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Particle Accelerators and
High Voltage Machines

COMPUTATION METHODS FOR AVF CYCLOTRON DESIGN STUDIES

M. M. Gordon
T. A. Welton

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Contract No. W-7405-eng-26

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M. M. Gordon and T. A. Welton

Date Issued
AUG 17 1959

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COMPUTATION METHODS FOR AVF CYCLOTRON DESIGN STUDIES

M. M. Gordon* and T. A. Welton

ABSTRACT

A development is given of the theory underlying the numerical methods which have been developed at ORNL for computing the properties of particle orbits in cyclotrons with azimuthally varying magnetic fields. Details of actual computer programs are not given, but such programs can be, and have been, simply laid out by the use of the ideas contained herein. A standard method for describing the magnetic field is given, and the general equations of motion in such a field are given in convenient form. These equations are specialized to the problem of determining equilibrium orbits and properties of small oscillations, and an iterative procedure is given for producing a magnetic field to yield isochronism (equal times for all equilibrium orbits). The emphasis is on a formulation which will lead to efficient use of a high-speed computer.

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*On leave of absence from the University of Florida Physics Department during the academic year 1957-58.

CHAPTER I. INTRODUCTION

It was pointed out in 1938 by Thomas[1] that constant orbital frequency could be made consistent with axial focusing in a cyclotron by the introduction of an azimuthal variation of the median plane magnetic field. Two three-sector electron models, using the Thomas principle have been built and successfully operated at the Lawrence Radiation Laboratory,[2] as early as 1950 and the original analysis by Thomas was much improved and extended by Judd.[3] More recently, the suggestion by Symon, Kerst, Jones, Laslett and Terwilliger[4] to enhance the axial focusing by spiralling the magnetic field pattern has made the fixed frequency cyclotron economically more attractive at moderately relativistic energies.

An extensive program has been in progress at the Oak Ridge National Laboratory since 1954 to study the feasibility of large (meson-producing) fixed-frequency cyclotrons of the Thomas type, probably with spiral introduced for economic reasons.[5] To gain some understanding of the problems involved in going to kinetic energies approaching the rest energy, a four-sector electron model (no spiral) has been constructed and successfully operated.[6] A more sophisticated model, with spiral, is presently under design.

In addition to these electron models, one actual machine of the Thomas type is in operation at Delft,[7] another is essentially complete at the University of Illinois, while others are under design at Oak Ridge National Laboratory, Lawrence Radiation Laboratory, and an imposing list of other institutions.

In view of the present and probable future interest in machines of this type, a systematic presentation of the computational philosophy evolved at ORNL seems to be of value, and will here be attempted.

In general, cyclotrons of the AVF type (azimuthally varying field, to cover both the Thomas and spiral possibilities) present severe computational problems. Because of the complicated magnetic field, and the difficulty of adjusting it, considerable premium is placed on accuracy and flexibility. The complete analytical treatment of orbits in such machines is excessively difficult, but a sequence of approximations can be made which increase simultaneously in accuracy and difficulty.[4,5,8] Unfortunately, the accuracy of a given method is hard to determine except by comparison with a more accurate method and the Oak Ridge philosophy has therefore been to set up numerical procedures which are probably of higher accuracy than really required. The more approximate analytical procedures are then used to guide as closely as possible the choice of cases for "exact" computation. At the same time, one of the essential complications in machine design is the fact that the actual magnetic field is neither analytically simple nor exactly determinable. This difficulty leads to the stringent requirement that a satisfactory numerical procedure must allow economical and accurate orbit tracing in a <u>real</u> field, which is imperfectly known. Thus, the possible choice for methods of field representation is much narrower than is the case if only general conceptual problems are to be subjected to accurate computation.

The area of utility for analytical calculations is continually renegotiable as accuracy requirements, computer availability, quality of

available programs, and quality and simplicity of available analytical procedures change relative to one another. No balanced presentation can therefore be given at this time, and this report claims only to give a reasonably detailed account of the structure and rationale of the best numerical procedures which we have so far developed.

In Chapter II we discuss the specification of the magnetic field. Because of the fixed-frequency condition we must deal with "non-scaling" fields and our procedures are specifically designed for handling such fields. In Chapter III we discuss procedures for computing general orbit properties in a specified magnetic field with or without acceleration. We have used these programs to investigate such problems as acceleration through resonances and beam deflection.(5) In Chapter IV we discuss the "equilibrium orbit" code which plays a key role in magnet design work. It is this code which calculates all the desired properties of the equilibrium orbits and linear oscillations for a given field. In Chapter V we discuss procedures for fulfilling the fixed-frequency condition in a systematic way.

The present report does not contain detailed coding instructions but only the general analysis required for writing codes suitable to specific computers. The procedures described represent our best ideas on these matters as they have developed out of our past experience.

The complete system of cyclotron codes now in use here was specifically designed for the Oracle, a good general purpose computer, but unfortunately unique so that codes are not directly useful to other people. An IBM-704 has now been installed here, however, and we are presently rewriting all our codes for this machine. As these codes are finished and tested, they will be made available through special reports.

A. Units

Throughout our work we use the so-called "cyclotron units". These are a set of dimensionless, relativistic units which are well suited to cyclotron work. The rest mass, m_o, of the particle is taken as the unit of mass, and the velocity of light, c, is taken as the unit of velocity as is usually the case for relativistic units. Since we are dealing here with a machine having a fixed-frequency rf accelerating system, it is quite natural to define a time unit in terms of this frequency. For this purpose we define the unit of angular velocity as ω_o, the ideal average angular velocity of the particle. The unit of time then is $\omega_o^{-1} = \tau_o/2\pi$ where τ_o is the ideal rotation period of the particle. Since the machine will not be perfectly isochronous, and since the best mean frequency can only be determined by detailed computation, ω_o must be chosen more or less arbitrarily.

If e is the charge of the particle, then the unit of magnetic field is given by $b = m_o c\, \omega_o/e$. The unit of length is clearly $a = c/\omega_o$, so that if B_o is the central field in units of b, and r is the orbit radius in units of a, then for low energy ions $B_o r = \beta$. If a field is reasonably isochronous over a range of ion energies (including zero), it is then convenient to let b equal the central field. We then have $r = \beta$ as a reasonable estimate of the equilibrium orbit radius for specified energy. This definition is actually very convenient in practice. For proton machines, one then finds for the "resonant frequency":

$$1/\tau_o = (15.246b)\ \text{Mc/sec};$$

and for the "cyclotron length unit":

$$a = c/\omega_o = (123.21b) \text{ inches} = (313.0b) \text{ cm};$$

where b is given in units of 10^4 gauss.

For ions with charge Ze and mass A (not exactly integral) times the proton mass, these relations generalize to:

$$1/\tau_o = \frac{Z}{A}(15.246b) \text{ mc/sec};$$

$$a = \frac{A}{Z}(123.21b) \text{ inches} = \frac{A}{Z}(313.0b) \text{ cm}.$$

CHAPTER II. MAGNETIC FIELD

The magnetic field in an AVF cyclotron must fulfill two basic requirements. The particle orbits in this field must have adequate vertical focusing and in addition, the rotation period must be sufficiently independent of energy to allow acceleration with the available voltage. At the same time, it is most desirable that the detailed form of the field be as little restricted as possible, so as to avoid unnecessary restrictions on the arrangement of iron and copper.

There are two methods of specifying the magnetic field which we have found useful. The first method is to give the field over a restricted portion of the machine in terms of simple functions. This procedure is most suitable for general theoretical investigations and for preliminary design studies. The second method is to give the field at individual points on a polar mesh covering the entire machine. The field values here are obtained either from coil calculations or directly from measurements on model magnets, combined with a suitable smoothing procedure. This latter method is particularly well suited to detailed design work. In either method certain approximations are made which we shall discuss below.

