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STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY

PART III.

Walter Kofink

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STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY

III. Distribution of the Roots of the Characteristic Equation in the Case of Anisotropic Scattering

Walter Kofink

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STUDIES OF THE SPHERICAL HARMONICS METHOD IN NEUTRON TRANSPORT THEORY

III. Distribution of the Roots of the Characteristic Equation in the Case of Anisotropic Scattering

Walter Kofink*

A. The Characteristic Equation in the Case of Anisotropic Scattering.

1. Derivation of the Characteristic Equation.

In this section we consider solutions of the Boltzmann equation in plane geometry of the form

$$f(\zeta,\mu) = g(\mu)\, e^{ikz} = g(\mu)\, e^{i\zeta/\kappa} \qquad \left(\kappa = \frac{\Sigma}{k},\ \zeta = \Sigma z\right) \tag{1}$$

admitting, however, that κ may be not only pure imaginary (for $c < 1$, $a = 0$) or real (for $c > 1$, $a = 0$) but that it may be a complex number $\kappa = \kappa_1 + i\kappa_2$ under some circumstances (e.g., in time dependent problems). We may ask the question, how many solutions of the form (1) exist in every point c,a of our c,a-plane,[1] where c is the constant of multiplication and a is the constant of linear anisotropy. The answer to this question is given by an investigation of the number of roots of the characteristic equation for every c,a value. Hence we derive first the characteristic equation, which yields the numbers κ for solutions of the form (1) of the Boltzmann equation

$$\begin{aligned} \mu \frac{\partial f(\zeta,\mu)}{\partial \zeta} + f(\zeta,\mu) &= \frac{1}{2}(1-\gamma_a) \int_{-1}^{+1} f(\zeta,\mu')\, d\mu' + \frac{3}{2}(1-\gamma_t)\, \mu \int_{-1}^{+1} f(\zeta,\mu')\, \mu' d\mu' \\ &= \frac{1}{2} c \int_{-1}^{+1} f(\zeta,\mu')\, d\mu' + \frac{1}{2}\, \frac{a}{1-c}\, \mu \int_{-1}^{+1} f(\zeta,\mu')\, \mu' d\mu' \,. \end{aligned} \tag{2}$$

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*On leave of absence from the Institute of Technology, Karlsruhe, Germany. This work is supported by the Bunderminster fuer Atomkernenergie und Wasserwirtschaft, Bad Godesberg, Germany.

(1) W. Kofink, "Studies of the Spherical Harmonics Method in Neutron Transport Theory", ORNL-2334 (1957).

By integration over μ from -1 to +1 one finds a relation between the first and the second integral

$$\frac{\partial}{\partial\zeta}\int_{-1}^{+1}\mu f(\zeta,\mu)\,d\mu = (c-1)\int_{-1}^{+1} f(\zeta,\mu)\,d\mu\ . \tag{3}$$

If we use now the supposition (1) for $f(\zeta,\mu)$, we obtain for the Boltzmann equation (2)

$$(1 + i\,\frac{\mu}{\kappa})\,g(\mu) = \frac{1}{2}c\int_{-1}^{+1} g(\mu')\,d\mu' + \frac{1}{2}\,\frac{a}{1-c}\,\mu\int_{-1}^{+1} g(\mu')\,\mu' d\mu' \tag{2a}$$

and for the relation (3)

$$\frac{i}{k}\int_{-1}^{+1}\mu g(\mu)\,d\mu = (c-1)\int_{-1}^{+1} g(\mu)\,d\mu\ . \tag{3a}$$

Using (3a) one eliminates the second integral in (2a) and one obtains, as is well known,

$$g(\mu) = \frac{\kappa(c + ia\kappa\mu)}{2(\kappa + i\mu)}\int_{-1}^{+1} g(\mu')\,d\mu'\ . \tag{4}$$

The last integral is a constant and a final integration over μ from -1 to +1 yields the <u>characteristic equation</u>

$$1 = \frac{\kappa}{2}\int_{-1}^{+1}\frac{c + ia\kappa\mu}{\kappa + i\mu}\,d\mu \quad (= I_1 + iI_2)\ , \tag{5}$$

which determines those values of κ as function of (c,a), for which (4) is fulfilled.

2. Numbers of Roots in Different Regions of the c,a Plane

B. Davison[2] has proved that there are (in our notation)

$$2 \text{ roots if } \frac{a}{c} > -1 \tag{6a}$$

$$4 \text{ roots if } \frac{a}{c} < -1 \text{ and } c > 1 \tag{6b}$$

[2] B. Davison, "Milne Problems in a Multiplying Medium with a Linearly Anisotropic Scattering," National Research Council of Canada, CTR-358 (1946).

$$0 \text{ roots if } \frac{a}{c} < -1 \quad \text{and} \quad c < 1 \tag{6c}$$

of the characteristic equation (5). c is always supposed to be larger than 0. We intend in the following to recognize further details of the behavior of these roots in their distribution over the c,a-plane.

First we show the well-known fact that with κ also $-\kappa$ is a root of (5). By converting the part of the integral from $\mu = -1$ to $\mu = 0$ in an integral from $\mu = 0$ to 1, one obtains for (5)

$$1 = \kappa^2 \int_0^1 \frac{c + a\mu^2}{\kappa^2 + \mu^2} d\mu \; (= I_1 + iI_2) \; . \tag{5a}$$

In this form it contains only κ^2 and $\pm\kappa$ are both roots. Equation (5a) shows also that with κ^2 also $(\kappa^*)^2$ is the square of a root. Because the left side is real, also the right side must be real, and its complex conjugate must have the same real value,

$$1 = (\kappa^*)^2 \int_0^1 \frac{c + a\mu^2}{(\kappa^*)^2 + \mu^2} d\mu \quad (= I_1 - iI_2) \; . \tag{5b}$$

Hence, if $\kappa = \kappa_1 + i\kappa_2$ is a root, $\kappa^* = \kappa_1 - i\kappa_2$ is a root of (5) also. We have, as a consequence of (5a) and (5b) together, that with κ also $-\kappa$, κ^*, $-\kappa^*$ are roots. The meaning of this is that for a complex root $\kappa = \kappa_1 + i\kappa_2$, these other 3 roots exist also at the same point c,a of the c,a-plane. If the real part of a root is 0, however, we have only two roots, $\kappa = i\kappa_2 \equiv -\kappa^*$ and $-\kappa = -i\kappa_2 \equiv \kappa^*$ and, if the imaginary part of a root is 0, we also have only two roots, $\kappa = \kappa_1 \equiv \kappa^*$ and $-\kappa = -\kappa_1 \equiv -\kappa^*$. The <u>classification of Davison</u> shows therefore that we have in the area $a > -c$ ($c > 0$ always) of the a,c-plane either $\kappa_1 = 0$ and <u>two imaginary roots</u> $i\kappa_2$, $-i\kappa_2$ or $\kappa_2 = 0$ and <u>two real roots</u> κ_1, $-\kappa_1$. Actually this area is divided by a straight line through $c = 1$ with the equation $\gamma_a\gamma_t \equiv 1 - c - \frac{a}{3} = 0$ in the c,a plane. This straight line cuts the other boundary line $a = -c$ at $c = \frac{3}{2}$,

$a = -\frac{3}{2}$. On the left side of the straight line $1 - c - \frac{a}{3} = 0$ we will find always $\kappa_1 = 0$ and the two roots will be specified by $\pm \kappa_2$; on the right side of $1 - c - \frac{a}{3} = 0$ we will find always $\kappa_2 = 0$ and the two roots will be specified by $\pm \kappa_1$.

The <u>area of four roots</u> is divided in a more complicated manner. It will be shown that there are in fact three additional possibilities to have four roots besides the possibility of a complex root mentioned above. It will turn out that we have in a first c,a area two pairs of different κ_2's, say $\pm \kappa_2'$ and $\pm \kappa_2''$ (but $\kappa_1' = \kappa_1'' = 0$), in a second area one pair $\pm \kappa_2$ (but $\kappa_1 = 0$) and one pair $\pm \kappa_1$ (but $\kappa_2 = 0$), and in a third area two pairs of κ_1's, say $\pm \kappa_1'$ and $\pm \kappa_1''$ (but $\kappa_2' = \kappa_2'' = 0$) and finally the area mentioned above with a complex κ, i.e., the roots $\pm \kappa$, $\pm \kappa^*$.

3. <u>The Five Different Cases of Behavior of the Characteristic Equation</u>.

