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TRANSIENT TEMPERATURE IN INFINITE PLATES, INFINITE CYLINDERS, AND SPHERES FOLLOWING A SIMULTANEOUS STEP CHANGE IN INTERNAL HEAT GENERATION RATE, COOLANT TEMPERATURE AND HEAT TRANSFER COEFFICIENT

L. G. Epel

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Duhamel's Theorem (Forcing Function)

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TRANSIENT TEMPERATURES IN INFINITE PLATES, INFINITE CYLINDERS, AND SPHERES FOLLOWING A SIMULTANEOUS STEP CHANGE IN INTERNAL HEAT GENERATION RATE, COOLANT TEMPERATURE AND HEAT TRANSFER COEFFICIENT

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NOMENCLATURE

Symbols

Bi	Biot number $= \frac{hx_o}{k}$ or $\frac{hr_o}{k}$
C_1, C_2, C_3, C_4	constants of integration
E	infinite series defined in the text
F	first term in E
g	rate of internal heat generation
G	first term in H
h	heat transfer coefficient
H	infinite series defined in the text
k	thermal conductivity
M_m, M_n	roots of transcendental equation
r	radius
R	function of r alone
t	temperature
T	temperature difference $= t - t_f$ (T_1 and T_2 are <u>variables</u>)
x	distance
X	function of x alone

Greek

α	thermal diffusivity $= \frac{k}{\rho C_p}$, where ρ is density and C_p is specific heat
θ	time
λ	separation constant
τ	function of θ and x, or θ and r
ϕ	Fourier modulus $= \frac{\alpha\,\theta}{x_o^2}$ or $\frac{\alpha\,\theta}{r_o^2}$
$\Psi(M_n)$	a function equal to $\int_o^{M_n} \xi^3 J_o(\xi)\, d\xi$

NOMENCLATURE (CONTD.)

Subscripts

0	refers to exterior dimensions of body
1	refers to conditions before the step changes take place
2	refers to conditions after the step changes take place
c	refers to cylinder
f	refers to coolant fluid
p	refers to plate
s	refers to sphere

INTRODUCTION

In a previous report[1] the transient temperatures in infinite plates, infinite cylinders and spheres during convective cooling following an instantaneous cessation of uniform heat generation in the body were investigated. It was assumed in that study that the coolant temperature and heat transfer coefficient at the surface of the body in question was constant. The solution was obtained in infinite series form and the actual values of the series were evaluated for a wide range of parameters.

A more general problem, of which the aforementioned one is a special case, is the question of transient temperatures after a simultaneous step change in internal heat generation rate, coolant temperature and heat transfer coefficient. The solution of this problem involves two series. One series is precisely the one generated in the simpler case; the second series is one which comes about in the study of coolant temperature change only, with no internal heat generation at any time. This second series has been evaluated by Prof. M. P. Heisler[2] for a range of variables comparable to that used in evaluating the first series. Thus, knowing the form of the general solution, use can be made of the already published values of both of the infinite series to arrive at a numerical answer.

The similarities in the development for the three geometries are emphasized in this development by assigning the same number to corresponding equations with merely primes or double primes distinguishing the geometry involved.

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1. L. G. Epel, Transient Temperatures in Infinite Plates, Infinite Cylinders, and Spheres During Convective Cooling from Initially Parabolic Temperature Profiles, ORNL-2511 (May 5, 1958).

2. M. P. Heisler, Temperature Charts for Induction and Constant-Temperature Heating, Trans. A.S.M.E., Vol. 69, pp. 227-236, April 1947.

THE INFINITE PLATE

Fourier's equation for this case is:

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \theta} \tag{1}$$

The boundary conditions are:

$$\frac{\partial T_2}{\partial x} = 0 \quad \text{at} \quad x = 0 \qquad \theta = \theta \tag{a}$$

$$h_2 T_2 = -k\frac{\partial T_2}{\partial x} \quad \text{at} \quad x = x_o \qquad \theta = 0 \tag{b}$$

The initial condition is obtained by solving Poisson's equation in rectangular coordinates and applying the boundary conditions

$$\frac{dT_1}{dx} = 0 \qquad \text{at} \quad x = 0$$

$$h_1 T_1 = -k\frac{dT_1}{dx} \quad \text{at} \quad x = x_o$$

This yields

$$T_2 = \frac{g_1 x_o^2}{2k}\left[\frac{2}{Bi_1} + 1\right] + (t_{f1} - t_{f2}) - \frac{g_1 x^2}{2k} \quad \text{at} \quad \theta = 0. \tag{c}$$

Assume a solution of Eq. (1) of the form

$$T_2(x,\theta) = \mathcal{T}(x,\theta) + X(x). \tag{2}$$

With this assumed form Eq. (1) becomes

$$\frac{\partial^2 \mathcal{T}}{\partial x^2} + \frac{\partial^2 X}{\partial x^2} + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial \mathcal{T}}{\partial \theta}. \tag{3}$$