A. Field Specification

We use polar coordinates (r, θ, z) in all our orbit calculations and refer to the z-direction as "axial". The median plane $(z = 0)$ is the plane of symmetry of the magnetic field such that $B_z(r, \theta, z)$ is an even function of z, while $B_r(r, \theta, z)$ and $B_\theta(r, \theta, z)$ are both odd functions of z. In other words, the magnetic field is everywhere normal to the median plane. Polar

coordinates are particularly appropriate for two reasons. First, the field of a cyclotron is usually designed to be invariant under rotation through an angle $\frac{2\pi}{N}$ about the axis of the machine, and this symmetry is inconveniently represented in rectangular coordinates unless N = 2 or 4. Second, a most convenient and economical method of orbit integration is to eliminate time and use a cyclic coordinate, such as θ, for the independent variable. Since this variable will advance by a constant increment for the numerical integration, interpolation in θ is made unnecessary if the field is determined from stored values.

The magnetic field in the median plane plays an important role so that we shall give it a special designation as follows:

$$B(r, \theta) \equiv B_z(r, \theta, 0). \tag{2.1}$$

All the essential properties of the machine depend rather directly on this function.

The magnetic field in the space occupied by the beam can be expressed in terms of the scalar potential ψ defined as follows:

$$\underset{\sim}{B} = \nabla \psi \; ; \quad \nabla^2 \psi = 0. \tag{2.2}$$

As a result of the symmetry of the field about the median plane, $\psi(r, \theta, z)$ is an odd function of z. Because of the importance of the median plane field, $B(r, \theta)$, it proves advantageous to express ψ entirely in terms of this function. This is done as follows:

$$\psi = \sum_n \frac{(-1)^n}{(2n+1)!} (L^n B) z^{2n+1} \tag{2.3}$$

where the sum extends from $n = 0$ to infinity and L represents the two-dimensional (r, θ) Laplacian operator. Thus, a complete specification of $B(r, \theta)$ is in principle sufficient to specify the entire magnetic field at all points off the median plane within the beam gap.

It is essential for our purposes that the function $B(r, \theta)$ be left as general as possible. As a result we cannot sum the above series to obtain a simple formula for ψ which will accurately represent the field for large z values. Fortunately, however, the conditions which obtain in an AVF cyclotron are such that only relatively small z values are significant. The reason for this is that since adequate axial focusing is provided for in this type of machine, it is then possible, as well as desirable, to have a relatively small beam gap. In addition, since the value of ν_z is relatively small $(0 < \nu_z < 0.5)$, the derivatives of $B(r, \theta)$ are not too large. Under these conditions we can often justify approximating the above series by using only the first term; that is:

$$\psi \cong \psi_1 = zB(r, \theta). \tag{2.4}$$

Such an approximation is very convenient and we make extensive use of it. It is important, however, to check its validity in all cases where this approximation might significantly affect the results of orbit calculations. For the type of machines so far considered, we have convinced ourselves that (2.4) is a valid approximation. The ratio of succeeding terms in the series is roughly $(Nz)^2$, where N is the number of sectors in the machine, so that one can estimate the error of the above approximation. It is in fact true that for the machine types which we have considered, the change of ν_z with axial

amplitude has always been inappreciable for the largest amplitude of interest. Orbit codes now being designed for the IBM-704 here will contain provisions for including the next higher order term in the series for ψ so that it will then be possible to make a direct check on the accuracy of (2.4) whenever this seems important.

The resonances which occur in an AVF cyclotron will be discussed in later reports, but it seems appropriate at this point to bring up certain questions relevant to the approximation (2.4). First of all let us note that, so far as motion confined to the median plane is concerned, the equations based on (2.4) are exact so that radial resonances can be treated precisely. As for the vertical motion, use of (2.4) restricts us to equations which are practically linear in z. As a result, only the coupling resonances of lowest order (e.g., $\nu_r = 2\ \nu_z$, $(N - 1)\ _r + 2\ \nu_z = N$, etc.) can be investigated. Fortunately, however, these resonances are by far the most significant for the axial motion. Furthermore, our experience with comparable radial resonances leads us to believe that the higher-order coupling resonances are not significant considering the small gap available in this type of machine.

There is another, more subtle, defect arising from the use of the approximation (2.4) for ψ; namely, that the equations of motion cannot then be derived from a Lagrangian. This defect is generally not serious except possibly where coupling resonances are involved. The theory of these resonances shows that only "sum" resonances (such as $(N - 1)\ \nu_r + 2\ \nu_z = N$) can lead to real axial instability, whereas the "difference" resonances (such as $\nu_r - 2\ \nu_z = 0$) lead to successive interchanging of energy between radial and vertical oscillations ("beam turn-over"). If the equations of motion cannot be derived from a

Lagrangian, then one may find the effects of a difference resonance giving rise to an apparent axial instability which is in fact only a "turn-over". This defect is corrected by a modification in the orbit codes in which B_r and B_θ are still calculated from (2.4), but B_z is calculated using the first two terms of the series (2.3). In this way the system has a Lagrangian which is correct to order z^2. Again because of the small beam gap involved, this modification is generally unnecessary, since the initial axial growth will be correctly given, and turn-over will occur outside the available gap, for interesting radial amplitudes.

In summary, we feel that sufficient flexibility and accuracy can be achieved for orbit studies in an AVF cyclotron by having available three methods for computing the magnetic fields. For the great majority of calculations, the approximation (2.4) is used. When desirable or necessary, the second term of the series (2.3) is also included, either to calculate B_z alone or to calculate all three field components.

B. Median Plane Field

As noted before, the median plane field $B(r, \theta)$ can be specified at discrete points on a polar mesh as far as calculations with a digital computer are concerned. For analytical calculations, however, a specific formula is required. The equations which result from these analytical calculations furnish valuable guides for the corresponding computer calculations. It is therefore quite desirable to have computer codes with facilities for using $B(r, \theta)$ in an analytical form. We shall discuss here the analytical form of $B(r, \theta)$ in order to establish a conventional notation for future reference.

The function $B(r, \theta)$ can be represented by a Fourier series as follows:

$$B(r, \theta) = \sum B_n(r) \cos n N(\theta - \mathcal{S}_n(r)) \tag{2.5}$$

where the sum extends from zero to infinity and N is the number of sectors, or periodic elements, in the magnetic field. It is possible for "N" to change as, for example, in one machine now being designed here which has four sectors at the center and eight sectors at the outside. In such a case, we consider $N = 4$ and that all $B_n(r)$ with odd n values go rapidly to zero as r is increased past the transition radius.

The function $B_o(r)$ is referred to as the "average field", that is,

$$B_o(r) = \langle B(r, \theta) \rangle \tag{2.6}$$

where angular brackets are used to denote an average over θ with r held constant. It is predominantly this function which determines the rotation period of the particle as a function of energy. In Chapter V we shall discuss how $B_o(r)$ can be determined so that a condition of "isochronism" results. The "field index" k is defined as follows:

$$k = (r/B_o)\ (dB_o/dr). \tag{2.7}$$

According to the "smooth approximation", the frequency of the radial oscillations ν_r is given by:

$$\nu_r^2 = 1 + k. \tag{2.8}$$

This simple formula is often rather inaccurate, but it is useful for obtaining estimates.