The integrand of the integral in (5) is a complex quantity and the equation (5) means that the real part of the right side is 1 and the imaginary part is 0. By adding and subtracting equations (5a) and (5b) one obtains for the real part I_1 and the imaginary part I_2 the following equations:

$$I_1 = \int_0^1 (c+a\mu^2) \frac{(\kappa\kappa^*)^2 + \frac{1}{2}\left(\kappa^2 + (\kappa^*)^2\right)\mu^2}{(\kappa^2+\mu^2)((\kappa^*)^2+\mu^2)} d\mu$$

$$= \int_0^1 (c+a\mu^2) \frac{(\kappa_1^2+\kappa_2^2)^2 + (\kappa_1^2-\kappa_2^2)\mu^2}{(\kappa_1^2-\kappa_2^2+\mu^2)^2 + 4\kappa_1^2\kappa_2^2} d\mu = 1 \tag{7a}$$

and

$$I_2 = \frac{\kappa^2 - (\kappa^*)^2}{2i} \int_0^1 \frac{(c+a\mu^2)\mu^2 d\mu}{(\kappa^2+\mu^2)((\kappa^*)^2+\mu^2)}$$

$$= 2\kappa_1\kappa_2 \int_0^1 \frac{(c+a\mu^2)\mu^2 d\mu}{(\kappa_1^2-\kappa_2^2+\mu^2)^2 + 4\kappa_1^2\kappa_2^2} = 0 . \tag{7b}$$

Equation (7b) shows that $I_2 = 0$ can be fulfilled in five different ways. If we put

$$J = \int_0^1 \frac{(c + a\mu^2)\ \mu^2 d\mu}{(\kappa_1^2 - \kappa_2^2 + \mu^2)^2 + 4\kappa_1^2\kappa_2^2} \tag{8}$$

we may distinguish the following nontrivial cases, satisfying $I_2 = 0$:

$$\begin{aligned}
&1) \quad \kappa_1 = 0, \quad \kappa_2 \neq 0, \quad J \neq 0 \\
&2) \quad \kappa_1 = 0, \quad \kappa_2 \neq 0, \quad J = 0 \\
&3) \quad \kappa_2 = 0, \quad \kappa_1 \neq 0, \quad J \neq 0 \\
&4) \quad \kappa_2 = 0, \quad \kappa_1 \neq 0, \quad J = 0 \\
&5) \quad \kappa_1 \neq 0, \quad \kappa_2 \neq 0, \quad J = 0
\end{aligned} \tag{9}$$

It will turn out that 1) and 3), where $J \neq 0$, give the two-root solutions and the characteristic equations which are usually considered for $c < 1$ or $c > 1$ in the area $a > -c$ of the c,a plane. They give further the four-root solutions in the area $a < -c$, which were described as the three cases a) two pairs of κ_2's, b) one pair of κ_2 and one pair of κ_1, and c) two pairs of κ_1's as solutions; the corresponding other component of κ is always 0.

Furthermore, 2) and 4) with the additional condition $J = 0$ represent together a curve in the $a < -c$ area of the c,a plane, called the envelope, which separates the region described above from the region with complex roots. This last region will be related finally to case 5). It is bordered on the right side by the envelope, to the left side by a vertical straight line from the point $c = 1$, $a = -1$ to $c = 1$, $a = -\infty$ in the c,a plane. This line is the right side border of the area with 0 roots also.

4. Discussion of Case 1)

Considering case 1) more closely, we fulfill (7b) by $\kappa_1 = 0$. Equation (7a), the characteristic equation of the exponentially increasing and decreasing

solutions (1)

$$f(\zeta,\mu) = g(\mu)\, e^{\pm\zeta/\kappa_2} , \tag{10}$$

takes then the form

$$I_1 = \kappa_2^2 \int_0^1 \frac{c + a\mu^2}{\kappa_2^2 - \mu^2}\, d\mu \tag{11}$$

$$= -a\kappa_2^2 + \kappa_2(c+a\kappa_2^2)\, \frac{1}{2} \log \frac{\kappa_2 + 1}{\kappa_2 - 1} = 1 \tag{12}$$

The integral (11) converges only for $\kappa_2 > 1$, if $c + a\kappa_2^2 \neq 0$. If we admit $c + a\kappa_2^2 = 0$ equation (12) yields $\kappa_2^2 = -\frac{1}{a}$; this is only possible if a is negative $a = -|a|$, $\kappa_2^2 = \frac{1}{|a|}$, $\kappa_2 = \pm\frac{1}{\sqrt{|a|}}$. The condition $c + a\kappa_2^2 = 0$ gives $c = 1$. If we put $\kappa_2 = 1$, the condition becomes $c + a = 0$, $a = -c$ and (11) yields $c = 1$ again. Therefore we have on the vertical straight line from the point $c = 1$, $a = -1$ to $c = 1$, $a = -\infty$ two solutions $\kappa_2 = \pm\frac{1}{\sqrt{|a|}}$; this line is the border between the regions of four roots and no roots. But on the straight line $a = -c$ from the point $c = 0$, $a = 0$ to $c = 1$, $a = -1$ we have no root with the exception of the endpoint $c = 1$, $a = -1$, where we have two roots $\kappa_2 = \pm 1$. This last straight line is the upper boundary of the region with no root.

Returning to our old notation[3] we may put $\kappa_2 = \lambda$ and we may call

$$Ch(\lambda) = 1 - I_1 = 1 + a\lambda^2 - \lambda(c+a\lambda^2)\, \mathrm{cotanh}^{-1}\lambda \tag{13}$$

the characteristic function. $Ch(\lambda_{0\infty}) = 0$ is then identical with the characteristic equation (12) and the characteristic equation in l.c. [3]

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[3] W. Kofink, "Studies of the Spherical Harmonics Method in Neutron Transport Theory", Part II, ORNL-2358, equation (58), (1957).

$Ch(\lambda_{0\infty}) = 0$ is a straight line in the c,a plane:

$$c \cdot (\lambda_{0\infty} \operatorname{cotanh}^{-1} \lambda_{0\infty}) + a\left[\lambda_{0\infty}^2 (\lambda_{0\infty} \operatorname{cotanh}^{-1} \lambda_{0\infty} - 1)\right] = 1 . \tag{12a}$$

the coefficients depend only on the value of $\lambda_{0\infty}$; therefore we have a set of straight lines in the c,a-plane with the parameter $\lambda_{0\infty}$ running from $\lambda_{0\infty} = 1 + \epsilon$ (ϵ positive, arbitrary small) to $\lambda_{0\infty} = \infty$. The point in which a special straight line with the parameter $\lambda_{0\infty}$ cuts the c-axis (this means for vanishing anistropy of the scattering), may be denoted by c_0. It is

$$c_0 = \frac{1}{\lambda_{0\infty} \operatorname{cotanh}^{-1} \lambda_{0\infty}} . \tag{14}$$

Introducing c_0 in equation (12a) we may express the anistropy-parameter a by

$$a = \frac{1}{\lambda_{0\infty}^2} \frac{c_0 - c}{1 - c_0} . \tag{12b}$$

In numerical calculations one would prefer to choose $\lambda_{0\infty}$ and to calculate c_0 by equation (14). Then one proceeds on the special straight line with the parameter $\lambda_{0\infty}$ by choosing c values on the left or the right side of c_0 and one will obtain the corresponding positive or negative a values by equation (12b).

For $\lambda_{0\infty} \to \infty$ we have $c_0 = 1$, because $\lambda_{0\infty} \operatorname{cotanh}^{-1} \lambda_{0\infty} \to 1$. The limit of $\lambda_{0\infty}^2 (\lambda_{0\infty} \operatorname{cotanh}^{-1} \lambda_{0\infty} - 1)$ is $\frac{1}{3}$, and the straight line corresponding to $\lambda_{0\infty} = \infty$ is

$$c + \frac{a}{3} = 1 . \tag{15}$$

It cuts the other limiting straight line of the Davison criterium (6a) for two roots $a = -c$ at the point $c = \frac{3}{2}$, $a = -\frac{3}{2}$. The sector of the c,a-plane between these straight lines contains therefore the set of straight lines (each denoted by a different $\lambda_{0\infty}$), on which we have two roots of equation (12). With increasing $\kappa_2 = \lambda_{0\infty}$ the straight lines become always steeper against the c-axis (see Figure 1).

