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A THERMAL PROBLEM ASSOCIATED WITH UNDERGROUND STORAGE OF RADIOACTIVE WASTES

Julian Crowell
F. L. Parker

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ORNL-3002

Contract No. W-7405-eng-26

HEALTH PHYSICS DIVISION

A THERMAL PROBLEM ASSOCIATED WITH UNDERGROUND STORAGE OF RADIOACTIVE WASTES

Julian Crowell and F. L. Parker

Submitted as a thesis to the Faculty of the Graduate School of Vanderbilt University in partial fulfillment of the requirements for the degree of Master of Science in physics.

DATE ISSUED

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ACKNOWLEDGMENTS

The authors are endebted to Dr. Rufus Ritchie and Dr. H. P. Yockey of Oak Ridge National Laboratory and Mr. Verner Anderson of the Oak Ridge Diffusion Plant for helpful suggestions; to Dr. W. G. Holladay of Vanderbilt University for suggestions and corrections; and to the Oak Ridge Institute of Nuclear Studies for making the study possible through a Graduate Research Fellowship and an AEC Radiological Physics Fellowship granted to the senior author.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	ii
TABLE OF CONTENTS	iii
LIST OF ILLUSTRATIONS	v
INTRODUCTION	1
I. THE GENERAL BOUNDARY-VALUE PROBLEM OF AN INFINITE MEDIUM HEATED BY A SPHERE AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME	4
II. TWO SPECIAL CASES OF THE GENERAL PROBLEM OF AN INFINITE MEDIUM HEATED BY A SPHERE AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME	9
The Thermal Properties of the Sphere Considered to be the Same as Those of the Infinite Medium	
The Contents of the Sphere Considered to be Well-Stirred, i.e. to Have an Infinite Conductivity	
III. SOME NUMERICAL RESULTS ON THE SPHERICAL PROBLEM WITH Q(t) SET EQUAL TO A SUM OF FISSION PRODUCTS AND THE INFINITE MEDIUM ASSUMED TO BE AN INFINITE SALT DEPOSIT	19
IV. THE PROBLEM OF AN INFINITE MEDIUM IN CONTACT AT $r = a$ WITH AN INFINITELY LONG CYLINDER FILLED WITH A WELL-STIRRED FLUID IN WHICH HEAT IS GENERATED AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME	30
V. CONCLUSIONS	36
Appendix	
I. THE LAPLACE TRANSFORMATION	38
II. SIMPSON'S RULE	41
III. FORTRAN AND THE IBM-704 PROGRAM	42
IV. NUMERICAL DATA IN TABULAR FORM	56
V. THE STEADY STATE	63

Appendix Page
VI. POINT-SOURCE SOLUTIONS 65

BIBLIOGRAPHY 68

LIST OF ILLUSTRATIONS

Figure		Page
1.	Contour of Integration for the Inversion Integral	13
2.	Temperature History of a Sphere of Radius Five Feet Filled With One-Year-Old Fission Products	24
3.	Temperature History of a Sphere of Radius Ten Feet Filled With One-Year-Old Fission Products	25
4.	Temperature History of a Sphere of Radius Five Feet Filled With Six-Year-Old Fission Products	26
5.	Temperature History of a Sphere of Radius Ten Feet Filled With Six-Year-Old Fission Products	27
6.	Temperature History of a Sphere of Radius Five Feet Filled With Eleven-Year-Old Fission Products	28
7.	Temperature History of a Sphere of Radius Ten Feet Filled With Eleven-Year-Old Fission Products	29

INTRODUCTION

The predicted nuclear-power economy in the United States of America for the years 1960 to 2000 presents the disturbing problem of disposing of radioactive reactor wastes. Culler and McLain[1] have presented curves predicting an accumulated total activity of about 350,000,000,000 curies by the year 2000. This figure assumes a heat generating capacity of 700,000 megawatts and a burnup of 4,000 megawatt days per ton. If nuclear fuels are to be a feasible replacement for fossil fuels, a safe, reliable method of disposal must be found for nuclear wastes.

Perhaps the most suitable environment for the disposal of radioactive wastes is underground deposits of rock salt.[2] Rock salt is impervious to water, and its plasticity makes it structurally safe under ordinary conditions. The heat generated by fission products stored in a rock salt cavity conceivably could cause a rise in temperature great enough to make salt flow plastically or even to melt. It is of interest, therefore, to investigate the temperature history of a salt cavity filled with radioactive wastes.

The problem studied in this thesis is that of an infinite medium

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[1] F. L. Culler and S. McLain (ed.), Status Report on the Disposal of Radioactive Wastes, A Report by the Committee on Disposal and Dispersal of Radioactive Wastes for the National Academy of Sciences National Research Council (Oak Ridge: Oak Ridge National Laboratory, 1957), pp. 53, 58.

[2] F. L. Parker, L. Hemphill, and J. Crowell, Status Report on Waste Disposal in Natural Salt Formations, ORNL-2560 (Oak Ridge: Oak Ridge National Laboratory, 1958).

heated by a cavity filled with exponentially decaying sources. Carslaw and Jaeger,[1] Goldenberg,[2] and Goldenberg and Tranter[3] have considered the problem of an infinite medium heated by a spherical, constant heat source. Carslaw and Jaeger,[4] Jaeger,[5] Whitehead,[6] and Blackwell[7] have considered the problem of an infinite medium heated by an infinitely long, constant, cylindrical source. In the literature there appear to be no applications of exponentially decaying heat sources surrounded by an infinite medium.

Certain simplifying assumptions are made in the problem under consideration. The salt deposit surrounding the disposal volume is considered infinite in extent, and the usual rectangular-parallelepiped room and tunnel of salt mines are approximated by a sphere and an infinitely long cylinder. Further simplifications are the assumptions

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[1] H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, (London: Oxford University Press, 1950), pp. 287-288.

[2] H. Goldenberg, "A Problem in Radial Heat Flow," British Journal of Applied Physics, II (August, 1951), pp. 233-37.

[3] H. Goldenberg and C. J. Tranter, "Heat Flow in an Infinite Medium Heated by a Sphere," British Journal of Applied Physics, III (September, 1952), pp. 296-98.

[4] Carslaw and Jaeger, ibid. pp. 285-286.

[5] J. C. Jaeger, "Conduction of Heat in an Infinite Region Bounded Internally by a Circular Cylinder of a Perfect Conductor," Australian Journal of Physics, IX (June, 1956), pp. 167-79.

[6] S. Whitehead, "An Approximate Method for Calculating Heat Flow in an Infinite Medium by a Cylinder," The Proceedings of the Physical Society, LVI (November, 1944), pp. 357-66.

[7] J. H. Blackwell, "A Transient-Flow Method for Determination of Thermal Constants of Insulating Materials in Bulk," Journal of Applied Physics, XXV (February, 1954), pp. 137-44.

that the thermal conductivities of wastes and salt do not change with temperature, that heat is lost only by conduction, that there are no phase changes in wastes or salt, and that all beta and gamma radiation is absorbed in the wastes. The many physical uncertainties in a qualitative description of an underground, geologic system make these assumptions justifiable.

I. THE GENERAL BOUNDARY-VALUE PROBLEM OF AN INFINITE MEDIUM HEATED BY A SPHERE AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME

A homogeneous sphere of radius a is surrounded by an infinite, homogeneous medium. Heat is generated in the sphere at the rate Q(t) per unit time per unit volume. (The general term Q(t) will later be replaced by a sum of exponential terms.) The boundary-value problem can be stated as follows. The heat conduction equations in spherical coordinates are

$$\frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} = \frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r}\frac{\partial T_1}{\partial r} + \frac{Q(t)}{K_1}, \quad 0 \leq r < a, \quad t > 0, \tag{1}$$

and

$$\frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} = \frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r}\frac{\partial T_2}{\partial r}, \quad r > a, \quad t > 0, \tag{2}$$

where subscripts 1 and 2 refer to the heated sphere and to the infinite medium respectively, T is temperature, t is time, r is radial distance from the center of the sphere, K is thermal conductivity, and α is thermal diffusivity.

Initially the system is at zero temperature, i.e.

$$T_1(r,0) = T_2(r,0) = 0 \ . \tag{3}$$

The boundary conditions are that the temperature and heat flux are continuous at the interface between media 1 and 2 and that the temperatures remain finite at the center of the sphere and at infinity.

Stated mathematically,

$$T_1(a,t) = T_2(a,t) , \tag{4}$$

$$K_1 \frac{\partial T_1}{\partial r}(a,t) = K_2 \frac{\partial T_2}{\partial r}(a,t) , \tag{5}$$

and

$$T_1 \text{ finite as } r \to 0, \quad T_2 \text{ finite as } r \to \infty . \tag{6}$$

A convenient method for solving the differential equations of mathematical physics is the operational calculus of Heaviside, the modern form of which is the Laplace Transformation.[1] The Laplace Transform of a function f(t) is defined by the integration

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt .$$

If the Laplace Transform $\bar{f}(s)$ is known, the function f(t) can be found by consulting a table of Laplace Transforms or by performing the integration

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \bar{f}(z)\, dz .$$

A short discussion of the Laplace Transformation is given in Appendix I.

Application of the Laplace Transformation to equations (1), (2), (4), (5), and (6) yields the following subsidiary set of equations.

$$q_1^2 \bar{T}_1 = \frac{\partial^2 \bar{T}_1}{\partial r^2} + \frac{2}{r} \frac{\partial \bar{T}_1}{\partial r} + \frac{\bar{Q}(s)}{K_1} , \quad 0 \leq r < a , \tag{7}$$

---

[1] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics (London: Oxford University Press, 1948), pp. viii-xvi.

$$q_2^2\,\overline{T} = \frac{\partial^2\overline{T}_2}{\partial r^2} + \frac{2}{r}\frac{\partial\overline{T}_2}{\partial r}\ , \qquad r > a\ , \tag{8}$$

$$\overline{T}_1(a) = \overline{T}_2(a)\ , \tag{9}$$

$$K_1\frac{\partial\overline{T}_1}{\partial r}(a) = K\frac{\partial\overline{T}_2}{\partial r}(a)\ , \tag{10}$$

and

$$\overline{T}_1 \text{ finite as } r \to 0, \text{ and } \overline{T}_2 \text{ finite as } r \to \infty, \tag{11}$$

where $\overline{T}$ and $\overline{Q}$ are the Laplace Transforms of $T(r,t)$ and $Q(t)$, s is a real, positive number used in the application of the Laplace Transform, and q is written for $\sqrt{s/\alpha}$.

Equations (7) and (8) can profitably be rewritten

$$\frac{\partial^2(r\overline{T}_1)}{\partial r^2} - q_1^2\,r\,\overline{T}_1 = \frac{-r\overline{Q}(s)}{K_1}\ , \qquad 0 \le r < a\ , \tag{12}$$

$$\frac{\partial^2(r\overline{T}_2)}{\partial r^2} - q_2^2\,r\,\overline{T}_2 = 0\ , \qquad r > a\ . \tag{13}$$

The complementary function of equation (12) is

$$\overline{T}_1 = \frac{C'}{r}\sinh(q_1 r) + \frac{C''}{r}\cosh(q_1 r)\ ,$$

where C' and C" are arbitrary constants. The particular integral of (12) is

$$\overline{T}_1 = \overline{Q}(s)/q_1^2K_1$$

The general solution of (12) is the sum of the complementary function and the particular integral, i.e.

$$\overline{T}_1 = \frac{C'}{r}\sinh(q_1 r) + \frac{C''}{r}\cosh(q_1 r) + \overline{Q}(s)/q_1^2K_1\ ,$$

but according to condition (11), $\overline{T}_1$ must remain finite as $r \to 0$, so C" must be zero, and

$$\overline{T}_1 = \frac{C'}{r} \sinh(q_1 r) + \overline{Q}(s)/q_1^2 K_1 \quad . \tag{14}$$

Equation (13) has a general solution of the form

$$\overline{T}_2 = \frac{C'''}{r} e^{+q_2 r} + \frac{C''''}{r} e^{-q_2 r} \quad ,$$

where C"' and C"" are arbitrary constants. Condition (11) states that $\overline{T}_2$ must remain finite as $r \to \infty$, so C"' must be zero, and

$$\overline{T}_2 = \frac{C''''}{r} e^{-q_2 r} \quad . \tag{15}$$

The constants C' and C"" can be found by a simultaneous solution of boundary conditions (9) and (10).

$$C' = \frac{\overline{Q}(s)}{q_1^2 K_1} F(s,a) \quad ,$$

and

$$C'''' = \frac{e^{q_2 a} \overline{Q}(s) a}{q_1^2 K_1} G(s,a) \quad ,$$

where

$$F(s,a) = \frac{K_2(1 + q_2 a)}{K_1\left[\sinh(q_1 a) - q_1 a \cosh(q_1 a)\right] - (1 + q_2 a) K_2 \sinh(q_1 a)} \quad ,$$

and

$$G(s,a) = \frac{K_1\left[\sinh(q_1 a) - q_1 a \cosh(q_1 a)\right]}{K_1\left[\sinh(q_1 a) - q_1 a \cosh(q_1 a)\right] - (1 + q_2 a) K_2 \sinh(q_1 a)} \quad .$$

These values of C' and C"" are then substituted into equations (14) and (15) to give the following values of $\overline{T}_1$ and $\overline{T}_2$.

$$\overline{T}_1 = \frac{\overline{Q}(s)}{q_1^2 K_1} \left[1 + \frac{1}{r} \sinh(q_1 r)\, F(s,a) \right] , \tag{16}$$

and

$$\overline{T}_2 = \frac{e^{-q_2(r-a)}\, \overline{Q}(s) a}{q_1^2 r}\, G(s,a) . \tag{17}$$

The solutions for T_1 and T_2 should be obtainable from the Inversion Theorem for the Laplace Transform, but the complex integrands of the inversion integrals appear rather involved, and the goals of this paper will be sufficiently fulfilled by considering two special cases of the general problem of a heat source embedded in an infinite medium.

II. TWO SPECIAL CASES OF THE GENERAL PROBLEM OF AN INFINITE MEDIUM HEATED BY A SPHERE AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME

The Thermal Properties of the Sphere Considered to be the Same as Those of the Infinite Medium[1]

If the thermal conductivity and diffusivity of the sphere are supposed equal to the thermal conductivity and diffusivity of the infinite medium, equations (16) and (17) reduce to

$$\bar{T}_1 = \frac{\bar{Q}(s)}{sC_2}\left[1 - \frac{e^{-q_2 a}\sinh(q_2 r)}{q_2 r} - \frac{ae^{-q_2 a}\sinh(q_2 r)}{r}\right], \quad (18)$$

and

$$\bar{T}_2 = \frac{\bar{Q}(s)e^{-q_2 r}a}{sC_2 r}\left[\cosh(q_2 a) - \frac{1}{q_2 a}\sinh(q_2 a)\right], \quad (19)$$

where C_2 is the specific heat per unit volume of the infinite medium. ($C = K/\alpha$.)

Now if two new functions are defined

$$\bar{T}_1^*(s) = \bar{T}_1(s)/\bar{Q}(s), \quad \text{and} \quad \bar{T}_2^*(s) = \bar{T}_2(s)/\bar{Q}(s),$$

and if Q(t) is defined as some function whose Laplace Transform is tabulated, then $T_1(t)$ and $T_2(t)$ can be obtained from $T_1^*(t)$, $T_2^*(t)$, and Q(t) by means of Duhamel's Theorem (Appendix I), i.e.