The boundary conditions (a) and (b), and the initial condition (c), are satisfied respectively by letting

$$\frac{\partial \mathcal{T}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial X}{\partial x} = 0 \quad \text{at} \quad x = 0 \qquad \theta = \theta \tag{d}$$

$$h_2 \mathcal{T} = -k \frac{\partial \mathcal{T}}{\partial x} \quad \text{and} \quad h_2 X = -k \frac{\partial X}{\partial x} \quad \text{at} \quad x = x_o \qquad \theta = \theta \tag{e}$$

$$\mathcal{T} = \frac{g_1 x_o^2}{2k} \left[\frac{2}{Bi_1} + 1 \right] + (t_{f1} - t_{f2}) - \frac{g_1 x^2}{2k} - X \quad \text{at } \theta = 0 \tag{f}$$

Equation (3) is broken up into two equations as:

$$\frac{\partial^2 X}{\partial x^2} + \frac{g_2}{k} = 0 \tag{4}$$

$$\frac{\partial^2 \mathcal{T}}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \mathcal{T}}{\partial \theta} \tag{5}$$

Thus Eqs. (4) and (5) together with boundary conditions (d) and (e) and initial condition (f) constitute a boundary value problem equivalent to Eqs. (1), (a), (b), and (c). The general solution of Eq. (4) is

$$X = -\frac{g_2}{2k} x^2 + c_1 x + c_2. \tag{6}$$

Applying boundary conditions (d) and (e) yields the particular solution which in dimensionless form is

$$\frac{X}{\frac{g_2 x_o^2}{2k}} = \left[\frac{2}{Bi_2} + 1 \right] - \frac{x^2}{x_o^2} . \tag{7}$$

The general solution of Eq. (5) is

$$\mathcal{T} = \left(C_3 \sin \frac{\lambda}{\sqrt{\alpha}} x + C_4 \cos \frac{\lambda}{\sqrt{\alpha}} x\right) e^{-\lambda^2 \theta} , \tag{8}$$

where λ is a constant to be determined.

Applying conditions (d), (e), and (f) yields the particular solution which in dimensionless form is

$$\frac{\tau}{\frac{g_2 x_o^2}{2k}} = \sum_{n=1}^{\infty} \frac{\left\{\left[\frac{g_1}{g_2}\left(\frac{2}{Bi_1}+1\right) - \left(\frac{2}{Bi_2}+1\right)\right] + \left[\frac{t_{f1}-t_{f2}}{\frac{g_2 x_o^2}{2k}}\right]\right\} \sin M_n - \frac{\left(\frac{g_1}{g_2}-1\right)}{M_n^2}\left[2M_n \cos M_n + (M_n^2 - 2)\sin M_n\right]}{\left[\frac{M_n}{2} + \frac{\sin 2M_n}{4}\right]} \cos \frac{M_n}{x_o} x \, e^{-M_n^2 \phi}, \tag{9}$$

where the M_n's are the consecutive roots of the transcendental equation

$$M_n \tan M_n = Bi_2 \quad \text{and defined as} \quad M_n \equiv \frac{\lambda_n x_o}{\sqrt{\alpha}}. \tag{10}$$

Define

$$E_p \equiv \sum_{n=1}^{\infty} \frac{\left(\frac{2}{Bi_2}+1\right)\sin M_n - \frac{1}{M_n^2}\left[2M_n \cos M_n + (M_n^2 - 2)\sin M_n\right]}{\left[\frac{M_n}{2} + \frac{\sin 2M_n}{4}\right]} \cos \frac{M_n}{x_o} x \, e^{-M_n^2 \phi}, \tag{11}$$

$$H_p \equiv \sum_{n=1}^{\infty} \frac{\sin M_n}{\left[\frac{M_n}{2} + \frac{\sin 2M_n}{4}\right]} \cos \frac{M_n}{x_o} x \, e^{-M_n^2 \phi}. \tag{12}$$

With this notation, Eq. (9) becomes

$$\frac{\tau}{\frac{g_2 x_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_p + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 x_o^2}{2k}}\right] H_p + \frac{g_1}{g_2}\left[\frac{2}{Bi_1} - \frac{2}{Bi_2}\right] H_p \, . \quad (13)$$

Thus the final solution in dimensionless form is from Eq. (2):

$$\frac{T_2}{\frac{g_2 x_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_p + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 x_o^2}{2k}}\right] H_p + \frac{g_1}{g_2}\left[\frac{2}{Bi_1} - \frac{2}{Bi_2}\right] H_p$$

$$+ \left[\frac{2}{Bi_2} + 1\right] - \frac{x^2}{x_o^2} \, . \quad (14)$$

THE INFINITE CYLINDER

Fourier's equation for this case is:

$$\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial^2 T_2}{\partial r} + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \theta} \tag{1'}$$

The boundary conditions are:

$$\frac{\partial T_2}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad \theta = \theta \tag{a'}$$

$$h_2 T_2 = -k\frac{\partial T_2}{\partial r} \quad \text{at} \quad r = r_o \qquad \theta = \theta \tag{b'}$$

