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ON THE CONNECTION BETWEEN FLUX AND SLOWING-DOWN DENSITY

L. Dresner

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ON THE CONNECTION BETWEEN FLUX AND SLOWING-DOWN DENSITY

Lawrence Dresner

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ABSTRACT

Various approximate connections have been suggested between the flux and slowing-down density. The validity of these approximations is investigated by comparison with a particular class of slowing-down problems in which the resonance escape probability can be calculated exactly.

1. In order to facilitate easy solution of the linear Boltzmann equation three approximate expressions have been suggested connecting the lethargy flux, $\phi(u)$, and slowing-down density, $q(u)$. These expressions are all of the form[1]

$$q(u) = (\xi \Sigma_s + \theta \Sigma_a)\phi(u), \tag{1}$$

where ξ is the average lethargy gain per collision, and Σ_s and Σ_a are the macroscopic scattering and absorption cross sections, respectively. In the most elementary approximation $\theta = 0$, but the resulting expression is correct only for a non-absorbing medium, i.e., $\Sigma_a = 0$. If the ratio $\beta = \Sigma_a/\Sigma_s$ is both <u>small</u> and <u>slowly-varying</u> the effects of absorption may be included in first or second order in β by setting $\theta = \xi$ or γ, respectively, where $2\gamma\xi$ is the mean squared lethargy gain per collision. Equation 1 is a series of approximations to the case in which β is slowly-varying but not small,[2] whence q and ϕ are connected by the equation

$$q(u) = \frac{\Sigma_a}{\lambda}\phi(u) \tag{2}$$

where λ is given by

$$1 + \beta = \frac{\Sigma_t}{\Sigma_s} = \frac{1 - \alpha^{1-\lambda}}{(1 - \lambda)(1 - \alpha)} . \tag{3}$$

Here $\alpha = \left(\frac{A - 1}{A + 1}\right)^2$ where A is the mass of the moderating nucleus. Finally, since $dq/du = -\Sigma_a\phi$ in an infinite medium, it follows from Eq. 2 that the resonance escape probability is approximately given by

$$P(u) = \exp\left(- \int_0^u \lambda(u')du'\right) \tag{4}$$

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1. H. Soodak, F. Adler, and E. Greuling, "Kinetic Theory of Neutrons," AECD-3645, p. 365 (March 1955).
2. H. Hurwitz, Jr., KAPL-706 (Nov. 1951).

In order to test the validity of these approximate formulae a particular class of problems will be solved exactly, and comparisons made with the corresponding approximate results.

2. The infinite medium Boltzmann equation for one element with isotropic scattering in the center of mass, can be written as

$$F(u) + \beta(u)F(u) = \int_0^u K(u - u')F(u') + S(u) \tag{5a}$$

where $F(u)$ is the scattering density, $S(u)$ is the source density, and

$$\begin{aligned} K(U) &= (1 - \alpha)^{-1} e^{-U} \qquad 0 \leq U \leq \epsilon \\ &= 0 \qquad\qquad\qquad\qquad U > \epsilon \end{aligned} \tag{5b}$$

where $\epsilon \cong \ln\left(\frac{1}{\alpha}\right)$. If the absorption to scattering ratio, $\beta(u)$, varies exponentially, i.e., if

$$\beta(u) = \beta e^{-Ku} \tag{6}$$

Equation 5 can be solved exactly. To carry out the solution we proceed as follows:

If $F(p)$ denotes the Laplace transform of $F(u)$, etc., then after transformation Eq. 5 becomes

$$F(p) + \beta F(p + K) = K(p)F(p) + S(p) \tag{7}$$

Use has been made here of the convolution theorem, and of the shifting effect of an exponential factor. Equation 7 can be solved by iteration by writing

$$F_{n+1}(p) = \gamma(p) \left[S(p) - \beta F_n(p + K)\right] \tag{8a}$$

$$F_o(p) = \gamma(p)S(p) \tag{8b}$$

where $\gamma(p) = \left[1 - K(p)\right]^{-1}$. The result of such iteration is the series

$$F(p) = \gamma(p) \sum_{n=o}^{\infty} (-)^n \beta^n S(p + nK) \prod_{j=1}^{n} \gamma(p + jK) \tag{9}$$