The quantity, $B(r, \theta) - B_o(r)$, is referred to as the "flutter field". It is this part of the field which must provide sufficient vertical focusing to overcome the "relativistic defocusing" created by the isochronism condition ($k \cong p^2$). For many purposes it is convenient to introduce the "flutter functions" f_n defined as follows:

$$f_n(r) = B_n(r)/B_o(r). \tag{2.9}$$

The quantities $\mathcal{S}_n(r)$ are referred to as the "spiral functions" since $\theta = \mathcal{S}_n(r)$ is the equation of the spiral traced out by the maximum of the n'th Fourier component of the field. The spiral pitch angle, defined by the equation

$$\tan \alpha_n = r(d\ \mathcal{S}_n/dr) \tag{2.10}$$

is then the angle between the tangent to this spiral line and the radial line at a given point. According to the "smooth approximation", the vertical focusing frequency ν_z is given by the following equation:

$$\overset{2}{z} = -k + \sum \left(\frac{1}{2}\right) f_n^2 (1 + 2 \tan^2\alpha_n). \tag{2.11}$$

Since this formula is only approximate, it can, in practice, often be simplified by setting all $\alpha_n = \alpha$ where α is determined simply by the geometry of the field. Thus equation (2.11) is replaced by

$$\nu_z^2 = -k + F^2 (1 + 2 \tan^2\alpha) \tag{2.12}$$

where F^2 can most easily be determined from the equation

$$F^2 B_o^2 = \langle (B - B_o)^2 \rangle . \tag{2.13}$$

This is equivalent to

$$F^2 = \frac{1}{2} \sum_{n=1}^{\infty} f_n^2 \tag{2.14}$$

which obviates the necessity of making a Fourier analysis of $B(r, \theta)$. Equation (2.12) has been used extensively at this laboratory for preliminary design work.

Let us now consider the behavior of the flutter field near the center of the machine ($r \longrightarrow 0$). Suppose we write:

$$B(r,\theta) = \sum_{n=-\infty}^{\infty} A_n(r)\, e^{inN\theta}, \tag{2.15}$$

where comparison with (2.5) yields:

$$\left.\begin{array}{l} A_o = B_o \\ \left.\begin{array}{l} A_n + A_{-n} = B_n \cos nN\ \mathcal{J}_n \\ i(A_n - A_{-n}) = B_n \sin nN\ \mathcal{J}_n. \end{array}\right\} \quad n > 0 \end{array}\right\} \tag{2.16}$$

We assume that the current and iron nearest to the median plane is at a distance c from it. Therefore the two planes $z = \pm c$ enclose a region of analyticity for the potential $\psi(r,\theta,z)$. It is a reasonable assumption that $\psi(r,\theta,\pm c)$ possesses no worse than step discontinuities, and if $\psi(r,\theta,c)$ then be written as

$$\psi(r,\theta,c) = \sum_{n} \psi_n(r)\, e^{inN\theta}, \tag{2.17}$$

ψ_n can be written as

$$\psi_n(r) = \int_0^\infty dk\, C_n(k)\, J_{nN}(kr). \tag{2.18}$$

Because $\psi(r,\theta,z)$ must satisfy the Laplace equation for $c > z > -c$, we can write

$$\psi(r,\theta,z) = \sum_n \int_0^\infty dk\, \frac{C_n(k)}{\sinh kc} \sinh kz\, J_{nN}(kr)\, e^{inN\theta}. \tag{2.19}$$

The Fourier components of B can then be written

$$A_n(r) = \int_0^\infty \frac{k\, dk}{\sinh kc}\, C_n(k)\, J_{nN}(kr). \tag{2.20}$$

Because of the assumed degree of regularity of $\psi(r,\theta,c)$, $C_n(k)$ must approach zero more rapidly than $k^{-1/2}$ for large k and it is therefore apparent that the Bessel transform of $A_n(r)$ must approach zero exponentially for large k. It is then permissible to expand $J_{nN}(kr)$ in a power series in r, and obtain the series for $A_n(r)$ by a term-by-term integration over k. Thus

$$A_n(r) = r^{nN} \sum_{p=0}^{\infty} a_p \left(\frac{r}{c}\right)^{2p} \tag{2.21}$$

where the series is easily shown to converge for $r < c$ but in general to diverge for $r > c$. The method of field representation used should clearly have the flexibility to represent $A_n(r)$ properly for small r. Fortunately this requirement is made less stringent by the fact that only the terms $n = \pm 1$ will be important for small r, while N will never be other than 3 or 4.

Similar considerations apply to an arbitrary point in the field. In general, a series expansion of B around any point in the median plane can be

expected to converge only for distances less than c. This strongly suggests that a polynomial representation for $A_n(r)$ cannot be assumed to be valid over a range larger than $\Delta r = 2c$. Although a polynomial can always be found to pass through the field values over a large range, it must be expected that serious errors will thereby appear in the higher derivatives of B. Since these higher derivatives are crucial for the effects at non-linear resonances, considerable care must be exercised in the choice of method for representing B. Use of an analytical form of sufficient flexibility to really represent the field is of limited utility in actual design work because of the large amount of computer time used in evaluating algebraic expressions of the required complexity. It has been found adequate to represent all functions of r by giving their values at a series of uniformly spaced radius values (including <u>one</u> value for negative r, assuming the value at r = 0 to be given). Required intermediate values and derivatives can then be found by use of interpolation polynomials, the 4-point central Lagrange procedure having proved very generally useful. Considerable optimization on this question is still possible, although no large gains are expected.

CHAPTER III. GENERAL MOTION

In this chapter we discuss the basic equations used in our orbit computations. We consider first the general equations of motion of a particle in the magnetic field and then describe how the effects of the rf acceleration are included. The computer codes based on these equations are used for general orbit studies of such problems as non-linear resonances, beam deflection, and acceleration through resonances. Special procedures have been developed for computation of equilibrium orbit and linear motion properties. These will be discussed in the next chapter.

A. General Equations of Motion

The motion of the particle is described in terms of cylindrical polar coordinates (r,θ,z). The plane $z = 0$ is the median plane of the machine as discussed in the previous chapter. Rather than the time t, we use θ as our independent variable and express r, z, and t as functions of θ. This choice is particularly suitable here since the particle's position is then directly correlated with the magnetic field configuration. Furthermore, as far as motion in the time-independent magnetic field alone is concerned, the equation of the orbit is then independent of $t(\theta)$. Use of θ as the independent variable has the great advantage of rendering unnecessary any interpolation process in the θ-direction, since only a finite set of θ-values ever occur. All the required values can then be stored, with interpolation required only for r. At the same time, θ can be used conveniently as the independent variable only for orbits which stay sufficiently close to circles centered at the origin. The complete requirement is that θ always

increase with t. Orbits which do not encircle the origin cannot be calculated in this way, and any orbit will be badly falsified for which the radial component of momentum becomes a large fraction of the total momentum.

We use primes on quantities to denote their derivatives with respect to Θ. In this section we consider the motion of the particle in the magnetic field alone.

If s is the arc-length along the orbit of the particle, then

$$s' = (r^2 + r'^2 + z'^2)^{1/2} \tag{3.1}$$

and the time $t(\Theta)$ may be obtained from the equation

$$t' = s'/\beta \tag{3.2}$$

where β is the velocity of the particle. The radial and vertical momenta of the particle are given by

$$p_r = pr'/s'; \; p_z = p\, z'/s' \tag{3.3}$$

where p is the momentum. We introduce the quantity q defined as follows:

$$q = (p^2 - p_r^2 - p_z^2)^{1/2} = \frac{rp}{s'} . \tag{3.4}$$

To describe the motion of the particle, we use the six canonical variables r, pr, z, p_z, t, and $E = (p^2 + 1)^{1/2}$. The equations of motion can then be written in a quasi-canonical form as follows:

$$p_r' = q - rB_z + (r/q)\, p_z\, B_\theta \tag{3.5a}$$

$$r' = (r/q)\, p_r \tag{3.5b}$$

$$p_z' = rB_r - (r/q)\, p_r\, B_\theta \tag{3.5c}$$

$$z' = (r/q)\, p_z \tag{3.5d}$$

$$t' = E\,(r/q) \tag{3.5e}$$

where the magnetic field components are all functions of r, θ, and z as specified in the previous chapter. It should be noted that we have, for convenience, made the replacement $\underset{\sim}{B} \longrightarrow -\underset{\sim}{B}$ in order that t increase along with θ.