The initial condition is obtained by solving Poisson's equation in cylindrical coordinates and applying the boundary conditions

$$\frac{dT_1}{dr} = 0 \qquad \text{at} \qquad r = 0$$

$$h_1 T_1 = -k\frac{dT_1}{dr} \quad \text{at} \quad r = r_o$$

This yields

$$T_2 = \frac{g_1 r_o^2}{2k}\left[\frac{1}{Bi_1} + \frac{1}{2}\right] + (t_{f1} - t_{f2}) - \frac{g_1 r^2}{4k}. \tag{c'}$$

Assume a solution of Eq. (1') of the form

$$T_2(r,\theta) = \mathcal{T}(r,\theta) + R(r). \tag{2'}$$

With this assumed form Eq. (1') becomes

$$\frac{\partial^2 \mathcal{T}}{\partial r^2} + \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\left[\frac{\partial \mathcal{T}}{\partial r} + \frac{\partial R}{\partial r}\right] + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial \mathcal{T}}{\partial \theta}. \tag{3'}$$

The boundary conditions (a') and (b'), and the initial condition (c'), are satisfied respectively by letting

$$\frac{\partial \tau}{\partial r} = 0 \quad \text{and} \quad \frac{\partial R}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad \theta = \theta \tag{d'}$$

$$h_2 \tau = -k \frac{\partial \tau}{\partial r} \quad \text{and} \quad h_2 R = -k \frac{\partial R}{\partial r} \quad \text{at} \quad r = r_o \qquad \theta = \theta \tag{e'}$$

$$\tau = \frac{g_1 r_o^2}{2k} \left[\frac{1}{Bi_1} + \frac{1}{2} \right] + (t_{f1} - t_{f2}) - \frac{g_1 r^2}{4k} - R \quad \text{at } \theta = 0 \tag{f'}$$

Equation (3') is broken up into two equations as:

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \frac{g_2}{k} = 0 \tag{4'}$$

$$\frac{\partial^2 \tau}{\partial r^2} + \frac{1}{r} \frac{\partial \tau}{\partial r} = \frac{1}{\alpha} \frac{\partial \tau}{\partial \theta} \tag{5'}$$

Thus Eqs. (4') and (5') together with boundary conditions (d') and (e') and initial condition (f') constitute a boundary value problem equivalent to Eqs. (1'), (a'), (b'), and (c'). The general solution of Eq. (4') is

$$R = -\frac{g_2}{4k} r^2 + C_1 \ln r + C_2 \, . \tag{6'}$$

Applying boundary conditions (d') and (e') yields the particular solution which in dimensionless form is

$$\frac{R}{\frac{g_2 r_o^2}{2k}} = \left[\frac{1}{Bi_2} + \frac{1}{2} \right] - \frac{r^2}{2r_o^2} \, . \tag{7'}$$

The general solution of Eq. (5') is

$$\tau = \left[C_3 J_o \left(\frac{\lambda}{\sqrt{\alpha}} r\right) + C_4 Y_o \left(\frac{\lambda}{\sqrt{\alpha}} r\right) \right] e^{-\lambda^2 \theta} \, , \tag{8'}$$

where λ is a constant to be determined.

Applying conditions (d'), (e'), and (f') yields the particular solution which in dimensionless form is

$$\frac{\tau}{\frac{g_2 r_o^2}{2k}} = \sum_{n=1}^{\infty} \frac{\left[\frac{g_1}{g_2}\left(\frac{2}{Bi_1}+1\right) - \left(\frac{2}{Bi_2}+1\right) + 2\left(\frac{t_{f1}-t_{f2}}{\frac{g_2 r_o^2}{2k}}\right)\right]\frac{J_1(M_n)}{M_n} - \frac{\left(\frac{g_1}{g_2}-1\right)}{M_n^4}\Psi(M_n)}{\left[J_o^2(M_n)+J_1^2(M_n)\right]} J_o\left(\frac{M_n}{r_o} r\right) e^{-M_n^2 \phi}, \quad (9')$$

where the M_n's are the consecutive roots of the transcendental equation

$$M_n \frac{J_1(M_n)}{J_o(M_n)} = Bi_2 \quad \text{and defined as} \quad M_n \equiv \frac{\lambda_n r_o}{\sqrt{\alpha}}. \quad (10')$$

Define

$$E_c = \sum_{n=1}^{\infty} \frac{\left(\frac{2}{Bi_2}+1\right)\frac{J_1(M_n)}{M_n} - \frac{\Psi(M_n)}{M_n^4}}{\left[J_o^2(M_n)+J_1^2(M_n)\right]} J_o\left(\frac{M_n}{r_o} r\right) e^{-M_n^2 \phi}, \quad (11')$$

$$H_c = \sum_{n=1}^{\infty} \frac{\frac{2}{M_n} J_1(M_n)}{\left[J_o^2(M_n)+J_1^2(M_n)\right]} J_o\left(\frac{M_n}{r_o} r\right) e^{-M_n^2 \phi}. \quad (12')$$

With this notation, Eq. (9') becomes

$$\frac{\tau}{\frac{g_2 r_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_c + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right] H_c + \frac{g_1}{g_2}\left[\frac{1}{Bi_1} - \frac{1}{Bi_2}\right] H_c \, . \tag{13'}$$