---

[1]The numerical results of Chapter III are obtained using the physical constants of the infinite medium, i.e. the physical constants of the infinite salt deposit.

$$T_1 = \int_0^t T_1^*(\tau)Q(t-\tau)\,d\tau = \int_0^t T_1^*(t-\tau)Q(\tau)\,d\tau \quad ,$$

and

$$T_2 = \int_0^t T_2^*(\tau)Q(t-\tau)\,d\tau = \int_0^t T_2^*(t-\tau)Q(\tau)\,d\tau \quad .$$

Equations (18) and (19) can be rewritten

$$\overline{T}_1^* = \frac{\overline{Q}(s)}{sC_2}\left[1 - \frac{e^{-q_2(a-r)} - e^{-q_2(a+r)}}{2q_2 r} - a\,\frac{e^{-q_2(a-r)} - e^{-q_2(a+r)}}{2r}\right] ,$$

and

$$\overline{T}_2^* = \frac{\overline{Q}(s)a}{sC_2 r}\left[\frac{e^{-q_2(r-a)} + e^{-q_2(r+a)}}{2} - \frac{e^{-q_2(r-a)} - e^{-q_2(r+a)}}{2q_2 a}\right] .$$

The inverse transforms of $\overline{T}_1^*$ and $\overline{T}_2^*$ are tabulated in Appendix I, and it is seen that

$$T_1^* = \frac{1}{2C_2}\left\{\operatorname{erf}\left[\frac{a-r}{2\sqrt{\alpha_2 t}}\right] + \operatorname{erf}\left[\frac{a+r}{2\sqrt{\alpha_2 t}}\right] - \sqrt{\frac{4\alpha_2 t}{r^2\pi}} \times \left[e^{-(a-r)^2/4\alpha_2 t} - e^{-(a+r)^2/4\alpha_2 t}\right]\right\} ,$$

and

$$T_2^* = \frac{1}{2C_2}\left\{\operatorname{erf}\left[\frac{r+a}{2\sqrt{\alpha_2 t}}\right] - \operatorname{erf}\left[\frac{r-a}{2\sqrt{\alpha_2 t}}\right] - \sqrt{\frac{4\alpha_2 t}{r^2\pi}} \times \left[e^{-(r-a)^2/4\alpha_2 t} - e^{-(r+a)^2/4\alpha_2 t}\right]\right\} ,$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy \quad .$$

Application of Duhamel's Theorem yields

$$T_1 = \int_0^t \frac{Q(t-\tau)}{2C_2} \left\{ \mathrm{erf}\left[\frac{a-r}{2\sqrt{\alpha_2\tau}}\right] + \mathrm{erf}\left[\frac{a+r}{2\sqrt{\alpha_2\tau}}\right] - \sqrt{\frac{4\alpha_2\tau}{r^2\pi}} \times \right.$$

$$\left. \times \left[e^{-(a-r)^2/4\alpha_2\tau} - e^{-(a+r)^2/4\alpha_2\tau} \right] \right\} d\tau \ , \quad (20)$$

and

$$T_2 = \int_0^t \frac{Q(t-\tau)}{2C_2} \left\{ \mathrm{erf}\left[\frac{r+a}{2\sqrt{\alpha_2\tau}}\right] - \mathrm{erf}\left[\frac{r-a}{2\sqrt{\alpha_2\tau}}\right] - \sqrt{\frac{4\alpha_2\tau}{r^2\pi}} \times \right.$$

$$\left. \times \left[e^{-(r-a)^2/4\alpha_2\tau} - e^{-(r+a)^2/4\alpha_2\tau} \right] \right\} d\tau \quad . \quad (21)$$

Of special interest are the temperatures at $r = 0$ and at $r = a$. At these space points, the integrals in (20) and (21) become

$$T_{1_{r=0}} = \int_0^t \frac{Q(t-\tau)}{C_2} \left\{ \mathrm{erf}\left[\frac{a}{2\sqrt{\alpha_2\tau}}\right] - \frac{a}{\sqrt{\alpha_2\tau\,\pi}} e^{-a^2/4\alpha_2\tau} \right\} d\tau \quad , \quad (22)$$

and at $r = a$,

$$T_1 = T_2 = \int_0^t \frac{Q(t-\tau)}{2C_2} \left\{ \mathrm{erf}\left[\frac{a}{\sqrt{\alpha_2\tau}}\right] - \sqrt{\frac{4\alpha_2\tau}{\pi a^2}} \left[1 - e^{-a^2/\alpha_2\tau} \right] \right\} d\tau \ . \quad (23)$$

Numerical results for equations (20), (21), (22), and (23) are given in Chapter III and Appendix IV; Q(t) in these results is the heat produced by fission products.

The Contents of the Sphere Considered to be Well-Stirred, i.e. to Have Infinite Conductivity

If equations (16) and (17) are specialized once more, this time allowing α_1 and K_1 to become infinite, then

$$\lim_{K_1 \to \infty} \overline{T}_1 = \frac{a^2 \overline{Q}(s)}{C_2} \left[\frac{1}{a^2 \beta s + 3a\sqrt{\alpha_2 s} + 3\alpha_2} \right] , \qquad (24)$$

where $\beta = C_1/C_2$.

The inverse transform of (24) has not been tabulated and must be evaluated by applying the Inversion Theorem for the Laplace Transform, i.e.

$$T_1 = \frac{a^2}{2\pi i C_2} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\overline{Q}(z)\, e^{zt}\, dz}{a^2 \beta z + 3a\sqrt{\alpha_2 z} + 3\alpha_2} . \qquad (25)$$

According to Cauchy's Theorem, the path of integration between the terminals $\gamma - i\infty$ and $\gamma + i\infty$ is a matter of choice, so the usual procedure is to pick a semicircular path with radius large enough to include all the poles of the integrand.[1] In the integrand being considered, there is a branch point at z = 0 and a branch cut along the

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[1] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics (London: Oxford University Press, 1948), pp. 71-86.

negative real axis, so the line integral (25) can be replaced by an integral along the path ACDD'C'A' (Fig. 1) plus the residues of any poles of the integrand.

If a function $\overline{T}_1^*(s)$ is again defined

$$\overline{T}_1^*(s) = \overline{T}_1(s)/\,\overline{Q}(s) \quad ,$$

then

$$T_1^*(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{T}_1^*(z)\, e^{zt}\, dz$$

$$= I_{A'C'} + I_{C'D'} + I_{D'D} +$$

$$+ I_{DC} + I_{CA} +$$

$$+\sum \text{Residues},$$

where I is written for the integral

$$\frac{1}{2\pi i}\int \overline{T}_1^*\, e^{zt}\, dz.$$

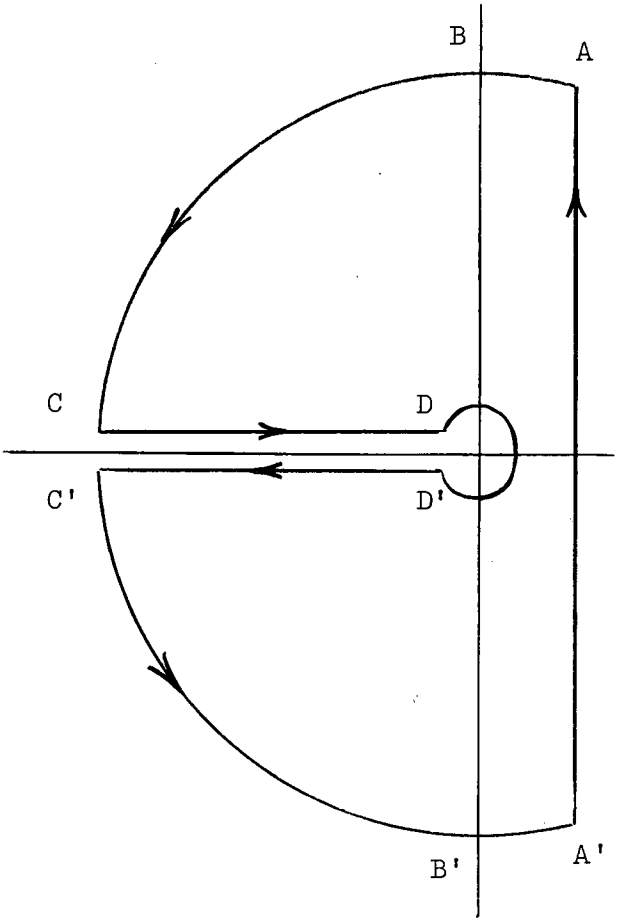

Fig. 1.—Contour of Integration for the Inversion Integral.

According to a lemma of Carslaw and Jaeger,[1] if

$$\left|\overline{T}_1^*(z)\right| < CR^{-k} \quad ,$$

when $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, $R > R_0$, where R_0, C, k are constants and $k > 0$, then $\int \overline{T}_1^*(z)\, e^{zt}\, dz$ taken over the arcs ABC and A'B'C' of the

[1] Ibid., 76.

circle of radius R (Fig. 1) tends to zero as $R \to \infty$, provided $t > 0$.

$\bar{T}_1$ does satisfy[1] this lemma, therefore

$$\lim_{R \to \infty} \int_{AC} \bar{T}_1^* e^{zt}\, dz = \lim_{R \to \infty} \int_{A'C'} \bar{T}_1^* e^{zt}\, dz = 0 .$$

The integral around the small circle at the origin is evaluated by letting $z = \epsilon e^{i\theta}$ and requiring $\epsilon \to 0$. This integral vanishes, i.e.

$$\int_{DD'} \bar{T}_1^* e^{zt}\, dz = \lim_{\epsilon \to 0} \frac{a^2}{2\pi i C_2} \int_{-\pi}^{\pi} \frac{e^{\epsilon(\cos\theta + i \sin\theta)t}\, ie^{i\theta}\, d\theta}{a^2\beta\, e^{i\theta} + 3a\sqrt{\alpha_2}\,\epsilon^{1/2} e^{i\theta/2} + 3\alpha_2} = 0 .$$

Thus $T_1^*(t)$ is equal to the contributions from the integrals along lines DC and C'D' plus the sum of the residues.

Along DC, let $z = (v^2\alpha_2/a^2)e^{i\pi} = -v^2\alpha_2/a^2$,

then $dz = (2v\alpha_2/a^2)e^{i\pi}\, dv = -(2v\alpha_2/a^2)\, dv$.

Therefore

$$I_{DC} = \frac{a^2}{2\pi i C_2} \int_0^{\infty} \frac{e^{-(v^2\alpha_2/a^2)t}(-2v\alpha_2/a^2)\, dv}{a^2\beta(-v^2\alpha_2/a^2) + 3a\sqrt{\alpha_2}(iv\sqrt{\alpha_2}/a) + 3\alpha_2} ,$$

which on simplification becomes

---

[1]The lemma is satisfied, for $\left|\bar{T}_1^*\right| = \left|a^2\beta R e^{i\theta} + 3a\sqrt{\alpha_2} R e^{i\theta/2} + 3\alpha_2\right|^{-1}$
$\leq \left|\mathrm{Im}(a^2\beta R e^{i\theta} + 3a\sqrt{\alpha_2 R}\, e^{i\theta/2} + 3\alpha_2)\right|^{-1} = \left|a^2\beta R \sin\theta + 3a\sqrt{\alpha_2 R}\, \sin(\theta/2)\right|^{-1}$
$< \left|a^2\beta R - 3a\sqrt{\alpha_2 R}\right|^{-1} < \left|a^2\beta - 3a\sqrt{\alpha_2}\right|^{-1} R^{-1} = CR^{-1}$.

$$I_{DC} = \frac{1}{\pi i C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\, v\, dv}{\beta v^2 - 3iv - 3} .$$

Similarly on C'D'

$$I_{C'D'} = \frac{1}{\pi i C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\, v\, dv}{-\beta v^2 - 3iv + 3} ,$$

and on combining I_{DC} and $I_{C'D'}$, it is seen that

$$T_1^*(t) = \frac{2}{3\pi C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\, v^2\, dv}{(\beta v^2/3 - 1)^2 + v^2} + \sum \text{Residues} .$$

The poles of $\bar{T}_1^*(z)$ are found by setting

$$a^2\beta z + 3a\sqrt{\alpha_2 z} + 3\alpha_2 = 0 . \tag{26}$$

The roots of this equation are[1]

$$z = \frac{3\alpha_2}{a^2\beta^2}\left[(3/2 - \beta) \pm \sqrt{3(3/4 - \beta)}\right] .$$

For physically possible values of β, i.e. about 1.5 to 2.5, these two roots will always be complex and can be written

$$z_1 = x + iy \text{ , and } z_2 = x - iy.$$

The sum of the residues at these two poles is

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[1]It was shown by Mr. Verner Anderson of Central Data Processing of Union Carbide Nuclear Company, Oak Ridge Gaseous Diffusion Plant, that neither root of (26) is spurious when the roots are complex conjugates. (Private communication)

$$\sum \text{Residues} = \left[\frac{a^2 e^{zt}}{C_2 \frac{d}{dz}(a^2\beta z + 3a\sqrt{\alpha_2 z} + 3\alpha_2)} \right]_{z = z_1,\, z_2}$$

$$= \frac{a}{C_2} \left[\frac{e^{z_1 t}}{a\beta + \frac{3}{2}\sqrt{\alpha_2/z_1}} + \frac{e^{z_2 t}}{a\beta + \frac{3}{2}\sqrt{\alpha_2/z_2}} \right] .$$

If z_1 and z_2 are written in polar form, i.e.

$$z_1 = R(\cos\theta + i \sin\theta) \quad , \text{ and } \quad z_2 = R(\cos\theta - i \sin\theta) \quad ,$$

where

$$\theta = \tan^{-1}(y/x) \quad , \text{ and } \quad R = \sqrt{x^2 + y^2} \quad , \text{ then}$$

$$\sum \text{Residues} = \frac{a}{C_2} \left[\frac{e^{xt}\left[2a\beta R \cos(yt) + 3\sqrt{\alpha_2 R} \cos(yt + \theta/2)\right]}{a^2\beta^2 + 9\alpha_2/4 + 3a\beta\sqrt{\alpha_2 R} \cos(\theta/2)} \right] .$$

The solution for $\overline{T}_1^*(t)$ is then

$$T_1^*(t) = \frac{2}{3\pi C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\, v^2\, dv}{(\beta v^2/3 - 1)^2 + v^2} +$$

$$+ \frac{a}{C_2} \left[\frac{e^{xt}\left[2a\beta R \cos(yt) + 3\sqrt{\alpha_2 R} \cos(yt + \theta/2)\right]}{a^2\beta^2 R + 9\alpha_2/4 + 3a\beta\sqrt{\alpha_2 R} \cos(\theta/2)} \right] . \qquad (27)$$

For the range of values of β, α_2, and a with which this paper deals; e.g. $\beta = 2$, $\alpha_2 = .07$ Ft^2/Hr , and $a \geq 5$ Ft; it will be noted that the real part of the poles z_1 and z_2 will be a very small negative number. For this reason the second term on the right in equation (27) will have its largest value when t is zero, i.e.

$$\sum \text{Residues} \leq \frac{a}{C_2}\left[\frac{2a\beta R + 3\sqrt{\alpha_2 R}\cos(\theta/2)}{a^2\beta^2 R + 9\alpha_2/4 + 3a\beta\sqrt{\alpha_2 R}\cos(\theta/2)}\right] .$$

This expression can be simplified by noting that $R = 3\alpha_2/a^2\beta$, thus

$$\sum \text{Residues} \leq \frac{1}{C_2}\left[\frac{2 + \sqrt{3/\beta}\cos(\theta/2)}{\beta + 3/4 + \sqrt{3\beta}\cos(\theta/2)}\right] .$$

If $\beta = 2$, and $C_2 = 27$ BTU/(FT^3-$^\circ$F), then

$$\theta = \tan^{-1}(y/x) = \tan^{-1}\left[\frac{3(\beta - 3/4)}{3/2 - \beta}\right] \cong 104^\circ ,$$

and

$$\sum \text{Residues} \leq \frac{1}{27}\left[\frac{2 + \sqrt{1.5}\cos(52^\circ)}{2 + .75 + \sqrt{6}\cos(52^\circ)}\right] \cong 0.02 .$$

Numerical calculations indicate that the sum of the residues is negligible compared with the first term in equation (27), so

$$T_1^*(t) \cong \frac{2}{3\pi C_2}\int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\, v^2\, dv}{(\beta v^2/3 - 1)^2 + v^2} .$$

In the case of the sphere filled with fission products, the heat generation is due to a sum of exponentially decaying elements, i.e.

$$Q(t) = \sum_{i=1}^{N} A_i e^{-\lambda_i t} .$$

If it is recalled that $\overline{T}_1(s) = \overline{T}_1^*(s)\overline{Q}(s)$, then by application of Duhamel's Theorem,

$$T_1(t) \cong \frac{2}{3\pi C_2} \sum_{i=1}^{N} A_i e^{-\lambda_i t} \int_0^{\infty} \frac{\left[1 - e^{(\lambda_i - v^2\alpha_2/a^2)t}\right] v^2 \, dv}{\left[(\beta v^2/3 - 1)^2 + v^2\right]\left[v^2\alpha_2/a^2 - \lambda_i\right]} \quad .(28)$$

If K_1 is allowed to approach infinity in the expression for the Laplace Transform of temperature T_2 , equation (17), then

$$\lim_{K_1 \to \infty} \overline{T}_2(s) = \frac{\overline{Q}(s)a^3}{C_2 r} \left[\frac{e^{-q_2(r-a)}}{a^2\beta s + 3a\sqrt{\alpha_2 s} + 3\alpha_2} \right] .$$

If the same procedure is used in finding T_2 as was used in finding T_1, it is seen that

$$T_2 \cong \frac{2a}{9\pi C_2 r} \sum_{i=1}^{N} A_i e^{-\lambda_i t} \int_0^{\infty} \left[\frac{1 - e^{(\lambda_i - v^2\alpha_2/a^2)t}}{(v^2\alpha_2/a^2 - \lambda_i)} \right] \times$$

$$\times \left[\frac{3v^2 \cos\left[\frac{v(r-a)}{a}\right] + (3v - \beta v^3) \sin\left[\frac{v(r-a)}{a}\right]}{(\beta v^2/3 - 1)^2 + v^2} \right] dv \quad . \qquad (29)$$

Equations (28) and (29) can be integrated numerically by approximating infinity with some large number or by transforming variables to obtain finite limits of integration.