The equation for E' is omitted from the above set of equations since for motion in the magnetic field alone, E and p are constant. In the next section we shall describe how the effects of the rf acceleration are introduced. For many purposes it is desirable to omit these effects entirely or to include them afterward.

Once the magnetic field is specified the above equations can be integrated to obtain the desired orbit information. In order to have meaningful initial values of the variables, it is usually advisable first to secure output data from the equilibrium orbit code described in the next chapter. Representing the equations of motion in the above form is well suited to the Runge-Kutta integration process generally used for orbit computations with a digital computer. In addition, the use of canonical pairs of variables permits one to interpret orbit data directly in terms of the motion of points in phase space. This is particularly important for accelerator work where

the overall behavior of the beam can be derived, through Liouville's theorem, by studying the corresponding motion in phase space of a few, selected orbits.

A word is necessary concerning the use of the Runge-Kutta process, which has the property that the truncation error destroys the conservative character of the motion. In actual fact, however, the interval size must be pushed to ridiculous extremes for any non-Hamiltonian character to be displayed. Thus, we are fully satisfied that 16 steps per sector are quite adequate for all but the most subtle theoretical purposes. The overwhelming advantage of the Runge-Kutta process lies in the fact that integrations can be begun at will without special preparation. As will be seen in Chapter IV, this feature yields a flexibility which allows very large savings of computer time.

B. Acceleration Effects

Over most of the machine the rf accelerating field acts as a small perturbation on the motion of the particle. Its effect can be neglected as a first approximation and then included afterward where it seems advisable.

The rf field acts on the particle only over a narrow θ interval where it crosses a "gap". As a result we can consider the electric force as if it were a periodic delta-function of θ and that its only effect is to change the energy of the particle. Under these conditions the rf field is inactive and the energy of the particle is constant until it reaches a "gap" position, at which point its energy changes discontinuously by an amount δE. If δ is the maximum possible energy gain per turn and γ is the number of rf gaps crossed in one revolution, then

$$\delta E = (\delta/\gamma)\ V\ (t-\Theta) \tag{3.6}$$

where $V(\omega t - \Theta)$ expresses the dependence of δE on the rf voltage wave form evaluated at the phase, $\omega t(\Theta) - \Theta$, appropriate to the time when the gap is crossed. For example, $V = \cos\ (\omega t - \Theta)$ for the simple sinusoidal wave form usually used. Under certain conditions it may be desirable to "spiral" the rf gaps to conform to the structure of the magnet. In this case, a discontinuity in p_r, in addition to that in E, occurs at each gap crossing which is given by:

$$\delta p_r = -(\delta p) \sin \alpha \tag{3.7}$$

where δ_p is the change in momentum corresponding to the change in energy δE. If $\Theta = \mathcal{S}(r)$ is the equation of the spiral line of the gaps, then α is given by $\tan \alpha = r(d\mathcal{S}/dr)$. Another possibility, which is of considerable theoretical utility is that of essentially continuous acceleration, by dividing the specified energy gain per turn among a number of radial gaps equal to the number of Runge-Kutta steps per revolution.

Near the center of the machine (low energies) the above approximate treatment will fail. In this region of the machine the rf electric forces play a significant role and special procedures are required to evaluate their effects realistically. Such procedures have been developed and worked into an orbit code specifically designed to study the central portion of an AVF cyclotron. This code will be described in a later report.

CHAPTER IV. EQUILIBRIUM ORBIT AND LINEAR MOTION

There are certain parameters associated with particle motion in a given field $B(r,\theta)$ which are of particular importance to the magnet designer. These are: ν_r and ν_z, the radial focusing frequencies; τ, the rotation period of the particle; and R, the mean orbit radius. In this chapter we shall describe the special procedures we have developed for efficient calculation of these parameters as a function of the momentum p. The computer code which carries out these computations is referred to as the "equilibrium orbit" code. The input for this code consists of the field $B(r, \theta)$ and a set of p values at which the results are desired. In addition to the above parameters this code will furnish all the detailed information desired on the equilibrium orbit and the radial and vertical linear oscillations about this orbit. Such information is particularly important in obtaining input data for general orbit studies using the programs described in the previous chapter.

The equilibrium orbit code consists of two parts. In the first part the equilibrium orbit is determined by a systematic iteration procedure. After this is done, the code then proceeds to calculate all the required parameters. These computations could be accomplished within the framework of the general orbit code (Chapter III). However, since this calculation is so important in itself, we have designed this special code in order to optimize both the speed and accuracy of these computations.

A. Equilibrium Orbit Determination

We restrict ourselves here to motion of a particle in the median plane ($z = p_z = 0$) with constant momentum p. The equilibrium orbit is sometimes

defined as that orbit which closes smoothly on itself. This definition is, however, not precise. We shall stipulate here that the particle is in the equilibrium orbit if and only if the values of (r, p_r) at the beginning and end of one sector are identical. That is, $r(\theta)$ for the equilibrium orbit is a periodic function having the same period as the sector structure of the magnetic field. The first problem that must be solved is the determination of this particular solution of the orbit equations.

$$q \longrightarrow Q = (p^2 - p_r^2)^{1/2} \tag{4.1}$$

and the differential equations (3.5a,b) for r and p_r become:

$$p_r' = Q - rB(r,\theta) \tag{4.2a}$$

$$r' = (r/Q)p_r. \tag{4.2b}$$

These differential equations are, of course, correct for any orbit in the median plane. The total integration interval for these and all the other differential equations in this code is one sector: $\Delta\theta = 2\pi/N$. The initial value of θ is arbitrary and, for convenience, we shall define this value as $\theta = 0$ so that the integrations proceed from $\theta = 0$ to $\theta = \theta_o \equiv 2\pi/N$. The determination of the equilibrium orbit reduces then to the determination of the initial values $r = r_o$, $p_r = p_{ro}$ which satisfy the following conditions:

$$r_o = r(\theta) \equiv r(\theta_o) \tag{4.3a}$$

$$p_{ro} = p_r(0) \equiv p_r(\theta_o). \tag{4.3b}$$

These all-important initial values can quite readily be determined by an iteration method which we shall describe in the remainder of this section. As we shall see, this procedure is based on an interpolation wherein it is assumed that the orbit resulting from a given trial value of (r_o, p_{ro}) differs from the true equilibrium orbit only in first-order (linear) effects. We shall refer to such an orbit as a "quasi-equilibrium orbit."

Since it is an important part of the iteration procedure, we consider first the linear radial motion about a given quasi-equilibrium orbit in the median plane. If x and p_x are the radial (first-order) displacement and corresponding momentum associated with the quasi-equilibrium orbit, then we can obtain the equations for x and p_x by making the replacements

$$r \longrightarrow r + x, \; p_r \longrightarrow p_r + p_x \tag{4.4}$$

in Eqs. (4.2a,b) and keeping only first-order terms in x and p_x. As a result we find

$$p_x' = -(p_r/Q)p_x - \frac{\partial}{\partial r} \quad rB(r,\theta) \quad x \tag{4.5a}$$

$$x' = (p_r/Q)x + (p^2 r/Q^3)p_x \tag{4.5b}$$

where r and p_r have the same values at each θ as those obtained from integrating Eqs. (4.2a,b); that is, $r(\theta)$ and $p_r(\theta)$ are the coordinates of the quasi-equilibrium orbit. We shall need two independent solutions of these equations and shall distinguish them by the subscripts 1 and 2. To generate these solutions we choose the following initial conditions:

$$\left.\begin{aligned} x_1(0) &= \delta x;\ p_{x1}(0) = 0 \\ x_2(0) &= 0;\ p_{x2}(0) = \delta p_x \end{aligned}\right\} \tag{4.6}$$

where δx and δp_x are arbitrary although they should be selected so as to optimize the accuracy of the final results. The integration of Eqs. (4.5a,b) to obtain (x_1, p_{x1}) and (x_2, p_{x2}) is not only an essential part of the iteration procedure for finding (r_o, p_{ro}), but also, as we shall see, it is an essential part of calculating the properties of the linear oscillations about the equilibrium orbit once this orbit has been found.