Thus the final solution in dimensionless form is from Eq. (2'):

$$\frac{T_2}{\frac{g_2 r_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_c + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right] H_c + \frac{g_1}{g_2}\left[\frac{1}{Bi_1} - \frac{1}{Bi_2}\right] H_c$$

$$+ \left[\frac{1}{Bi_2} + \frac{1}{2}\right] - \frac{r^2}{2r_o^2} \, . \tag{14'}$$

THE SPHERE

Fourier's equation for this case is:

$$\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r}\frac{\partial T_2}{\partial r} + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \theta} \tag{1''}$$

The boundary conditions are:

$$\frac{\partial T_2}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad \theta = \theta \tag{a''}$$

$$h_2 T_2 = -k\frac{\partial T}{\partial r} \quad \text{at} \quad r = r_o \quad \theta = \theta \tag{b''}$$

The initial condition is obtained by solving Poisson's equation in spherical coordinates and applying the boundary conditions

$$\frac{dT_1}{dr} = 0 \qquad \text{at} \quad r = 0$$

$$h_1 T_1 = -k\frac{dT_1}{dr} \quad \text{at} \quad r = r_o$$

This yields

$$T_2 = \frac{g_1 r_o^2}{2k}\left[\frac{2}{3\,Bi_1} + \frac{1}{3}\right] + (t_{f1} - t_{f2}) - \frac{g_1 r^2}{6k} \quad \text{at} \quad \theta = 0\,. \tag{c''}$$

Assume a solution of Eq. (1") of the form

$$T_2(r,\theta) = \tau\,(x,\theta) + R(r) \tag{2''}$$

With this assumed form Eq. (1") becomes

$$\frac{\partial^2 \tau}{\partial r^2} + \frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\left(\frac{\partial \tau}{\partial r} + \frac{\partial R}{\partial r}\right) + \frac{g_2}{k} = \frac{1}{\alpha}\frac{\partial \tau}{\partial \theta}\,. \tag{3''}$$

The boundary conditions (a") and (b"), and the initial condition (c"), are satisfied respectively by letting

$$\frac{\partial \tau}{\partial r} = 0 \quad \text{and} \quad \frac{\partial R}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad \theta = \theta \tag{d"}$$

$$h_2 \tau = -k \frac{\partial \tau}{\partial r} \quad \text{and} \quad h_2 R = -k \frac{\partial R}{\partial r} \quad \text{at} \quad r = r_o \quad \theta = \theta \tag{e"}$$

$$\tau = \frac{g_1 r_o^2}{2k} \left[\frac{2}{3\, Bi_1} + \frac{1}{3} \right] + (t_{f1} - t_{f2}) - \frac{g_1 r^2}{6k} - R \quad \text{at } \theta = 0 \tag{f"}$$

Equation (3") is broken up into two equations as:

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r} \frac{\partial R}{\partial r} + \frac{g_2}{k} = 0 \tag{4"}$$

$$\frac{\partial^2 \tau}{\partial r^2} + \frac{2}{r} \frac{\partial \tau}{\partial r} = \frac{1}{\alpha} \frac{\partial \tau}{\partial \theta} \tag{5"}$$

Thus Eqs. (4") and (5"), together with boundary conditions (d") and (e") and initial condition (f"), constitute a boundary value problem equivalent to Eqs. (1"), (a"), (b"), and (c"). The general solution of Eq. (4") is

$$R = -\frac{g_2}{6k} r^2 + \frac{C_1}{r} + C_2 \, . \tag{6"}$$

Applying boundary conditions (d") and (e") yields the particular solution which in dimensionless form is

$$\frac{R}{\frac{g_2 r_o^2}{2k}} = \left[\frac{2}{3\, Bi_2} + \frac{1}{3} \right] - \frac{r^2}{3 r_o^2} \, . \tag{7"}$$

The general solution of Eq. (5") is

$$\tau = \left(C_3 \frac{\sin \frac{\lambda}{\sqrt{\alpha}} r}{r} + C_4 \frac{\cos \frac{\lambda}{\sqrt{\alpha}} r}{r} \right) e^{-\lambda^2 \theta} \, , \tag{8"}$$

where λ is a constant to be determined.

Applying conditions (d"), (e"), and (f") yields the particular solution which in dimensionless form is

$$\frac{\tau}{\frac{g_2 r_o^2}{2k}} = \sum_{n=1}^{\infty} \frac{\left\{\frac{g_1}{g_2}\left[\frac{3}{M_n^2} - 1 + \frac{1}{Bi_1}\right] - \left[\frac{3}{M_n^2} - 1 + \frac{1}{Bi_2}\right] + \frac{3}{2}\frac{(t_{f1}-t_{f2})}{g_2 r_o^2/2k}\right\}\sin M_n - \cos M_n\left\{\frac{g_1}{g_2}\left[\frac{M_n}{Bi_1} + \frac{3}{M_n}\right] - \left[\frac{M_n}{Bi_2} + \frac{3}{M_n}\right] + \frac{3}{2} M_n \frac{t_{f1}-t_{f2}}{g_2 r_o^2/2k}\right\}}{\frac{3}{4} M_n \left[M_n - \frac{\sin 2M_n}{2}\right]}$$

$$\frac{r_o}{r} \sin \frac{M_n}{r_o} r \, e^{-M_n^2 \phi} , \qquad (9")$$

where the M_n's are the consecutive roots of the transcendental equation

$$1 - M_n \cot M_n = Bi_2 \qquad \text{and defined as} \qquad M_n \equiv \frac{\lambda_n r_o}{\sqrt{\alpha}} . \qquad (10")$$