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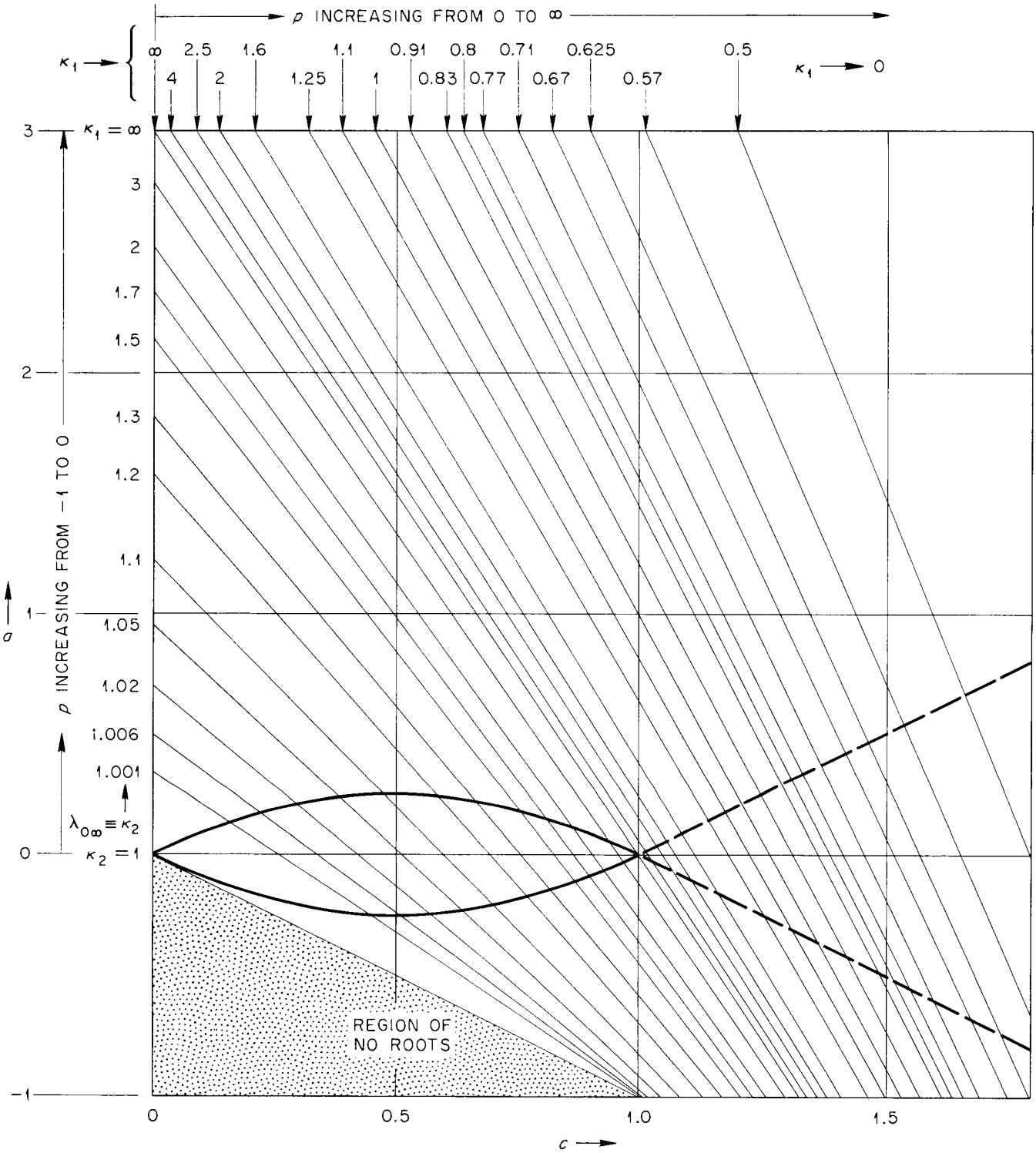

Fig. 1. Straight Line in the c, a-Plane, Along Which the Characteristic Roots κ_1 or κ_2 are Constant. For Stationary Solutions ($c < 1$) Only the Region Between the Solid Curves $a = \pm c$ ($c - 1$) is of Importance. Between the Dashed Lines $c > 1$ We have the Region of Solutions, for Which Only Scattering is Anisotropic, but Fission and Absorption are Assumed Isotropic.

Every straight line of the set is cut by every other straight line of the set in a point of the c,a-plane below the line $a = -c$, but the set has no common point for them altogether; this means the set forms an envelope below $a = -c$. It will be shown that this envelope satisfies the additional condition $J = 0$.

5. Discussion of Case 3.

We proceed to the consideration of Case 3. Here we fulfill the equation (7b) by $\kappa_2 = 0$. Equation (7a), the characteristic equation for sinusoidal solutions (1), takes the form

$$I_1 = \kappa_1^2 \int_0^1 \frac{c + a\mu^2}{\kappa_1^2 + \mu^2}\, d\mu \tag{16}$$

$$= a\kappa_1^2 + \kappa_1(c - a\kappa_1^2) \tan^{-1} \frac{1}{\kappa_1} = 1 . \tag{17}$$

These equations are valid for $\infty \geqslant \kappa_1^2 \geqslant 0$. If we replace λ by $i\lambda$ in (13) or if we call $\kappa_1 = \lambda$ in (17) we obtain

$$Ch(i\lambda) = 1 - I_1 = 1 - a\lambda^2 - \lambda(c - a\lambda^2) \operatorname{cotan}^{-1} \lambda \tag{18}$$

as the characteristic function for these cases, which include $c \geqslant 1$, $a = 0$. The roots of (18) are the characteristic values in the sinusoidal case of solutions (1).

By using the parameter

$$p = \frac{1}{\kappa_1^2 - \kappa_2^2} = \begin{cases} -\dfrac{1}{\kappa_2^2} & \text{if } \kappa_1 = 0 \quad \text{(in case 1)} \\[2ex] \dfrac{1}{\kappa_1^2} & \text{if } \kappa_2 = 0 \quad \text{(in case 3)} \end{cases} \tag{19}$$

both integrals (11) and (16) take the same form unifying the cases 1 and 3:

$$I_1 = \int_0^1 \frac{c + a\mu^2}{1 + p\mu^2}\, d\mu = 1 . \tag{20}$$

Equation (20) represents a set of straight lines in the c,a-plane with a parameter p.

The parameter p is permitted to run continuously through the interval $-1 < p < \infty$, giving a straight line in the a,c-plane for every fixed p as a solution of case 1 in the interval $-1 < p \leq 0$ and of case 3 in the interval $0 \leq p < \infty$. The second set of straight lines continues just the first one, their inclination against the c axis becoming steeper with increasing p (decreasing κ_1). For $p \to \infty$ ($\kappa_1 \to 0$) the angle of inclination goes to a right angle against the c axis (see Figure 1).

The integral J of equation (8) will be

$$J = \begin{cases} \displaystyle\int_0^1 \frac{(c+a\mu^2)\ \mu^2 d\mu}{(\mu^2-\kappa_2^2)^2} & \text{for } \kappa_1 = 0 \quad \text{(in case 1)} \qquad (21a) \\[2ex] \displaystyle\int_0^1 \frac{(c+a\mu^2)\ \mu^2 d\mu}{(\mu^2+\kappa_1^2)^2} & \text{for } \kappa_2 = 0 \quad \text{(in case 3)} \qquad (21b) \end{cases}$$

$$= p^2 \int_0^1 \frac{(c+a\mu^2)\ \mu^2 d\mu}{(1+p\mu^2)^2} \qquad \text{in both cases.} \qquad (21)$$

If one compares J with the condition of an envelope of the set of straight lines (20)

$$\frac{\partial I_1}{\partial p} = -\int_0^1 \frac{(c+a\mu^2)\ \mu^2 d\mu}{(1+p\mu^2)^2} = 0 \qquad (22)$$

one recognizes that this condition is identical with the vanishing of the second factor in equation (21):

$$\frac{1}{p^2} J = 0, \quad \text{or} \quad J = 0 \quad \text{for} \quad p \neq \infty \ (\kappa_1 \neq 0) \ . \qquad (23)$$

These are the cases (2) and (4) in the list (9). The points of the envelope in the c,a-plane satisfy both equations (20) and (22).

They yield for $p = 0$, i.e., for the point P_C of connection of both cases 1) and 3) in the c,a-plane:

$$c + \frac{a}{3} = 1 \tag{20a}$$

$$\frac{c}{3} + \frac{a}{5} = 0 . \tag{22a}$$

Hence the coordinates c_1, a_1 of P_C are

$$c_1 = \frac{9}{4}, \quad a_1 = -\frac{15}{4} \tag{24}$$

and Equation (20a) shows that P_C lies of course on the straight line with Equation (15). (see Figure 2).

6. Discussion of the Envelope (Cases 2 and 4).

With the denotation

$$K(p) = \int_0^1 \frac{d\mu}{1 + p\mu^2} \qquad (-1 < p < \infty) \tag{25}$$

we may express

$$\frac{\partial K}{\partial p} = -\int_0^1 \frac{\mu^2 d\mu}{(1+p\mu^2)^2} = \frac{1}{2p} \int_0^1 \mu \cdot \frac{d}{d\mu}\left(\frac{1}{1 + p\mu^2}\right) d\mu$$

$$= \frac{1}{2p(1+p)} - \frac{1}{2p} K(p) \tag{26}$$

and

$$\int_0^1 \frac{\mu^2 d\mu}{1 + p\mu^2} = \frac{1}{p}\int_0^1 \frac{1 + p\mu^2 - 1}{1 + p\mu^2} d\mu = \frac{1}{p}\left(1-K(p)\right) . \tag{27}$$

