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TRANSIENT TEMPERATURES IN INFINITE PLATES, INFINITE CYLINDERS, AND SPHERES DURING CONVECTIVE COOLING FROM INITIALLY PARABOLIC TEMPERATURE PROFILES

L. G. Epel

OAK RIDGE NATIONAL LABORATORY
operated by
UNION CARBIDE CORPORATION
for the
U.S. ATOMIC ENERGY COMMISSION

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ORNL-2511

Contract No. W-7405-eng-26

Reactor Projects Division

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DATE ISSUED

MAY 5 1958

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INTRODUCTION

The solution of many engineering problems requires knowledge of temperature fields in various solids as functions of time and bounding conditions. Many charts of the Gurney Lurie type that aid in the estimation of temperature profiles over a wide range of time, thermal properties, dimensions of body, and bounding conductance can be found in the literature. In all of these presentations, however, the assumption is made that the temperature of the body under consideration is uniform initially.

The increased importance of temperature structures in bodies, that under some circumstances can be considered as uniform internal heat generators, makes an analysis of the transient temperatures during convective cooling in such bodies desirable.

During steady state operation the part under consideration will attain an equilibrium temperature distribution in accordance with Poisson's equation. Subsequently, it may be caused to cease its generation of heat and if cooling continues the temperature profile will vary with time and introduce problems (such as thermal stress) which depend on the temperature structure for their ultimate solution.

It is the purpose of this paper to present the evaluations of the series which represent the solutions to the Fourier equation in rectangular, cylindrical and spherical coordinates for the case in which the initial temperature field is that owing to a spatially uniform heat source. In the development, the temperature of the cooling fluid is considered to be constant.

To emphasize the similarities in the development for the three geometries, corresponding equations have been assigned the same number with merely primes or double primes distinguishing the geometry involved.

NOMENCLATURE

Symbols

A	area
B_i	Biot number $= \frac{hx_o}{k}$ or $\frac{hr_o}{k}$
c	constant of integration
g	rate of internal heat generation
h	unit surface conductance
k	thermal conductivity
M_n, M_m	roots of transcendental equation
r	radius
R	function of r alone
t	temperature
T	temperature difference $= t - t_f$
x	distance
X	function of x alone

Subscripts

c	center
f	surrounding fluid
i	initial
o	outside
s	surface

Greek

α	thermal diffusivity $= \frac{k}{\rho c_p}$ where ρ is density and c_p is specific heat
θ	time
Θ	function of θ alone
λ	separation constant
ϕ	Fourier modulus $= \frac{\alpha\theta}{x_o^2}$ or $\frac{\alpha\theta}{r_o^2}$
Ψ	a function defined in the appendix

THE INFINITE PLATE

Initial Temperature Distribution

Poisson's equation for this case is:

$$\frac{d^2t}{dx^2} + \frac{g}{k} = 0 \tag{1}$$

The general solution of Eq (1) is:

$$t = c_1 + c_2x - \frac{gx^2}{2k} \tag{2}$$

The boundary conditions are:

$$\left.\frac{dt}{dx}\right]_{x=o} = 0 \quad \text{where } x = o \text{ refers to the center of the plate}$$

$$-k\left.\frac{dt}{dx}\right]_{x=x_o} = h(t_{x_o} - t_f)$$

Utilization of the boundary conditions yields the particular solution

$$t = gx_o\left(\frac{1}{h} + \frac{x_o}{2k}\right) + t_f - \frac{gx^2}{2k} \tag{3a}$$

or

$$\frac{t - t_f}{gx_o^2/2k} = \frac{2}{B_i} + 1 - \frac{x^2}{x_o^2} \tag{3b}$$

Transient Temperature Distribution

Fourier's equation for this case is:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta} \tag{4}$$

Assuming a product solution $T = X(x)\Theta(\theta)$ and choosing a separation constant of $-\lambda^2/\alpha$ the solution of Eq (4) becomes:

$$T = \left(c_3 \cos \frac{\lambda}{\sqrt{\alpha}} x + c_4 \sin \frac{\lambda}{\sqrt{\alpha}} x\right) e^{-\lambda^2 \theta} \tag{5}$$

Applying the boundary condition that $\left.\frac{\partial T}{\partial x}\right]_{\substack{x=o \\ \theta=\theta}} = 0$ yields

$$T = c_4 \cos \frac{\lambda}{\sqrt{\alpha}} x \, e^{-\lambda^2 \theta} \tag{6}$$

The boundary condition $\left.hT\right]_{\substack{x=x_o \\ \theta=\theta}} = -k \left.\frac{\partial T}{\partial x}\right]_{\substack{x=x_o \\ \theta=\theta_o}}$ becomes after some algebraic manipulation:

$$M_n \tan M_n = B_i \tag{7}$$

where $M_n \equiv \frac{\lambda x_o}{\sqrt{\alpha}}$

So Eq (6) becomes:

$$T = \sum_{n=1}^{\infty} C_n \cos \frac{M_n}{x_o} x \, e^{-M_n^2 \phi} \tag{8}$$

where C_n is a constant corresponding to the nth root, M_n, of Eq (7).

To evaluate the C_n's use is made of the initial condition:

$$T_i = gx_o\left[\frac{1}{h} + \frac{x_o}{2k}\right] - \frac{gx^2}{2k} = \sum_{n=1}^{\infty} C_n \cos \frac{M_n}{x_o} x \tag{9}$$

Multiply both sides of Eq (9) by $\cos \frac{M_m}{x_o} x \, dx$ and integrate from 0 to x_o.

Now

$$\int_o^{x_o} C_n \cos \frac{M_n}{x_o} x \cos \frac{M_m}{x_o} x \, dx$$

$$= C_n \frac{x_o(M_m \tan M_m - M_n \tan M_n)}{\dfrac{M_m^2 - M_n^2}{\cos M_m \cos M_n}} \qquad \text{if } M_n \neq M_m$$

$$= C_n \frac{x_o}{M_n} \left(\frac{M_n}{2} + \frac{\sin 2 M_n}{4}\right) \qquad \text{if } M_n = M_m$$

From Eq (7) it is seen that the value of the integral is zero if $M_n \neq M_m$. Hence, all terms but the nth vanish on the right of Eq (9) after the integration is performed.

Therefore

$$\int_o^{x_o} \left[gx_o\left(\frac{1}{h} + \frac{x_o}{2k}\right) - \frac{gx^2}{2k}\right] \cos \frac{M_n}{x_o} x \, dx =$$

$$C_n \frac{x_o}{M_n} \left(\frac{M_n}{2} + \frac{\sin 2 M_n}{4}\right) \tag{10}$$

The integral on the left of Eq (10) is evaluated as:

$$\frac{gx_o^2}{M_n}\left(\frac{1}{h}+\frac{x_o}{2k}\right)\sin M_n - \frac{gx_o^3}{2kM_n^3}\left[2M_n\cos M_n + (M_n^2-2)\sin M_n\right]$$

And therefore

$$C_n = \frac{gx_o^2}{2k}\,\frac{\left(\frac{2}{B_i}+1\right)\sin M_n - \frac{1}{M_n^2}\left[2\,M_n\cos M_n + (M_n^2-2)\sin M_n\right]}{\left[\frac{M_n}{2}+\frac{\sin 2\,M_n}{4}\right]} \qquad (11)$$

Hence the final form of the solution as expressed by Eq (8) is:

$$\frac{t-t_f}{\frac{gx_o^2}{2k}} = \sum_{n=1}^{\infty}\frac{\left(\frac{2}{B_i}+1\right)\sin M_n - \frac{1}{M_n^2}\left[2M_n\cos M_n + (M_n^2-2)\sin M_n\right]}{\left[\frac{M_n}{2}+\frac{\sin 2M_n}{4}\right]}\cos\frac{M_n}{x_o}x\,e^{-M_n^2\phi} \qquad (12)$$

THE INFINITE CYLINDER

Initial Temperature Distribution

Poisson's equation for this case is:

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{g}{k} = 0 \qquad (1')$$

The general solution of Eq (1') is:

$$t = c_1 + c_2 \ln r - \frac{g}{4k} r^2 \qquad (2')$$

The boundary conditions are:

$$\left.\frac{dt}{dr}\right]_{r=o} = 0$$

$$- k \left.\frac{dt}{dr}\right]_{r=r_o} = h(t_{r_o} - t_f)$$

Utilization of the boundary conditions yields the particular solution

$$t = \frac{gr_o}{2}\left(\frac{1}{h} + \frac{r_o}{2k}\right) + t_f - \frac{gr^2}{4k} \qquad (3a')$$

or

$$\frac{t - t_f}{gr_o^2/2k} = \frac{1}{B_i} + \frac{1}{2} - \frac{r^2}{2r_o^2} \qquad (3b')$$

Transient Temperature Distribution

Fourier's equation for this case is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta} \qquad (4')$$

Assuming a product solution $T = R(r)\Theta(\theta)$ and choosing a separation constant of $-\lambda^2/\alpha$ the solution of 4' becomes:

$$T = \left[c_3 J_o\left(\frac{\lambda}{\sqrt{\alpha}}\, r\right) + c_4 Y_o\left(\frac{\lambda}{\sqrt{\alpha}}\, r\right)\right] e^{-\lambda^2\theta} \qquad (5')$$

Applying the boundary condition that $\left.\frac{\partial T}{\partial r}\right]_{\substack{r=o\\ \theta=\theta}} = 0$ yields

$$T = c_3 J_o\left(\frac{\lambda}{\sqrt{\alpha}}\, r\right) e^{-\lambda^2\theta} \qquad (6')$$

The boundary condition $hT\Big]_{\substack{r=r_o\\ \theta=\theta_o}} = -\,k\left.\frac{\partial T}{\partial r}\right]_{\substack{r=r_o\\ \theta=\theta_o}}$ becomes after some algebraic manipulation:

$$M_n \frac{J_1(M_n)}{J_o(M_n)} = B_i \qquad (7')$$

where $M_n \equiv \frac{\lambda r_o}{\sqrt{\alpha}}$

So Eq (6') becomes

$$T = \sum_{n=1}^{\infty} C_n J_o\left(\frac{M_n}{r_o} r\right) e^{-M_n^2 \phi} \qquad (8')$$

where C_n is a constant corresponding to the nth root, M_n, of Eq (7').

To evaluate the C_n's use is made of the initial condition:

$$T_i = \frac{gr_o}{2}\left[\frac{1}{h} + \frac{r_o}{2k}\right] - \frac{gr^2}{4k} = \sum_{n=1}^{\infty} C_n J_o\left(\frac{M_n}{r_o}\, r\right) \qquad (9')$$

Multiply both sides of Eq (9') by $rJ_o\left(\frac{M_m}{r_o} r\right)$ dr and integrate from 0 to r_o.

Now

$$\int_o^{r_o} C_n r J_o\left(\frac{M_n}{r_o} r\right) J_o\left(\frac{M_m}{r_o} r\right) dr$$

$$= C_n r_o^2 \frac{M_m \frac{J_1(M_m)}{J_o(M_m)} - M_n \frac{J_1(M_n)}{J_o(M_n)}}{\frac{M_m^2 - M_n^2}{J_o(M_n)\ J_o(M_m)}} \qquad \text{if } M_n \neq M_m$$

$$= C_n \frac{r_o^2}{2} \left[J_o^2(M_n) + J_1^2(M_n)\right] \qquad \text{if } M_n = M_m$$

From Eq (7') it is seen that the value of the integral is zero if $M_n \neq M_m$. Hence all terms but the nth vanish on the right of Eq (9') after the integration is performed.

Therefore

$$\int_o^{r_o} \left[\frac{gr_o}{2}\left(\frac{1}{h} + \frac{r_o}{2k}\right) - \frac{gr^2}{4k}\right] r\ J_o\left(\frac{M_n}{r_o} r\right) dr =$$

$$C_n \frac{r_o^2}{2} \left[J_o^2(M_n) + J_1^2(M_n)\right] \tag{10'}$$

The integral on the left of Eq (10') is evaluated as:

$$\frac{gr_o^3}{2M_n}\left(\frac{1}{h}+\frac{r_o}{2k}\right) J_1(M_n) - \frac{gr_o^4}{4kM_n^4}\,\Psi(M_n)$$

and therefore

$$C_n = \frac{gr_o^2}{2k}\;\frac{\left(\frac{2}{B_i}+1\right)\frac{J_1(M_n)}{M_n} - \frac{\Psi(M_n)}{M_n^4}}{J_o^2(M_n) + J_1^2(M_n)} \tag{11'}$$

Hence, the final form of the solution as expressed by Eq (8') is:

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = \sum_{n=1}^{\infty} \frac{\left(\frac{2}{B_i}+1\right)\frac{J_1(M_n)}{M_n} - \frac{\Psi(M_n)}{M_n^4}}{J_o^2(M_n) + J_1^2(M_n)}\; J_o\left(\frac{M_n}{r_o}\,r\right) e^{-M_n^2\phi} \tag{12'}$$

THE SPHERE

Initial Temperature Distribution

Poisson's equation for this case is:

$$\frac{d^2t}{dr^2} + \frac{2}{r}\frac{dt}{dr} + \frac{g}{k} = 0 \qquad (1'')$$

The general solution of Eq (1") is:

$$t = c_1 + \frac{c_2}{r} - \frac{1}{6}\frac{g}{k}r^2 \qquad (2'')$$

The boundary conditions are:

$$\left.\frac{dt}{dr}\right]_{r=o} = 0$$

$$-k\left.\frac{dt}{dr}\right]_{r=r_o} = h(t_{r_o} - t_f)$$

Utilization of the boundary conditions yields the particular solution

$$t = \frac{gr_o}{3}\left(\frac{1}{h} + \frac{r_o}{2k}\right) + t_f - \frac{gr^2}{6k} \qquad (3a'')$$

or

$$\frac{t - t_f}{gr_o^2/2k} = \frac{2}{3B_i} + \frac{1}{3} - \frac{r^2}{3r_o^2} \qquad (3b'')$$

Transient Temperature Distribution

Fourier's equation for this case is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta} \qquad (4'')$$

Assuming a product solution $T = R(r)\Theta(\theta)$ and choosing a separation constant of $-\lambda^2/\alpha$ the solution of Eq (4") becomes:

$$T = \left(c_3 \frac{\sin \frac{\lambda}{\sqrt{\alpha}} r}{r} + c_4 \frac{\cos \frac{\lambda}{\sqrt{\alpha}} r}{r}\right) e^{-\lambda^2 \theta} \qquad (5")$$

Applying the boundary condition that $\left.\frac{\partial T}{\partial r}\right]_{\substack{r=o \\ \theta=\theta}} = 0$ yields

$$T = C_3 \frac{\sin \frac{\lambda}{\sqrt{\alpha}} r}{r} e^{-\lambda^2 \theta} \qquad (6")$$

The boundary condition $hT\Big]_{\substack{r=r_o \\ \theta=\theta_o}} = -k \left.\frac{\partial T}{\partial r}\right]_{\substack{r=r_o \\ \theta=\theta_o}}$ becomes after some algebraic manipulation:

$$1 - M_n \cot M_n = B_i \qquad (7")$$

where $M_n = \frac{\lambda r_o}{\sqrt{\alpha}}$

So Eq (6") becomes

$$T = \sum_{n=1}^{\infty} C_n \frac{\sin \frac{M_n}{r_o} r}{r} e^{-M_n^2 \phi} \qquad (8")$$

where C_n is a constant corresponding to the nth root, M_n, of Eq (7").

To evaluate the C_n's use is made of the initial condition:

$$T_i = \frac{gr_o}{3}\left(\frac{1}{h} + \frac{r_o}{2k}\right) - \frac{gr^2}{6k} = \sum_{n=1}^{\infty} C_n \frac{\sin \frac{M_n}{r_o} r}{r} \qquad (9")$$

Multiply both sides of Eq (9") by $r \sin \frac{M_m}{r_o} r\, dr$ and integrate from 0 to r_o.

Now

$$\int_o^{r_o} C_n \sin \frac{M_n}{r_o} r \sin \frac{M_m}{r_o} r dr$$

$$= C_n \frac{r_o \sin M_n \sin M_m}{M_n^2 - M_m^2} (M_m \cot M_m - M_n \cot M_n) \qquad \text{if } M_n \neq M_m$$

$$= C_n \frac{r_o}{2M_n} \left(M_n - \frac{\sin 2 M_n}{2} \right) \qquad \text{if } M_n = M_m$$

From Eq (7") it is seen that the value of the integral is zero if $M_n \neq M_m$. Hence, all terms but the nth vanish on the right of Eq (9") after the integration is performed.