Define

$$E_s \equiv \sum_{n=1}^{\infty} \frac{\left(\frac{3}{M_n^2} + \frac{1}{Bi_2} - 1\right)\sin M_n - \left(\frac{M_n}{Bi_2} + \frac{3}{M_n}\right)\cos M_n}{\frac{3}{4} M_n \left(M_n - \frac{\sin 2M_n}{2}\right)} \frac{r_o}{r} \sin \frac{M_n}{r_o} r \; e^{-M_n^2 \phi} , \qquad (11")$$

$$H_s \equiv \sum_{n=1}^{\infty} \frac{\frac{3}{2}\left[\sin M_n - M_n \cos M_n\right]}{\frac{3}{4} M_n \left[M_n - \frac{\sin 2M_n}{2}\right]} \frac{r_o}{r} \sin \frac{M_n}{r_o} r \; e^{-M_n^2 \phi} . \qquad (12")$$

With this notation, Eq. (9") becomes

$$\frac{\tau}{\frac{g_2 r_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_s + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right] H_s + \frac{g_1}{g_2}\left[\frac{2}{3\ Bi_1} - \frac{2}{3\ Bi_2}\right] H_s \ . \qquad (13")$$

Thus the final solution in dimensionless form is from Eq. (2"):

$$\frac{T_2}{\frac{g_2 r_o^2}{2k}} = \left[\frac{g_1}{g_2} - 1\right] E_s + \left[\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right] H_s + \frac{g_1}{g_2}\left[\frac{2}{3\ Bi_1} - \frac{2}{3\ Bi_2}\right] H_s$$

$$+ \left[\frac{2}{3\ Bi_2} + \frac{1}{3}\right] - \frac{r^2}{3r_o^2} \ . \qquad (14")$$

DISCUSSION

As mentioned in the introduction, the general solution in each case involves linear combinations of two infinite series, both of which have already been evaluated. Reference to the publications noted[3] will show that for the range of Biot numbers investigated, both infinite series are approximated to a high degree of accuracy by the first term in the series provided that the Fourier modulus is greater than about 0.2. This affords a great simplification in the form of the general solution since the infinite series can be replaced by a single term.

Let the first term (n = 1) in the infinite series denoted by E be represented by F, and the first term in the infinite series denoted by H be represented by G. Eqs. (14), (14'), and (14") can then be written respectively to a great degree of accuracy as

$$\frac{T_2}{\frac{g_2 x_o^2}{2k}} = \left\{ \left[\frac{g_1}{g_2} - 1\right] F_p + \left[\frac{g_1}{g_2}\left(\frac{2}{Bi_1} - \frac{2}{Bi_2}\right) + \left(\frac{t_{f1} - t_{f2}}{\frac{g_2 x_o^2}{2k}}\right)\right] G_p \right\} \cos \frac{M_1}{x_o} x \, e^{-M_1^2 \phi}$$

$$+ \left[\frac{2}{Bi_2} + 1\right] - \left(\frac{x}{x_o}\right)^2 , \qquad (15)$$

$$\frac{T_2}{\frac{g_2 r_o^2}{2k}} = \left\{ \left[\frac{g_1}{g_2} - 1\right] F_c + \left[\frac{g_1}{g_2}\left(\frac{1}{Bi_1} - \frac{1}{Bi_2}\right) + \left(\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right)\right] G_c \right\} J_o\left(\frac{M_1}{r_o} r\right) e^{-M_1^2 \phi}$$

$$+ \left[\frac{1}{Bi_2} + \frac{1}{2}\right] - \frac{1}{2}\left(\frac{r}{r_o}\right)^2 , \qquad (15')$$

3. L. G. Epel, op.cit., p. 22 and M. P. Heisler, op.cit., p. 232.

$$\frac{T_2}{\frac{g_2 r_o^2}{2k}} = \left\{ \left[\frac{g_1}{g_2} - 1 \right] F_s + \left[\frac{g_1}{g_2} \left(\frac{2}{3\,Bi_1} - \frac{2}{3\,Bi_2} \right) + \left(\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}} \right) \right] G_s \right\} \frac{r_o}{r} \sin \frac{M_1}{r_o} r\, e^{-M_1^2 \phi}$$

$$+ \left[\frac{2}{3\,Bi_2} + \frac{1}{3} \right] - \frac{1}{3} \left(\frac{r}{r_o} \right)^2 , \qquad (15'')$$

where the subscripts on F and G refer to the geometry in question. To facilitate computations, the F's and G's are presented as functions of Bi_2 in Figs. 1 and 2. Thus, the temperature at any point in the body under consideration at any time for any combination of initial and boundary conditions is found by obtaining F and G from Figs. 1 and 2, and then utilizing Eqs. (15), (15') or (15").