It is not possible to invert Eq. 9, of course, but the asymptotic form of F(u) for large u can be obtained by application of the Tauberian theorem. This theorem states that the asymptotic behavior of F(u) can be obtained by examining F(p) in the neighborhood of its pole with the largest real part. Since in the problem we are considering F(u) asymptotically becomes constant, the relavant pole of F(p) occurs at $p = 0$. As we shall see in the next section this pole is due to a pole of $\gamma(p)$ at the origin. Anticipating this result, and noting that asymptotically $q(u) = \xi F(u)$, we can write

$$P = \lim_{u \to \infty} P(u) = \xi \cdot \lim_{p \to o} \left[p\gamma(p)\right] \cdot \sum_{n=o}^{\infty} (-)^n \beta^n S(nK) \prod_{j=1}^{n} \gamma(jK) \tag{10}$$

if the source is normalized to unity.

3. In order to evaluate Eq. 10 numerically, it is necessary to compute $\gamma(p)$, and to specify a source S(u), and to compute S(p). From Eq. 5b it follows that

$$K(p) = \frac{J\left[\epsilon(p + 1)\right]}{J(\epsilon)} \tag{11a}$$

where

$$J(x) = \frac{1 - e^{-x}}{x} \tag{11b}$$

Thus

$$\gamma(p) = \frac{J(\epsilon)}{J(\epsilon) - J\left[\epsilon(p + 1)\right]} \tag{11c}$$

With regard to the source function $S(u)$, let us choose it to be the source provided by a constant collision density for $u < 0$. This corresponds to letting the neutrons slow down in the absence of absorption until they have assumed the constant asymptotic scattering density, and then letting them encounter a region of exponentially varying absorption with a sharp edge at $u = 0$. If $S(u)$ is normalized to correspond to unit slowing down density at $u = 0$, it has the form

$$\begin{aligned} S(u) &= \xi^{-1}(1 - \alpha)^{-1}(e^{-u} - \alpha) \qquad 0 \leq u \leq \epsilon \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad u > \epsilon \end{aligned} \tag{12}$$

From Eq. 12 it follows that

$$\xi \cdot S(p) = \frac{J\left[\epsilon(p + 1)\right]}{J(\epsilon)} - \alpha \frac{J(\epsilon p)}{J(\epsilon)} \tag{13}$$

For ease in carrying out the numerical calculations let us consider the case of very heavy moderators, i.e., let $\epsilon \to 0$, but let ϵp remain finite. (This is equivalent to noting that for $A >> 1$, if we choose ϵ as the unit of lethargy and ϵ^{-1} as the unit of p, all moderators behave in a formally identical fashion.) Then

$$\gamma(p) = \frac{p}{p - 1 + e^{-p}} \tag{14}$$

where now p is measured in units of ϵ^{-1}. Similarly

$$S(p) = 2\,\frac{p - 1 + e^{-p}}{p^2} \tag{15}$$

Inserting these last results in Eq. 10 gives after rearrangement

$$P = 1 - \sum_{n=1}^{\infty} (-)^{n+1} \beta^n \cdot \frac{2}{nK} \prod_{j=1}^{n-1} \frac{jK}{jK - 1 + e^{-jK}} \tag{16}$$