III. SOME NUMERICAL RESULTS ON THE SPHERICAL PROBLEM WITH Q(t) SET EQUAL TO A SUM OF FISSION PRODUCTS AND THE INFINITE MEDIUM ASSUMED TO BE AN INFINITE SALT DEPOSIT

In the problem of the spherical cavity filled with radioactive wastes, the heat generation term Q(t) becomes $\sum_{i=1}^{N} A_i e^{-\lambda_i t}$, where N is the number of fission products, A_i is the heat generation per unit time per unit volume for the ith fission product and λ_i is the disintegration constant of the ith fission product. If $\sum_{i=1}^{N} A_i e^{-\lambda_i t}$ is substituted for Q(t) in equations (20) and (21), the expressions for temperature inside and outside the cavity become

$$T_1(t) = \int_0^t \sum_{i=1}^{N} A_i \frac{e^{-\lambda_i (t-\tau)}}{2C_2} \left\{ \operatorname{erf}\left[\frac{a-r}{2\sqrt{\alpha_2 \tau}}\right] + \operatorname{erf}\left[\frac{a+r}{2\sqrt{\alpha_2 \tau}}\right] - \sqrt{\frac{4\alpha_2 \tau}{r^2 \pi}} \times \right.$$

$$\left. \times \left[e^{-(a-r)^2/4\alpha_2 \tau} - e^{-(a+r)^2/4\alpha_2 \tau} \right] \right\} d\tau \,, \qquad (30)$$

and

$$T_2(t) = \int_0^t \sum_{i=1}^{N} A_i \frac{e^{-\lambda_i (t-\tau)}}{2C_2} \left\{ \operatorname{erf}\left[\frac{r+a}{2\sqrt{\alpha_2 \tau}}\right] - \operatorname{erf}\left[\frac{r-a}{2\sqrt{\alpha_2 \tau}}\right] - \sqrt{\frac{4\alpha_2 \tau}{r^2 \pi}} \times \right.$$

$$\times \left[e^{-(r+a)^2/4\alpha_2\tau} - e^{-(r-a)^2/4\alpha_2\tau} \right] \Bigg\} d\tau \ . \qquad (31)$$

Simpson's Rule (Appendix II) and the Fortran coding system on the IBM-704 computer (Appendix III) have been used to calculate temperature histories for cavities of five and ten feet radii. Calculations were made for radial distances of $r = 0$, $r = a/2$, $r = a$, $r = 2a$, and $r = 10a$ for each cavity. The physical constants of rock salt used in the calculation are given in Table 1.

TABLE 1

PHYSICAL CONSTANTS OF ROCK SALT

Thermal Conductivity[1] (150 °C)	8.58×10^{-3} cal/(cm-sec-°C)
Specific Gravity[2]	2.160
Specific Heat[3] (Per Unit Mass)	0.22 Btu/(lb-°F)

In terms of Btu, °F, feet, and hours these units become $K_2 = 2.07$ Btu/(Ft-Hr-°F), $\alpha_2 = 0.067$ Ft^2/Hr, and $C_2 = 27$ Btu/(Ft^3-°F).

The summation of integrals (30) or (31) over all the possible fission products in wastes just removed from a reactor would be a time-consuming proposition, even on an electronic computer. In order to avoid such a lengthy calculation, the heat generation from reactor wastes was

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[1] Francis Birch and Harry Clark, "The Thermal Conductivity of Rocks and its Dependence on Temperature and Composition," American Journal of Science, CCXXXVIII (August, 1940), p. 552.

[2] Ibid., p. 543.

[3] International Critical Tables, Vol. V, National Research Council (New York: McGraw-Hill Book Company, Inc., 1929), p. 100.

approximated by six exponential terms. This method of approximation was suggested by the method of analysis of composite decay curves given in standard references on atomic physics.[1] The data for the approximation were obtained from a heat decay curve calculated by J. O. Blomeke of Oak Ridge National Laboratory and given by Culler and McLain in their Status Report on the Disposal of Radioactive Wastes.[2] The decay curve is for a natural or slightly enriched uranium reactor operated to 10,000 Megawatt-days/(ton uranium) at 33 Megawatts/(ton uranium) and assuming 800 gallons of wastes per ton of uranium processed. Blomeke's graph shows not only the composite heat generation, but the individual heat contributions from Sr^{90} and Cs^{137} and the total heat generation from the rare earths. By successively subtracting the Cs^{137}, Sr^{90}, and rare earth curves from the composite curve and then by separating the heat generation due to rare earths and the heat generation due to the remainder of the fission products into two exponential terms each, a six-exponential approximation to the composite curve is obtained.

The six "elements" and their disintegration constants are given below in tabular form. Also given are the rates of heat generation of these elements after being out of the reactor for one year.

Temperature histories are given in figures 2-7, pages 24-29, and in tables in Appendix IV for "Blomeke's reactor wastes" that have been cooled for one, six, and eleven years respectively before being placed in a spherical cavity in a salt deposit. The temperatures given are not

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[1] Irving Kaplan, Nuclear Physics, (Cambridge, Mass.: Addison-Wesley Publishing Company, Inc., 1956), pp. 196-197.

[2] Culler and McLain, Status Report on the Disposal of Radioactive Wastes, ORNL 57-3-114, p. 68.

actual temperatures, but temperature rises. In order to obtain the actual temperatures it is necessary to add the initial temperature of the wastes-salt configuration to the calculated temperatures.

TABLE II

SIX-EXPONENTIAL APPROXIMATION OF FISSION PRODUCT HEAT GENERATION FOR ONE-YEAR-OLD WASTES DILUTED TO 800 GALLONS OF WASTES PER TON OF URANIUM. REACTOR OPERATED TO 10,000 MWD/TON URANIUM.

i	Element	A_i (Btu/Ft^3-Hr)	λ_i (Hr)$^{-1}$
1	$Cs^{137} + Ba^{137}$	5.65	2.98×10^{-6}
2	$Sr^{90} + Y^{90}$	6.35	2.83×10^{-6}
3	"Long-lived rare earths"	23.29	2.73×10^{-5}
4	"Short-lived rare earths"	23.76	2.14×10^{-4}
5	"Long-lived Remainder"	34.11	9.31×10^{-5}
6	"Short-lived Remainder"	20.00	2.93×10^{-4}

Maximum temperature rises at the wastes-salt interface for the cavity of radius five feet are 317, 64, and 42 °F for the one, six, and eleven-year-old wastes respectively. In the sphere of radius ten feet, the maximum temperature rises are 984, 216, and 154 °F respectively. It appears that a "room" in a salt deposit will not melt, even if it contains one-year-old wastes.

At the center of the sphere of radius five feet the maximum temperature rises are 528, 102, and 66 °F for one, six, and eleven-year-old wastes respectively. The corresponding temperature rises in the sphere of radius ten feet are 1734, 369, and 245 °F. On the basis

of these center temperature rises, it appears that one-year-old, and possibly six-year-old, aqueous wastes will boil when placed in underground deposits.

It must be remembered that these calculated temperature rises are intended only to represent orders of magnitude, and the three and four significant figures given in the curves and tables are not expected to have a high degree of accuracy.

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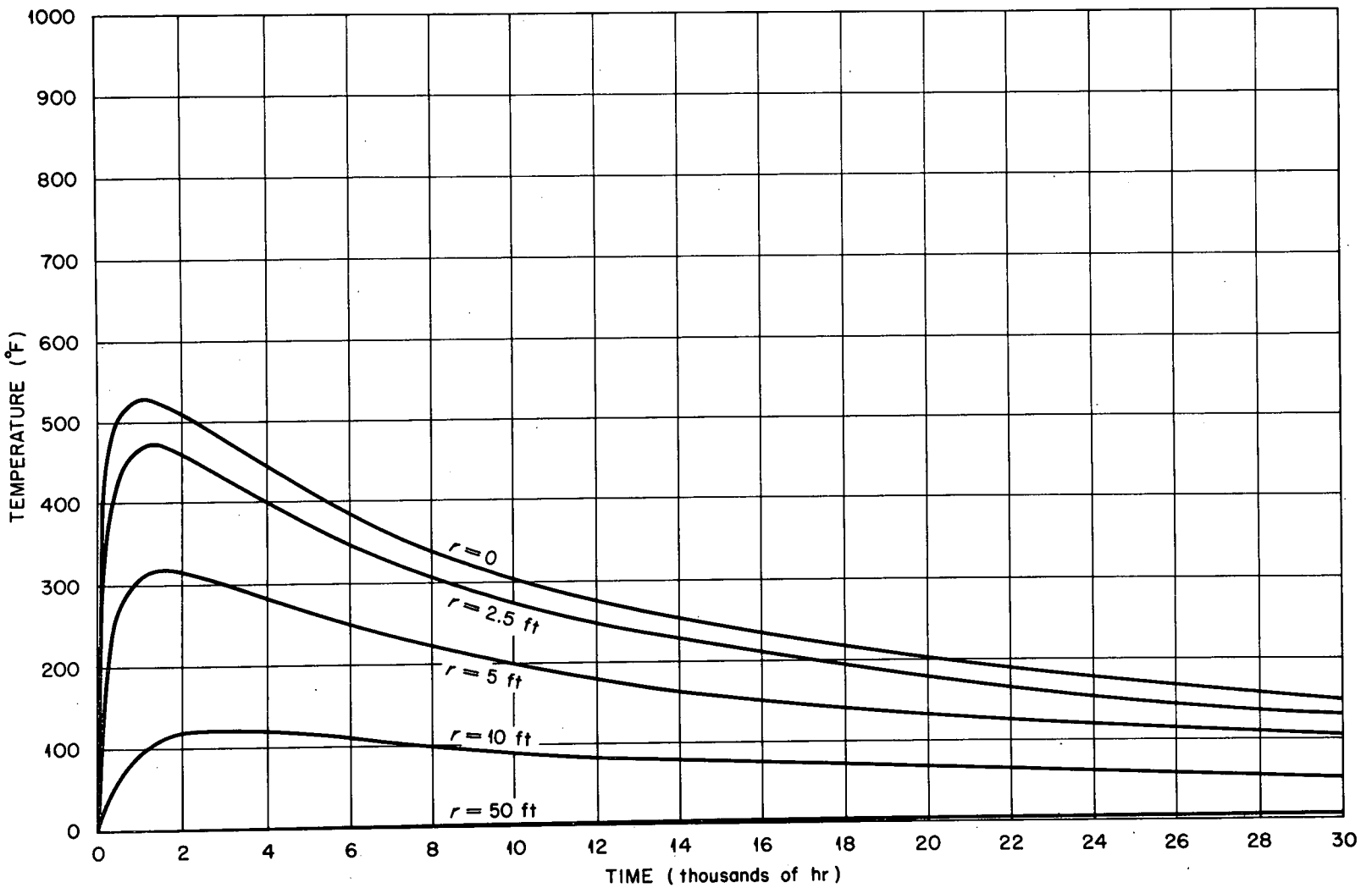

Fig. 2. 1 Year Old Waste – Sphere of 5-ft Radius, Surrounded by an Infinite Bed of Salt.

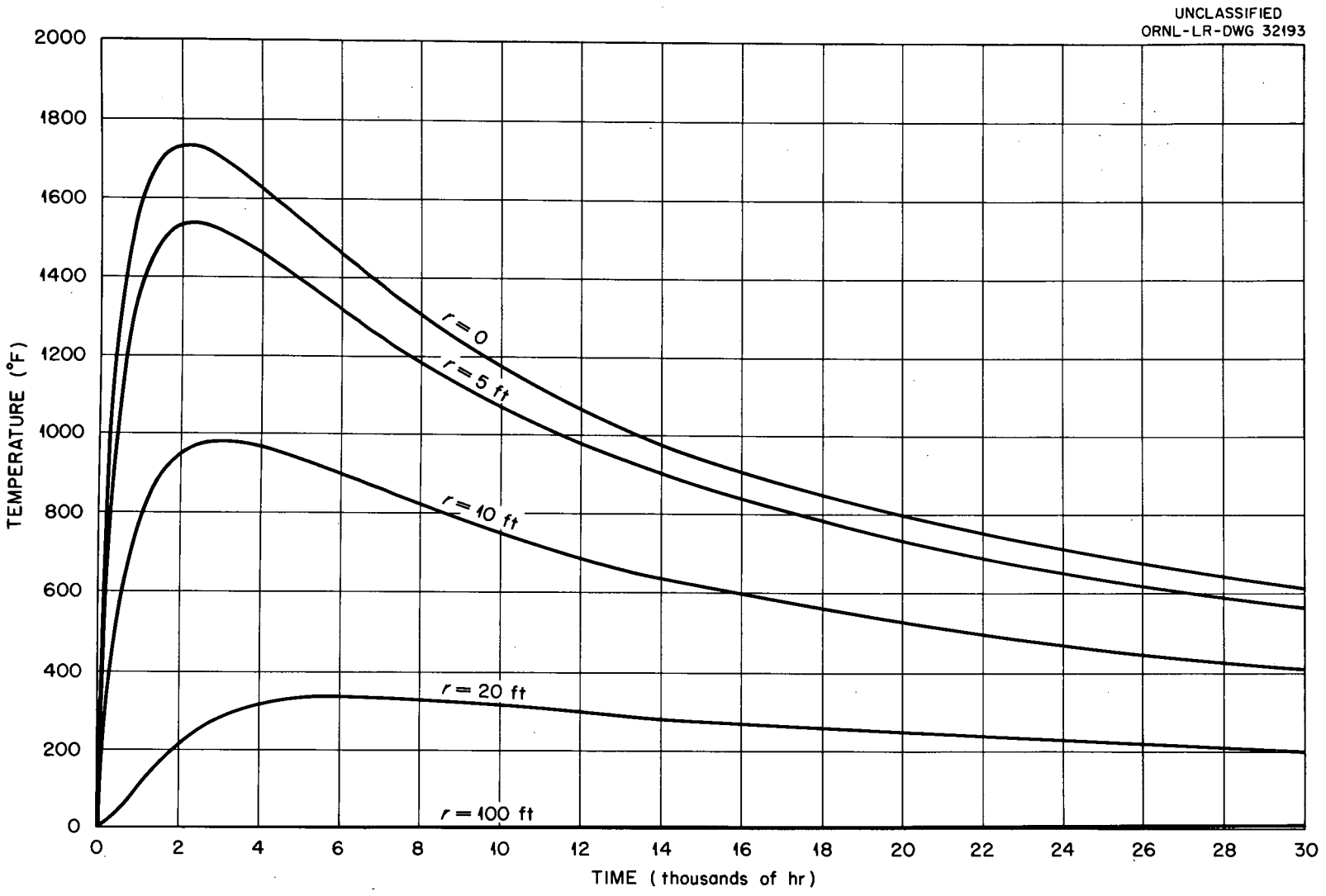

Fig. 3. 1 Year Old Waste—Sphere of 10-ft Radius, Surrounded by an Infinite Bed of Salt.

