y\int_{-1}^{1} f_n(x,y)\,y\,dy + \frac{P_{2n}(y)}{P_{2n}(0)}\, f_n^{II}(x,0) \qquad \text{for } x \leqslant 0.$$

(6.9) $\lim_{x\to-\infty} e^{\frac{x}{y}} f_n(x,y) = 0$ for $y > 0$, when n is sufficiently large (more precisely, when n is so large that $\lambda_1 = \lambda_1^{(2n)} > 1$).

$$(6.10)\qquad f_n(0,-\mu_j) = 0 \quad (1 \leqslant j \leqslant n).$$

Proof. (6.7) can be directly verified by using the first expression of (2.21). Also (6.10) is satisfied because of (2.22).

Let

$$(6.11)\qquad R_n(x,y) = y\,\frac{\partial f_n(x,y)}{\partial x} + f_n(x,y) - \frac{c}{2}\int_{-1}^{1} f_n(x,y)\,dy$$

$$-\frac{1}{2}\,\frac{a'}{1-c}\,y\int_{-1}^{1} f_n(x,y)\,y\,dy. \qquad (x \leqslant 0)$$

By (2.21) and Lemma 6.1, we have

$$R_n(x,y) = -y\sum_{i=0}^{n}\alpha_i\, e^{\frac{x}{\lambda_i}}\left\{a\lambda_i + \frac{c + a\lambda_i^2}{P_{2n}(\lambda_i)}\,\frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y+\lambda_i}\right\}$$

$$-\sum_{i=0}^{n}\alpha_i\lambda_i\, e^{\frac{x}{\lambda_i}}\left\{a\lambda_i + \frac{c + a\lambda_i^2}{P_{2n}(\lambda_i)}\,\frac{P_{2n}(y) - P_{2n}(-\lambda_i)}{y+\lambda_i}\right\}$$

$$-c\sum_{i=0}^{n}\alpha_i\, e^{\frac{x}{\lambda_i}} + a\,y\sum_{i=0}^{n}\alpha_i\lambda_i\, e^{\frac{x}{\lambda_i}}$$

or

$$R_n(x, y) = \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} (c + a\lambda_i^2) \left[1 - \frac{P_{2n}(y)}{P_{2n}(\lambda_i)}\right]$$

$$- a \sum_{i=0}^{n} \alpha_i \lambda_i^2 e^{\frac{x}{\lambda_i}} - c \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} .$$

Hence

$$R_n(x, y) = - \sum_{i=0}^{n} \alpha_i e^{\frac{x}{\lambda_i}} (c + a\lambda_i^2) \frac{P_{2n}(y)}{P_{2n}(\lambda_i)}$$

$$= \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(x, 0) \qquad \text{for } x \leqslant 0,$$

and thus proving (6.8).

In order to prove (6.9), we recall $\lambda_0 = -\lambda_1$ and the fact (Theorem 3.3) that $\lambda_1 = \lambda_1^{(2n)} > 1$ for n sufficiently large. For $0 < y \leqslant 1$, we have $\frac{1}{y} + \frac{1}{\lambda_0} \geqslant 1 - \frac{1}{\lambda_1}$ and therefore $\frac{1}{y} + \frac{1}{\lambda_0} > 0$ for n sufficiently large. Hence, for n sufficiently large, we have

$$\lim_{x \to -\infty} e^{\frac{x}{y} + \frac{x}{\lambda_i}} = 0 \quad \text{for } 0 < y \leqslant 1 \text{ and } 0 \leqslant i \leqslant n;$$

which implies

$$\lim_{x \to -\infty} e^{\frac{x}{y}} f_n(x, y) = 0 \qquad \text{for } 0 < y \leqslant 1.$$

In the next result, we confine ourselves to the region $x \leqslant 0$.

<u>Theorem 6.2</u>. <u>Assume that for</u> $-\infty < x \leqslant 0$, $-1 \leqslant y \leqslant 1$, <u>equation</u> (2.1) <u>has a solution</u> f <u>satisfying the boundary conditions</u> (2.2) <u>and</u> (2.3). <u>If</u>, <u>for</u> $x \leqslant 0$, <u>we have</u>

(6.12) $$\lim_{n \to \infty} \int_{-1}^{1} f_n(x, y)\, dy = \int_{-1}^{1} f(x, y)\, dy$$

<u>and</u>

(6.13) $$\lim_{n \to \infty} \int_{-1}^{1} y\, f_n(x, y)\, dy = \int_{-1}^{1} y\, f(x, y)\, dy$$

both with uniform convergence on $-\infty < x \leqslant 0$, then:

(6.14) $f(x, y) = \lim_{n\to\infty} f_n^I(x, y)$ holds for $x \leqslant 0$, $0 \leqslant y \leqslant 1$;

(6.15) $f(x, y) = \lim_{n\to\infty} \left[f_n^I(x, y) - e^{-\frac{x}{y}} f_n^I(0, y) \right]$ holds for $x \leqslant 0$, $-1 \leqslant y < 0$,

provided $G_{2n}(y) = 0$ is satisfied by at most a finite number of indices n.