For a given trial value of (r_o, p_{ro}) we have now six differential equations for the quantities r, pr, x_1, p_{x1}, x_2, and p_{x2} which give the coordinates of the quasi-equilibrium orbit and the linearized motion about it. An inspection of these equations will show directly that the most efficient procedure is to integrate all six equations simultaneously, since in this way a given set of fields and derivatives is utilized as many times as possible before going on to the next set. Needless to say, no economy of <u>storage</u> is claimed for this procedure, the saving being wholly one of <u>time</u>. From the results of this integration we can define the elements of the matrix (J) and the ϵ's as follows:

$$x_1(\theta_o) = J_{11}\ \delta x;\ p_{x1}(\theta_o) = J_{21}\ \delta x \tag{4.7a}$$

$$x_2(\theta_o) = J_{12}\ \delta p_x;\ p_{x2}(\theta_o) = J_{22}\ \delta p_x \tag{4.7b}$$

$$\epsilon_1 = r(\theta_o) - r_o;\quad \epsilon_2 = p_r(\theta) - p_{ro} \tag{4.7c}$$

$$\epsilon = \left|\epsilon_1\right| + \left|\epsilon_2\right| \tag{4.7d}$$

The significance of the matrix (J) will be discussed later. Since both ϵ_1 and ϵ_2 would be zero if the correct (r_o, p_{ro}) were used, it follows that ϵ is a measure of the error in the determination of the equilibrium orbit. As a result we use the smallness of ϵ to test whether a satisfactory value of (r_o, p_{ro}) has been found.

If the value of ϵ is not small enough, we obtain an improved value of (r_o, p_{ro}) by the following procedure. For the given (r_o, p_{ro}) value, we assume that the resulting quasi-equilibrium orbit differs from the true equilibrium orbit only in first-order (linear) effects. As a result, we write the equation for the quasi-equilibrium orbit as follows:

$$\left.\begin{aligned} r &= r^* + a_1 x_1 + a_2 x_2 \\ p_r &= p_r^* + a_1 p_{x1} + a_2\, p_{x2} \end{aligned}\right\} \tag{4.8}$$

where (r^*, p_r^*) are the coordinates of the true equilibrium orbit. From Eqs. (4,6) and (4.7) we then have the following equations for determining the constants a_1 and a_2:

$$\begin{aligned} (J_{11} - 1)\, a_1 \delta x + J_{12}\, a_2 \delta p_x &= \quad 1 \\ J_{21}\, a_1 \delta x + (J_{22} - 1)\, a_2 \delta p_x &= \quad 2 \end{aligned} \tag{4.9}$$

and, hence, the equations for r_o^* and p_{ro}^* are:

$$r_o^* = r_o - a_1 \delta x; \; p_{ro}^* = p_{ro} - a_2\, \delta p_x \tag{4.10}$$

where (r_o^*, p_{ro}^*) is then the improved trial value of (r_o, p_{ro}). With this new set of initial conditions the entire procedure can be repeated. These iterations are continued until a sufficiently small value of ϵ is obtained.

The iteration procedure just outlined converges quite rapidly. Since the procedure is correct to first-order, if the error in a given trial value of (r_o, p_{ro}) is ϵ, then the error in the improved value will be of order ϵ^2. Thus, the errors on successive iterations in (r_o, p_{ro}) will be of order $\epsilon, \epsilon^2, \epsilon^4, \epsilon^8, \ldots$.

The total time required for the iteration procedure to converge depends entirely on the quality of the initial choice for (r_o, p_{ro}). If no information is available for guidance then the following choice should be made:

$$r_o = \beta, \; p_{ro} = 0, \tag{4.11}$$

assuming the central isochronous field to be equal to the field unit b, previously defined. This choice is a good one wherever the orbits are nearly circular as is certainly the case near the center of the machine or when a large number of sectors N is involved. (See Section A of Chapter V). When this code is being run at a succession of p values, as is usually the case, a considerable saving in time can be achieved by using an extrapolation procedure to obtain the initial (r_o, p_{ro}) trial value. If $r_o(p)$ and $p_{ro}(p)$ are the final values found for these quantities as a function of p, then the corresponding values of

$$(r_o - \beta)/\beta; \; p_{ro}/p \tag{4.12}$$

are accumulated and used as variables in the extrapolation procedure since these latter quantities are slowly varying functions of p. The procedure we have adopted uses (4.11) for the first (smallest) p value, then a one-point formula with (4.12) for the next p value, then a two-point formula, and so on.

After the first few p values have been run, the extrapolation procedure changes to one using a fixed number of points (three or four depending on the coarseness of the p interval) with the values of (4.12) obtained from those p values immediately preceding the new one. It is quite normal in design work to run this code at twenty or more p values and we have found that using this extrapolation procedure cuts the running time by a factor of two or three.

B. Linear Motion Properties

Since the convergence rate of the above iteration procedure for determining (r_o, p_{ro}) is known, it is possible to establish the test on ϵ of Eq. (4.7d) such that the "correct" (r_o, p_{ro}) is determined without requiring an additional integration of the equations (4.2a,b). For example, if it is desired that the equilibrium orbit be determined with an accuracy such that $\epsilon < 10^{-10}$, then the ϵ -test should actually be made on the condition $\epsilon < 10^{-5}$ (or 10^{-6} if a margin of safety is desired) since it is known that the improved value of (r_o, p_{ro}) derived from (4.10) will then be in error by only the square of this quantity. The code will still make another integration of Eqs. (4.2a,b) so that the final value of ϵ can be checked, but before this last integration is performed certain changes have to be made.*

On the final integration through a sector we determine all the properties of the equilibrium orbit and the linear radial and axial oscillations about it. The two differential equations (4.2a,b) are integrated with the now

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*It is at this point that the utility of the very flexible Runge-Kutta process becomes most obvious.

"correct" initial values (r_o, p_{ro}) so that the resulting functions r and p_r give the coordinates of the equilibrium orbit. Two sets of the differential equations (4.5a,b), one for (x_1, p_{x1}) and the other for (x_2, p_{x2}), are also integrated as before with initial conditions given by (4.6). Since the (r, p_r) values are now the correct equilibrium orbit coordinates, then the resulting (x, p_x) values will characterize correctly the linear radial oscillations about the equilibrium orbit. To these six differential equations (which are the same as those used in the iteration procedure above) we now add six more, all of which are to be integrated simultaneously.

The linearized equations for the vertical motion, as derived from Eqs. (3.5c,d), are as follows:*

$$p_z' = z\left[r\left(\frac{\partial B}{\partial r}\right) - (p_r/Q)\left(\frac{\partial B}{\partial \theta}\right)\right] \tag{4.13a}$$

$$z' = (r/Q)p_z. \tag{4.13b}$$

Here again we need two sets of these equations in order to generate two independent solutions. If we distinguish these solutions by subscripts 1 and 2, then the initial conditions are as follows:

$$\left.\begin{aligned} z_1(0) &= \delta z;\ p_{z1}(0) = 0 \\ z_2(0) &= 0;\ p_{z2}(0) = \delta p_z \end{aligned}\right\} \tag{4.14}$$

where, as before with δx and δp_x, the values of δz and δp_z, though arbitrary, should be selected for optimum accuracy. These four additional differential equations will yield complete information about the linear

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*Note that the simplest representation for the z-dependence of the field now becomes exact for the properties of present interest.

axial motion about the equilibrium orbit.