Therefore

$$\int_o^{r_o} \left[\frac{gr_o}{3} \left(\frac{1}{h} + \frac{r_o}{2k} \right) - \frac{gr^2}{6k} \right] r \sin \frac{M_n}{r_o} r\, dr =$$

$$C_n \frac{r_o}{2M_n} \left(M_n - \frac{\sin 2 M_n}{2} \right) \qquad (10")$$

The integral on the left of Eq (10") is evaluated as:

$$\frac{gr_o^3}{3M_n^2}\left[\frac{r_o}{k}\left(\frac{3}{M_n^2}\sin M_n - \frac{3}{M_n}\cos M_n - \sin M_n\right) + \frac{1}{h}(\sin M_n - M_n \cos M_n)\right]$$

And therefore

$$C_n = \frac{gr_o^2}{2k}\,\frac{4r_o}{3M_n(M_n - \frac{\sin 2M_n}{2})}\left[\left(\frac{3}{M_n^2} + \frac{1}{B_i} - 1\right)\sin M_n - \left(\frac{M_n}{B_i} + \frac{3}{M_n}\right)\cos M_n\right] \quad (11")$$

Hence the final form of the solution as expressed by Eq (8") is:

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = \sum_{n=1}^{\infty} \frac{\left(\frac{3}{M_n^2} + \frac{1}{B_i} - 1\right)\sin M_n - \left(\frac{M_n}{B_i} + \frac{3}{M_n}\right)\cos M_n}{\frac{3}{4}M_n\left(M_n - \frac{\sin 2M_n}{2}\right)}\,\frac{r_o}{r}\sin\frac{M_n}{r_o}r\,e^{-M_n^2\phi} \quad (12")$$

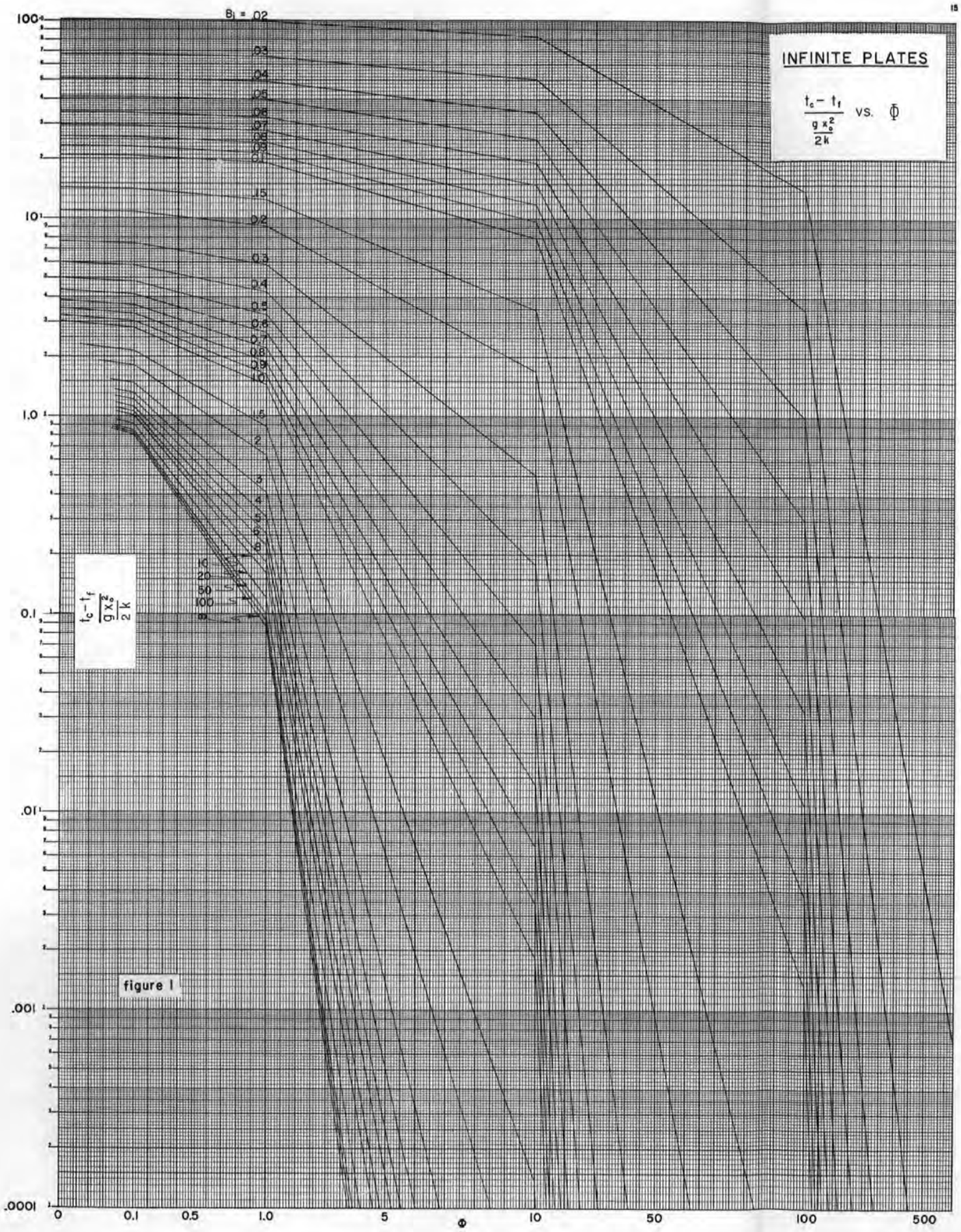

INFINITE PLATES
$\frac{t_c - t_f}{\frac{g x_o^2}{2k}}$ vs. Φ
Bi = .02
.03
.04
.05
.06
.07
.08
.09
.1
.15
.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.5
2
3
4
5
6
8
10
20
50
100
∞
$\frac{t_c - t_f}{\frac{g x_o^2}{2k}}$
100
10
1.0
0.1
.01
.001
.0001
0
0.1
0.5
1.0
5
10
50
100
500
Φ