Proof. We rewrite equations (2.1) and (6.8) as

$$(6.16) \qquad y \frac{\partial f(x,y)}{\partial x} + f(x, y) = F(x, y), \qquad (x \leqslant 0, -1 \leqslant y \leqslant 1),$$

$$(6.17) \qquad y \frac{\partial f_n(x,y)}{\partial x} + f_n(x, y) = F_n(x, y), \qquad (x \leqslant 0, -1 \leqslant y \leqslant 1);$$

where

$$(6.18) \qquad F(x, y) = \frac{c}{2} \int_{-1}^{1} f(x, t)dt + \frac{a\,y}{2(1-c)} \int_{-1}^{1} f(x, t)\, tdt,$$

$$(6.19) \qquad F_n(x, y) = \frac{c}{2} \int_{-1}^{1} f_n(x, t)\, dt + \frac{a\,y}{2(1-c)} \int_{-1}^{1} f_n(x, t)\, tdt + \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(x, 0).$$

From (6.16), (6.17), we have

$$(6.20) \qquad f(x, y) = e^{-\frac{x}{y}} \left[f(0, y) + \frac{1}{y} \int_0^x e^{\frac{s}{y}} F(s, y)\, ds \right],$$

$$(6.21) \qquad f_n(x, y) = e^{-\frac{x}{y}} \left[f_n(0, y) + \frac{1}{y} \int_0^x e^{\frac{s}{y}} F_n(s, y)\, ds \right]$$

for $x \leqslant 0$, $-1 \leqslant y \leqslant 1$ and $y \neq 0$. From (6.18), (6.19), we have the identity

$$(6.22) \qquad F(x, y) - F_n(x, y) + \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(x, 0) =$$

$$= \frac{c}{2} \int_{-1}^{1} \left[f(x, t) - f_n(x, t) \right] dt + \frac{a\,y}{2(1-c)} \int_{-1}^{1} t \left[f(x, t) - f_n(x, t) \right] dt$$

for $x \leqslant 0$, $-1 \leqslant y \leqslant 1$.

We discuss now three separate cases according to whether $y > 0$ or $y = 0$ or $y < 0$.

(Case 1) $0 < y \leqslant 1$.

As f satisfies the boundary condition (2.2), we derive from (6.20)

$$f(0, y) + \frac{1}{y} \int_0^{-\infty} e^{\frac{s}{y}} F(s, y)\, ds = 0 \qquad (0 < y \leqslant 1)$$

and therefore

$$f(x, y) = \frac{1}{y} e^{-\frac{x}{y}} \int_{-\infty}^{x} e^{\frac{s}{y}} F(s, y)\, ds, \qquad (x \leqslant 0,\ 0 < y \leqslant 1).$$

Similarly, from (6.9) and (6.21), we infer that

$$f_n(x, y) = \frac{1}{y} e^{-\frac{x}{y}} \int_{-\infty}^{x} e^{\frac{s}{y}} F_n(s, y)\, ds, \qquad (x \leqslant 0,\ 0 < y \leqslant 1)$$

holds for all $n \geqslant n_0$, where n_0 is so large that $\lambda_1^{(2n_0)} > 1$. Hence

$$(6.23) \qquad f(x, y) = f_n^{I}(x, y) + f_n^{II}(x, y) + \frac{1}{y} e^{-\frac{x}{y}} \int_{-\infty}^{x} e^{\frac{s}{y}} \left[F(s, y) - F_n(s, y)\right] ds$$

is valid for $x \leqslant 0$, $0 < y \leqslant 1$ and $n \geqslant n_0$.

When $n \geqslant n_0$, we have $\frac{1}{y} + \frac{1}{\lambda_i} > 0$ for $0 \leqslant i \leqslant n$, $0 < y \leqslant 1$; and therefore

$$\int_{-\infty}^{x} e^{s(\frac{1}{y} + \frac{1}{\lambda_i})}\, ds = \frac{\lambda_i\, y}{y + \lambda_i} e^{x(\frac{1}{y} + \frac{1}{\lambda_i})} \quad \text{for } 0 \leqslant i \leqslant n,\ 0 < y \leqslant 1.$$

Hence, according to the definition (6.2) of f_n^{II},

$$\int_{-\infty}^{x} e^{\frac{s}{y}} f_n^{II}(s, 0)\, ds = - P_{2n}(0) \sum_{i=0}^{n} \frac{\alpha_i\, (c + a\lambda_i^2)}{P_{2n}(\lambda_i)} \int_{-\infty}^{x} e^{s(\frac{1}{y} + \frac{1}{\lambda_i})}\, ds$$

$$= - y\, e^{\frac{x}{y}} P_{2n}(0) \sum_{i=0}^{n} \frac{\alpha_i\, \lambda_i\, (c + a\lambda_i^2)}{(y + \lambda_i)\, P_{2n}(\lambda_i)} e^{\frac{x}{\lambda_i}}$$