The last two of the twelve differential equations are those for τ, the rotation period of a particle in the equilibrium orbit, and for R, the mean radius of this orbit. These equations are as follows: (cf. Eq. (3.5e))

$$\tau' = (N/2\pi)\ E\ (r/Q) \tag{4.15}$$

$$R' = (N/2\pi)\ r \tag{4.16}$$

with initial conditions $\tau(0) = 0$, $R(0) = 0$. The factor $(N/2\pi)$ is a normalizing constant which arises from the fact the equations are integrated through one sector ($\Delta\theta = 2\pi/N$).

When the final integration through a sector of all twelve differential equations is completed, all information required for the output data is available. The final value of ϵ can be obtained from Eq. (4.7d) and the elements of the matrix (J) can be calculated from Eqs. (4.7a,b) as done before. By analogy we can define the elements of a matrix (K) which are calculated from the results of the z-motion integrations as follows:

$$z_1(\theta_o) = K_{11}\ \delta z;\ p_{z1}(\theta_o) = K_{21}\ \delta z \tag{4.17a}$$

$$z_2(\theta_o) = K_{12}\ \delta p_z;\ p_{z2}(\theta_o) = K_{22}\ \delta p_z. \tag{4.17b}$$

Since the linear differential equations for x and z are of the general Mathieu-Hill variety, it follows then from Floquet's theorem that the focusing frequencies ν_r and ν_r are given by:

$$2\cos(\nu_r\theta_o) = J_{11} + J_{22} \tag{4.18a}$$

$$2\cos(\nu_z\theta_o) = K_{11} + K_{22} \tag{4.18b}$$

where $(\nu_r\theta_o)$ and $(\nu_z\theta_o)$ are usually denoted by σ_r and σ_z. (Since $\cos\sigma_z$ may be greater than unity, it is essential that the arc-cos routine used for obtaining σ_z be alternatively an arc-cosh routine as well.) The matrices (J) and (K) are transformation matrices which propagate the solutions through successive sectors. As a result, the matrix for transforming the solutions through n sectors is, for example:

$$\left.\begin{aligned} (J)^n &= A_n(J) - A_{n-1} \\ \text{where} \quad A_n &= (\sin n\sigma)/\sin\sigma \end{aligned}\right\} \tag{4.19}$$

the same being true for (K). One additional condition on these matrices is the Wronskian relation:

$$\det(J) = \det(K) \equiv 1. \tag{4.20}$$

These relations can be used for checking the accuracy and consistency of the results. These matrices can be used to evaluate the characteristics of the "invariant ellipses" associated with the linear motion as we shall show in Chapter VI.

From the results of integrating Eq. (4.15) we obtain $\tau(\theta_o) \equiv \tau$, the rotation period of the particle in units of τ_o, the ideal value. This τ value is discussed at length in the next chapter. From the integration of Eq. (4.16) we obtain the value of $r(\theta) \equiv R$, the mean radius of the equilibrium orbit. For an isochronous field R is less than, though nearly equal to β, if the central isochronous field is b.

Thus we have arrived at a system of straightforward procedures for

evaluating all the desired properties of the equilibrium orbits and associated linear motions for a given median plane field $B(r,\theta)$. The final value obtained for ϵ is a measure of the error not only in the equilibrium orbit determination, but also in all the other parameters. Since only motion in the median plane and purely linear axial motion is involved here, the values of B_r, B_θ, and B_z used are quite correct.

As noted in the introductory remarks, it is possible to carry out the calculations performed by the equilibrium orbit code within the framework of the general orbit code (Chapter III). This indeed has been the procedure used so far in all our Oracle codes. The determination of the equilibrium orbit is carried out using Eqs. (3.5a,b) with $z = p_z = 0$. These equations are integrated through one sector three times in succession. The initial conditions for the first integration are $r = r_o$, $p_r = p_{ro}$ which are, as above, the trial values for the equilibrium orbit coordinates. From the results of this integration ϵ_1 and ϵ_2 are determined through Eq. (4.7c) as before. The initial conditions for the second and third integrations are $r_1 = r_o + \delta x$, $p_{r1} = p_{ro}$ and $r_2 = r_o$, $p_{r2} = p_{ro} + \delta p_x$. From the results of these integrations the elements of the matrix (J) are determined, by analogy to (4.7a,b) as follows:

$$\left.\begin{aligned} r_1(\theta_o) &= r_o + J_{11}\delta x;\ p_{r1}(\theta_o) = p_{ro} + J_{12}\,\delta px \\ r_2(\theta_o) &= r_o + J_{21}\delta x;\ p_{r2}(\theta_o) = p_{ro} + J_{22}\,\delta px. \end{aligned}\right\} \tag{4.21}$$

With these results one then obtains an improved value of (r_o, p_{ro}) through Eqs. (4.9, 4.10). This process is repeated as required to determine the

equilibrium orbit to the desired accuracy. The results of the final iteration also give the final values of the matrix (J). When this iteration process is completed, the three differential equations (3.5c, d, e) for p_z, z, and t, together with those for r and p_r, are then all integrated two more times through a sector. The initial values for (r, px) each time are (r_o, p_{ro}) while those for (z, pz) are first (δx, 0) and then (0, δP_z). From the results of these integrations the matrix (K) can be determined via Eqs. (4.17a,b) as before, and the value of τ can be determined from the integration of t' from $t(\theta_o) = (2\pi/N)\tau$.

Although the foregoing procedure has worked quite well, it has one basic drawback, namely, that the accuracy of the results is both limited and hard to evaluate. This comes about from the choice of the parameters δx, δp_x, δz, δp_z, which must try to satisfy two contradictory requirements. These parameters must be small enough so that the non-linear effects they produce are negligible and, at the same time, they must be large enough to yield an adequate number of significant figures in the final results. A compromise must therefore be made in the choice of the parameters. Once the choice is made, one is then faced with the difficult problem of evaluating the non-linear effects in each situation in order to estimate errors. This problem is completely avoided in the new type of code since Eqs. (4.5a,b) and (4.13a,b) for the radial and axial oscillations are linear by construction. Another, more apparent, advantage in the new procedures is the gain in speed

which can be obtained from integrating all the differential equations simultaneously.

An IBM-704 version of the equilibrium orbit code has recently been completed which uses all of the new procedures. (This code will be described in detail in a separate report.) The average time required by this code to complete all the computations for one p value is about six seconds, if 16 Runge-Kutta steps per sector are used.

CHAPTER V. ISOCHRONISM

We define isochronism as that field condition for which $\tau \equiv 1$ for all p values, where τ is the rotation period of the particle in units of τ_o, the ideal value. In the previous chapter we described how τ, along with other quantities, can be calculated as a function of p for a given median plane field $B(r,\theta)$. In this chapter we shall describe procedures whereby $B_o(r)$, the average field, can be determined so as to give isochronism.

If the machine must accelerate different ions or the same ion to different final energies, then it is essential that $B_o(r)$ be re-determined for each operating condition. In practice, the required adjustments might be achieved through a suitable set of concentric, circular, pole-face windings.

In certain circumstances it may be desirable or necessary to forego isochronism over part of the machine. For example, if the flutter field is too feeble at the center of the machine, it may be necessary to adjust $B_o(r)$ in order to achieve adequate axial focusing. The amount of phase slip which occurs in one revolution is given by $2\pi(\tau - 1)$ so that the knowledge of $\tau(p)$ vs. p enables one to calculate the phase as a function of energy. The procedures detailed below apply specifically to the achievement of the condition $\tau(p) = 1$, but it will be apparent how these procedures can be modified so that $\tau(p)$ will follow any prescribed curve. As a result, it is then possible to determine $B_o(r)$ so that phase slippage occurring in one part of the machine is compensated for in another.