By application of (25) and (27) to the characteristic Equation (20) one obtains

$$I_1 = \frac{a}{p} + \left(c - \frac{a}{p}\right) K(p) \tag{28}$$

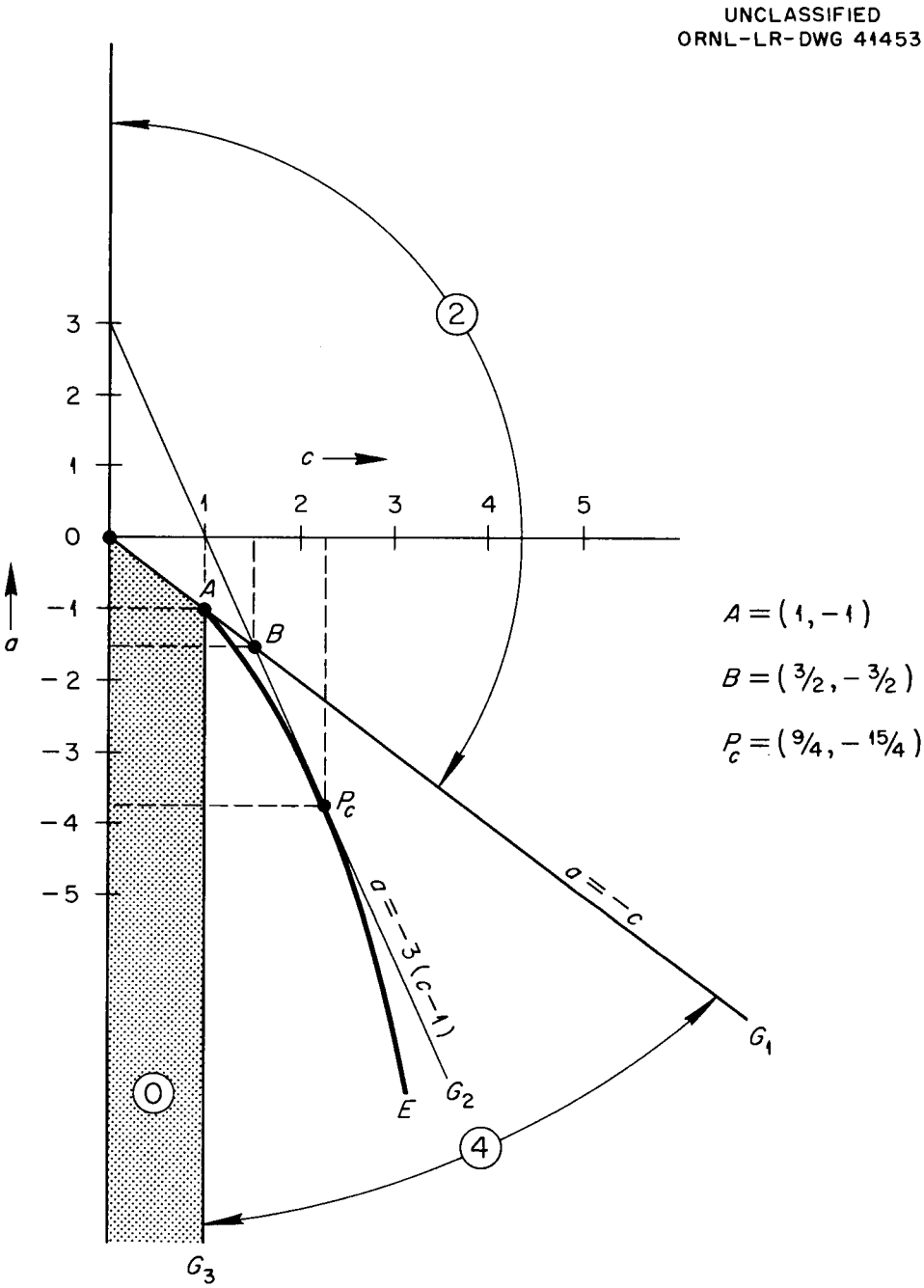

Fig. 2. A Schematical Picture Showing in the Small Circle the Number of Roots in Different Regions of the c, a-Plane and some Details Discussed in the text.

and further by application of (26) to the derivative of (28)

$$\frac{\partial I_1}{\partial p} = -\frac{a}{p^2} + \frac{a}{p^2} K(p) + (c - \frac{a}{p}) \frac{\partial K}{\partial p}$$

$$= \frac{1}{2p^2} \left\{ (3a-pc)\ K(p) + \frac{p(c-2a) - 3a}{1 + p} \right\} \tag{29}$$

or

$$J = -p^2 \frac{\partial I_1}{\partial p} = \frac{1}{2} \left\{ (pc-3a)\ K(p) - \frac{p(c-2a) - 3a}{1 + p} \right\} . \tag{30}$$

The points $c = c_E$, $a = a_E$ on the envelope fulfill the two Equations (20) and (23) in the representation with the parameter p

$$(pc_E - a_E)\ K(p) + a_E - p = 0 \tag{31}$$

$$(1+p)(pc_E - 3a_E)\ K(p) + 3a_E - p(c_E - 2a_E) = 0 . \tag{32}$$

The equation of the envelope in the c,a-plane would result from Equations (31) and (32) by elimination of the parameter p.

A more convenient representation for numerical calculations is the parameter-representation $(-1 < p < \infty)$

$$c_E = \frac{3 + 2p - 3(1+p)\ K(p)}{1 + (1+p)\ K(p)\big(1-2K(p)\big)}$$

$$a_E = p\ \frac{1 - (1+p)\ K(p)}{1 + (1+p)\ K(p)\big(1-2K(p)\big)} \tag{33}$$

with

$$K(p) = \begin{cases} \dfrac{1}{2\sqrt{|p|}} \log \dfrac{1+\sqrt{|p|}}{1-\sqrt{|p|}} = \kappa_2 \operatorname{cotanh}^{-1} \kappa_2 \quad \text{for } -1 < p < 0, \sqrt{|p|} = \dfrac{1}{\kappa_2} & (34a) \\ \text{case 2) of list (9)} \\ \dfrac{1}{\sqrt{p}} \tan^{-1} \sqrt{p} = \kappa_1 \operatorname{cotan}^{-1} \kappa_1 \quad \text{for } 0 < p < \infty, \sqrt{p} = \dfrac{1}{\kappa_1} & (34b) \\ \text{case 4) of list (9)} \end{cases}$$

$$= 1 - \frac{p}{3} + \frac{p^2}{5} - \frac{p^3}{7} + - \ldots \quad \text{for } p^2 < 1 \text{ in both cases.} \tag{34c}$$

In the environment of $p = 0$ the following quotients for c_E and a_E may be derived using (34c):

$$c_E = \frac{9}{4} \frac{1 - \frac{3}{7} p + \frac{5}{21} p^2 - \frac{5}{33} p^3 + \frac{15}{143} p^4 - + \ldots}{1 - \frac{5}{7} p + \frac{18}{35} p^2 - \frac{446}{1155} p^3 + \frac{6323}{21021} p^4 - + \ldots}$$

$$a_E = -\frac{15}{4} \frac{1 - \frac{1}{5} p + \frac{3}{35} p^2 - \frac{1}{21} p^3 + \frac{1}{33} p^4 - + \ldots}{1 - \frac{5}{7} p + \frac{18}{35} p^2 - \frac{446}{1155} p^3 + \frac{6323}{21021} p^4 - + \ldots} . \tag{35}$$

For $p \longrightarrow 0$ of course we come back to the values (24) $c_E \longrightarrow c_1 = \frac{9}{4}$ and $a_E \longrightarrow a_1 = -\frac{15}{4}$, which is the point P_c on the envelope E (see Figure 3).

The following Table I gives some points on the envelope for case 2), where $\kappa_1 = 0$ and $J = 0$, specified by the parameter $\kappa_2 = \frac{1}{\sqrt{|p|}}$:

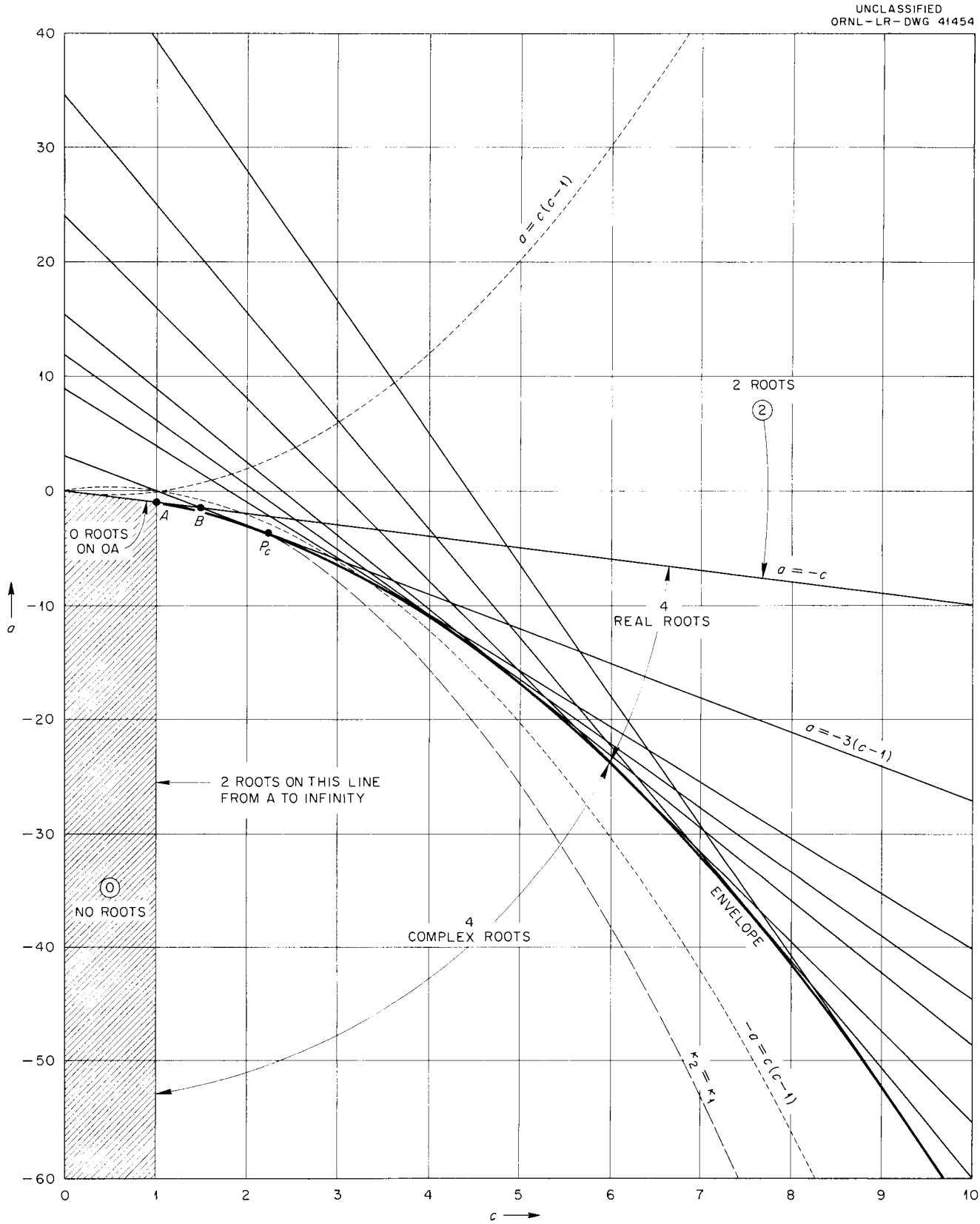

Fig. 3. It Should Give an Impression how the κ_1 = Const-Curves Build Up the Lower Part of the Envelope Beginning at P_c. The κ_2 = Const-Curves are not Drawn Here. The Dotted Curves $\pm\, a = c\,(c-1)$ Include all those c, a-points, for which Absorption, Fission and Scattering of Single Neutrons Obey the Same Non-Negative Probability Law.

Table I

κ_2	c_E	a_E
1	1	- 1
1.001	1.033	- 1.043
1.002	1.053	- 1.070
1.004	1.082	- 1.112
1.006	1.106	- 1.147
1.008	1.126	- 1.178
1.01	1.144	- 1.206
1.02	1.215	- 1.321
1.05	1.353	- 1.565
1.1	1.498	- 1.847
1.2	1.674	- 2.228
1.3	1.785	- 2.490
1.4	1.863	- 2.684

κ_2	c_E	a_E
1.5	1.922	- 2.834
1.6	1.968	- 2.953
1.7	2.004	- 3.049
1.8	2.033	- 3.129
1.9	2.057	- 3.195
2	2.078	- 3.249
2.5	2.143	- 3.436
3	2.177	- 3.534
4	2.213	- 3.639
5	2.227	- 3.681
7.5	2.242	- 3.725
↓	↓	↓
∞	2.25	- 3.75

$$-\frac{a_E}{c_E} = \frac{1 - \dfrac{\kappa_2^2 - 1}{\kappa_2} \operatorname{cotanh}^{-1} \kappa_2}{1 + 3\,(\kappa_2^2 - 1)\left[1 - \kappa_2 \operatorname{cotanh}^{-1} \kappa_2\right]}, \quad \frac{1}{c_E} = \kappa_2 \operatorname{cotanh}^{-1} \kappa_2 - \frac{a_E}{c_E}\,\kappa_2^2\,(1 - \kappa_2 \operatorname{cotanh}^{-1} \kappa_2). \tag{33a}$$

A further Table II shows the continuation of the envelope for case 4), where $\kappa_2 = 0$ and $J = 0$, in some points belonging to values of the parameter $\kappa_1 = 1/\sqrt{p}$:

Table II

κ_1	c_E	a_E
∞	2.25	- 3.75
↓	↓	↓
20	2.254	- 3.761
10	2.255	- 3.766
5	2.275	- 3.827
4	2.289	- 3.869
10/3	2.306	- 3.922
2.5	2.349	- 4.053
2	2.402	- 4.219
1.6	2.480	- 4.474
1.25	2.610	- 4.912
10/9	2.693	- 5.202
1	2.782	- 5.521
10/11	2.874	- 5.865
5/6	2.971	- 6.236
0.8	3.021	- 6.431

κ_1	c_E	a_E
10/13	3.071	- 6.632
5/7	3.174	- 7.051
2/3	3.279	- 7.495
0.625	3.387	- 7.961
4/7	3.551	- 8.704
0.5	3.833	- 10.05
0.4	4.416	- 13.15
1/3	5.016	- 16.78
0.25	6.241	- 25.58
0.2	7.485	- 36.40
0.16	9.053	- 52.76
0.125	11.26	- 80.93
0.1	13.79	- 120.6
0.05	26.49	- 439.3
0.02	64.67	- 2595
0.01	128.33	- 10189

$$-\frac{a_E}{c_E} = \frac{1 - \frac{\kappa_1^2 + 1}{\kappa_1} \operatorname{cotan}^{-1} \kappa_1}{1 - 3\,(\kappa_1^2+1)(1 - \kappa_1 \operatorname{cotan}^{-1} \kappa_1)}, \quad \frac{1}{c_E} = \kappa_1 \operatorname{cotan}^{-1} \kappa_1 + \frac{a_E}{c_E}\,\kappa_1^2\,(1 - \kappa_1 \operatorname{cotan}^{-1} \kappa_1) \tag{33b}$$

For small κ_1 (large positive p) asymptotically holds:

$$K(p) \sim \frac{\pi}{2}\,\frac{1}{\sqrt{p}} = \frac{\pi}{2}\,\kappa_1$$

$$\left.\begin{aligned} c_E &\sim \frac{4}{\pi}\sqrt{p} = \frac{4}{\pi}\,\frac{1}{\kappa_1} \\ a_E &\sim -p = -\frac{1}{\kappa_2} \end{aligned}\right\} \quad a_E \sim -\frac{\pi^2}{16}\,c_E^2 \,. \tag{36}$$

In fact it is possible to eliminate the parameter p from the Equations (31) and (32) because K(p) is the only transcendent function in both equations. The elimination of K(p) from both equations

$$K(p) = \frac{p - a_E}{pc_E - a_E} = \frac{pc_E - 3a_E - 2p\,a_E}{(1+p)(pc_E - 3a_E)}$$

yields a quadratic equation for p (if $p \neq 0$):

$$c_E p^2 - \left[c_E^2 - c_E - a_E(c_E-3)\right] p + a_E\left[a_E + 3(c_E-1)\right] = 0 \,. \tag{37}$$

By this equation c_E, a_E, p are connected on the envelope. One has to keep in mind that only <u>one</u> of the three variables can be considered as a free variable because Equation (37) is equivalent to only one of the two Equations (31) and (32). Hence we have (37) always to consider together with the characteristic Equation (31) for instance. Solving the quadratic Equation (37) one obtains

$$p = \frac{c_E^2 - c_E - a_E(c_E-3) \pm \sqrt{(c_E-1)\left[c_E(c_E-a_E)^2 - (c_E+3a_E)^2\right]}}{2c_E} \tag{37a}$$

The sign of the root has to be determined always in such a manner that Equation (31) is satisfied.

There is an interesting relation between the Equation (37) on the envelope and the function

$$h(c,a;u) = c + \left[c^2 - c + a(3-c)\right] u^2 + a\left[a + 3(c-1)\right] u^4 \tag{38}$$

which was defined in the Equation (51) of l. c.[3] $h(c,a;u)$ occurs in the integrand of all integrals involved in the different quantities derived there. If one compares the Equations (37) and (38), one recognizes that

$$h(c_E, a_E; -\frac{1}{\sqrt{-p}}) = 0 \tag{39}$$

$h(c,a;u)$ was defined only for the case 1) in l. c.[3] So we restrict ourselves to this case, in which $\frac{1}{\sqrt{-p}} = \kappa_2 = \lambda_{0\infty}$. Then (39) has the meaning that

$$h(c_E, a_E; \lambda_{0\infty}) = 0 \tag{39a}$$

on the envelope.

7. <u>The Three Different Regions with Four Roots above the Envelope</u>.

We discuss now more thoroughly the four-root-solutions below the straight line $a = -c$ and the envelope. They were mentioned in the discussion after the list 9).