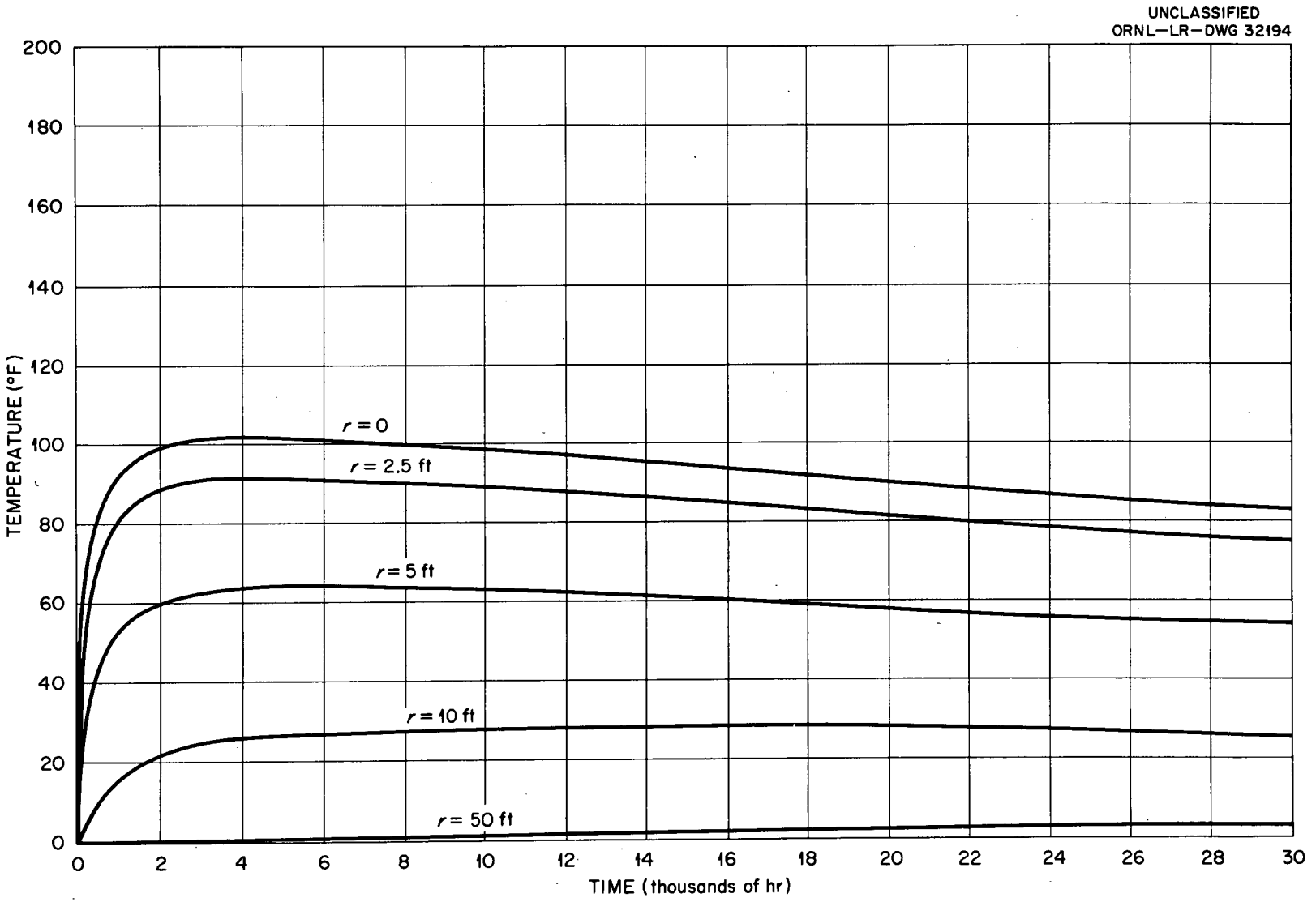

Fig. 4. 6 Year Old Waste — Sphere of 5-ft Radius. Surrounded by an Infinite Bed of Salt.

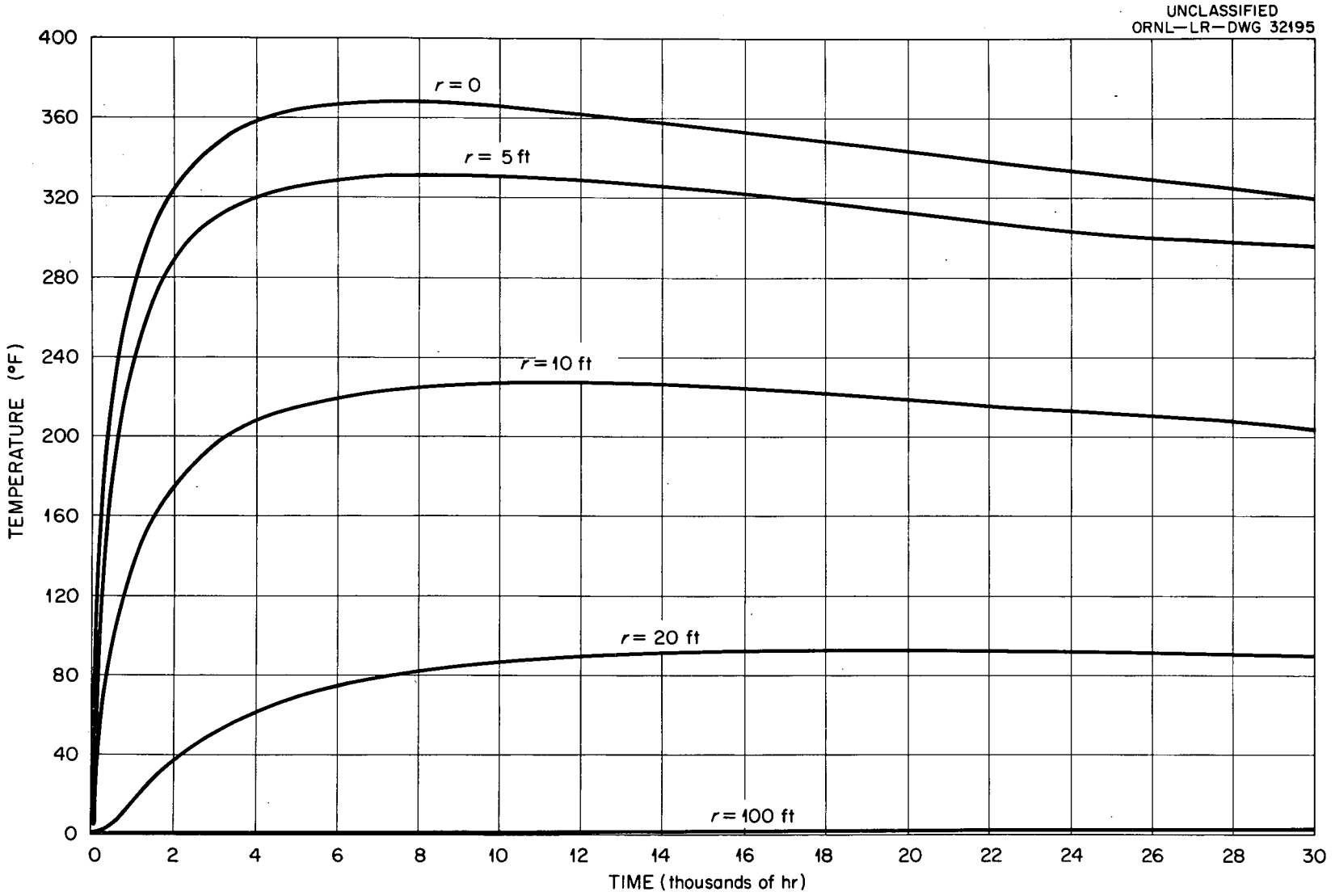

Fig. 5. 6 Year Old Waste–Sphere of 10-ft Radius. Surrounded by an Infinite Bed of Salt.

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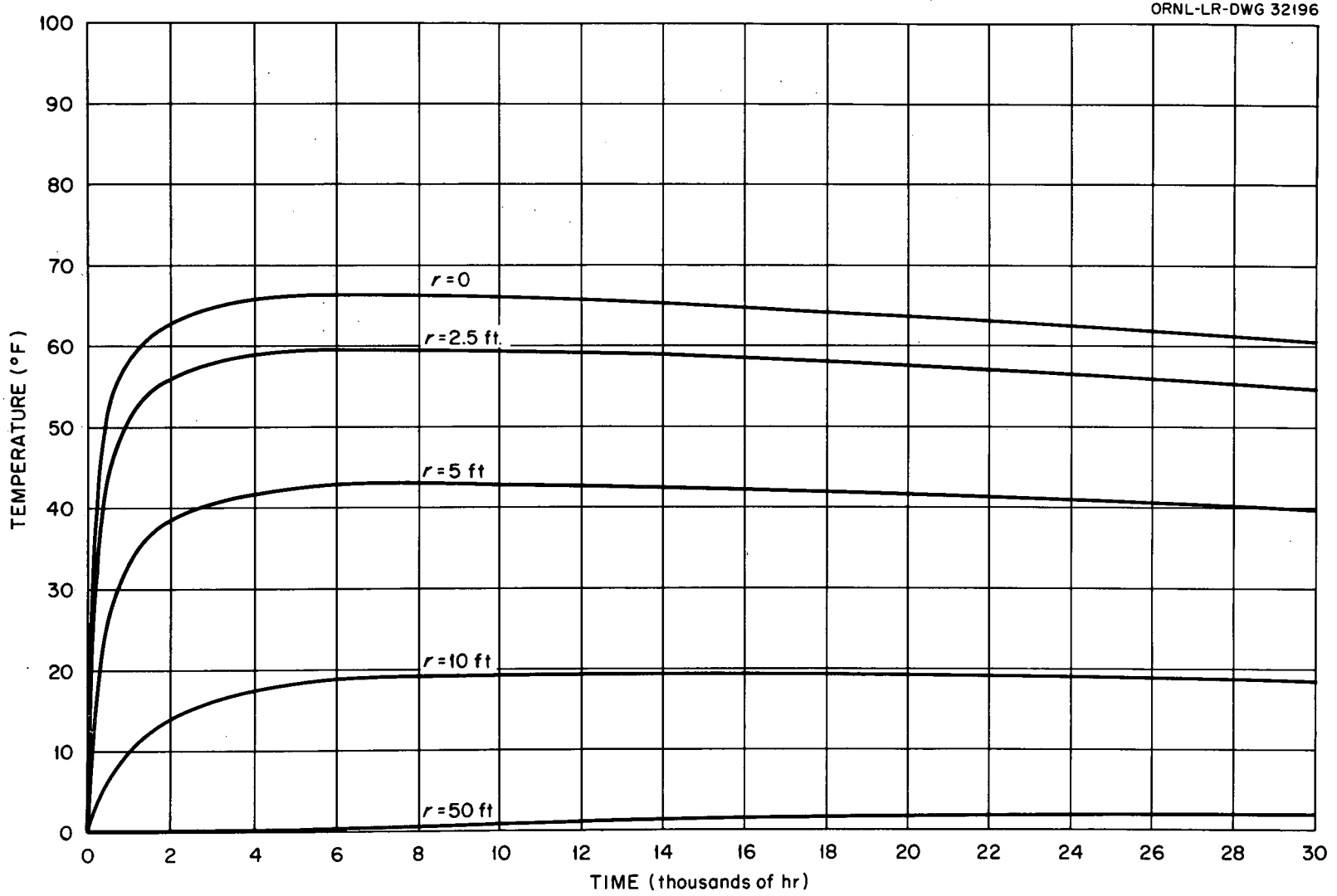

Fig. 6. 11 Year Old Waste-Sphere of 5-ft Radius. Surrounded by an Infinite Bed of Salt

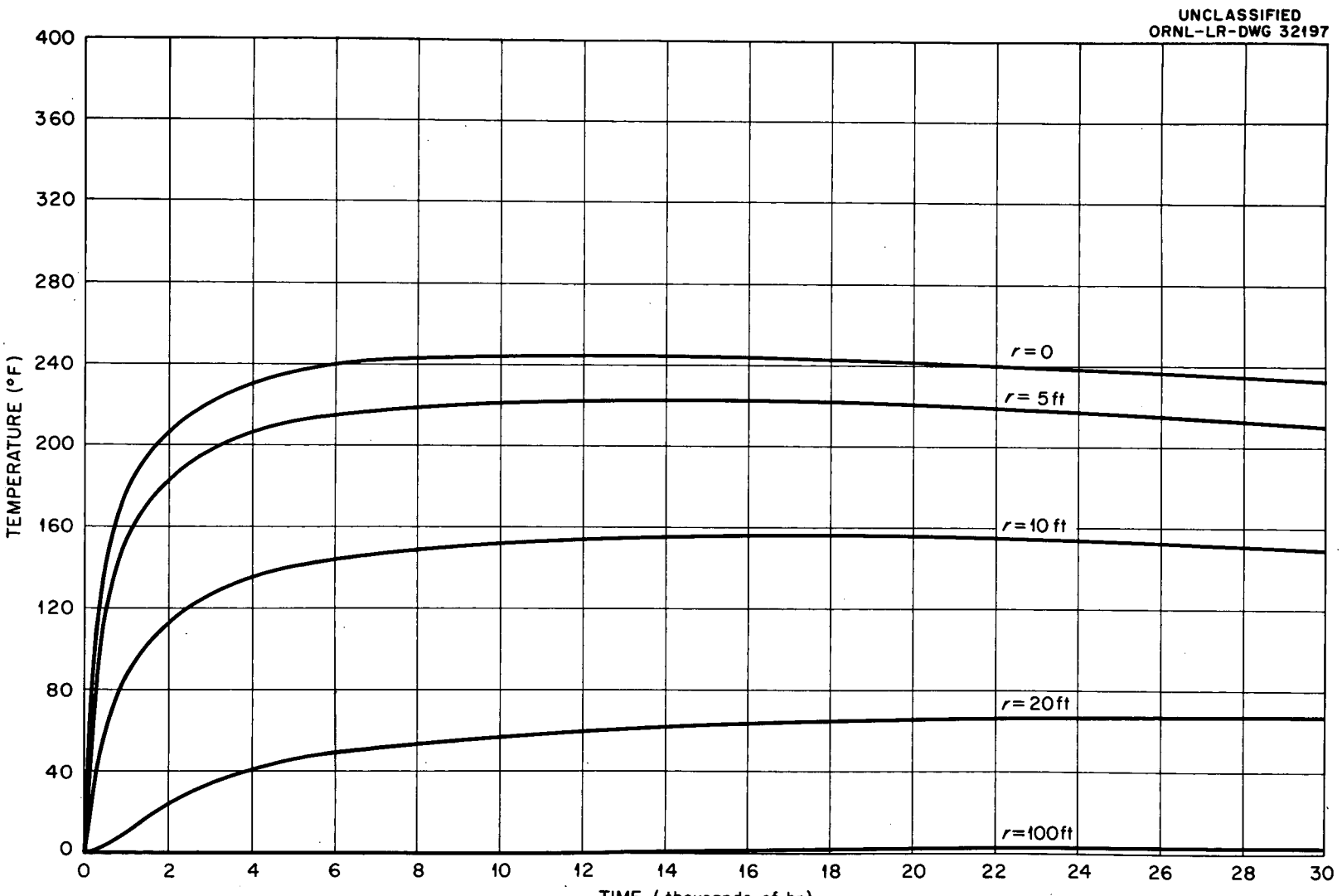

Fig. 7. 11 Year Old Waste—Sphere of 10-ft Radius. Surrounded by an Infinite Bed of Salt.