or

$$(6.24) \qquad f_n^{II}(x, y) = \frac{1}{y} e^{-\frac{x}{y}} \frac{P_{2n}(y)}{P_{2n}(0)} \int_{-\infty}^{x} e^{\frac{s}{y}} f_n^{II}(s, 0)\, ds$$

holds for $x \leqslant 0$, $0 < y \leqslant 1$ and $n \geqslant n_0$. Then (6.23) can be written

$$(6.25) \quad f(x, y) = \lim_{n\to\infty} \Big\{ f_n^{I}(x, y) + \\ + \frac{1}{y} e^{-\frac{x}{y}} \int_{-\infty}^{x} e^{\frac{s}{y}} \left[F(s, y) - F_n(s, y) + \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(s, 0) \right] ds \Big\},$$

$$(x \leqslant 0,\ 0 < y \leqslant 1).$$

Using (6.22) and the uniform convergence in (6.12), (6.13), we have

$$\lim_{n\to\infty} \int_{-\infty}^{x} e^{\frac{s}{y}} \left[F(s, y) - F_n(s, y) + \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(s, 0) \right] ds = 0,$$

$$(x \leqslant 0,\ 0 < y \leqslant 1);$$

which combined with (6.25) yields the desired result

$$f(x, y) = \lim_{n\to\infty} f_n^{I}(x, y) \quad (x \leqslant 0,\ 0 < y \leqslant 1).$$

(Case 2) $y = 0$.

From (6.16) - (6.19), we write

$$f(x, 0) = F(x, 0) - F_n(x, 0) + f_n(x, 0)$$

$$= \frac{c}{2} \int_{-1}^{1} \left[f(x, t) - f_n(x, t) \right] dt - f_n^{II}(x, 0) + f_n(x, 0)$$

or

$$f(x, 0) = f_n^{I}(x, 0) + \frac{c}{2} \int_{-1}^{1} \left[f(x, t) - f_n(x, t) \right] dt.$$

Then by (6.12), we get $f(x, 0) = \lim_{n\to\infty} f_n^{I}(x, 0)$ for $x \leqslant 0$.

(Case 3) $-1 \leqslant y < 0$, and $G_{2n}(y) = 0$ is satisfied by at most a finite number of n's.

By (2.3), $f(0, y) = 0$ for $y < 0$. So (6.20), (6.21) imply

$$(6.26) \qquad f(x, y) = f_n^I(x, y) - e^{-\frac{x}{y}} f_n^I(0, y) +$$

$$+ e^{-\frac{x}{y}} \left\{ \frac{1}{y} \int_0^x e^{\frac{s}{y}} \left[F(s, y) - F_n(s, y) \right] ds + e^{\frac{x}{y}} f_n^{II}(x, y) - f_n^{II}(0, y) \right\},$$

$$(x \leqslant 0, \ -1 \leqslant y < 0).$$

If $y < 0$ is not a zero of G_{2n}, then $\frac{1}{y} + \frac{1}{\lambda_i} \neq 0$ for $0 \leqslant i \leqslant n$ (where $\lambda_i = \lambda_i^{(2n)}$) and by (6.2):

$$\int_0^x e^{\frac{s}{y}} f_n^{II}(s, 0)\, ds = - P_{2n}(0) \sum_{i=0}^{n} \frac{\alpha_i(c + a\lambda_i^2)}{P_{2n}(\lambda_i)} \int_0^x e^{s(\frac{1}{y} + \frac{1}{\lambda_i})}\, ds$$

$$= - P_{2n}(0) \sum_{i=0}^{n} \frac{\alpha_i(c + a\lambda_i^2)}{P_{2n}(\lambda_i)} \frac{\lambda_i y}{y + \lambda_i} \left[e^{x(\frac{1}{y} + \frac{1}{\lambda_i})} - 1 \right].$$

Hence, if $y < 0$ is not a zero of G_{2n}, then

$$(6.27) \qquad \frac{1}{y} \frac{P_{2n}(y)}{P_{2n}(0)} \int_0^x e^{\frac{s}{y}} f_n^{II}(s, 0)\, ds = e^{\frac{x}{y}} f_n^{II}(x, y) - f_n^{II}(0, y), \qquad (x \leqslant 0).$$

Combining this fact with (6.26), we infer that

$$(6.28) \qquad f(x, y) = \lim_{n \to \infty} \left\{ f_n^I(x, y) - e^{-\frac{x}{y}} f_n^I(0, y) \right.$$

$$\left. + \frac{1}{y} e^{-\frac{x}{y}} \int_0^x e^{\frac{s}{y}} \left[F(s, y) - F_n(s, y) + \frac{P_{2n}(y)}{P_{2n}(0)} f_n^{II}(s, 0) \right] ds \right\}$$

holds for $x \leqslant 0$, $-1 \leqslant y < 0$, provided $G_{2n}(y) = 0$ is satisfied by only a finite number of n's. Finally, in view of (6.22) and the uniform convergence in (6.12), (6.13), the desired relation (6.15) follows immediately from (6.28).

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