figure 1

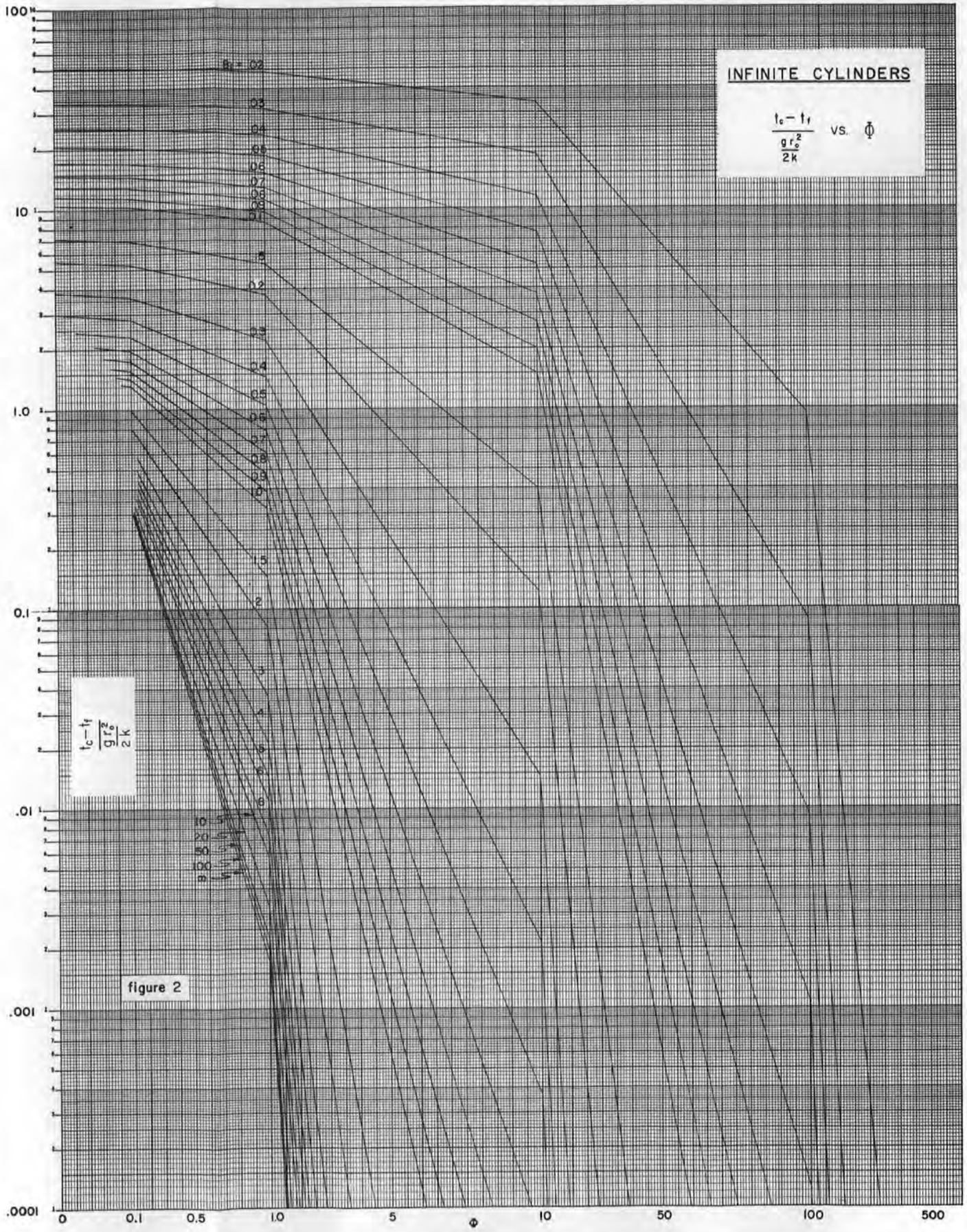

INFINITE CYLINDERS
$\frac{t_c - t_f}{\frac{g r_o^2}{2k}}$ vs. Φ
$\frac{t_c - t_f}{\frac{g r_o^2}{2k}}$
$B_i = .02$
.03
.04
.05
.06
.07
.08
.09
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.5
2
3
4
5
6
10
20
30
100
∞
100
10
1.0
0.1
.01
.001
.0001
0
0.1
0.5
1.0
5
10
50
100
500
Φ

figure 2

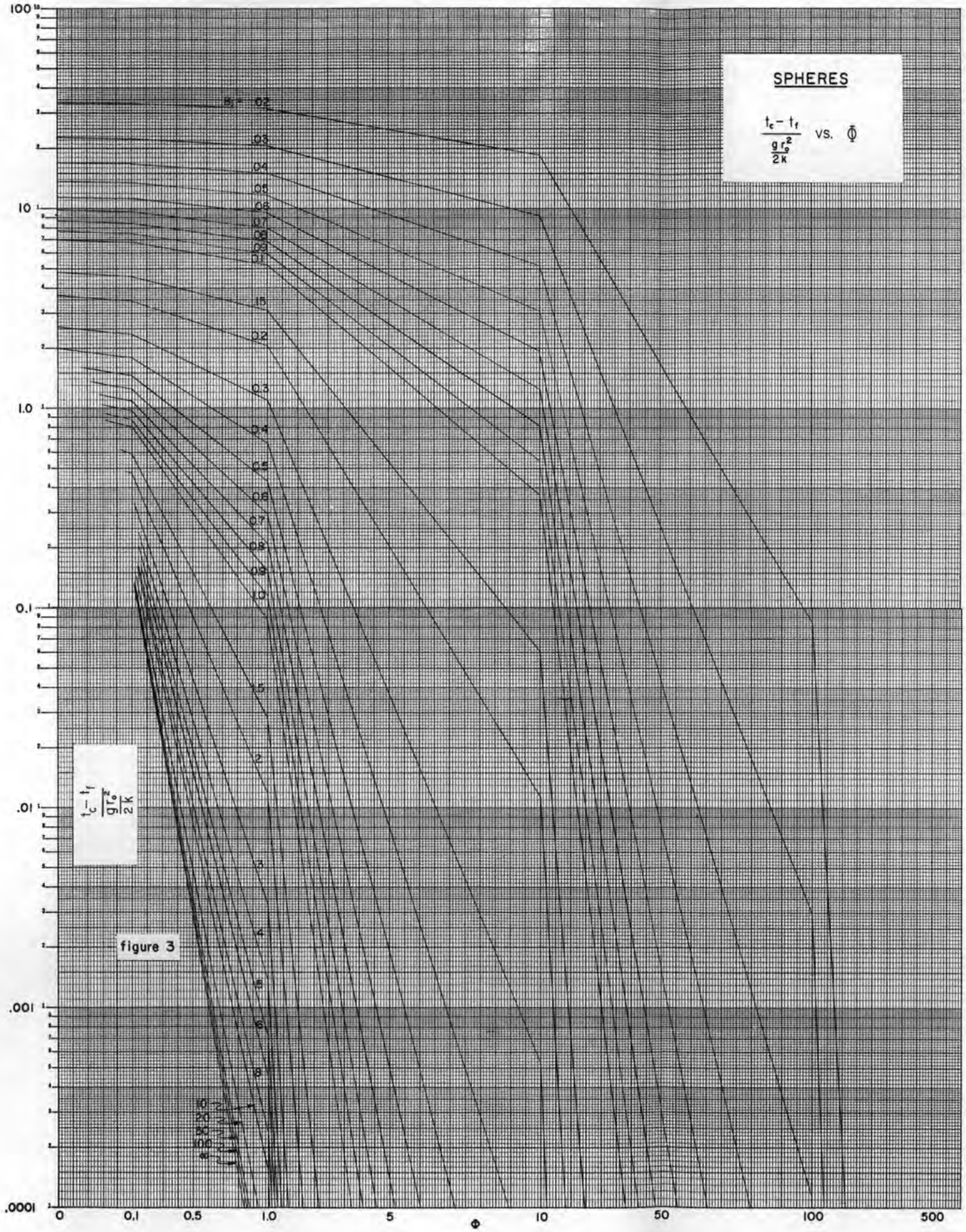

SPHERES
$\frac{t_c - t_f}{\frac{g r_o^2}{2k}}$ vs. Φ
$\frac{t_c - t_f}{\frac{g r_o^2}{2k}}$
Φ
100
10
1.0
0.1
.01
.001
.0001
0
0.1
0.5
1.0
5
10
50
100
500