IV. THE PROBLEM OF AN INFINITE MEDIUM IN CONTACT AT r = a WITH AN INFINITELY LONG CYLINDER FILLED WITH A WELL-STIRRED FLUID IN WHICH HEAT IS GENERATED AT THE RATE Q(t) PER UNIT TIME PER UNIT VOLUME

An infinitely long cylinder surrounded by an infinite medium is filled with a well-stirred fluid generating heat at the rate Q(t) per unit time per unit volume. The conduction equation in cylindrical co-ordinates for the infinite medium is

$$\frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} = \frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} \quad , \qquad r > a \quad . \tag{32}$$

Initially the system is at zero temperature, i.e.

$$T_1(r,0) = T_2(r,0) = 0 \quad . \tag{33}$$

The boundary conditions are that the temperatures of fluid and infinite medium must be equal at their interface, that the temperature of the medium must go to zero at infinity, and that the heat generated per unit time in the fluid must equal the heat absorbed per unit time by the fluid plus the heat lost per unit time by the cylinder. Stated symbolically, these conditions are

$$T_1(a,t) = T_2(a,t) \quad , \qquad t > 0 \quad , \tag{34}$$

$$T_2 \to 0 \quad \text{as} \quad r \to \infty \, , \tag{35}$$

and

$$\pi a^2 Q(r) = \pi a^2 C_1 \frac{\partial T_1}{\partial t} - 2\pi a K_2 \left[\frac{\partial T_2}{\partial r}\right]_{r = a} ,$$

or

$$Q(t) = C_1 \frac{\partial T_1}{\partial t} - \frac{2K_2}{a}\left[\frac{\partial T_2}{\partial r}\right]_{r = a} . \tag{36}$$

On application of the Laplace Transform, the following subsidiary equations are obtained.

$$\frac{1}{\alpha_2} \overline{T}_2 s = \frac{\partial^2 \overline{T}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}_2}{\partial r} , \qquad r > a , \tag{37}$$

$$\overline{T}_1(a) = \overline{T}_2(a) , \tag{38}$$

$$\overline{Q}(s) = C_1 \overline{T}_1 s - (2K_2/a)\left[\frac{\partial \overline{T}_2}{\partial r}\right]_{r = a} , \tag{39}$$

and

$$\overline{T}_2 \to 0 \quad \text{as} \quad r \to \infty . \tag{40}$$

Equation (37) has the general solution

$$\overline{T}_2 = C' I_o(q_2 r) + C'' K_o(q_2 r) ,$$

where C' and C" are arbitrary constants, and I_o and K_o are modified, zero-order Bessel functions of the first and second kinds respectively. The ratio $\sqrt{s/\alpha_2}$ is again defined as q_2.

By making use of boundary conditions (38), (39), and (40), it is found that the Laplace Transform of the temperature throughout the well-stirred liquid is

$$\overline{T}_1(s) = \frac{\overline{Q}(s)\, K_o(q_2 a)}{C_1 s\, K_o(q_2 a) + (2C_2 \sqrt{\alpha_2 s}/a)\, K_1(q_2 a)} , \tag{41}$$

where K_1 is the modified, first-order Bessel function of the second kind.

Similarly, the Laplace Transform of the temperature outside the cylinder is

$$\bar{T}_2(s) = \frac{\bar{Q}(s)\, K_o(q_2 r)}{C_1 s\, K_o(q_2 a) + (2C_2\sqrt{\alpha_2 s}/a)\, K_1(q_2 a)} \quad . \tag{42}$$

Once again $\bar{T}_1^*$ and $\bar{T}_2^*$ are defined

$$\bar{T}_1^* = \bar{T}_1 / \bar{Q} \quad , \quad \text{and} \quad \bar{T}_2^* = \bar{T}_2 / \bar{Q} \quad .$$

The solutions for T_1^* and T_2^* are found from the Inversion Theorem for the Laplace Transform, for example

$$T_1^* = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{zt}\, K_o(\mu_2 a)\, dz}{C_1 z\, K_o(\mu_2 a) + (2C_2\sqrt{\alpha_2 z}/a)\, K_1(\mu_2 a)} \quad , \tag{43}$$

and

$$T_2^* = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{zt}\, K_o(\mu_2 r)\, dz}{C_1 z\, K_o(\mu_2 a) + (2C_2\sqrt{\alpha_2 z}/a)\, K_1(\mu_2 a)} \quad , \tag{44}$$

where $\mu_2 = \sqrt{z/\alpha_2}$.

Expressions (43) and (44) have branch points at $z = 0$ and branch cuts along the negative, real axis. Furthermore, as it will be shown in the following discussion, neither expression has poles on or within the contour of figure 1, page 13.

When the denominators of (43) and (44) are equated to zero, the resulting equations are of the general form[1]

$$f(w)K_n'(w) - g(w)K_n(w) = 0 \quad ,$$

---

[1]Note that $K_o'(w) = -K_1(w)$.

where f and g are analytic functions of w, and K_n and K_n' are the nth order, modified Bessel function of the second kind and its derivative respectively. According to Erdélyi and Kermack,[1] the equation just given has no roots $w = \xi + i\eta$ with $\xi \geq 0$ provided $\mathrm{Re}\left[g(w)/f(w)\right] \geq 0$ for $\xi \geq 0$.

In equations (43) and (44), $w = \sqrt{z}$, and it is desired to find the roots[2] of

$$\frac{2C_2}{b} K_o'(wb) - C_1 w\, K_o(wb) = 0 \quad ,$$

where $b = a/\sqrt{\alpha_2}$.

It is seen that

$$\mathrm{Re}\left[g(w)/f(w)\right] = \frac{\beta}{2b}\,\mathrm{Re}(w) \geq 0 \quad \text{for} \quad \mathrm{Re}(w) \geq 0 \quad ;$$

thus, there are no poles of (43) and (44) on or within the contour of figure 1, and the solutions for T_1 and T_2 can be reduced to infinite integrals along lines DC and C'D' together with contributions from the small circle at the origin.

The substitutions $z = (v^2\alpha_2/a^2)e^{i\pi}$ and $z = (v^2\alpha_2/a^2)e^{-i\pi}$ are now made on the lines DC and C'D' respectively, and by making use of the

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[1] A. Erdélyi and W. O. Kermack, "Note on the Equation $f(z)K_n'(z) - g(z)K_n(z) = 0$," Proceedings of the Cambridge Philosophical Society, XLI (1945), pp. 74-75.

[2] It is only necessary to find the roots for $\mathrm{Re}(w) \geq 0$ because $-\pi < \arg z < \pi$; therefore, $-\pi/2 < \arg w < \pi/2$.

relationship[1] that

$$K_n(ze^{\pm i\pi/2}) = \pm \frac{1}{2} i\pi e^{\mp n\pi i/2} \left[-J_n(z) \pm Y_n(z)\right] ,$$

where J_n and Y_n are nth-order Bessel functions of the first and second kinds respectively, it is seen that

$$\int_{DC} \bar{T}_1^* e^{zt} dz = \frac{1}{\pi i C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t} \left[J_o(v) - iY_o(v)\right] dv}{\beta v \left[J_o(v) - iY_o(v)\right] - 2\left[J_1(v) - iY(v)\right]} ,$$

and

$$\int_{C'D'} \bar{T}_1^* e^{zt} dz = \frac{1}{\pi i C_2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t} \left[J_o(v) + iY_o(v)\right] dv}{\beta v \left[-J_o(v) - iY_o(v)\right] + 2\left[J_1(v) + iY_1(v)\right]} .$$

The sum of these two integrals is

$$I_{DC} + I_{C'D'} = \frac{8}{C_2\pi^2} \int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t} dv}{v\left\{\left[\beta v J_o(v) - 2J_1(v)\right]^2 + \left[\beta v Y_o(v) - 2Y_1(v)\right]^2\right\}} .$$

The integral $I_{D'D}$ around the small circle at the origin is evaluated by writing $z = (\epsilon\alpha_2/a^2)e^{i\theta}$ and taking the limit as $\epsilon \to 0$, i.e.

$$I_{D'D} = \lim_{\epsilon \to 0} \frac{1}{2\pi C_2} \int_{-\pi}^{\pi} \frac{e^{(\epsilon\alpha_2/a^2)t(\cos\theta + i\sin\theta)} K_o(\sqrt{\epsilon}e^{i\theta/2}) d\theta}{\beta K_o(\sqrt{\epsilon}e^{i\theta/2}) + 2\epsilon^{-1/2}e^{-i\theta/2}K_1(\sqrt{\epsilon}e^{i\theta/2})}$$

$$= 0 .$$

Therefore,

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[1] H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, (London: Oxford University Press, 1950), p. 376.

$$T_1^*(t) = \frac{8}{C_2\pi^2}\int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\,dv}{v\left\{\left[\psi(v)\right]^2 + \left[\Delta(v)\right]^2\right\}} ,$$

where $\psi(v) = \beta v Y_o(v) - 2Y_1(v)$, and $\Delta(v) = \beta v J_o(v) - 2J_1(v)$.

By Duhamel's Theorem, when $Q(t) = \sum_{i=1}^{N} A_i e^{-\lambda_i t}$,

$$T_1(t) = \frac{8}{C_2\pi^2}\sum_{i=1}^{N} A_i e^{-\lambda_i t}\int_0^\infty \frac{\left\{1 - e^{(\lambda_i - v^2\alpha_2/a^2)t}\right\}dv}{\left\{v^2\alpha_2/a^2 - \lambda_i\right\} v\left\{\left[\psi(v)\right]^2 + \left[\Delta(v)\right]^2\right\}} . \tag{45}$$

By following a procedure exactly the same as that for finding T_1, it can be shown that

$$T_2^*(t) = \frac{2}{C_2\pi}\int_0^\infty \frac{e^{-(v^2\alpha_2/a^2)t}\left[J_o(vr/a)\psi(v) - Y_o(v)\Delta(v)\right]dv}{\left[\psi(v)\right]^2 + \left[\Delta(v)\right]^2} ,$$

and by Duhamel's Theorem,

$$T_2(t) = \frac{2}{C_2\pi}\sum_{i=1}^{N} A_i e^{-\lambda_i t}\int_0^\infty \left[\frac{1 - e^{(\ _i - v^2\alpha_2/a^2)t}}{v^2\alpha_2/a^2 - \lambda_i}\right] \times$$

$$\times \left[\frac{J_o(vr/a)\psi(v) - Y_o(vr/a)\Delta(v)}{\left[\psi(v)\right]^2 + \left[\Delta(v)\right]^2}\right] dv . \tag{46}$$

Equations (45) and (46) can be integrated numerically by approximating infinity with some large number or by transforming variables in order to obtain finite limits of integration.

CONCLUSIONS

The Laplace Transformation has been used to obtain solutions to the heat conduction equations for the sphere and the infinitely long cylinder filled with exponentially decaying heat sources and surrounded by an infinite medium. In the case of the sphere, the solutions have been limited to two definite physical situations: (1) the problem in which the thermal properties of the sphere are assumed equal to the thermal properties of the infinite medium (equations (30) and (31)); (2) the problem in which the thermal conductivity of the sphere is assumed infinite (equations (28) and (29)). Solutions to the cylindrical problem have been obtained only for the case in which the thermal conductivity of the cylinder is assumed infinite (equations (45) and (46)).

The original intent of the authors was to obtain numerical results for all the solutions mentioned above, but lack of time limited the calculations to the spherical problem in which sphere and infinite medium are assumed to have equal thermal properties. The results have been given in Chapter III and in Appendix IV for a sphere filled with one, six, and eleven-year-old reactor wastes from a reactor operated to 10,000 Mwd/(ton uranium) at 33 Mw/(ton uranium). The wastes from one ton of uranium have been diluted to 800 gallons. A great many simplifying assumptions were made in arriving at the solutions, so the numerical results are intended only to indicate orders of magnitude. The many physical uncertainties in the geologic description of an underground region make unrealistic more complete mathematical analysis. It seems safe to conclude, however, that

salt deposits enclosing radioactive wastes will not melt. It does seem likely that the wastes may boil in some instances.

It is hoped that in the future numerical results can be obtained for the sphere and the infinitely long cylinder whose thermal conductivities are infinite. These results should help to refine the "orders of magnitudes" of the results already given. The writer also would like to obtain solutions to the heat conduction equations for the region bounded internally by an elliptical cylinder containing exponentially decaying heat sources.

APPENDIX I

The Laplace Transformation

Churchill[1] and Carslaw and Jaeger[2] give very good presentations of the Laplace Transformation. The brief discussion and table which follow can be found in either of these sources.

If a function f(t), defined for all positive values of t, is multiplied by e^{-st} and integrated with respect to t from zero to infinity, a new function

$$\bar{f}(s) = L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$

is obtained. This new function $\bar{f}(s)$ is called the Laplace Transform of f(t). For example if

$$f(t) = 1 \quad , \text{ then } \bar{f}(s) = \int_0^\infty e^{-st}\, dt = 1/s \quad ,$$

or if

$$f(t) = e^{at} \quad , \text{ then } \bar{f}(s) = \int_0^\infty e^{-(s-a)t}\, dt = 1/(s-a) \quad .$$

A list of theorems on the Laplace Transformation and a table

---

[1] R. V. Churchill, Modern Operational Mathematics in Engineering (New York: McGraw-Hill Book Company, Inc., 1944)

[2] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics (London: Oxford University Press, 1948)

of Laplace Transforms used in this paper are given below.

THEOREM I. $L\left[f_1 + f_2\right] = L\left[f_1\right] + L\left[f_2\right]$.

THEOREM II. $L\left[\frac{\partial f}{\partial t}\right] = sL\left[f\right] - f_0$,

where f_0 is the value of f at $t = 0$.

THEOREM III. $L\left[\frac{\partial^n f}{\partial x^n}\right] = \frac{\partial^n \bar{f}}{\partial x^n}$.

THEOREM IV.

$$L\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] = L\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right]$$
$$= L\left[f_1(t)\right] L\left[f_2(t)\right] .$$

Theorem IV is known as the Faltung or Superposition Theorem and as Duhamel's Theorem.

It is possible to reduce a partial differential equation in f(t) to an ordinary differential equation in $\bar{f}(s)$ by means of the Laplace Transformation. Once the ordinary differential equation has been solved for $\bar{f}(s)$, it is possible to determine f(t) by means of the Inversion Theorem for the Laplace Transform. This theorem states that

$$f(t) = L^{-1}\left[\bar{f}(s)\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\,\bar{f}(z)\,dz ,$$

where z is used instead of s to indicate that the integration is in the complex plane. The number γ is chosen large enough so that all the singularities of $\bar{f}(z)$ lie to the left of the line $(\gamma-i\infty, \gamma+i\infty)$. It

is often possible to find $L^{-1}\ \bar{f}(s)$ in a table of transforms.