It should be noted that the value of τ is calculated for particles in the equilibrium orbit, although in an actual machine the particles will

be executing oscillations about the equilibrium orbit. Fortunately, the first-order effect of the oscillations on the orbit period will average out to zero over a sufficient number of revolutions. At the center of the machine where ν_r is close to unity these effects will nearly average to zero during each revolution.

A. Approximations for $B_o(r)$

In this section we shall present approximate analytical formulas for $B_o(r)$. Such formulas are important when orbit calculations are to be carried out using analytically specified fields. In addition, these formulas are quite useful for preliminary design studies. In the next section we shall describe numerical methods for determining $B_o(r)$ where high precision is required.

Approximate expressions for $B_o(r)$ are obtained from the following two basic equations:

$$p = \langle s'(r,\theta)\ B\ (r,\theta)\rangle \tag{5.1}$$

$$\beta\tau = \langle s'(r,\theta)\rangle \tag{5.2}$$

where $s' = (r^2 + r'^2)^{1/2}$ and the averaging is carried out along the equilibrium orbit. The first equation follows from the simple periodicity of the equilibrium orbit, while the second is an expression of the length of this orbit.

The simplest approximation is to ignore the effect of the flutter field. In this case, the equilibrium orbits are circles and the above equations yield

the following result:

$$p = rB_o(r); \ \beta = r. \tag{5.3}$$

For isochronism ($\gamma \equiv 1$), one then has

$$\beta = r; \ B_o(r) = E(r) \equiv (1 - r^2)^{-1/2} \tag{5.4}$$

where $E(\beta)$ is the total energy of the particle. This approximation will be good whenever the flutter is small and the number of sectors large.

A considerable improvement on the above approximation can be obtained by considering to first-order the effect of the flutter field. For $\beta(r,\theta)$ given by (2.5), the first-order approximation to the equilibrium orbit is

$$r(\theta) \equiv R(1 + \rho)$$

$$\rho \cong \rho_1 = \sum_n (f_n/n^2N^2) \cos nN(\theta - \zeta_n) \tag{5.5}$$

where f_n and ζ_n are evaluated at $r = R$, the mean radius of the orbit. Using this expression for $r(\theta)$ to evaluate Eqs. (5.1) and (5.2) one eventually obtains the following approximation for $B_o(r)$:

$$B_o(r) = E(\beta) \left[1 + 2g_1 + 2g_2\right]^{-1}$$

$$\beta(r) = r(1 + g_1)$$

$$g_1(r) = \sum_n (f_n/2nN)^2 \tag{5.6}$$

$$g_2(r) = \sum_n (f_n/2nN)^2 k_n$$

$$k_n(r) = r/B_n \, dB_n/dr$$

By considering higher order terms in $r(\theta)$ it can be shown that this expression is correct to order N^{-2} except for terms involving products of three f's

(such as $f_1^2 f_2$, $f_1 f_2 f_3$, etc.). Experience has shown that (5.6) will give isochronism to within a few parts in 10^4 for N = 6 or 8, and to within a few parts in 10^3 for N = 3 or 4. (It is worth noting in passing that for a three-sector, weak-spiral machine, B_o will actually decrease with increase in r near the center.) The accuracy provided by (5.6) will be sufficient for most purposes. Where greater accuracy is required, the method described in the next section can be used.

B. Iteration Methods

In this section we shall describe an iteration method for determining $B_o(r)$ which is capable of making the field $B(r,\theta)$ isochronous to any degree of accuracy desired. This method is numerical rather than analytical in nature and so will apply best to cases where $B(r,\theta)$, or at least $B_o(r)$ is supplied as a table of numerical values. For simplicity we shall assume here that the flutter field remains unchanged and that only the average field $B_o(r)$ is altered during successive iterations. That is, the question to be answered is as follows: for a given flutter field what is the correct $B_o(r)$ which must be added in order to achieve isochronism? The computer code which calculates $B_o(r)$ by the interation method should be an appendage to the equilibrium orbit code (Chapter IV) since information must pass back and forth between the two.

For a given median plane field having $B_o(r)$ as its average, the equilibrium orbit code will provide values of R and τ as a function of p. We shall assume that the p values used are such that the resultant R values cover the machine in sufficient detail. Let us consider now that R rather than p is the independent variable; that is, $\tau = \tau(R)$, $p = p(R)$. We can

then compute for each R value given the quantities δ_1 and δ_2 defined as follows:

$$\beta\tau/R = 1 + \delta_1/p^2 \tag{5.7}$$

$$RB_o(R)/p = 1 - \delta_2/p^2 \tag{5.8}$$

The first equation is a relation between β (and hence p) and R for a given τ, while the second is a relation between the functions $B_o(R)$ and $p(R)$. Insofar as δ_1 and δ_2 depend on the properties of the flutter field, they can be considered as functions of R alone. Since, however, they also depend on B_o and its derivatives, we must write, in general, $\delta_i = \delta_i(R, p(R))$ (i = 1 or 2). The analysis of the previous section can be used to show that the correction terms to the equations $(B\ /R = 1)$ and $(RB_o/p = 1)$ depend, to first-order only on $g_1(R)$, $g_2(R)$ and their radial derivatives given in (5.6). Since these quantities are quadratic in the f_n^2 and since $f_n \sim p^{-1}$, it follows that the correction terms vary as p^{-2} to first-order. Thus, δ_1 and δ_2 in Eqs. (5.7) and (5.8) are approximately independent of p or B_o.

These considerations suggest the following procedure for obtaining an improved value of $B_o(R)$. We first set $\tau = 1$ in Eq. (5.7) and solve the resulting equation for $\beta_1(R) = \beta\ (R,1)$:

$$\beta_1/R = 1 + \delta_1/p_1^2 \tag{5.9}$$

where δ_1 has the same value as that obtained from Eq. (5.7) for the given R. Since $p_1 = \beta_1/\sqrt{1 - \beta_1^2}$, Eq. (5.9) is a cubic equation for β_1 and can readily

be solved. The resulting function $p_1(R)$ is approximately the correct $p(R)$ required for isochronism. We then insert these values into Eq. (5.8) to obtain the improved value of $B_0(R)$ which we denote as $B_0^*(R)$; that is

$$B_0^*(R) = (p_1/R) \left[1 - \delta_2/p_1^2\right] \tag{5.10}$$

where δ_2 is the same as that derived from Eq. (5.8). In this way we obtain a table of $B_0^*(R)$ for the given set of R values. This table can then be used for interpolation to obtain the improved B_0 values at whatever r values are desired.

Since the error in the δ's and hence in the improved $B_0(r)$ is of "second-order", the procedure described above can be iterated repeatedly to obtain successively better approximations to the ideal condition of isochronism. At each step in the iteration process, it is, of course, necessary to rerun the field, with the new $B_0(r)$, through the equilibrium orbit code in order to obtain a new set of R and τ values. It is important to note that for any reasonable starting value of $B_0(r)$ (for example $E(r)$, or even a constant), the very first improved value will be at least as good as that obtained by using Eq. (5.6). Thus, the choice of starting value for $B_0(r)$ is not at all critical. This iteration method is being incorporated in the equilibrium orbit code for the 704 but has not yet been tested. We feel confident that it will converge rapidly.