4. It is possible to evaluate the series in Eq. 16 directly for small β, but the convergence for large β fails. In particular for $K = 1$, the radius of convergence in $|\beta|$ of series (16) is unity. In this particular case ($K = 1$) however we can obtain an approximate expression for the sum of the series which can be analytically continued beyond $|\beta| = 1$. To do this we choose a value of N such that e^{-N} can be neglected in comparison with unity, and write

$$\prod_{j=1}^{n-1} \frac{j}{j - 1 + e^{-j}} = \prod_{j=1}^{N-1} \frac{j}{j - 1 + e^{-j}} \cdot \frac{n - 1}{N - 1} \tag{17}$$

where e^{-j} has been neglected for $n \geq N$. Thus Eq. 16 becomes

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1} . \beta^n . \frac{2}{n} \cdot \prod_{j=1}^{n-1} \frac{j}{j - 1 + e^{-j}}$$

$$- \frac{2}{N - 1} \prod_{j=1}^{N-1} \frac{j}{j - 1 + e^{-j}} \sum_{n=N+1}^{\infty} (-)^{n+1} \beta^n \left(\frac{n - 1}{n} \right) \tag{18}$$

The series in the last term of Eq. 18 can be summed using the following well known series:

$$\sum_{n=N+1}^{\infty} (-)^{n+1} \beta^n = (-)^N \frac{\beta^{N+1}}{1 + \beta} \tag{19a}$$

$$\sum_{n=N+1}^{\infty} (-)^{n+1} \frac{\beta^n}{n} = \ln(1 + \beta) - \sum_{n=1}^{N} (-)^{n+1} \frac{\beta^n}{n} \tag{19b}$$

Hence, for K = 1,

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1} \beta^n C_n - \frac{N}{N-1} C_N \left\{ (-)^N \frac{\beta^{N+1}}{1+\beta} - \ln(1+\beta) + \sum_{n=1}^{N} (-)^{n+1} \frac{\beta^n}{n} \right\} \tag{20a}$$

where

$$C_n = \frac{2}{n} \prod_{j=1}^{n-1} \frac{j}{j - 1 + e^{-j}} \; ; \quad C_1 = 2 \tag{20b}$$

5. A similar procedure is possible when $K = \frac{1}{2}$. In this case

$$\prod_{j=1}^{n-1} \frac{\frac{1}{2} j}{\frac{1}{2} j - 1 + e^{-\frac{1}{2} j}} = \prod_{j=1}^{N-1} \frac{\frac{1}{2} j}{\frac{1}{2} j - 1 + e^{-\frac{1}{2} j}} \cdot \frac{(n-1)(n-2)}{(N-1)(N-2)} \tag{21}$$

where $e^{-\frac{1}{2} N} \ll 1$. Thus

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1} \beta^n C_n - \frac{N}{(N-1)(N-2)} C_N \sum_{n=N+1}^{\infty} (-)^{n+1} \beta^n \frac{(n-1)(n-2)}{n} \tag{22a}$$

where now

$$C_n = \frac{4}{n} \prod_{j=1}^{n-1} \frac{\frac{1}{2} j}{\frac{1}{2} j - 1 + e^{-\frac{1}{2} j}} \; ; \quad C_1 = 4 \tag{22b}$$

Using Eqs. 19a, 19b, and the equation

$$\sum_{n=N+1}^{\infty} (-)^{n+1}\beta^n \cdot n = -\beta \frac{d}{d\beta} \sum_{n=N+1}^{\infty} (-\beta)^n = (-)^N \frac{(N+1)\beta^{N+1} + N\beta^{N+2}}{(1+\beta)^2} \tag{23}$$

we can write

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1}\beta^n C_n - \frac{N}{(N-1)(N-2)} C_N \left\{ (-)^n \frac{(N-2)\beta^{N+1} + (N-3)\beta^{N+2}}{(1+\beta)^2} \right.$$

$$\left. + 2\ln(1+\beta) - 2 \sum_{n=1}^{N} (-)^{n+1} \frac{\beta^n}{n} \right\} \tag{24}$$