A SHORT TABLE OF LAPLACE TRANSFORMS

$\bar{f}(s)$	$f(t)$
$1/s$	1
$1/s^2$	t
$s^{-3/2}$	$2\sqrt{t/\pi}$
$1/(s-a)$	e^{at}
$s^{-1}\ e^{-k\sqrt{s}}$, $(k \geq 0)$	$\mathrm{erfc}(k/2\sqrt{t})$
$s^{-1/2}\ e^{-k\sqrt{s}}$, $(k \geq 0)$	$\frac{1}{\sqrt{\pi t}}\ e^{-k^2/4t}$
$s^{-3/2}\ e^{-k\sqrt{s}}$, $(k \geq 0)$	$2\sqrt{t/\pi}\ e^{-k^2/4t} - k\ \mathrm{erfc}(k/2\sqrt{t})$

APPENDIX II

Simpson's Rule

Simpson's Rule is used for the numerical evaluation of integrals in this paper. Simpson's Rule states[1] that if $y_0, y_1, y_2, \ldots, y_{2m}$, are the values of $y = f(x)$ at equally spaced points $x_0, x_1, x_2, \ldots, x_{2m}$, with interval h, then

$$\int_{X_0}^{X_{2m}} y\,dx = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{2m-1} + y_{2m}) - \frac{my^{(4)}h^5}{90} ,$$

where $\frac{my^{(4)}h^5}{90}$ is a remainder term, and $y^{(4)}$ denotes the value of the fourth derivative at some point between x_0 and x_{2m}.

Hildebrand[2] says that if two calculations are made, one (I_n) with n and one (I_{2n}) with 2n intervals, the error in I_{2n} can be estimated by $(I_{2n} - I_n)/15$.

---

[1] William E. Milne, Numerical Calculus (Princeton, N. J.: Princeton University Press, 1949), p. 120.

[2] F. B. Hildebrand, Advanced Calculus for Engineers (New York: Prentice-Hall, Inc., 1949), p. 118.

APPENDIX III

Fortran and the IBM-704 Program

The IBM-704 Electronic Data-Processing Machine[1] is a large-scale, high-speed, electronic calculator controlled by an internally stored program, i.e. set of instructions. This machine has three types of ferromagnetic memory: magnetic core, magnetic drum, and magnetic tapes. On most problems the machine executes instructions at the rate of about 40,000 per second.

The IBM Mathematical Formula Translating System, "Fortran,"[2] is an automatic coding system for the IBM-704. "Fortran" is an interpretive program which produces an object program in machine language from a source program written in "Fortran" language which closely approximates ordinary mathematical symbolism.

The "Fortran" source program for calculating integrals (30) and (31) is given on pages 43-46. This program was translated by the "Fortran" system which then produced a machine-language program given on pages 47-55.

It is readily seen from a comparison of the length of the source and object programs that the "Fortran" system represents a powerful tool for mathematicians, physicists, and engineers.

---

[1] IBM-704 Electronic Data-Processing Machine Manual of Operation (New York: International Business Machines Corporation, 1955)

[2] Fortran Automatic Coding System for the IBM-704 (New York: International Business Machines Corporation, 1955)

```
C      TEMPERATURE HISTORY OF A SPHERICAL CAVITY CONTAINING
C     1FISSION-PRODUCTS SURROUNDED BY AN INFINITE SALT MEDIUM
       DIMENSIONR(5),B(10),C(75),D(10),A(10),E(400),
      1DEUXA(10),TROISA(10),S(400),Q1(10),Q2(10),QM1(10),QM2(10)
       FIDDLEF(X,Z)=ERRORF(X/(SQRTF(.2792*Z)))
      1-(X/(SQRTF(3.141592654*.06980*Z)))*(EXPF(-X**2/
      2(4.*.06980*Z)))
       FADDLEF(X,Y,Z)=ERRORF((X-Y)/(2.*(SQRTF(.0698*Z))))
      1+ERRORF((X+Y)/(2.*(SQRTF(.0698*Z))))-(2./Y)*
      2(SQRTF(.06980*Z/3.141592654))*((EXPF(-(X-Y)**2/(4.
      3*.06980*Z)))-(EXPF(-(X+Y)**2/(4.*.06980*Z))))
       FUDDLEF(X,Y,Z)=ERRORF((X+Y)/(2.*(SQRTF(.0698*Z))))
      1-ERRORF((Y-X)/(2.*(SQRTF(.0698*Z))))-(2./Y)
      2*(SQRTF(.06980*Z/3.141592654))*((EXPF(-(Y-X)**2/
      3(4.*.06980*Z)))-(EXPF(-(Y+X)**2/(4.*.06980*Z))))
       READ55,(R(I),I=1,2),(B(J),J=1,5),(C(K),K=1,18),
      1(D(L),A(L),L=1,6)
 0055 FORMAT(2E6.0/5E7.1/12E6.0/6E6.0/6E12.6/6E12.6)
       WRITEOUTPUTTAPE1,610
 0610 FORMAT(80H1  TEMPERATURE HISTORY OF A SPHERICAL CAVITY CONTAINING
      1ONE-YEAR-OLD FISSION-   /66H                    PRODUCTS SURROUNDED
      2 BY AN INFINITE SALT DEPOSIT/68H0  THERMAL CONDUCTIVITIES OF SALT
      3AND FISSION-PRODUCTS ASSUMED EQUAL)
       KOUNT=0
       E(1)=1.0
       DO 650 M=2,400
 0650 E(M)=E(M-1)+1.0
 0475 DO65I=1,2
       DO65J=1,5
       P=B(J)*R(I)
       DO65K=1,18
       IF(K-1)680,690,680
 0690 DELTA1=40.0
       DELTA2=50.0
       INT1=39
       INT2=38
 0315 T1=0.0
       T2=0.0
       DO265L=1,6
       Q1(L)=0.0
       Q2(L)=0.0
       IF(ABSF(P-R(I))-1.0)105,105,115
 0105 SIMSON=A(L)/(2.*27.00)*(EXPF(-D(L)*C(K)))
 0135 V=FADDLEF(R(I),P,C(K))
       GOTO155
 0115 IF(P-R(I))500,105,510
 0500 SIMSON=A(L)/27.00*(EXPF(-D(L)*C(K)))
       IF(P-1.0)125,125,110
 0110 SIMSON=SIMSON/2.
       GO TO 135
 0125 V=FIDDLEF(R(I),C(K))*2.0
       GOTO155
 0510 SIMSON=0.0
       V=FUDDLEF(R(I),P,C(K))
 0155 SIMSON=SIMSON+1./(2.*27.00)*V*A(L)
       H=DELTA1
```

```
      SAMSON=SIMSON
0285  DO165M=1,INT1,2
      S(M)=E(M)*C(K)/H
      IF(P-1.0)175,175,185
0175  SIMSON=SIMSON+4./27.00*A(L)*(EXPF(-D(L)*(C(K)-S(M))))
     1*(FIDDLEF(R(I),S(M)))
      GOTO165
0185  IF(P-(R(I)+1.0))195,195,205
0195  SIMSON=SIMSON+2./27.00*A(L)*(EXPF(-D(L)*(C(K)-S(M))))
     1*(FADDLEF(R(I),P,S(M)))
      GOTO165
0205  SIMSON=SIMSON+2./27.00*A(L)*(EXPF(-D(L)*(C(K)-
     1S(M))))*(FUDDLEF(R(I),P,S(M)))
0165  CONTINUE
      DO215M=2,INT2,2
      S(M)=E(M)*C(K)/H
      IF(P-1.0)225,225,235
0225  SIMSON=SIMSON+2./27.00*A(L)*(EXPF(-D(L)*(C(K)-S(M))))
     1*(FIDDLEF(R(I),S(M)))
      GOTO215
0235  IF(P-(R(I)+1.0))245,245,255
0245  SIMSON=SIMSON+1./27.00*A(L)*(EXPF(-D(L)*(C(K)-S(M))))
     1*(FADDLEF(R(I),P,S(M)))
      GOTO215
0255  SIMSON=SIMSON+1./27.00*A(L)*(EXPF(-D(L)*(C(K)-S(M))))
     1*(FUDDLEF(R(I),P,S(M)))
0215  CONTINUE
      IF(H-DELTA1)275,275,520
0275  Q1(L)=Q1(L)+SIMSON*C(K)/(3.*H)
      T1=T1+Q1(L)
      INT1=INT1+10
      INT2=INT2+10
      H=DELTA2
      SIMSON=SAMSON
      GOTO285
0520  Q2(L)=Q2(L)+SIMSON*C(K)/(3.*H)
      T2=T2+Q2(L)
      INT1=INT1-10
0265  INT2=INT2-10
      COMPAR=ABSF(T2-T1)
      IF(COMPAR-0.5)295,295,305
0305  DELTA1=DELTA1+10.0
      DELTA2=DELTA2+10.0
      INT1=INT1+10
      INT2=INT2+10
      GOTO315
0680  DELTA1=20.0
      DELTA2=30.0
      INT1=19
      INT2=18
0920  T1=0.0
      T2=0.0
      DO 900 L=1,6
      QM1(L)=0.0
      QM2(L)=0.0
      IF (ABSF(P-R(I))-1.0) 700,700,710
```

```
0700 SIMSON=A(L)/54.*(EXPF(-D(L)*(C(K)-C(K-1))))*
    1FADDLEF(R(I),P,C(K-1))+(1./54.)*FADDLEF(R(I),P,C(K))*A(L)
     GO TO 720
0710 IF (P-R(I)) 730,700,740
0730 SIMSON=A(L)/54.*(EXPF(-D(L)*(C(K)-C(K-1))))
     IF (P-1.0) 750,750,760
0750 SIMSON=SIMSON*2.*FIDDLEF(R(I),C(K-1))+(1./27.)*
    1FIDDLEF(R(I),C(K))*A(L)
     GO TO 720
0760 SIMSON=SIMSON*FADDLEF(R(I),P,C(K-1))+(1./54.)*
    1FADDLEF(R(I),P,C(K))*A(L)
     GO TO 720
0740 SIMSON=(1./54.)*FUDDLEF(R(I),P,C(K))*A(L)+A(L)/54.*EXPF(-D(L)*
    1(C(K)-C(K-1)))*FUDDLEF(R(I),P,C(K-1))
0720 H=DELTA1
     SAMSON=SIMSON
0890 DO 790 M=1,INT1,2
     S(M)=C(K-1)+(C(K)-C(K-1))*E(M)/H
     IF (P-1.0) 770,770,780
0770 SIMSON=SIMSON+.14814815*A(L)*(EXPF(-D(L)*
    1(C(K)-S(M))))*(FIDDLEF(R(I),S(M)))
     GO TO 790
0780 IF (P-(R(I)+1.0)) 800,800,810
0800 SIMSON=SIMSON+.07407407*A(L)*(EXPF(-D(L)*
    1(C(K)-S(M))))*(FADDLEF(R(I),P,S(M)))
     GO TO 790
0810 SIMSON=SIMSON+.07407407*A(L)*(EXPF(-D(L)*
    1(C(K)-S(M))))*(FUDDLEF(R(I),P,S(M)))
0790 CONTINUE
     DO 820 M=2,INT2,2
     S(M)=C(K-1)+(C(K)-C(K-1))*E(M)/H
     IF (P-1.0) 830,830,840
0830 SIMSON=SIMSON+.07407407*A(L)*(EXPF(-D(L)*
    1(C(K)-S(M))))*(FIDDLEF(R(I),S(M)))
     GO TO 820
0840 IF(P-(R(I)+1.0))850,850,860
0850 SIMSON=SIMSON+1./27.*A(L)*(EXPF(-D(L)*(C(K)-
    1S(M))))*FADDLEF(R(I),P,S(M))
     GO TO 820
0860 SIMSON=SIMSON+1./27.*A(L)*(EXPF(-D(L)*(C(K)-
    1S(M))))*FUDDLEF(R(I),P,S(M))
0820 CONTINUE
     IF (H-DELTA1) 870,870,880
0870 QM1(L)=QM1(L)+Q2(L)*EXPF(-D(L)*(C(K)-C(K-1)))+SIMSON*(C(K)-
    1C(K-1))/(3.*H)
     T1=T1+QM1(L)
     INT1=INT1+10
     INT2=INT2+10
     H=DELTA2
     SIMSON=SAMSON
     GO TO 890
0880 QM2(L)=QM2(L)+Q2(L)*EXPF(-D(L)*(C(K)-C(K-1)))+SIMSON*(C(K)-
    1C(K-1))/(3.*H)
     T2=T2+QM2(L)
     INT1=INT1-10
0900 INT2=INT2-10
```

```
      COMPAR=ABSF(T2-T1)
      IF(COMPAR-0.5) 960,960,910
0960  DO 970 L=1,6
0970  Q2(L)=QM2(L)
      GO TO 295
0910  DELTA1=DELTA1+10.0
      DELTA2=DELTA2+10.0
      INT1=INT1+10
      INT2=INT2+10
      GO TO 920
0295  T=T2
      IF(SENSESWITCH1)325,335
0325  IF(K-1)345,355,345
0355  PRINT365,R(I),P,C(K),T,DELTA2
0365  FORMAT(5H0  A=F5.1,6H    R=F6.1,43H         TIME      TEMPERATURE
     1       DELTA2/1H0,F34.0,F13.1,F16.0)
      GOTO65
0345  PRINT375,C(K),T,DELTA2
0375  FORMAT(1H0,F34.0,F13.1,F16.0)
      GOTO65
0335  IF(K-1)385,395,385
0395  WRITEOUTPUTTAPE1,365,R(I),P,C(K),T,DELTA2
      GOTO65
0385  WRITEOUTPUTTAPE1,375,C(K),T,DELTA2
0425  IF(K-18)65,435,65
0435  WRITEOUTPUTTAPE1,600
0600  FORMAT(1H1)
0065  CONTINUE
      KOUNT=KOUNT+1
      IF(KOUNT-2)445,455,465
0445  DO485L=1,6
      DEUXA(L)=A(L)*(EXPF(-D(L)*43800.))
0485  A(L)=DEUXA(L)
      WRITEOUTPUTTAPE1,620
0620  FORMAT(80H1  TEMPERATURE HISTORY OF A SPHERICAL CAVITY CONTAINING
     1FIVE-YEAR OLD FISSION-  /66H                    PRODUCTS SURROUNDED
     2 BY AN INFINITE SALT DEPOSIT/68H0  THERMAL CONDUCTIVITIES OF SALT
     3AND FISSION-PRODUCTS ASSUMED EQUAL)
      PRINT 660
0660  FORMAT (29H1PASS GO  COLLECT 500 DOLLARS)
      GOTO475
0455  DO495L=1,6
      TROISA(L)=A(L)*(EXPF(-D(L)*43800.))
0495  A(L)=TROISA(L)
      WRITEOUTPUTTAPE1,630
0630  FORMAT(80H1  TEMPERATURE HISTORY OF A SPHERICAL CAVITY CONTAINING
     1TEN-YEAR-OLD FISSION-   /66H                    PRODUCTS SURROUNDED
     2 BY AN INFINITE SALT DEPOSIT/68H0  THERMAL CONDUCTIVITIES OF SALT
     3AND FISSION-PRODUCTS ASSUMED EQUAL)
      PRINT 660
      GOTO475
0465  ENDFILE1
      STOP7777
```

STORAGE FOR VARIABLES APPEARING IN DIMENSION OR EQUIVALENCE SENTENCES

	DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT
E	32657	77621	D	32677	77645	DEUXA	32257	77001	C	32752	77760	B	32762	77772
A	32667	77633	Q1	31837	76135	Q2	31827	76123	QM1	31817	76111	QM2	31807	76077
R	32767	77777	S	32237	76755	TROISA	32247	76767						