All straight lines, belonging to different κ_2 (but $\kappa_1 = 0$) traverse the crucial line $a = -c$, below which four solutions exist, between the two points $A(c = 1, a = -1)$ and $B(c = \frac{3}{2}, a = -\frac{3}{2})$. They touch the envelope E for increasing κ_2 consecutively between A and P_C $(c = \frac{9}{4}, a = -\frac{15}{4})$. Let us consider now a straight line with a special parameter κ_2'. All straight lines with a parameter $\kappa_2 < \kappa_2'$ will traverse it before (smaller c and - a) its touching point with the envelope; all straight lines with a parameter $\kappa_2 > \kappa_2'$ will traverse it after (larger c and - a) its touching point with the envelope. The straight line $\kappa_2 = \infty$ with the equation $a = -3(c-1)$ is the last of this set with $\kappa_2 > \kappa_2'$. Hence it is the boundary of the region, in which a straight line κ_2' is cut successively by all other straight lines of the same set. To every point c,a in the triangle ABP_C with the curved

basis AP_C on the envelope belong two parameter-values κ_2' and κ_2'', which label the two straight lines traversing one another in the point c,a. They indicate four roots $\pm \kappa_2'$ and $\pm \kappa_2''$ (but $\kappa_1 = 0$).

An analoguous consideration shows that the sector of the c,a-plane G_1BG_2 with the cusp in B and the border lines $a = - c$ and $a = - 3(c-1)$ is occupied by the solutions with two roots $\pm \kappa_2$ (but $\kappa_1 = 0$) <u>and</u> two roots $\pm \kappa_1$ (but $\kappa_2 = 0$). In every point c,a of this sector two straight lines traverse one another, one belonging to the set $\kappa_2(\kappa_1 = 0)$ and the other to set $\kappa_1(\kappa_2 = 0)$. In the region G_2P_CE with the cusp in P_C between the lower part of the envelope E and the straight line $a = - 3(c-1)$ we find the four solutions belonging to $\pm \kappa_1'$, $\pm \kappa_1''$ (but $\kappa_2 = 0$). In every point a,c of it two straight lines labeled by the different parameters κ_1' and κ_1'' traverse one another. (see Figure 4).

8. <u>The Region of Four Complex Roots below the Envelope (Case 5)</u>.

The last area with four roots, to be discussed, is the sector G_3AE with the cusp in A, the straight line AG_3 ($c = 1$ from $a = - 1$ to $a = - \infty$) and the envelope AE as boundaries. It contains the examples of case 5) of the list (9), in which κ is a complex root and J is 0. By Equations (5a) and (5b) was recognized that $\pm \kappa^*$, $- \kappa$ are roots, if κ is a complex root.

Below Equation (12) it was shown that on the vertical straight line AG_3 the roots $\kappa_2 = \pm \frac{1}{\sqrt{|a|}}$ (but $\kappa_1 = 0$) exist. Hence <u>two</u> roots lie on every point of the boundary between the regions of four roots and of 0-roots. And this line can be considered as a continuation of the "κ_2-part" P_CA of the envelope, because $\kappa_1 = 0$ holds likewise on AG_3 and P_CA; the κ_2 values decrease from $\kappa_2 = 1$ at A to $\kappa_2 = 0$ at $a = - \infty$ on G_3. In fact κ_2 increases continuously on G_3AP_C from $\kappa_2 = 0$ to $\kappa_2 = \infty$ (but $\kappa_1 = 0$); and on the remainder of the envelope κ_1 decreases from $\kappa_1 = \infty$ at P_C to $\kappa_1 = 0$ at infinity on the envelope E (but $\kappa_2 = 0$).

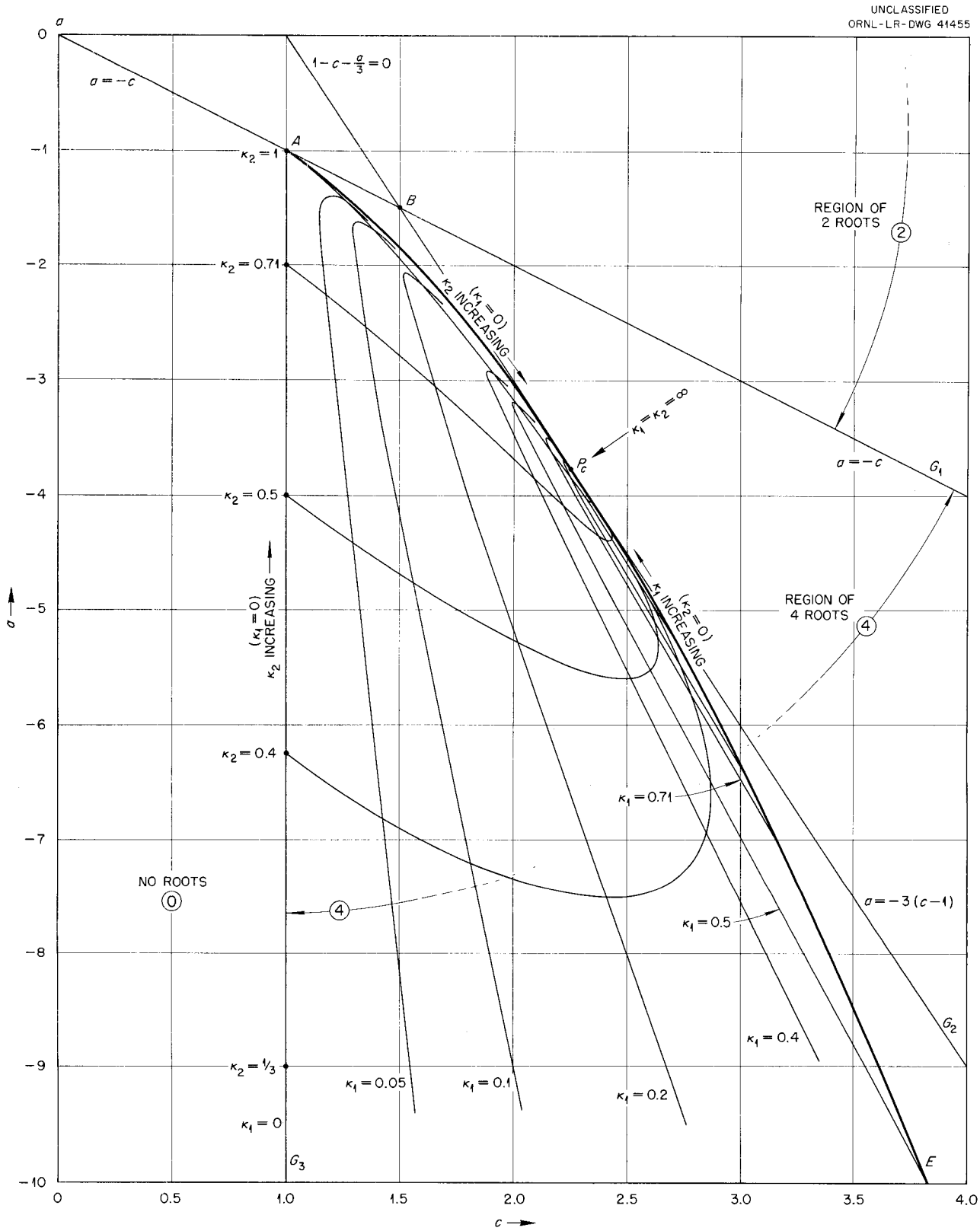

Fig. 4. The Complex Roots of the Characteristic Equation for Anisotropic Scattering with Multiplication Between the Envelope E and the Straight Line G_3.

If one considers now the region of four roots with this boundary one finds curves of constant κ_2 by starting at that point on G_3AP_C, which belongs to this special κ_2. On the boundary is $\kappa_1 = 0$ in this point. With increasing κ_1 the point moves into the interior of the region with complex roots. It goes to P_C, if κ_1 increases to ∞, as a tangent to the envelope. The set of curves of constant κ_2 enters the point P_C for $\kappa_1 \longrightarrow \infty$ in the same direction, in which κ_1 increases to infinity on the envelope itself. Because the curves κ_2 = constant start for $\kappa_2 > 0.8$ at considerably higher a values than that of P_C they show the form of a hook, which gets an always sharper form the larger is κ_2 = constant. The curves of constant κ_1 show a similar behavior. They start on the lower part of the envelope P_CE at the point labeled by the value κ_1 (but $\kappa_2 = 0$). With increasing κ_2 the point moves into the interior of the region. It describes a curve with the form of a hook and enters the point P_C for $\kappa_2 \longrightarrow \infty$ on a tangent to the envelope in the same direction in which κ_2 increases on the envelope. This direction is opposite to that of the curves of constant κ_2.