figure 3

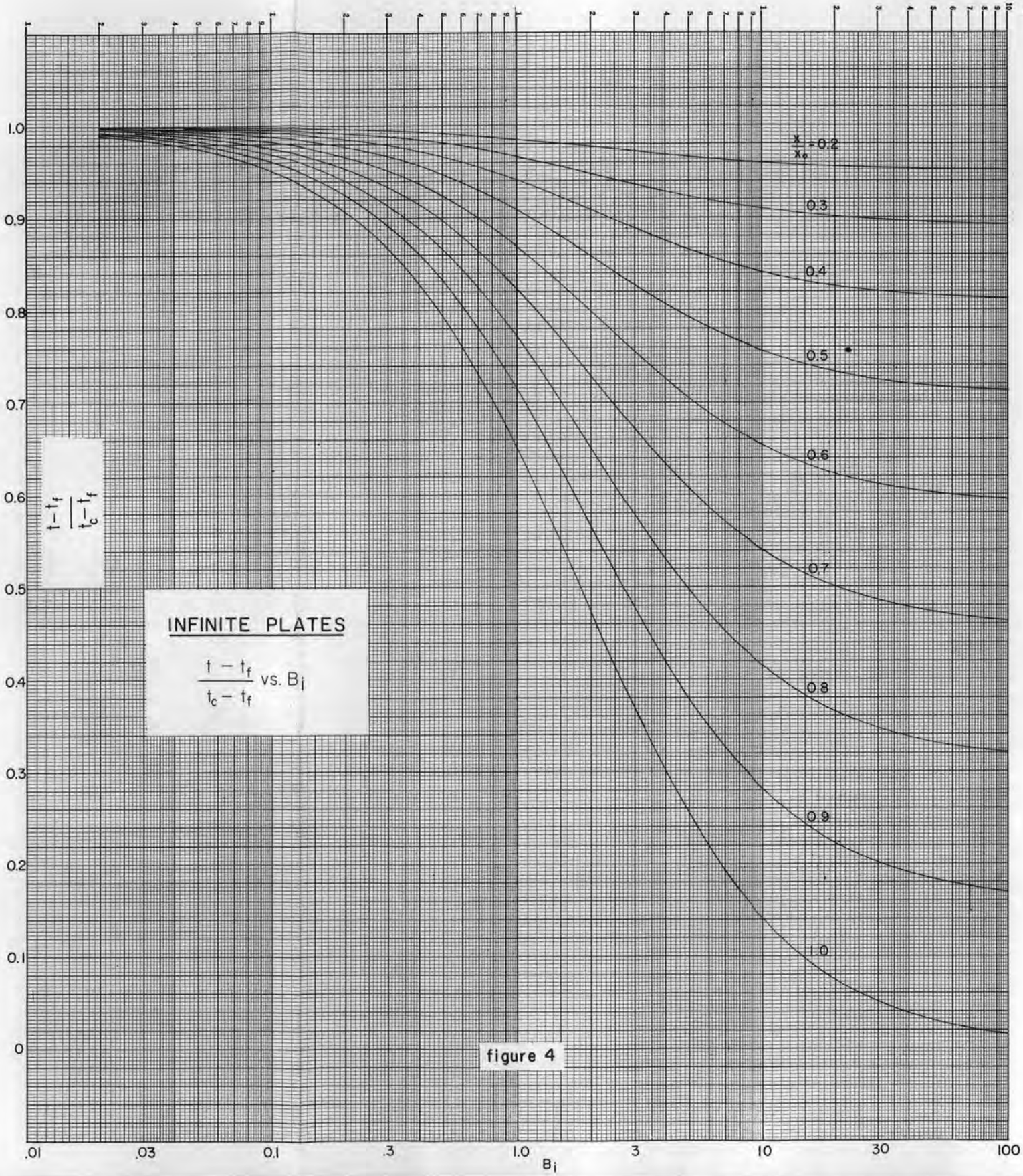

INFINITE PLATES
$\frac{t - t_f}{t_c - t_f}$ vs. B_i
$\frac{t - t_f}{t_c - t_f}$
$\frac{x}{x_o} = 0.2$
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
.01
.03
0.1
.3
1.0
3
10
30
100
B_i

figure 4

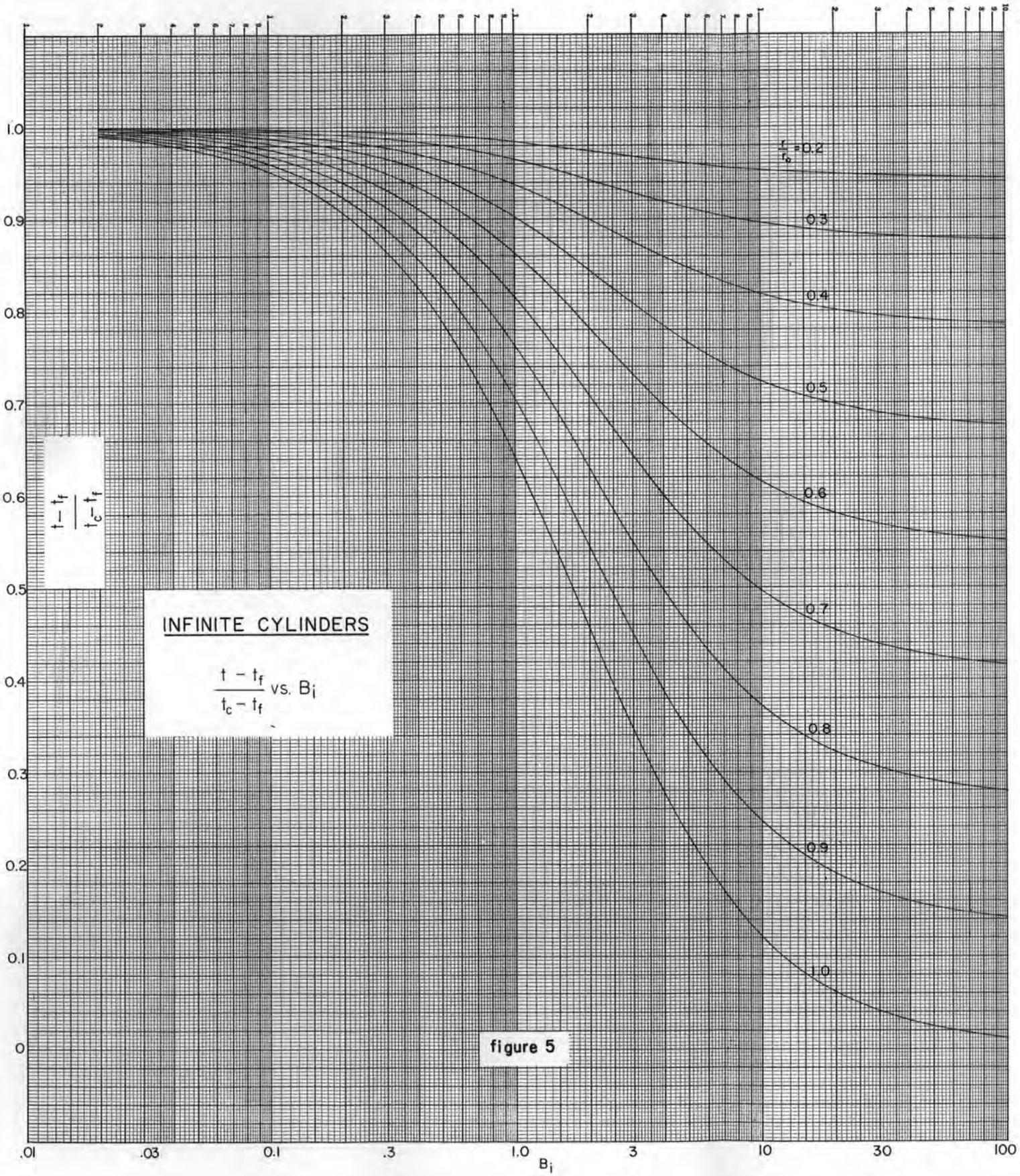

INFINITE CYLINDERS
$\frac{t - t_f}{t_c - t_f}$ vs. B_i
$\frac{t - t_f}{t_c - t_f}$
$\frac{r}{r_o}$ = 0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
.01
.03
0.1
.3
1.0
3
10
30
100
B_i