The case in which the Biot number becomes infinite is included for consideration since this is equivalent to fixing the surface temperature of the solid at the same temperature as the cooling fluid. If it is desired to specify the temperature at the surface of the body rather than the fluid temperature this is accomplished by assigning the surface temperature to the fluid and considering the Biot number as infinite. This artifice may be used either before or after the step change occurs, or both before and after it occurs. Care should be taken to adjust the proper Biot number and fluid temperature (subscript 1 or 2) in order to simulate the problem properly.

Finally, it should be pointed out that the problem in which the body is covered with one or more layers of cladding, bonding materials, surface scales, insulation, etc., can be solved easily by use of Figs. 1 and 2. It is only necessary to consider the layers of the various materials as comprising fractions of the total film resistance and finding the total effective resistance that this imaginary film affords to the flow of heat emanating from the heat generating body. The reciprocal of this total resistance is then the effective pseudo-heat-transfer coefficient applied at the surface of the generating portion of the configuration. For easy

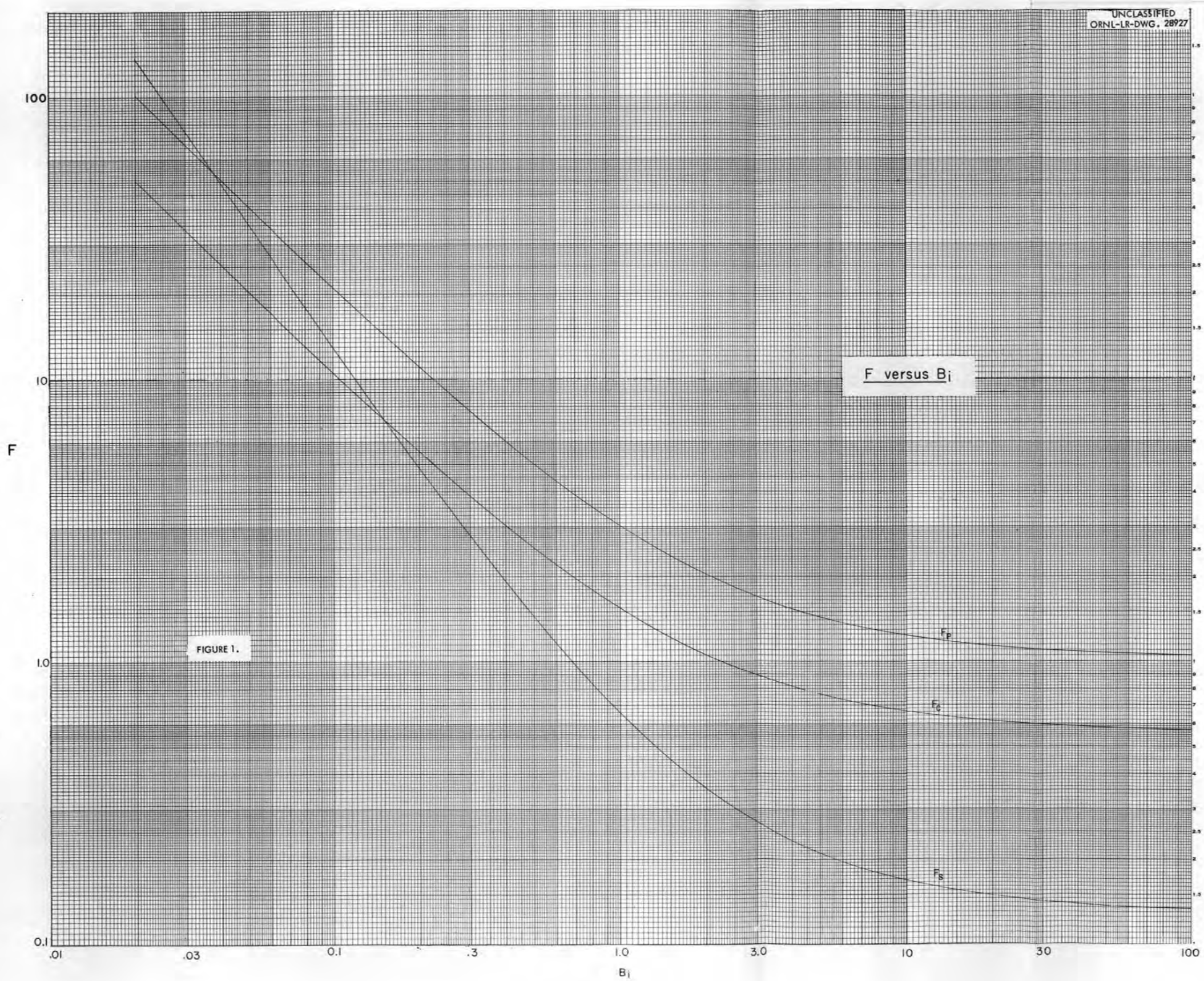
reference the formulae for these coefficients are presented below for the three geometries:

$$h' = \frac{1}{\frac{x_a}{k_a} + \frac{x_b}{k_b} + \frac{x_c}{k_c} + \cdots\cdots\cdots\cdots + \frac{x_n}{k_n} + \frac{1}{h}}, \tag{16}$$