Curves, on which the roots $\kappa_2 = \alpha\kappa_1$, with a fixed (of course real) quotient α of their imaginary and real parts lie, start also in P_C for $\kappa_1 = \infty$. They leave that point into the interior of the region with complex roots, if κ_1 decreases, in a definite angle, depending on the constant α, against the envelope. Only if α is 0 or ∞ they leave P_C in the direction of the tangent on the envelope or in the opposite direction - those cases were discussed before. We consider now the cases, for which α is different from 0 and ∞. When κ_1 decreases $\longrightarrow$ o, they can never meet the boundary G_3AP_CE of the region with complex roots in a finite point, because on that boundary is one of the numbers κ_1, κ_2 always 0 ($\kappa_1 = 0$ on G_3AP_C, $\kappa_2 = 0$ on P_CE), but the other is always different from 0 in points with finite c,a. Hence all those curves run to infinity in the c,a-plane between AG_3 and the envelope AE. An example of such a curve is $\kappa_2 = \kappa_1$ (or $\kappa_2 = -\kappa_1$ because the problem depends on κ^2 only). Table III shows its behavior; it is drawn on Figure 3.

Table III

$\kappa_1 = \kappa_2$	c	a
∞	9/4	-15/4
5	2.250011	- 3.750136
4	2.250155	- 3.750716
10/3	2.250332	- 3.751517
2.5	2.250618	- 3.753515
2	2.2522	- 3.7607
1.6	2.2563	- 3.7787
1.25	2.2663	- 3.8247
10/9	2.2757	- 3.8679
1	2.2882	- 3.9260
10/11	2.3042	- 4.0010
5/6	2.3239	- 4.0942
0.8	2.3353	- 4.1481

$\kappa_1 = \kappa_2$	c	a
$1/\sqrt{2}$	2.379	- 4.359
0.625	2.440	- 4.660
0.5	2.603	- 5.516
0.4	2.849	- 6.925
1/3	3.121	- 8.654
0.25	3.701	- 12.96
0.2	4.305	- 18.33
0.16	5.076	- 26.47
0.125	6.168	- 40.54
0.1	7.426	- 60.38
0.05	13.76	- 219.7
0.02	32.84	- 1298.
0.01	64.67	- 5094

The curve in the c,a-plane belonging to the Table III gives on every point c,a the four roots $\kappa = \kappa_1(1+i)$, $\kappa_1(1-i)$, $-\kappa_1(1+i)$, $-\kappa_1(1-i)$ as a solution of the real part $I_1 = 1$ and the imaginary part $I_2 = 0$ of the characteristic equation (see Equations 7a and 7b):

$$I = \kappa_1^2 \left\{ 2a - \kappa_1 \, a \log \frac{1 + 2\kappa_1^2 + 2\kappa_1}{1 + 2\kappa_1^2 - 2\kappa_1} + \frac{c}{\kappa_1} \left[\tan^{-1} \frac{\kappa_1 + 1}{\kappa_1} - \tan^{-1} \frac{\kappa_1 - 1}{\kappa_1} \right] \right\} = 1 \qquad (40)$$

and

$$I_2 = \kappa_1\left\{2\kappa_1 a - \frac{1}{4}\left(c+2a\kappa_1^2\right)\log\frac{1+2\kappa_1^2+2\kappa_1}{1+2\kappa_1^2-2\kappa_1} + \frac{1}{2}\left(c-2a\kappa_1^2\right)\left[\tan^{-1}\frac{\kappa_1+1}{\kappa_1} - \tan^{-1}\frac{\kappa_1-1}{\kappa_1}\right]\right\} = 0. \tag{41}$$

In numerical calculations it is convenient to calculate $-\frac{a}{c}$ and $\frac{1}{c}$ as functions of the parameter $\kappa_1 = \kappa_2$:

$$-\frac{a}{c} = \frac{2T_{\kappa_1} - L_{\kappa_1}}{2\kappa_1\left[4 - \kappa_1\left(2T_{\kappa_1}+L_{\kappa_1}\right)\right]} \tag{42}$$

$$\frac{1}{c} = \kappa_1\left[T_{\kappa_1} - \frac{a}{c}\,\kappa_1\left(\kappa_1 L_{\kappa_1}-2\right)\right] \tag{43}$$

with the abbreviations

$$L_{\kappa_1} = \log\frac{1+\left(\frac{\kappa_1+1}{\kappa_1}\right)^2}{1+\left(\frac{\kappa_1-1}{\kappa_1}\right)^2} \quad \text{and } T_{\kappa_1} = \tan^{-1}\frac{\kappa_1+1}{\kappa_1} - \tan^{-1}\frac{\kappa_1-1}{\kappa_1}.$$

For small κ_1 holds asymptotically $c \sim \frac{2}{\pi\kappa_1}$, $a \sim -\frac{1}{2\kappa_1^2}$, $a \sim -\frac{\pi^2}{8}c^2$. (44)

By comparison of the last equation with the corresponding Equation (36) one recognizes that the $\kappa_1 = \kappa_2$ curve runs twice as fast to $a = \infty$ for increasing c as the envelope. In the general case of a complex root $\kappa = \kappa_1 + i\kappa_2$ one has to solve the Equations (7a) and (7b)

$$I_1 = \frac{\kappa_1^2+\kappa_2^2}{2}\left\{2a - \kappa_2 a\log\frac{\kappa_1^2+(\kappa_2+1)^2}{\kappa_1^2+(\kappa_2-1)^2}\right.$$

$$\left. + \frac{1}{\kappa_1}\left[c - \left(\kappa_1^2-\kappa_2^2\right)a\right]\left(\tan^{-1}\frac{\kappa_2+1}{\kappa_1} - \tan^{-1}\frac{\kappa_2-1}{\kappa_1}\right)\right\} = 1 \tag{45}$$

and

$$I_2 = 2\kappa_1\kappa_2 a - \frac{\kappa_1}{4}\left[c - (\kappa_1^2 - 3\kappa_2^2)\, a\right] \log \frac{\kappa_1^2 + (\kappa_2+1)^2}{\kappa_1^2 + (\kappa_2-1)^2}$$

$$+ \frac{\kappa_2}{2}\left[c - (3\kappa_1^2 - \kappa_2^2)\, a\right]\left(\tan^{-1}\frac{\kappa_2+1}{\kappa_1} - \tan^{-1}\frac{\kappa_2-1}{\kappa_1}\right) = 0 \qquad (46)$$

and one finds quickly a and c for every given $\kappa = \kappa_1 + i\kappa_2$ in the case 5) of list (9) from

$$-\frac{a}{c} = \frac{\kappa_2 T_\kappa - \frac{\kappa_1}{2} L_\kappa}{4\kappa_1\kappa_2 + \frac{\kappa_1}{2}(\kappa_1^2 - 3\kappa_2^2)\, L_\kappa - \kappa_2(3\kappa_1^2 - \kappa_2^2)\, T_\kappa} \qquad (47)$$

and

$$\frac{1}{c} = \frac{\kappa_1^2 + \kappa_2^2}{2\kappa_1}\left\{T_\kappa - \frac{a}{c}\left[\kappa_1(\kappa_2 L_\kappa - 2) + (\kappa_1^2 - \kappa_2^2)\, T_\kappa\right]\right\} \qquad (48)$$

with the abbreviations

$$L_\kappa = \log \frac{1 + \left(\frac{\kappa_2+1}{\kappa_1}\right)^2}{1 + \left(\frac{\kappa_2-1}{\kappa_1}\right)^2} \quad \text{and } T_\kappa = \tan^{-1}\frac{\kappa_2+1}{\kappa_1} - \tan^{-1}\frac{\kappa_2-1}{\kappa_1}.$$

For $\kappa_2 = \kappa_1$ one comes back of course to Equations (40) - (43).

B. The Position of the Largest Root of the Characteristic Equation in P_L-Spherical Harmonics Approximation

One likes to distinguish the largest root λ_o of the characteristic equation in P_L approximation -see for instance W. Kofink, ORNL Report 2334, Equation (12)-

$$(1+a\lambda_\kappa^2)\ P_{L+1}(\lambda_\kappa) - \lambda_\kappa(c+a\lambda_\kappa^2)\ W_L(\lambda_\kappa) = 0 \tag{49}$$

from the other smaller roots λ_s ($s = 1,2,3,\ldots \frac{L-1}{2}$). One is accustomed to call the partial solution, belonging to λ_o, the asymptotic solution and to call the other solutions, belonging to the λ_s's, the transient solutions, because only λ_o passes over the boundary of the interval $-1 \leq \lambda \leq 1$, if the degree of approximation L goes to infinity. λ_o goes then to $\lambda_{o\infty}$, the solution of the exact characteristic equation

$$Ch(\lambda_{0\infty}) = 1 + a\lambda_{0\infty}^2 - \lambda_{0\infty}(c+a\lambda_{0\infty}^2)\ \text{cotanh}^{-1}\ \lambda_{0\infty} = 0\ . \tag{13a}$$

$\lambda_{0\infty}$ is the mean diffusion length in units of the mean free path.
Hence the question arises, at which degree of approximation $L = L_1$ passes the root λ_o of (49) over the boundary $\lambda = 1$ (or $\lambda_o = -\lambda_o$ over $\lambda = -1$) of the interval $-1 \leq \lambda \leq 1$?