figure 5

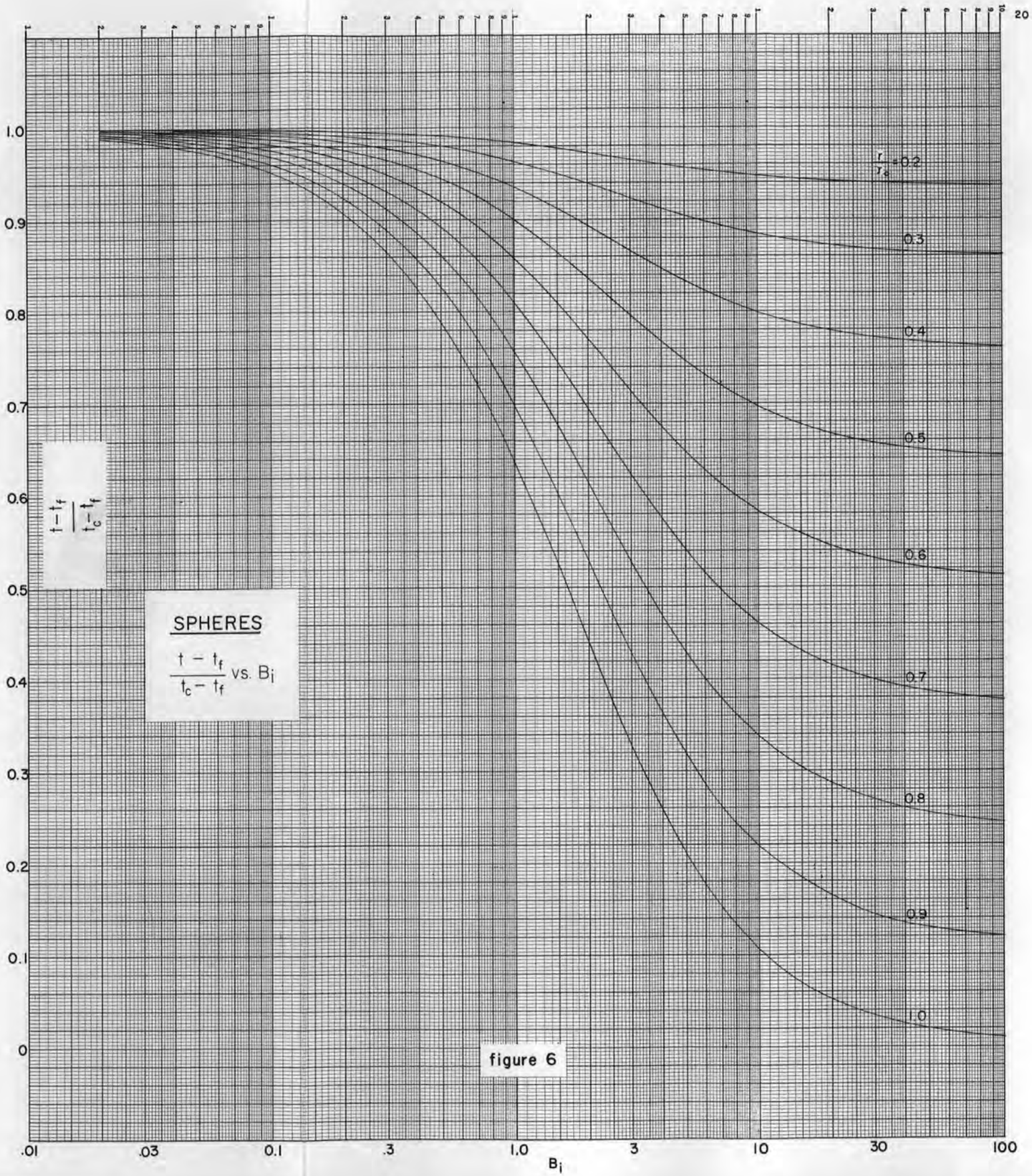
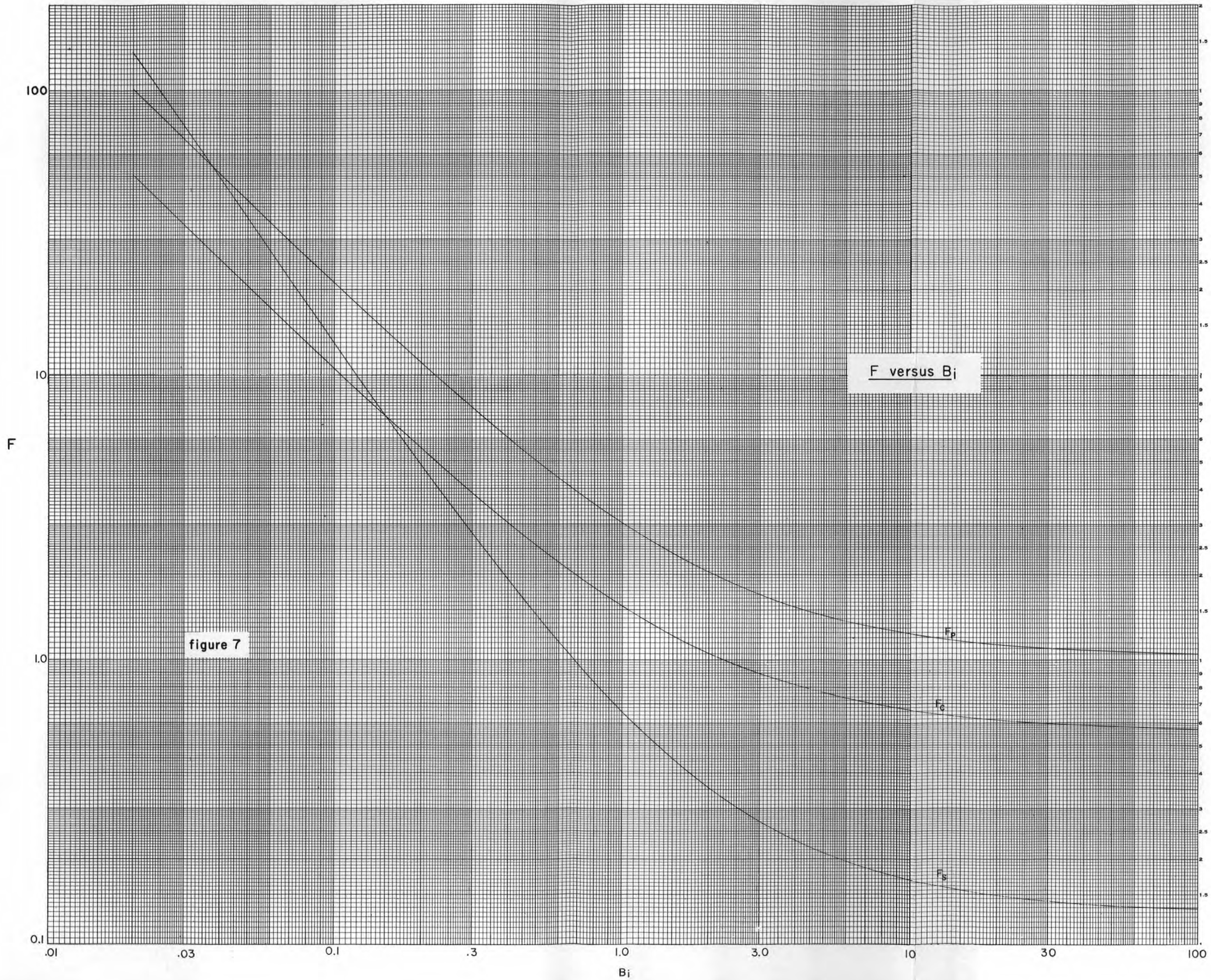

SPHERES
$\frac{t - t_f}{t_c - t_f}$ vs. B_i
$\frac{t - t_f}{t_c - t_f}$
$\frac{r}{r_o}$ = 0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
.01
.03
0.1
.3
1.0
3
10
30
100
20
B_i

figure 6


F versus Bi
figure 7
F
100
10
1.0
0.1
.01
.03
0.1
.3
1.0
3.0
10
30
100
Bi
Fp
Fc
Fs

Discussion

The numerical evaluation of the series in Eqs. (12), (12') and (12") for the dimensionless temperature was performed on the IBM 704 digital computer. The results for a wide range of parameters are shown on Figs. 1 through 6. Due to certain characteristics of the infinite sum, it was possible to present all the information on two charts for each geometry chosen. The first chart for each geometry leads to the temperature at the center of the body; the second chart yields the temperature at any other point.

It will be noticed that the dimensionless temperatures are given by the sum of an infinite series of three factors. The first factor in each term is a function of the Biot number, the second factor depends on the Biot number and position in the body, the third factor is determined by the Biot number and the time. When the dimensionless temperature is plotted against time for any given Biot number and position in the body, the result is in general a curve representing the series $\sum_{n=1}^{\infty} K_n e^{-M_n^2 \phi}$ where K_n is the nth constant depending only on Biot number and position in the body. Reference to the temperature-time charts for the center temperatures indicates that over the range considered the curve so obtained is in fact, a straight line. This can only mean that the series converge so rapidly that the first term alone adequately describes the infinite series[1].

1. The lines were discontinued when the time was so small that the second term in the infinite series for the center temperature became about one percent as great as the first term.