6. Let us finally consider the case of K = 2. In this case

$$\prod_{j=1}^{n-1} \frac{2j}{2j - 1 + e^{-2j}} = \prod_{j=1}^{N-1} \frac{2j}{2j - 1 + e^{-2j}} \cdot \frac{(2N-3)!!}{(2N-2)!!} \cdot \frac{(2n-2)!!}{(2n-3)!!} \tag{25}$$

where $e^{-2N} \ll 1$. Thus

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1}\beta^n C_n - NC_N \frac{(2N-3)!!}{(2N-2)!!} \sum_{n=N+1}^{\infty} (-)^{n+1}$$

$$\times\ \beta^n \frac{1}{n} \frac{(2n-2)!!}{(2n-3)!!} \tag{26a}$$

where

$$C_n = \frac{1}{n} \prod_{j=1}^{n-1} \frac{2j}{2j - 1 + e^{-2j}} \ ; \quad C_1 = 1 \tag{26b}$$

and $x!! = x(x-2)(x-4) \ldots 2$ or 1. The series 26a can be summed explicitly with the use of the following series derived in Appendix A:

$$\sum_{n=1}^{\infty} (-)^{n+1}\beta^n \frac{1}{n}\frac{(2n-2)!!}{(2n-3)!!} = 2\sqrt{\frac{\beta}{1+\beta}}\ \text{arcsinh}\sqrt{\beta} - (\text{arcsinh}\sqrt{\beta})^2 \tag{27}$$

One then finds

$$P = 1 - \sum_{n=1}^{N} (-)^{n+1}\beta^n C_n - NC_N \frac{(2N-3)!!}{(2N-2)!!}\left[2\sqrt{\frac{\beta}{1+\beta}}\ \text{arcsinh}\sqrt{\beta}\right.$$

$$\left. - (\text{arcsinh}\sqrt{\beta})^2 - \sum_{n=1}^{N} (-)^{n+1}\frac{\beta^n}{n}\frac{(2n-2)!!}{(2n-3)!!}\right] \tag{28}$$

6. In order to evaluate P from Eq. 4 it is necessary to calculate $\int_0^\infty \lambda(u)du$, where λ is given by

$$1 + \beta e^{-Ku} = \frac{J[\epsilon(1-\lambda)]}{J(\epsilon)} \xrightarrow[\epsilon\to 0]{} \frac{e^{\lambda}-1}{\lambda} \tag{29}$$

and where the limit as $\epsilon \to 0$ is taken in a manner similar to the limits in Eqs. 14 and 15. λ is thus measured in units of ϵ^{-1}. It is possible to change the variable of integration from u to λ by use of Eq. 29, obtaining

$$\int_0^\infty \lambda(u)du = \frac{1}{K}\int_0^{\lambda_o} d\lambda\ \frac{\lambda\frac{d}{d\lambda}\left(\frac{e^{\lambda}-1}{\lambda}\right)}{\left(\frac{e^{\lambda}-1}{\lambda}-1\right)} \tag{30a}$$

where

$$1 + \beta = (e^{\lambda_o} - 1)/\lambda_o \tag{30b}$$

Expansion of the integrand in series and term-by-term integration gives

$$\int_0^\infty \lambda(u)du = \frac{1}{K}\left[\lambda_o + \frac{\lambda_o^2}{6} + \frac{\lambda_o^3}{54} + \frac{\lambda_o^4}{1080} - \frac{\lambda_o^5}{16200} - \frac{\lambda_o^6}{81648} + \ldots\right] \tag{31}$$

7. Plotted in Fig. 1 is an exact calculation of P vs β for K = 1, as well as four approximate calculations based on Eqs. 1 and 4. The curves labelled "Fermi," "Wigner," and "Goertzel-Grueling" correspond to Eq. 1 with $\theta = 0$, ξ, and $\gamma(\sim \frac{2}{3}\xi)$, respectively. The curve labelled "Hurwitz" corresponds to Eq. 4. For $P \gtrsim 0.20$ the Hurwitz and Goertzel-Greuling approximations are essentially exact and represent a substantial improvement over the Wigner or Fermi approximations. For $0.20 \gtrsim P \gtrsim 0.03$ all of the approximations based on Eq. 4 fail to some extent, although that of Hurwitz is best. For smaller values of K and the same P for K = 1 one would expect better agreement with Hurwitz approximation since the ratio Σ_a/Σ_s is smaller and more slowly varying. As K increases one might expect the curve labelled "exact" to approach Wigner's approximation, since the latter is correct for a very thin absorbing spike. Figure 2, which is similar to Fig. 1, except that K = 2, shows this effect clearly. Here the curve labelled "exact" lies above the Hurwitz approximation, whereas in Fig. 1 it lies below it.