If we write Equation (49) for λ_o in the form

$$c\ \lambda_0 W_L(\lambda_0) + a\ \lambda_0^2\big(\lambda_0 W_L(\lambda_0) - P_{L+1}(\lambda_0)\big) = 1 \tag{50}$$

we see that (50) is the equation of a straight line in the c,a plane in the same sense as the exact Equation (13a) is. The coefficients of this straight line depend on L and λ_o in this interpretation. If one puts $\lambda_o = 1$, one has a relation between c,a and the degree of approximation $L = L_1$, where λ_o passes the boundary:

$$cW_{L_1}(1) + a\big(W_{L_1}(1) - P_{L_1+1}(1)\big) = 1\ . \tag{51}$$

By putting $c = 1$ and $a = -1$ this equation is always fulfilled. Therefore the bundle of straight lines (51) has the common point $(1, -1)$ in the (c,a) plane. To determine one of them completely we need a further point in the (c,a) plane. We take as that point the cut c_1 of it with the c axis for $a = o$ (this means for isotropic scattering), which is simply

$$c_1 = \frac{1}{W_{L_1}(1)} \quad . \tag{52}$$

To evaluate this numerically we remark that

$$P_{L+1}(1) = 1 \quad \text{and}^{(4)} \quad W_L(1) = k_{L+1} = \sum_{n=0}^{L_1} \frac{1}{n+1} \; . \tag{53}$$

Hence we may write the equation of the straight lines (51) for anisotropic scattering

$$c \cdot k_{L_1+1} + a(k_{L_1+1} - 1) = 1 \; . \tag{51a}$$

Table IV shows c_1 as function of L_1. In a problem with <u>isotropic scattering</u> <u>$c = c_o$ must be larger than c_1 of the table</u>, if one wants that λ_o is larger than 1 in P_{L_1} approximation of the spherical harmonics method. c_o has the same meaning as in Equation (14). Another interpretation of the table is: The degree of approximation L should be larger than L_1 in a problem with $c_o = c_1$ to obtain the desirable distinction of λ_o from the other roots λ_s of Equation (49).

Table IV

L_1	1	3	5	7	9	11	13	15
c_1	2/3	0.480	0.408	0.368	0.341	0.322	0.308	0.296

L_1	17	19	21	23	25	27	29	31
c_1	0.286	0.280	0.271	0.265	0.259	0.255	0.250	0.246

4) E. W. Hobson, <u>The Theory of Spherical and Ellipsoidal Harmonics</u>, Cambridge University Press (1931), p. 55.

We see for instance from the table that for $c_0 = 0.5$ P_3-approximation is sufficient, but for $c_0 = 0.25$ P_{31}-approximation would be necessary to shift $\pm \lambda_0$ outside of the interval $-1 \leq \lambda \leq 1$.

For anisotropic scattering, $a \neq 0$ (with the exclusion of the case of no absorption $c = 1$), the simple condition $c_0 > c_1$ of isotropic scattering must be replaced by a statement, on which side of a crucial straight line with the Equation (51a) one has to remain. Using (52) and (53) to introduce c_1 in Equation (51a) the equation of this crucial straight line takes the simple form

$$c + a(1-c_1) = c_1 . \tag{51b}$$

To every L_1 belongs such a line. On the right side of it in the c,a plane is $\lambda_0 > 1$, on the line is $\lambda_0 = 1$ and on the left side is $\lambda_0 < 1$ in P_{L_1} approximation. See Figure 5. The inclination of the straight lines is in such a manner that in the case of preferred forward scattering (with positive a) a lower L_1 is required than in the case of preferred backward scattering (with negative a) at the same c for $\lambda_0 > 1$.

These statements are restricted of course to the region in the c,a plane above the line $a = -c$, i.e., to the region of two roots $\pm \lambda_{0\infty}$ of the exact characteristic Equation (13a). They may be extended to cases $c_0 > 1$ in this region.

For small c one may use an asymptotic formula[5] for large L_1 to calculate c_1 in the coefficients of the straight line, labeled by L_1,

$$\frac{1}{c_1} = k_{L_1+1} \sim \log(L_1+1) + C + \frac{1}{2(L_1+1)} - \frac{1}{12(L_1+1)^2} + - \tag{54}$$

with $C = 0,577216$ (Euler-Mascheroni's constant).

If we keep only the most important terms of (54) for very small c_1 (that is high absorption $\gamma_a \approx 1$, $c_1 \approx 0$), the degree of approximation L_1, at which λ_0 passes

(5) E. Jahnke, F. Emde, Tables of Functions, 4th Ed., Dover Publ. (1945), p. 19.

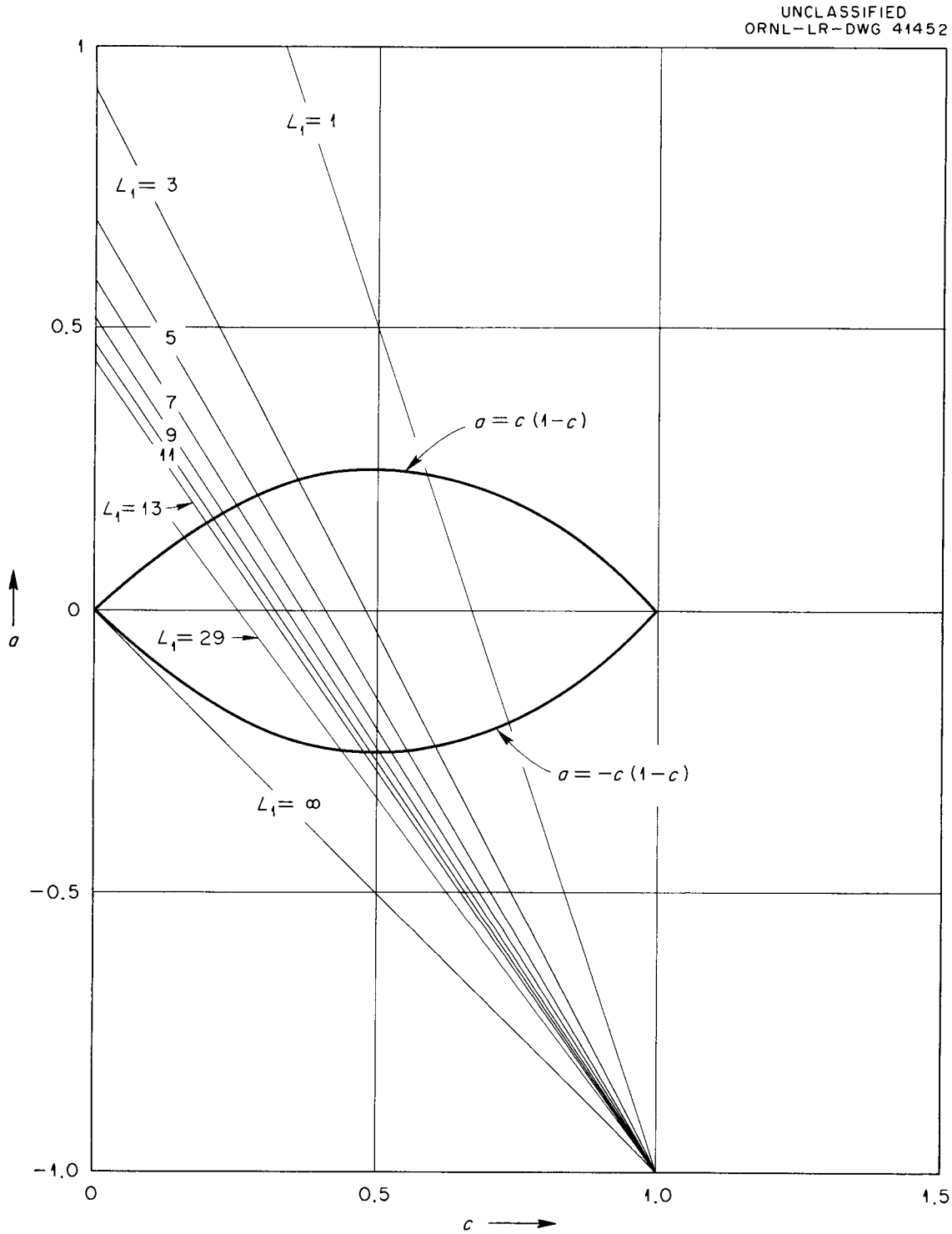

Fig. 5. A Straight Line Denoted by the Degree L_1 of the Spherical Harmonics Approximation Divides the c, a-Plane in Two Parts. On the Right Side of the Line the Asymptotic Root λ_0 of the Characteristic Equation is > 1, on the Left Side < 1 in this Approximation.

1 for isotropic scattering is asymptotically

$$L_1 \sim 0.5615 \cdot e^{1/c_1} . \tag{55}$$

For <u>linear anisotropic</u> scattering c_1 in Table IV and in the Equation (55) must be replaced by

$$\frac{c_1 + a_1}{1 + a_1}$$

where $a_1 = 3\gamma_a(1 - \gamma_t)$ is the constant of anisotropy, defined in l. c.[1], p. 2.

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