STORAGE FOR VARIABLES WHICH DO NOT APPEAR IN DIMENSION OR EQUIVALENCE SENTENCES

	DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT
INT2	31797	76065	INT1	31796	76064	H	31795	76063	DELTA2	31794	76062	DELTA1	31793	76061
COMPAR	31792	76060	KOUNT	31791	76057	K	31790	76056	P	31789	76055	SAMSON	31788	76054
SIMSON	31787	76053	T1	31786	76052	T2	31785	76051	T	31784	76050	V	31783	76047

NAMES OF FORTRAN FUNCTIONS WITH CORRESPONDING INTERNAL FORMULA NUMBERS AND OCTAL LOCATIONS

	IFN	LOC		IFN	LOC		IFN	LOC		IFN	LOC		IFN	LOC
FIDDLE	2	02242	FADDLE	3	02315	FUDDLE	4	02427						

EXTERNAL FORMULA NUMBERS WITH CORRESPONDING INTERNAL FORMULA NUMBERS AND OCTAL LOCATIONS

EFN	IFN	LOC	EFN	IFN	LOC	EFN	IFN	LOC	EFN	IFN	LOC	EFN	IFN	LOC
55	20	00000	610	23	00000	650	27	00061	475	28	00066	690	33	00113
315	37	00123	105	43	00144	135	44	00162	115	46	00175	500	47	00201
110	49	00217	125	51	00223	510	53	00237	155	55	00253	285	58	00270
175	61	00304	185	63	00342	195	64	00350	205	66	00407	165	67	00445
225	71	00464	235	73	00522	245	74	00530	255	76	00570	215	77	00626
275	79	00634	520	86	00662	265	89	00700	305	92	00717	680	97	00734
920	101	00744	700	107	00765	710	109	01041	730	110	01045	750	112	01065
760	114	01124	740	116	01164	720	117	01237	890	119	01243	770	122	01262
780	124	01321	800	125	01327	810	127	01367	790	128	01426	830	132	01446
840	134	01505	850	135	01513	860	137	01552	820	138	01610	870	140	01616
880	147	01661	900	150	01714	960	153	01732	970	154	01733	910	156	01740
295	161	01755	325	163	01761	355	164	01766	355	166	02005	365	167	00000
345	169	02010	345	171	02024	375	172	00000	335	174	02027	395	175	02034
395	177	02053	385	179	02056	385	181	02072	425	182	02074	435	183	02101
435	184	02110	600	185	00000	65	186	02112	445	189	02133	485	191	02146
620	194	00000	660	197	00000	455	199	02175	495	201	02210	630	204	00000
465	208	02237	465	208	02237									