$$h' = \frac{1}{\frac{r_o}{k_a} \ln \frac{r_a}{r_o} + \frac{r_o}{k_b} \ln \frac{r_b}{r_a} + \frac{r_o}{k_c} \ln \frac{r_c}{r_b} + \cdots\cdots + \frac{r_o}{k_n} \ln \frac{r_n}{r_{n-1}} + \frac{r_o}{r_n h}}, \tag{16'}$$

$$h' = \frac{1}{\frac{r_o^2(r_a-r_o)}{r_o\, r_a\, k_a} + \frac{r_o^2(r_b-r_a)}{r_a\, r_b\, k_b} + \frac{r_o^2(r_c-r_b)}{r_b\, r_c\, k_c} + \cdots\cdots + \frac{r_o^2(r_n-r_{n-1})}{r_{n-1}\, r_n\, k_n} + \frac{r_o^2}{r_n^2\, h}}, \tag{16''}$$

where a, b, c .... n, stand for the first, second, third, ... n^{th} layers of material.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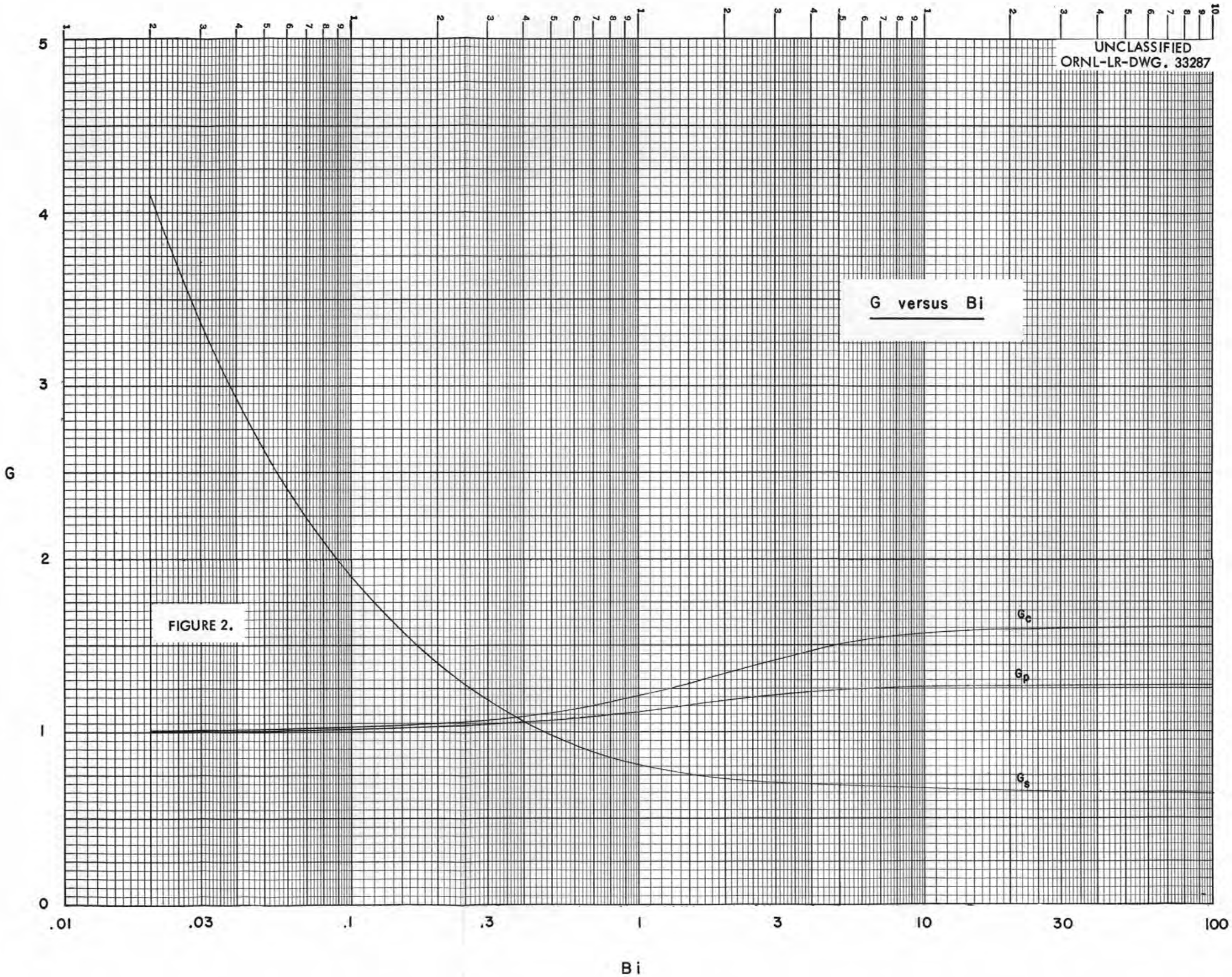
F versus B_i
F_p
F_c
F_s
F
100
10
1.0
0.1
.01
.03
0.1
.3
1.0
3.0
10
30
100
B_i

FIGURE 1.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G versus Bi
G
Bi
G_c
G_p
G_s
0
1
2
3
4
5
.01
.03
.1
.3
1
3
10
30
100

FIGURE 2.