This leads to a convenient method of expressing the values of these series for the range reported. Considering only the first term in the infinite sum, Eqs. (12), (12') and (12") become

$$\frac{t - t_f}{\frac{gx_o^2}{2k}} = \left\{ \frac{\left(\frac{2}{B_i} + 1\right) \sin M_1 - \frac{1}{M_1^2}\left[2M_1 \cos M_1 + (M_1^2 - 2) \sin M_1\right]}{\frac{M_1}{2} + \frac{\sin 2 M_1}{4}} \right\} \cos\left(\frac{M_1 x}{x_o}\right) e^{-M_1^2 \phi} \qquad (13)$$

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = \left\{ \frac{\left(\frac{2}{B_i} + 1\right) \frac{J_1(M_1)}{M_1} - \frac{\Psi(M_1)}{M_1^4}}{J_o^2(M_1) + J_1^2(M_1)} \right\} J_o\left(\frac{M_1 r}{r_o}\right) e^{-M_1^2 \phi} \qquad (13')$$

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = \left\{ \frac{\left(\frac{3}{M_1^2} + \frac{1}{B_i} - 1\right) \sin M_1 - \left(\frac{M_1}{B_i} + \frac{3}{M_1}\right) \cos M_1}{\frac{3}{4} M_1 \left(M_1 - \frac{\sin 2 M_1}{2}\right)} \right\} \frac{r_o}{r} \sin\left(\frac{M_1 r}{r_o}\right) e^{-M_1^2 \phi} \qquad (13")$$

where M_1 in Eqs. (13), (13') and (13") represent the first root of the transcendental Eqs. (7), (7') and (7") respectively[2]. The factor in brackets in each of these equations is cumbersome to evaluate and so their values have been computed on the electronic computer and are reported as F_p, F_c and F_s respectively. Substituting F's for the terms in brackets, the equations become

2. The first six roots of Eqs. (7), (7') and (7") can be found in Appendix IV, tables I, III and II respectively of Conduction of Heat in Solids, H. S. Carslaw and J. C. Jaeger, Oxford University Press, London, 1st edition, 1947, pp 377-379.

$$\frac{t - t_f}{\frac{gx_o^2}{2k}} = F_p \cos\left(\frac{M_1 x}{x_o}\right) e^{-M_1^2 \phi} \tag{14}$$

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = F_c \, J_o\left(\frac{M_1 r}{r_o}\right) e^{-M_1^2 \phi} \tag{14'}$$

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = F_s \frac{r_o}{r} \sin\left(\frac{M_1 r}{r_o}\right) e^{-M_1^2 \phi} \tag{14''}$$

The temperature for values of Biot number, position, and time not shown in Figs. 1 through 6 is determined by finding F from Fig. 7 and using Eqs. (14), (14'), or (14"). The only restriction is that times less than those used in Figs. 1 through 3 for any particular Biot number should not be considered.

It is obvious that, as the surface conductance becomes infinite, t_s approaches t_f. For this reason the evaluation of series (12), (12') and (12") was carried out with $B_i \rightarrow \infty$. This corresponds to the case in which the temperature of the surface of the body instead of the temperature of the surrounding fluid is known. The temperature at any point within the body at any time can be found by considering t_f to be equal to the known surface temperature and utilizing the charts for $B_i \rightarrow \infty$. This one additional evaluation then yields the solutions to the many problems which involve cooling from an initially parabolic temperature distribution while the surface temperature is kept fixed.

One further application of the charts merits comment. In many practical applications, the body being studied may be clad with a material having, in general, different thermal properties than the rest of the solid and not generating heat within itself at any time. Temperatures within the body can still be found from the charts if one adopts the viewpoint that the clad is part of the film resistance. With this artifice in mind it is only necessary to redefine the surface conductance to be the reciprocal of the sum of the resistances of the clad and film and to proceed as before. This concept can, of course, be extended to encompass the cases in which there is more than one layer of clad, one over the other, bonding materials between clads, surface scales, insulation, etc. For convenience, the formulae for the pseudo-surface-conductance are listed below for the three geometries.

$$h' = \frac{1}{\frac{x_a}{k_a} + \frac{x_b}{k_b} + \frac{x_c}{k_c} + \ldots + \frac{x_n}{k_n} + \frac{1}{h}} \tag{15}$$

$$h' = \frac{1}{\frac{r_o}{k_a} \ln \frac{r_a}{r_o} + \frac{r_o}{k_b} \ln \frac{r_b}{r_a} + \frac{r_o}{k_c} \ln \frac{r_c}{r_b} + \ldots + \frac{r_o}{k_n} \ln \frac{r_n}{r_{n-1}} + \frac{r_o}{r_n h}} \tag{15'}$$

$$h' = \frac{1}{\frac{r_o^2(r_a - r_o)}{r_o r_a k_a} + \frac{r_o^2(r_b - r_a)}{r_a r_b k_b} + \frac{r_o^2(r_c - r_b)}{r_b r_c k_c} + \ldots + \frac{r_o^2(r_n - r_{n-1})}{r_{n-1} r_n k_n} + \frac{r_o^2}{r_n^2 h}} \tag{15''}$$

In Eqs. (15), (15') and (15") the subscripts a, b, c, ... n, stand for the first, second, third, ... n^{th} layers of material. It should be noted that these formulae are based on the surface area of the generating material (subscript "o") even though the surface conductance is generally appropriately applied to the surface at which the film acts. This was necessary because of the way the surface conductance boundary condition is applied in the analysis.

Illustrative Example

For those who do not wish to concern themselves with the mathematical analysis presented in the foregoing pages, but simply want to make use of the results, an illustrative example is included to point out the general method.

Consider a 3/4 in. diameter fuel rod operating in a gas-cooled reactor under steady state conditions. The heat generation rate in the fuel (UO_2) is constant over the cross section of the cylindrical element and equal to 2×10^6 Btu/ft^{3}hr. The surface conductance is invariable at 180 Btu/hr ft^2 ^{o}F. There is a 20 mil stainless steel clad (assume no contact resistance between fuel and clad) and a scale deposit on the outside of the clad which has an effective conductance based on the fuel surface of 2000 Btu/hr ft^2 ^{o}F. If the thermal conductivity, specific heat, and density of the UO_2 are 1.23 Btu/hr ft ^{o}F, 0.085 Btu/lb ^{o}F, and 650 lb/ft^3 respectively, and the thermal conductivity of the stainless steel clad is 14.8 Btu/hr ft ^{o}F, determine the difference between the fuel center, fuel surface and the coolant temperature one minute after heat generation ceases.

Solution - To enter the charts, all that is required is the value of the Fourier Modulus and Biot number.

$$\phi = \frac{\alpha\theta}{r_o^2} = \frac{k\theta}{\rho c_p r_o^2} = \frac{1.23 \quad \frac{1}{60}}{650 \quad 0.085 \ (0.03125)^2} = 0.380$$

In order to evaluate the Biot number first calculate the effective surface conductance based on the fuel outside diameter. From Eq. (15') with just one layer of material, the clad, the surface conductance is:

$$h' = \frac{1}{\frac{r_o}{k_{clad}} \ln \frac{r_{clad}}{r_o} + \frac{r_o}{r_{clad} h} + \frac{1}{h_{scale}}} = \frac{1}{\frac{0.03125}{14.8} \ln \frac{0.03292}{0.03125} + \frac{0.03125}{0.03292(180)} + \frac{1}{2000}} =$$

$$\frac{1}{0.0001098 + 0.005274 + 0.0005000} = 170 \text{ Btu/ft}^2\text{hr}^o\text{F}$$

Thus

$$B_i = \frac{h'r_o}{k} = \frac{170 \quad 0.03125}{1.23} = 4.32$$

With the known Biot number and Fourier Modulus enter Figs. 2 and 5 to obtain

$$\frac{t_c - t_f}{\frac{gr_o^2}{2k}} = 0.190$$

and $$\frac{t - t_f}{t_c - t_f} = 0.261 \qquad \text{for} \qquad \frac{r}{r_o} = 1.0$$

Since

$$\frac{gr_o^2}{2k} = \frac{2x10^6(0.03125)^2}{2 \text{ x } 1.23} = 794 \ ^oF$$

$$t_c - t_f = 0.190 \frac{gr_o^2}{2k} = 0.190 \text{ x } 794 = 151 \ ^oF$$

and

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = \frac{t - t_f}{t_c - t_f} \text{ x } \frac{t_c - t_f}{\frac{gr_o^2}{2k}} = 0.261 \text{ x } 0.190 = 0.0496 \text{ for } \frac{r}{r_o} = 1.0$$

Hence

$$t - t_f = 0.0496 \text{ x } \frac{gr_o^2}{2k} = 39.4 \ ^oF \text{ for } \frac{r}{r_o} = 1.0$$

Rather than using the charts, use can be made of Fig. 7 and Eq. (14').
For the center temperature, Eq. (14') becomes

$$\frac{t_c - t_f}{\frac{gr_o^2}{2k}} = F_c \; e^{-M_1^2 \phi}$$

F_c is seen to be 0.798, from Fig. 7, and M_1 is approximately 1.9342 as obtained from the tables by straight line interpolation.
Therefore

$$t_c - t_f = \frac{gr_o^2}{2k} \; F_c \; e^{-M_1^2 \phi}$$

$$= 794 \; (0.798) \; e^{-(1.9342)^2 0.380}$$

$$= 152^{\circ}F$$

For the temperature at $\frac{r}{r_o} = 1$, Eq. (14') becomes

$$\frac{t - t_f}{\frac{gr_o^2}{2k}} = F_c \; J_o \; (M_1) \; e^{-M_1^2 \phi}$$

Therefore

$$t - t_f = \frac{gr_o^2}{2k} \; F_c \; J_o \; (M_1) \; e^{-M_1^2 \phi}$$

$$= 794(0.798)0.262 \; e^{-(1.9342)^2 0.380}$$

$$= 39.8^{\circ}F$$

The results as obtained by both methods agree with each other to within about one percent.