The resonance escape probability P is a better index of the applicability of the Hurwitz approximation than the absorption-to-scattering parameter, β. For $K \leq 2$ and $P \gtrsim 0.20$ Hurwitz' approximation is probably adequate. For $K \leq 1$ and $P \gtrsim 0.20$ it gives essentially exact results. Considering that K = 2 corresponds to a cross section which falls by a factor of e(=2.718) in a lethargy interval ξ, one sees that the applicability of the Hurwitz approximation is very wide indeed.

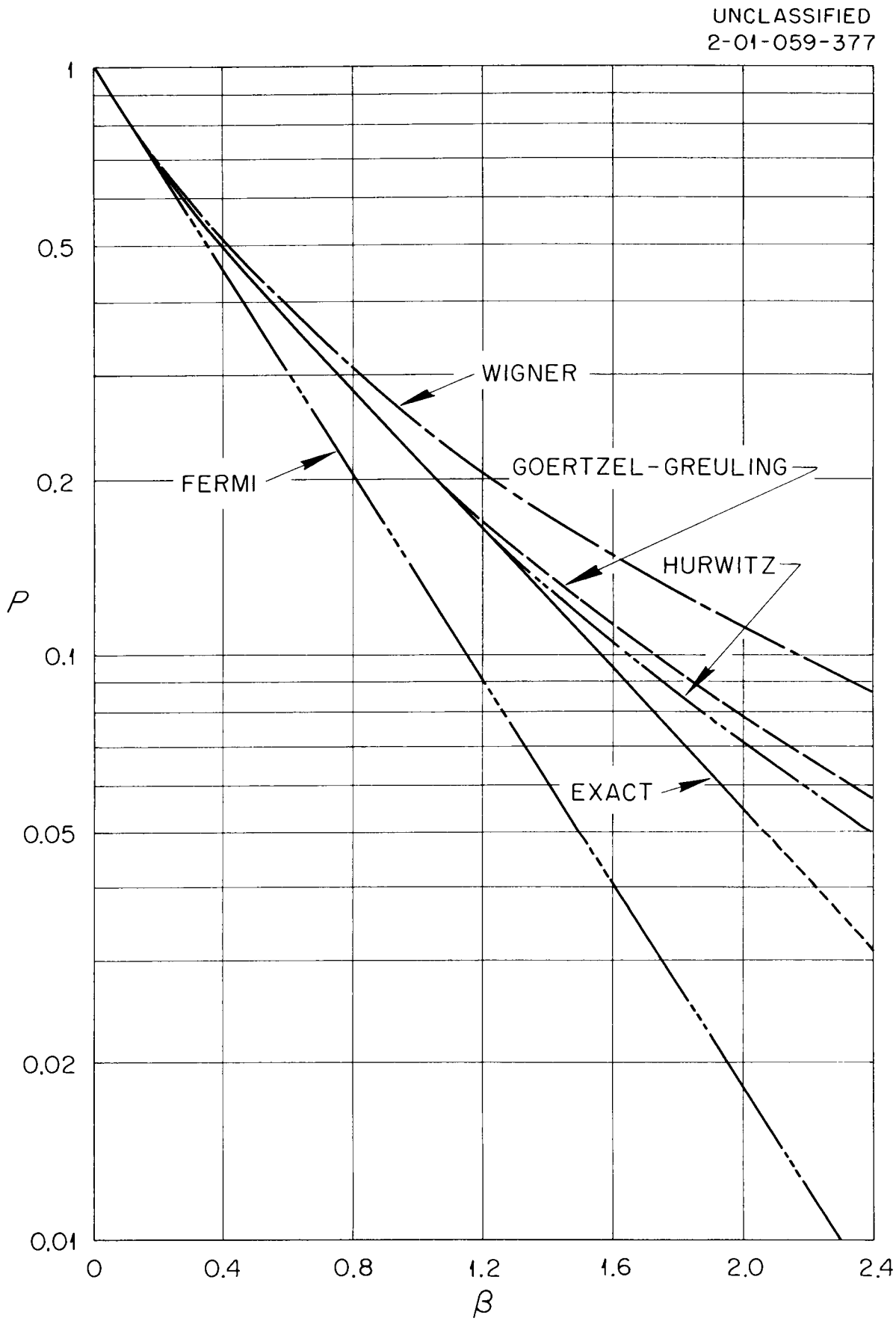

Fig. 1. P vs. β for $K = 1$.

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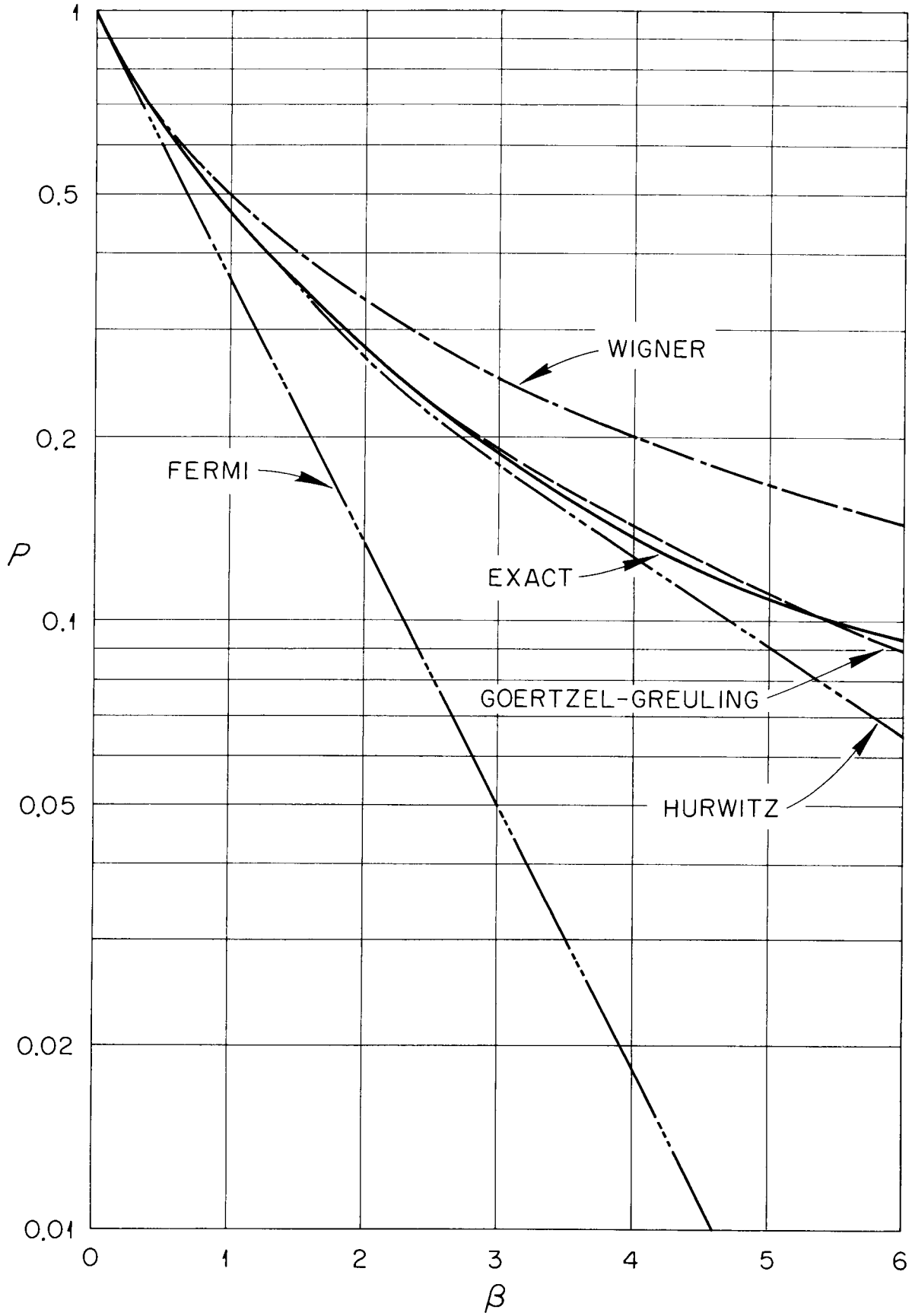