ILLUSTRATIVE EXAMPLE

The foregoing analysis may appear somewhat abstract to the user who does not wish to follow the derivation in detail, but simply wishes to make use of the results. For this benefit an example is included here to present the procedure in solving a problem in general.

Consider a 3/4-in-dia fuel rod operating in a gas-cooled reactor under steady state conditions. The heat generation rate in the fuel is constant over the cross section of the cylindrical element and equal to 2×10^6 Btu/ft^{3}hr; the heat transfer coefficient is 180 Btu/hr ft^2 °F. There is a 20-mil stainless steel clad (assume no contact resistance between fuel and clad) and a scale deposit on the outside of the clad which has an effective conductance based on the fuel surface area of 2000 Btu/hr ft^2 °F. The thermal conductivity, specific heat, and density of the fuel are 1.23 Btu/hr ft °F, 0.085 Btu/lb °F, and 650 lb/ft^3 respectively, and the thermal conductivity of the stainless steel clad is 14.8 Btu/hr ft °F. At a given time the reactor power is halved, the coolant flow is decreased so that the heat transfer coefficient is subsequently 120 Btu/hr ft^2 °F, and the coolant temperature decreases 100°F. Estimate the temperature difference between a point 1/10 of the way from the center of the fuel rod and the new coolant temperature 1 min after the perturbations take place. Assume that the changes take place simultaneously and instantaneously.

The solution is given formally by Eq. (15'). To use this form it is necessary to know Bi_1, Bi_2, and ϕ. To ascertain the Biot numbers one must know the effective heat transfer coefficients based on the fuel outside diameter. To this end use is made of Eq. (16') as follows:

$$h_1' = \frac{1}{\frac{r_o}{k_{clad}} \ln \frac{r_{clad}}{r_o} + \frac{r_o}{r_{clad}\, h_1} + \frac{1}{h_{scale}}}$$

$$= \frac{1}{\frac{0.03125}{14.8} \ln \frac{0.03292}{0.03125} + \frac{0.03125}{0.03292(180)} + \frac{1}{2000}} = 170 \text{ Btu/ft}^2 \text{ hr } ^\circ\text{F}$$

$$h_2' = \frac{1}{\frac{r_o}{k_{clad}} \ln \frac{r_{clad}}{r_o} + \frac{r_o}{r_{clad}\, h_2} + \frac{1}{h_{scale}}}$$

$$= \frac{1}{\frac{0.03125}{14.8} \ln \frac{0.03292}{0.03125} + \frac{0.03125}{0.03292(120)} + \frac{1}{2000}} = 117 \text{ Btu/ft}^2 \text{ hr } ^\circ\text{F}$$

Hence

$$Bi_1 = \frac{h_1' \, r_o}{k} = \frac{170(0.03125)}{1.23} = 4.32$$

$$Bi_2 = \frac{h_2' \, r_o}{k} = \frac{117(0.03125)}{1.23} = 2.97$$

Also

$$\phi = \frac{\alpha\, \theta}{r_o^2} = \frac{1.23\ (1/60)}{650(0.085)(0.03125)^2} = 0.380$$

Notice that the values of F and G that must be used are the ones corresponding to Bi_2. In this case $F_c = 0.900$ and $G_c = 1.42$ from Figs. 1 and 2 respectively. M_1 is found by interpolation in tables[4] to be 1.7830.

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4. See for example, Carslaw and Jaeger, Conduction of Heat in Solids, Oxford University Press, London, 1947, pp. 377-379.

Substituting into Eq. (15')

$$\frac{T_2}{\frac{g_2 r_o^2}{2k}} = \left\{ \left[\frac{g_1}{g_2} - 1\right] F_c + \left[\frac{g_1}{g_2}\left(\frac{1}{Bi_1} - \frac{1}{Bi_2}\right) + \left(\frac{t_{f1} - t_{f2}}{\frac{g_2 r_o^2}{2k}}\right)\right] G_c \right\} J_o\left(\frac{M_1}{r_o} r\right) e^{-M_1^2 \phi}$$

$$+ \left[\frac{1}{Bi_2} + \frac{1}{2}\right] - \frac{1}{2}\left(\frac{r}{r_o}\right)^2 ,$$

$$\frac{T_2}{\frac{10^6 \times 0.03125^2}{2 \times 1.23}} = \left\{ \left[\frac{2 \times 10^6}{10^6} - 1\right] 0.900 + \left[\frac{2 \times 10^6}{10^6}\left(\frac{1}{4.32} - \frac{1}{2.97}\right)\right.\right.$$

$$\left.\left. + \left(\frac{100}{\frac{10^6 \times 0.03125^2}{2 \times 1.23}}\right)\right] 1.42 \right\} J_o(1.7830 \times 0.1)\, e^{-1.7830^2 (0.380)}$$

$$+ \left[\frac{1}{2.97} + \frac{1}{2}\right] - \frac{1}{2}\left(\frac{1}{10}\right)^2 ,$$

$$T_2 = t_2 - t_{f2} = 443°F.$$

Therefore, the point in question is seen to be 443°F above the new fluid temperature.

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