Appendix A

Values of the Function $\Psi(M_n)$

$$\Psi(M_n) = \int_0^{M_n} \xi^3 J_0(\xi)\, d\xi$$

M_n	$\Psi(M_n)$	M_n	$\Psi(M_n)$
.00000	.000000	1.78870	1.340736
.19950	.000393	1.90810	1.575035
.28140	.001517	1.98980	1.725860
.34380	.003391	2.04900	1.829397
.39600	.005952	2.09370	1.897031
.44170	.008864	2.12860	1.949582
.61700	.033606	2.15660	1.981496
.74650	.067967	2.17950	2.008923
.85160	.111716	2.25090	2.087761
.94080	.160651	2.28800	2.118985
1.01840	.212427	2.32610	2.143121
1.08730	.266530	2.34550	2.152089
1.14900	.326606	2.35720	2.156356
1.20480	.382473	2.38090	2.161790
1.25580	.444693	2.40480	2.163777
1.59940	.985691		

$$\Psi(M_n) = \int_0^{M_n} \xi^3 J_0(\xi)\, d\xi$$

M_n	$\Psi(M_n)$	M_n	$\Psi(M_n)$
3.83170	-11.872452	4.29100	-22.742893
3.83690	-11.911503	4.46340	-27.743086
3.84210	-11.950711	4.60180	-31.765594
3.84730	-11.990077	4.71310	-34.919163
3.85250	-12.029600	4.80330	-37.371275
3.85770	-12.069279	4.87720	-39.280715
3.88350	-12.579701	4.93840	-40.759262
3.90910	-13.098296	4.98970	-41.945465
3.93440	-13.623727	5.03320	-42.846311
3.95940	-14.155567	5.17730	-45.524829
3.98410	-14.693368	5.25680	-46.701640
4.00850	-15.236694	5.34100	-47.627169
4.03250	-15.784066	5.38460	-47.972593
4.05620	-16.336061	5.41120	-48.121088
4.07950	-16.891698	5.46520	-48.363988
		5.52010	-48.447918

$$\Psi(M_n) = \int_0^{M_n} \xi \; {}^3J_0(\xi)\, d\xi$$

M_n	$\Psi(M_n)$	M_n	$\Psi(M_n)$
7.01560	30.741407	7.28840	59.298149
7.01840	30.836806	7.41030	72.863893
7.02130	30.936473	7.52010	84.486221
7.02410	31.032098	7.61770	95.322164
7.02700	31.132001	7.70390	104.225950
7.02980	31.227855	7.77970	112.191688
7.04400	32.708265	7.84640	118.505445
7.05820	34.197018	7.90510	124.220734
7.07230	35.687593	7.95690	129.057090
7.08640	37.185926	8.14220	144.187917
7.10040	38.678881	8.25340	151.692018
7.11430	40.166087	8.37710	158.103497
7.12820	41.659909	8.44320	160.567626
7.14210	43.160162	8.48400	161.833925
7.15580	44.646991	8.56780	163.601609
		8.65370	164.235955

$$\Psi(M_n) = \int_0^{M_n} \xi^3 J_0(\xi)\, d\xi$$

M_n	$\Psi(M_n)$	M_n	$\Psi(M_n)$
10.17350	-52.761884	10.36580	-87.071554
10.17540	-52.924936	10.45660	-111.912698
10.17740	-53.098614	10.54230	-134.322891
10.17940	-53.272398	10.62230	-156.041683
10.18130	-53.435740	10.69640	-175.875991
10.18330	-53.609713	10.76460	-192.346814
10.19310	-54.470445	10.82710	-207.929774
10.20290	-55.333507	10.88420	-222.161841
10.21270	-56.198835	10.93630	-234.754100
10.22250	-57.066327	11.13670	-278.899383
10.23220	-57.925199	11.26770	-303.253220
10.24190	-58.786034	11.42210	-326.264932
10.25160	-59.648781	11.50810	-335.641756
10.26130	-60.513338	11.56210	-340.575259
10.27100	-61.379601	11.67470	-347.679947
		11.79150	-350.221217

$$\Psi(M_n) = \int_0^{M_n} \xi^3 J_0(\xi)\, d\xi$$

M_n	$\Psi(M_n)$	M_n	$\Psi(M_n)$
13.32370	93.908921	13.47190	145.495282
13.32520	94.167195	13.54340	183.266079
13.32670	94.425551	13.61250	216.579050
13.32820	94.684006	13.67860	249.204956
13.32970	94.942542	13.74140	280.950832
13.33120	95.201165	13.80080	311.491321
13.33870	96.495532	13.85660	339.704487
13.34620	97.791906	13.90900	365.677540
13.35370	99.090319	13.95800	386.570814
13.36110	100.369834	14.15760	476.722064
13.36860	101.671913	14.29830	531.852869
13.37610	102.975737	14.47480	588.954011
13.38350	104.260344	14.57740	614.454168
13.39100	105.567451	14.64330	627.219892
13.39840	106.855172	14.78340	647.194770
		14.93090	654.504819

Appendix B

Values of the Function F_p

B_i	F_p	B_i	F_p
0.02	101.00096	0.80	3.5209369
0.03	67.668003	0.90	3.2447681
0.04	51.001599	1.00	3.0239434
0.05	41.002000	1.50	2.3625499
0.06	34.335519	2.00	2.0324363
0.07	29.573973	3.00	1.7022194
0.08	26.002912	4.00	1.5366648
0.09	23.225620	5.00	1.4369781
0.10	21.003791	6.00	1.3703489
0.15	14.338765	8.00	1.2866407
0.20	11.006945	10.00	1.2362181
0.30	7.6766723	15.00	1.1686135
0.40	6.0126293	20.00	1.1346527
0.50	4.6859648	50.00	1.0732329
0.60	4.3506265	100.00	1.0526516
0.70	3.8763919	∞	1.0320501

Appendix C

Values of the Function F_c

B_i	F_c	B_i	F_c
0.02	50.501076	3.00	0.89662616
0.04	25.506464	4.00	0.81555377
0.06	17.171535	5.00	0.76689009
0.08	13.004749	6.00	0.73387078
0.10	10.513592	7.00	0.71076465
0.20	5.5106619	8.00	0.69269638
0.30	3.8545841	9.00	0.67954061
0.40	3.0248618	10.00	0.66842967
0.50	2.5298840	15.00	0.63354819
0.60	2.2022478	20.00	0.61575298
0.70	1.9693062	30.00	0.59772871
0.80	1.7914034	40.00	0.58862252
0.90	1.6569183	50.00	0.58311772
1.00	1.5459509	100.00	0.57210521
2.00	1.0554381	∞	0.56093505

Appendix D

Values of the Function F_s

B_i	F_s	B_i	F_s
0.02	137.69647	0.80	0.83333272
0.03	75.415130	0.90	0.73350941
0.04	49.279643	1.00	0.65702198
0.05	35.465949	1.50	0.44712195
0.06	27.144106	2.00	0.35432307
0.07	21.665154	3.00	0.27062359
0.08	17.838883	4.00	0.23233081
0.09	15.037234	5.00	0.21044358
0.10	12.914998	6.00	0.19626427
0.15	7.2377709	8.00	0.17893964
0.20	4.8388229	10.00	0.16872653
0.30	2.7873053	16.00	0.15362311
0.40	1.9133671	21.00	0.14769407
0.50	1.4450420	51.00	0.13664370
0.60	1.1588410	101.00	0.13285092
0.70	0.96821242	∞	0.12900622

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