Fig. 2. P vs. β for $K=2$.

APPENDIX A

Consider first the function S(x) defined by

$$S(x) = \sum_{n=1}^{\infty} \frac{(2n - 2)!!}{(2n - 3)!!} x^{2n} \tag{A.1}$$

It is easily verified that

$$x^3 \int \frac{dx}{x} \frac{dS}{dx} = \sum_{n=1}^{\infty} \frac{(2n - 2)!!}{(2n - 3)!!} \cdot \frac{2n}{2n - 1} \cdot x^{2n+2} = S(x) - x^2 \tag{A.2}$$

Differentiation of Eq. A.2 yields the inhomogeneous first order linear differential equation

$$\frac{dS}{dx} + \frac{3S(x)}{x(x^2 - 1)} = \frac{x}{x^2 - 1} \tag{A.3}$$

An integrating factor for the homogeneous equation is

$$I = \exp\left(3 \int \frac{dx}{x(x^2 - 1)} \right) \tag{A.4}$$

The integral can be carried out by the method of partial fractions and Eq. A.4 gives for I

$$I = x^{-3}(1 - x^2)^{+\frac{3}{2}} \tag{A.5}$$

A particular solution of the inhomogeneous equation is

$$S(x) = - x^{+3}(1 - x^2)^{-\frac{3}{2}} \int dx \frac{(1 - x^2)^{\frac{1}{2}}}{x^2} \tag{A.6}$$

The integral can be evaluated with the help of the substitution $x = \sin\theta$ which yields

$$S(x) = \frac{x^2}{1 - x^2} + \frac{x^3}{(1 - x^2)^{\frac{3}{2}}} \arcsin x \tag{A.7}$$

which can be verified by direct expansion. Finally setting $x = i\sqrt{y}$ yields

$$\sum_{n=1}^{\infty} \frac{(2n - 2)!!}{(2n - 3)!!} (-)^{n+1} y^n = \frac{y}{1 + y} - \left(\frac{y}{1 + y}\right)^{\frac{3}{2}} \operatorname{arcsinh} \sqrt{y} \tag{A.8}$$

which can again be verified by direct expansion. From Eq. A.8 it follows that

$$\sum_{n=1}^{\infty} \frac{(2n - 2)!!}{(2n - 3)!!} (-)^{n+1} \frac{y^n}{n} = \int_0^y \frac{dy}{y} \left\{ \frac{y}{1 + y} - \left(\frac{y}{1 + y}\right)^{\frac{3}{2}} \operatorname{arcsinh} \sqrt{y} \right\} \tag{A.9a}$$

$$= 2 \sqrt{\frac{y}{1 + y}} \operatorname{arcsinh} \sqrt{y} - (\operatorname{arcsinh} \sqrt{y})^2 \tag{A.9b}$$

if one makes the substitution $y = \sinh^2 z$ and integrates by parts. Equation A.9b can again be verified by direct expansion.

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