The convergence rate of the iteration method could be improved by a modification which takes into account the dependence of the δ's on $p(R)$. To do this, however, we cannot use the R values obtained from the equilibrium

orbit output, but must instead interpolate the values of $p(R)$ and $\tau(R)$ each time so as to obtain these functions at a set of pre-assigned and fixed r values. (This is made necessary by the fact that the values of R will change each time $B_o(r)$ changes.) For convenience we can choose this set of r values to be the same as those at which the function $B_o(r)$ is tabulated. If on successive iterations the values of $p(R)$ obtained at a given r value are: $p'(r)$, $p''(r)$, ..., then the corresponding δ's obtained from (5.7) and (5.8) will be $\delta_i(r,p')$, $\delta_i(r,p'')$, ... ($i = 1$ or 2). In solving (5.9) then for β_1, an iteration scheme should be used in which the appropriate value of $\delta_1(r,p_1)$ to be employed at each step is obtained through an interpolation into the accumulated set of values of $\delta_1(r,p)$. Having thereby obtained β_1, and hence p_1, the value of $\delta_2(r,p_1)$ to be used in solving (5.10) for B_o^* can likewise be determined by interpolation. Although this modification in the above iteration method is considerably more complicated, it should increase the rate of convergence. Whether such complication is necessary will be determined from tests (yet to be performed) of these iteration methods.

In the process of changing B_o to achieve isochronism the values of ν_z will also change as can be seen from Eq. (2.11). The values of ν_z are usually not rigidly specified so that these changes may not be significant. In addition, if the value of B_o at the start of the iterations is close enough to the final value, then the changes in ν_z will be very small. If, however, the values of ν_z obtained from the corrected B_o are not satisfactory, then the flutter field should be changed to readjust the ν_z values.

Following this adjustment a new $B_o(r)$ should be calculated again from the iteration scheme. By working back and forth between adjusting ν_z and readjusting B_o, satisfactory values of both ν_z and τ can finally be obtained. Such a procedure has been used at ORNL as an essential part of the design procedure for the radial sector electron cyclotron and has been found to work quite satisfactorily.[6]

CHAPTER VI. DETAILS OF THE SMALL OSCILLATIONS

The equilibrium orbit code whose structure was sketched in Chapter IV provides complete information on the small amplitude radial and axial excursions from the equilibrium orbit. In this chapter will be given for completeness the relations between the raw data produced at each Runge-Kutta step for x_1, x_2, z_1, z_2, p_{x1}, p_{x2}, p_{z1}, p_{z2}, and the quantities required to start a betatron oscillation at given θ with given radial and axial amplitudes and phases. This processed output from the equilibrium orbit code is essential if non-linear orbit properties are to be determined conveniently by orbit tracing with a general orbit code.

Let (M) refer to either of the matrices J or K defined in Chapter IV. Let $X(\theta)$ and $Y(\theta)$ refer either to $x(\theta)$ and $p_x(\theta)$ or $z(\theta)$ and $p_z(\theta)$. We write:

$$\begin{aligned} X(\theta) &= M_{11}(\theta)\, X(0) + M_{12}(\theta)\, Y(0) \\ Y(\theta) &= M_{21}(\theta)\, X(0) + M_{22}(\theta)\, Y(0) \end{aligned} \tag{6.1}$$

where θ has been taken as zero at the azimuth where the Runge-Kutta process is begun in going through a sector. In the notation of Chapter IV, $M \equiv M(\theta_0)$, and the relations (4.7) and 4.17) are generalized to become:

$$\begin{aligned} X_1(\theta) &= M_{11}(\theta)\, \delta X \\ Y_1(\theta) &= M_{21}(\theta)\, \delta X \\ X_2(\theta) &= M_{12}(\theta)\, \delta Y \\ Y_2(\theta) &= M_{22}(\theta)\, \delta Y \end{aligned} \tag{6.2}$$

An elementary solution of (6.1) satisfies:

$$X_{\pm}(\theta + \theta_o) = e^{\pm i\sigma} X_{\pm}(\theta)$$
$$Y_{\pm}(\theta + \theta_o) = e^{\pm i\sigma} Y_{\pm}(\theta). \tag{6.3}$$

From (6.1) it follows that:

$$X_{\pm}(0) = \Lambda_{\pm} M_{12} \tag{6.4a}$$

$$Y_{\pm}(0) = \Lambda_{\pm} (-M_{11} + e^{\pm i\sigma}) \tag{6.4b}$$

with:

$$\cos\sigma = 1/2\ (M_{11} + M_{22}). \tag{6.5}$$

These elementary solutions can be found at aribtrary values of θ by use of (6.1), which yields:

$$X_{\pm}(\theta) = M_{11}(\theta)\ X_{\pm}(0) + M_{12}(\theta)\ Y_{\pm}(0)$$
$$Y_{\pm}(\theta) = M_{21}(\theta)\ X_{\pm}(0) + M_{22}(\theta)\ Y_{\pm}(0). \tag{6.6}$$

A general solution, with arbitrary phase ψ can be constructed by combining the two solutions as follows:

$$X(\theta) = X_{+}\ e^{i\psi} + X_{-}\ e^{-i\psi}$$
$$Y(\theta) = Y_{+}\ e^{i\psi} + Y_{-}\ e^{-i\psi} \tag{6.7}$$

The multiplicative constants $\Lambda_{\pm}$ are taken to be:

$$\Lambda_{+} = \Lambda_{-} = 1/2\sqrt{\frac{a}{M_{12}\ \sin\sigma}} \tag{6.8}$$

The constant $\mathcal{A}$ is proportional to the square of the amplitude of oscillation and is just the area of the phase figure produced by the motion. Thus:

$$\mathcal{A} = \int dX \int dY \tag{6.9}$$

where the double integral is to be extended over that region of the phase space bounded by the (X,Y) values which the motion can achieve for a given Θ.

If (6.4), (6.6), (6.7), and (6.8) are combined, we obtain:

$$\begin{aligned} X(\Theta) &= \sqrt{\frac{\mathcal{A}}{E}} \left[A(\Theta) \cos \psi + B(\Theta) \sin \psi\right] \\ Y(\Theta) &= \sqrt{\frac{\mathcal{A}}{E}} \left[C(\Theta) \cos \psi + D(\Theta) \sin \psi\right] \end{aligned} \tag{6.10}$$

The coefficients A, B, C, D, and E are given by:

$$A(\Theta) = M_{12}\, M_{11}(\Theta) - 1/2(M_{11} - M_{22})\, M_{12}(\Theta) \tag{6.11a}$$

$$B(\Theta) = -\sin\sigma\, M_{12}(\Theta) \tag{6.11b}$$

$$C(\Theta) = M_{12}\, M_{21}(\Theta) - 1/2\, (M_{11} - M_{22})\, M_{22}(\Theta) \tag{6.11c}$$

$$D(\Theta) = -\sin\sigma\, M_{22}(\Theta) \tag{6.11d}$$

$$E = M_{12} \sin\sigma . \tag{6.11e}$$

The form (6.10) allows initial conditions to be specified at any value of Θ corresponding to any required phase and amplitude for the oscillation, in the linearized approximation. The required coefficients, defined in (6.11), are easily calculated from the result of the equilibrium orbit code.

Another use of these formulae is the choice of suitable axes for the phase plots. Clearly the choice of $\sqrt{\frac{a}{\pi}}\cos\psi$ and $\sqrt{\frac{a}{\pi}}\sin\psi$ as equally scaled rectangular axes will result in phase plots of superior appearance. From (6.10), we obtain:

$$\begin{aligned} \sqrt{\frac{a}{\pi}}\cos\psi &= \frac{1}{\sqrt{E}}\left[-D(\theta)\,X(\theta) + B(\theta)\,Y(\theta)\right] \\ \sqrt{\frac{a}{\pi}}\sin\psi &= \frac{1}{\sqrt{E}}\left[C(\theta)\,X(\theta) - A(\theta)\,Y(\theta)\right]. \end{aligned} \tag{6.12}$$

If A, B, C, and D are obtained at the θ-value desired for phase-plotting, the use of the linear combinations given in (6.12) for the rectangular axes will yield circles for small amplitudes and will cause the polar angle in the plot to be identical with the phase ψ.

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