SUBROUTINES OBTAINED FROM LIBRARY

	DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT		DEC	OCT
(DBC)	1565	03035	(CSH)	2082	04042	(BDC)	2163	04163	(FIL)	2193	04221	(LEV)	2206	04236
(RTN)	2233	04271	(SPH)	2713	05231	(STH)	2809	05371	EXP	2817	05401	SQRT	2880	05500
ERROR	2901	05525												

```
5A      CAL *
        XIT (LEV)
        ETM
        CAL (DBC)
        SLW 1
        CAL (CSH)
5D1     NTR 8)1N,0,81
6A      LXD 2)+2,4
7A      ETM
        NTR R+1,4
        LTM
7A1     TXI *+1,4,1
7A2     TXL 7A,4,2
9A      LXD 2)+2,2
10A     ETM
        NTR B+1,2
        LTM
10A1    TXI *+1,2,1
10A2    TXL 10A,2,5
12A     LXD 2)+2,1
13A     ETM
        NTR C+1,1
        LTM
13A1    TXI *+1,1,1
13A2    TXL 13A,1,18
15A     LXD 2)+2,4
16A     ETM
        NTR D+1,4
        NTR A+1,4
        LTM
16A1    TXI *+1,4,1
16A2    TXL 16A,4,6
18A     LTM
19A     CAL *
        XIT (RTN)
21A     CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (STH)
21D1    NTR 8)J2,0,1
22A     CAL *
        XIT (FIL)
24A     CLA 2)+1
        STO KOUNT
25A     CLA 3)+5
        STO E
26A     LXD 2),4
27A     CLA E+2,4
        FAD 3)+5
        STO E+1,4
27A1    TXI *+1,4,1
27A2    TXL 27A,4,400
28A     LXD 2)+2,4
        SXD C)G0,4
        TRA 29A
E)C     SXD C)G0,4
29A     LXD 2)+2,2
        SXD C)G1,2
        TRA 30A
E)D     SXD C)G1,4
D)60D   LXD C)G1,2
D)40D   LXD C)G0,4
30A     LDQ B+1,2
        FMP R+1,4
        STO P
31A     LXD 2)+2,1
        PXD 0,1
        STO K
32A     CLA K
        SUB 2)+2
32A1    TZE 33A
        TPL 97A
        TRA 97A
33A     CLA 3)+6
        STO DELTA1
34A     CLA 3)+7
        STO DELTA2
35A     CLA 2)+3
        STO INT1
36A     CLA 2)+4
        STO INT2
37A     CLA 3)+8
        STO T1
38A     CLA 3)+8
        STO T2
39A     LXD 2)+2,4
        SXD C)G3,4
40A     CLA 3)+8
        STO Q1+1,4
41A     CLA 3)+8
        STO Q2+1,4
42A     CLA P
        LXD C)G0,2
        FSB R+1,2
        SSP
        FSB 3)+5
42A1    TZE 43A
        TPL 46A
43A     LDQ D+1,4
        FMP C+1,1
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+1
        LDQ 3)+4
        FMP 3)+9
        STO 1)+2
        CLA A+1,4
        FDP 1)+2
        FMP 1)+1
        STO SIMSON
44A     CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+1,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO V
45A     TRA 55A
46A     CLA P
        FSB R+1,2
46A1    TZE 43A
        TPL 53A
47A     LDQ D+1,4
        FMP C+1,1
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        FDP 3)+9
        FMP A+1,4
        STO SIMSON
48A     CLA P
        FSB 3)+5
48A1    TZE 51A
        TPL 49A
        TRA 51A
49A     CLA SIMSON
        FDP 3)+4
        STQ SIMSON
50A     TRA 44A
51A     CLA R+1,2
        STO 4)1
        CLA C+1,1
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        LDQ 1)+1
        FMP 3)+4
        STO V
52A     TRA 55A
53A     CLA 3)+8
        STO SIMSON
54A     CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+1,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO V
55A     LDQ 3)+4
        FMP 3)+9
        STO 1)+1
        LDQ A+1,4
        FMP 3)+5
        FDP 1)+1
        FMP V
        FAD SIMSON
        STO SIMSON
56A     CLA DELTA1
        STO H
57A     CLA SIMSON
        STO SAMSON
58A     LXD 2)+2,2
        CLA INT1
        STD 67A2
59A     CLA E+1,2
        FDP H
        FMP C+1,1
        STO S+1,2
60A     CLA P
        FSB 3)+5
60A1    TZE D)40S
        TPL D)40T
D)40S   LXD C)G0,4
61A     CLA R+1,4
        STO 4)1
        CLA S+1,2
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+3
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
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62A     TRA 67A
D)140T  LXD C)G0,4
63A     CLA R+1,4
        FAD 3)+5
        CHS
        FAD P
63A1    TZE 64A
        TPL 66A
64A     CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+4
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
65A     TRA 67A
66A     CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+4
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
67A     BSS
67A1    TXI *+1,2,2
67A2    TXL 59A,2
68A     LXD 2),2
        SXD C)G5,2
        CLA INT2
        STD 77A2
69A     CLA E+1,2
        FDP H
        FMP C+1,1
        STO S+1,2
70A     CLA P
        FSB 3)+5
70A1    TZE D)413
        TPL D)414
D)413   LXD C)G0,4
71A     CLA R+1,4
        STO 4)1
        CLA S+1,2
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+4
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
75A     TRA 77A
E)16    SXD C)G5,2
76A     CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+5
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
72A     TRA 77A
D)1414  LXD C)G0,4
73A     CLA R+1,4
        FAD 3)+5
        CHS
        FAD P
73A1    TZE 74A
        TPL E)16
74A     CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+5
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
77A     BSS
77A1    TXI *+1,2,2
77A2    TXL 69A,2
78A     CLA H
        FSB DELTA1
78A1    TZE 79A
        TPL 86A
79A     LDQ 3)+10
        FMP H
        STO 1)+1
        CLA SIMSON
        FDP 1)+1
        FMP C+1,1
        FAD Q1+1,4
        STO Q1+1,4
80A     CLA T1
        FAD Q1+1,4
        STO T1
81A     CLA INT1
        ADD 2)+5
        STO INT1
82A     CLA INT2
        ADD 2)+5
        STO INT2
83A     CLA DELTA2
        STO H
84A     CLA SAMSON
        STO SIMSON
85A     TRA 58A
86A     LDQ 3)+10
        FMP H
        STO 1)+1
        CLA SIMSON
        FDP 1)+1
        FMP C+1,1
        FAD Q2+1,4
        STO Q2+1,4
87A     CLA T2
        FAD Q2+1,4
        STO T2
88A     CLA INT1
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        SUB 2)+5
        STO INT1
89A     CLA INT2
        SUB 2)+5
        STO INT2
89A1    TXI *+1,4,1
        SXD C)G3,4
89A2    TXL 40A,4,6
90A     CLA T2
        FSB T1
        SSP
        STO COMPAR
91A     CLA COMPAR
        FSB 3)+11
91A1    TZE 161A
        TPL 92A
        TRA 161A
92A     CLA DELTA1
        FAD 3)+12
        STO DELTA1
93A     CLA DELTA2
        FAD 3)+12
        STO DELTA2
94A     CLA INT1
        ADD 2)+5
        STO INT1
95A     CLA INT2
        ADD 2)+5
        STO INT2
96A     TRA 37A
97A     CLA 3)+13
        STO DELTA1
98A     CLA 3)+14
        STO DELTA2
99A     CLA 2)+6
        STO INT1
100A    CLA 2)+7
        STO INT2
101A    CLA 3)+8
        STO T1
102A    CLA 3)+8
        STO T2
103A    LXD 2)+2,4
        SXD C)G3,4
104A    CLA 3)+8
        STO QM1+1,4
105A    CLA 3)+8
        STO QM2+1,4
106A    CLA P
        LXD C)G0,2
        FSB R+1,2
        SSP
        FSB 3)+5
106A1   TZE 107A
        TPL 109A
107A    CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+1,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA 3)+5
        FDP 3)+15
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+1
        STO 1)+2
        CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+2,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+3
        CLA C+1,1
        FSB C+2,1
        STO 1)+4
        LDQ D+1,4
        FMP 1)+4
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+5
        LDQ 1)+3
        FMP A+1,4
        FDP 3)+15
        FMP 1)+5
        FAD 1)+2
        STO SIMSON
108A    TRA 117A
109A    CLA P
        FSB R+1,2
109A1   TZE 107A
        TPL 116A
110A    CLA C+1,1
        FSB C+2,1
        STO 1)+1
        LDQ D+1,4
        FMP 1)+1
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        FDP 3)+15
        FMP A+1,4
        STO SIMSON
111A    CLA P
        FSB 3)+5
111A1   TZE 112A
        TPL 114A
112A    CLA R+1,2
        STO 4)1
        CLA C+1,1
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA 3)+5
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+1
        STO 1)+2
        CLA R+1,2
        STO 4)1
        CLA C+2,1
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+3
        FMP SIMSON
        STO 7)
        LDQ 7)
        FMP 3)+4
        FAD 1)+2
        STO SIMSON
113A    TRA 117A
114A    CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+1,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA 3)+5
        FDP 3)+15
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+1
        STO 1)+2
        CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+2,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+3
        LDQ SIMSON
        FMP 1)+3
        FAD 1)+2
        STO SIMSON
115A    TRA 117A
116A    CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+2,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB C+2,1
        STO 1)+2
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP A+1,4
        FDP 3)+15
        FMP 1)+3
        STO 1)+4
        CLA R+1,2
        STO 4)1
        CLA P
        STO 4)1+1
        CLA C+1,1
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+5
        CLA 3)+5
        FDP 3)+15
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+5
        FAD 1)+4
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        STO SIMSON
117A    CLA DELTA1
        STO H
118A    CLA SIMSON
        STO SAMSON
119A    LXD 2)+2,2
        SXD C)G5,2
        CLA INT1
        STD 128A2
120A    CLA C+1,1
        FSB C+2,1
        FDP H
        FMP E+1,2
        FAD C+2,1
        STO S+1,2
121A    CLA P
        FSB 3)+5
121A1   TZE D)41P
        TPL D)41Q
D)41P   LXD C)G0,4
122A    CLA R+1,4
        STO 4)1
        CLA S+1,2
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+16
        STO 7)
        LDQ 7)
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
123A    TRA 128A
D)41Q   LXD C)G0,4
124A    CLA R+1,4
        FAD 3)+5
        CHS
        FAD P
124A1   TZE 125A
        TPL 127A
125A    CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+17
        STO 7)
        LDQ 7)
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
126A    TRA 128A
127A    CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+17
        STO 7)
        LDQ 7)
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
128A    BSS
128A1   TXI *+1,2,2
128A2   TXL 120A,2
129A    LXD 2),2
        CLA INT2
        STD 138A2
130A    CLA C+1,1
        FSB C+2,1
        FDP H
        FMP E+1,2
        FAD C+2,1
        STO S+1,2
131A    CLA P
        FSB 3)+5
131A1   TZE D)420
        TPL D)421
D)420   LXD C)G0,4
132A    CLA R+1,4
        STO 4)1
        CLA S+1,2
        STO 4)1+1
        SXD 6)+4,4
        TSX FIDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+17
        STO 7)
        LDQ 7)
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
133A    TRA 138A
D)421   LXD C)G0,4
134A    CLA R+1,4
        FAD 3)+5
        CHS
        FAD P
134A1   TZE 135A
        TPL 137A
135A    CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FADDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+5
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
136A    TRA 138A
137A    CLA R+1,4
        STO 4)1
        CLA P
        STO 4)1+1
        CLA S+1,2
        STO 4)1+2
        SXD 6)+4,4
        TSX FUDDLE,4
        LXD 6)+4,4
        STO 1)+1
        CLA C+1,1
        FSB S+1,2
        STO 1)+2
        LXD C)G3,4
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
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        LXD 6)+4,4
        STO 1)+3
        LDQ 1)+1
        FMP 3)+5
        FDP 3)+9
        FMP A+1,4
        STO 7)
        LDQ 7)
        FMP 1)+3
        FAD SIMSON
        STO SIMSON
138A    BSS
138A1   TXI *+1,2,2
138A2   TXL 130A,2
139A    CLA H
        FSB DELTA1
139A1   TZE 140A
        TPL 147A
140A    LDQ 3)+10
        FMP H
        STO 1)+1
        CLA C+1,1
        FSB C+2,1
        STO 1)+2
        FDP 1)+1
        FMP SIMSON
        STO 1)+3
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+4
        LDQ Q2+1,4
        FMP 1)+4
        FAD QM1+1,4
        FAD 1)+3
        STO QM1+1,4
141A    CLA T1
        FAD QM1+1,4
        STO T1
142A    CLA INT1
        ADD 2)+5
        STO INT1
143A    CLA INT2
        ADD 2)+5
        STO INT2
144A    CLA DELTA2
        STO H
145A    CLA SAMSON
        STO SIMSON
146A    TRA 119A
147A    LDQ 3)+10
        FMP H
        STO 1)+1
        CLA C+1,1
        FSB C+2,1
        STO 1)+2
        FDP 1)+1
        FMP SIMSON
        STO 1)+3
        LDQ D+1,4
        FMP 1)+2
        CHS
        SXD 6)+4,4
        TSX EXP,4
        LXD 6)+4,4
        STO 1)+4
        LDQ Q2+1,4
        FMP 1)+4
        FAD QM2+1,4
        FAD 1)+3
        STO QM2+1,4
148A    CLA T2
        FAD QM2+1,4
        STO T2
149A    CLA INT1
        SUB 2)+5
        STO INT1
150A    CLA INT2
        SUB 2)+5
        STO INT2
150A1   TXI *+1,4,1
        SXD C)G3,4
150A2   TXL 104A,4,6
151A    CLA T2
        FSB T1
        SSP
        STO COMPAR
152A    CLA COMPAR
        FSB 3)+11
152A1   TZE 153A
        TPL 156A
153A    LXD 2)+2,4
154A    CLA QM2+1,4
        STO Q2+1,4
154A1   TXI *+1,4,1
154A2   TXL 154A,4,6
155A    TRA 161A
156A    CLA DELTA1
        FAD 3)+12
        STO DELTA1
157A    CLA DELTA2
        FAD 3)+12
        STO DELTA2
158A    CLA INT1
        ADD 2)+5
        STO INT1
159A    CLA INT2
        ADD 2)+5
        STO INT2
160A    TRA 101A
161A    CLA T2
        STO T
162A    PSE 113
        TRA 174A
163A    CLA K
        SUB 2)+2
163A1   TZE 164A
        TPL 169A
        TRA 169A
164A    CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (SPH)
164D1   NTR 8)BD
165A    ETM
        LXD C)G0,4
        NTR R+1,4
        NTR P
        NTR C+1,1
        NTR T
        NTR DELTA2
        LTM
166A    CAL *
        XIT (FIL)
168A    TRA 186A
169A    CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (SPH)
169D1   NTR 8)BN
170A    ETM
        NTR C+1,1
        NTR T
        NTR DELTA2
        LTM
171A    CAL *
        XIT (FIL)
173A    TRA 186A
174A    CLA K
        SUB 2)+2
174A1   TZE 175A
        TPL 179A
        TRA 179A
175A    CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (STH)
175D1   NTR 8)BD,0,1
176A    ETM
        LXD C)G0,4
        NTR R+1,4
        NTR P
        NTR C+1,1
        NTR T
        NTR DELTA2
        LTM
177A    CAL *
        XIT (FIL)
178A    TRA 186A
179A    CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (STH)
179D1   NTR 8)BN,0,1
180A    ETM
        NTR C+1,1
        NTR T
        NTR DELTA2
        LTM
181A    CAL *
        XIT (FIL)
182A    CLA K
        SUB 2)+7
182A1   TZE 183A
        TPL 186A
        TRA 186A
183A    CAL *
        XIT (LEV)
        ETM
        CAL (BDC)
        SLW 1
        CAL (STH)
183D1   NTR 8)IO,0,1
184A    CAL *
        XIT (FIL)
186A    BSS
186A1   TXI *+1,1,1
        SXD C)G2,1
        SXD K,1
186A2   TXL 32A,1,18
        LXD C)G1,4
186A4   TXI *+1,4,1
186A5   TXL E)D,4,5
        LXD C)G0,4
186A7   TXI *+1,4,1
186A8   TXL E)C,4,2
187A    CLA KOUNT
        ADD 2)+2
        STO KOUNT
188A    CLA KOUNT
        SUB 2)
188A1   TZE 199A
        TPL 208A
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189A   LXD 2)+2,4
190A   LDQ D+1,4
       FMP 3)+18
       CHS
       SXD 6)+4,4
       TSX EXP,4
       LXD 6)+4,4
       STO 1)+1
       LDQ A+1,4
       FMP 1)+1
       STO DEUXA+1,4
191A   CLA DEUXA+1,4
       STO A+1,4
191A1  TXI *+1,4,1
191A2  TXL 190A,4,6
192A   CAL *
       XIT (LEV)
       ETM
       CAL (BDC)
       SLW 1
       CAL (STH)
192D1  NTR 8)JC,0,1
193A   CAL *
       XIT (FIL)
195A   CAL *
       XIT (LEV)
       ETM
       CAL (BDC)
       SLW 1
       CAL (SPH)
195D1  NTR 8)KK
196A   CAL *
       XIT (FIL)
198A   TRA 28A
199A   LXD 2)+2,4
200A   LDQ D+1,4
       FMP 3)+18
       CHS
       SXD 6)+4,4
       TSX EXP,4
       LXD 6)+4,4
       STO 1)+1
       LDQ A+1,4
       FMP 1)+1
       STO TROISA+1,4
201A   CLA TROISA+1,4
       STO A+1,4
201A1  TXI *+1,4,1
201A2  TXL 200A,4,6
202A   CAL *
       XIT (LEV)
       ETM
       CAL (BDC)
       SLW 1
       CAL (STH)
202D1  NTR 8)JM,0,1
203A   CAL *
       XIT (FIL)
205A   CAL *
       XIT (LEV)
       ETM
       CAL (BDC)
       SLW 1
       CAL (SPH)
205D1  NTR 8)KK
206A   CAL *
       XIT (FIL)
207A   TRA 28A
208A   WEF 1
209A   HPR 4095
       TRA 209A
2A     LDQ 4)1+1
       FMP 3)+3
       STO 7)
       LDQ 7)
       FMP 3)+2
       STO 1)1
       LDQ 4)1
       FMP 4)1
       FDP 1)1
       STQ 1)1+1
       CLS 1)1+1
       SXD 7),4
       TSX EXP,4
       LXD 7),4
       STO 1)1+2
       LDQ 4)1+1
       FMP 3)+1
       STO 7)
       LDQ 7)
       FMP 3)+2
       SXD 7),4
       TSX SQRT,4
       LXD 7),4
       STO 1)1+3
       CLA 4)1
       FDP 1)1+3
       FMP 1)1+2
       STO 1)1+4
       LDQ 3)
       FMP 4)1+1
       SXD 7),4
       TSX SQRT,4
       LXD 7),4
       STO 1)1+5
       CLA 4)1
       FDP 1)1+5
       STQ 1)1+6
       CLA 1)1+6
       SXD 7),4
       TSX ERROR,4
       LXD 7),4
       FSB 1)1+4
       TRA 1,4
3A     LDQ 4)1+2
       FMP 3)+3
       STO 7)
       LDQ 7)
       FMP 3)+2
       STO 1)1
       CLA 4)1
       FAD 4)1+1
       STO 1)1+1
       STO 1)
       LDQ 1)
       FMP 1)
       FDP 1)1
       STQ 1)1+2
       CLS 1)1+2
       SXD 7),4
       TSX EXP,4
       LXD 7),4
       STO 1)1+3
       CLA 4)1
       FSB 4)1+1
       STO 1)1+4
       STO 1)
       LDQ 1)
       FMP 1)
       FDP 1)1
       STQ 1)1+5
       CLS 1)1+5
       SXD 7),4
       TSX EXP,4
       LXD 7),4
       FSB 1)1+3
       STO 1)1+6
       CLA 3)+2
       FDP 3)+1
       FMP 4)1+2
       SXD 7),4
       TSX SQRT,4
       LXD 7),4
       STO 1)1+7
       CLA 3)+4
       FDP 4)1+1
       FMP 1)1+6
       STO 7)
       LDQ 7)
       FMP 1)1+7
       STO 1)1+8
       LDQ 3)+2
       FMP 4)1+2
       SXD 7),4
       TSX SQRT,4
       LXD 7),4
       STO 1)1+9
       LDQ 3)+4
       FMP 1)1+9
       STO 1)1+10
       CLA 1)1+1
       FDP 1)1+10
       STQ 1)1+11
       CLA 1)1+11
       SXD 7),4
       TSX ERROR,4
       LXD 7),4
       STO 1)1+12
       CLA 1)1+4
       FDP 1)1+10
       STQ 1)1+13
       CLA 1)1+13
       SXD 7),4
       TSX ERROR,4
       LXD 7),4
       FAD 1)1+12
       FSB 1)1+8
       TRA 1,4
4A     LDQ 4)1+2
       FMP 3)+3
       STO 7)
       LDQ 7)
       FMP 3)+2
       STO 1)1
       CLA 4)1+1
       FAD 4)1
       STO 1)
       LDQ 1)
       FMP 1)
       FDP 1)1
       STQ 1)1+1
       CLS 1)1+1
       SXD 7),4
       TSX EXP,4
       LXD 7),4
       STO 1)1+2
       CLA 4)1+1
       FSB 4)1
       STO 1)1+3
       STO 1)
       LDQ 1)
       FMP 1)
       FDP 1)1
       STQ 1)1+4
       CLS 1)1+4
       SXD 7),4
       TSX EXP,4
       LXD 7),4
       FSB 1)1+2
       STO 1)1+5
       CLA 3)+2
       FDP 3)+1
       FMP 4)1+2
       SXD 7),4
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        TSX SQRT,4
        LXD 7),4
        STO 1)1+6
        CLA 3)+4
        FDP 4)1+1
        FMP 1)1+5
        STO 7)
        LDQ 7)
        FMP 1)1+6
        STO 1)1+7
        LDQ 3)+2
        FMP 4)1+2
        SXD 7),4
        TSX SQRT,4
        LXD 7),4
        STO 1)1+8
        LDQ 3)+4
        FMP 1)1+8
        STO 1)1+9
        CLA 1)1+3
        FDP 1)1+9
        STQ 1)1+10
        CLA 1)1+10
        SXD 7),4
        TSX ERROR,4
        LXD 7),4
        STO 1)1+11
        CLA 4)1
        FAD 4)1+1
        FDP 1)1+9
        STQ 1)1+12
        CLA 1)1+12
        SXD 7),4
        TSX ERROR,4
        LXD 7),4
        FSB 1)1+11
        FSB 1)1+7
        TRA 1,4
2)      OCT +000002000000
        OCT +000000000000
        OCT +000001000000
        OCT +000047000000
        OCT +000046000000
        OCT +000012000000
        OCT +000023000000
        OCT +000022000000
3)      OCT +177435715153
        OCT +202622077324
        OCT +175435715153
        OCT +203400000000
        OCT +202400000000
        OCT +201400000000
        OCT +206500000000
        OCT +206620000000
        OCT +000000000000
        OCT +205660000000
        OCT +202600000000
        OCT +200400000000
        OCT +204500000000
        OCT +205500000000
        OCT +205740000000
        OCT +206660000000
        OCT +176457320460
        OCT +175457320453
        OCT +220526140000
6)      OCT +233000000000
        OCT +000000077777
        OCT +000000000000
        OCT +000001000000
        OCT +000000000000
8)1N    BCD 1(2E6.0
        BCD 1/5E7.1
        BCD 1/12E6.
        BCD 10/6E6.
        BCD 10/6E12
        BCD 1.6/6E1
        BCD 12.6)
        OCT -377777777777
8)J2    BCD 1(80H1
        BCD 1 TEMPE
        BCD 1RATURE
        BCD 1 HISTO
        BCD 1RY OF
        BCD 1A SPHE
        BCD 1RICAL
        BCD 1CAVITY
        BCD 1 CONTA
        BCD 1INING
        BCD 1ONE-YE
        BCD 1AR-OLD
        BCD 1 FISSI
        BCD 1ON-
        BCD 1/66H
        BCD 1
        BCD 1
        BCD 1      P
        BCD 1RODUCT
        BCD 1S SURR
        BCD 1OUNDED
        BCD 1 BY AN
        BCD 1 INFIN
        BCD 1ITE SA
        BCD 1LT DEP
        BCD 1OSIT/6
        BCD 18H0  T
        BCD 1HERMAL
        BCD 1 CONDU
        BCD 1CTIVIT
        BCD 1IES OF
        BCD 1 SALT
        BCD 1AND FI
        BCD 1SSION-
        BCD 1PRODUC
        BCD 1TS ASS
        BCD 1UMED E
        BCD 1QUAL)
        OCT -377777777777
8)BD    BCD 1(5H0
        BCD 1A=F5.1
        BCD 1,6H
        BCD 1 R=F6.
        BCD 11,43H
        BCD 1
        BCD 1 TIME
        BCD 1     TE
        BCD 1MPERAT
        BCD 1URE
        BCD 1
        BCD 1DELTA2
        BCD 1/1H0,F
        BCD 134.0,F
        BCD 113.1,F
        BCD 116.0)
        OCT -377777777777
8)BN    BCD 1(1H0,F
        BCD 134.0,F
        BCD 113.1,F
        BCD 116.0)
        OCT -377777777777
8)IO    BCD 1(1H1)
        OCT -377777777777
8)JC    BCD 1(80H1
        BCD 1 TEMPE
        BCD 1RATURE
        BCD 1 HISTO
        BCD 1RY OF
        BCD 1A SPHE
        BCD 1RICAL
        BCD 1CAVITY
        BCD 1 CONTA
        BCD 1INING
        BCD 1FIVE-Y
        BCD 1EAR OL
        BCD 1D FISS
        BCD 1ION-
        BCD 1/66H
        BCD 1
        BCD 1
        BCD 1      P
        BCD 1RODUCT
        BCD 1S SURR
        BCD 1OUNDED
        BCD 1 BY AN
        BCD 1 INFIN
        BCD 1ITE SA
        BCD 1LT DEP
        BCD 1OSIT/6
        BCD 18H0  T
        BCD 1HERMAL
        BCD 1 CONDU
        BCD 1CTIVIT
        BCD 1IES OF
        BCD 1 SALT
        BCD 1AND FI
        BCD 1SSION-
        BCD 1PRODUC
        BCD 1TS ASS
        BCD 1UMED E
        BCD 1QUAL)
        OCT -377777777777
8)KK    BCD 1 (29H1
        BCD 1PASS G
        BCD 10  COL
        BCD 1LECT 5
        BCD 100 DOL
        BCD 1LARS)
        OCT -377777777777
8)JM    BCD 1(80H1
        BCD 1 TEMPE
        BCD 1RATURE
        BCD 1 HISTO
        BCD 1RY OF
        BCD 1A SPHE
        BCD 1RICAL
        BCD 1CAVITY
        BCD 1 CONTA
        BCD 1INING
        BCD 1TEN-YE
        BCD 1AR-OLD
        BCD 1 FISSI
        BCD 1ON-
        BCD 1/66H
        BCD 1
        BCD 1
        BCD 1      P
        BCD 1RODUCT
        BCD 1S SURR
        BCD 1OUNDED
        BCD 1 BY AN
        BCD 1 INFIN
        BCD 1ITE SA
        BCD 1LT DEP
        BCD 1OSIT/6
        BCD 18H0  T
        BCD 1HERMAL
        BCD 1 CONDU
        BCD 1CTIVIT
        BCD 1IES OF
        BCD 1 SALT
        BCD 1AND FI
        BCD 1SSION-
        BCD 1PRODUC
        BCD 1TS ASS
        BCD 1UMED E
```

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BCD 1QUAL)
OCT -377777777777
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APPENDIX IV

Numberical Data in Tabular Form

The data represented graphically in figures 2-7 are given here in tabular form. The wastes used are those from a uranium reactor operated to 10,000 Mwd/ton at 33 Mw/ton. The wastes are diluted to 800 gallons of wastes per ton of uranium processed.

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 5 FEET CONTAINING ONE-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURE (^{o}F)				
	r=0 ft	r=2.5 ft	r=5 ft	r=10 ft	r=50 ft
500	495.4	435.4	270.0	60.2	0.0
1,000	527.9	469.6	308.8	94.6	0.0
1,500	525.8	470.4	317.4	110.5	0.0
2,000	513.6	461.0	315.9	118.3	0.1
3,000	480.6	433.2	302.3	122.9	0.7
4,000	446.5	403.5	284.7	121.4	1.6
5,000	415.0	375.7	267.0	117.5	2.5
7,000	361.9	328.3	235.5	107.9	4.3
10,000	303.2	275.6	199.3	94.3	6.2
15,000	240.8	219.2	159.4	77.4	7.6
20,000	201.8	183.9	134.2	66.0	8.0
25,000	175.2	159.6	116.7	57.8	7.9
30,000	155.7	141.9	103.8	51.7	7.7
40,000	128.8	117.5	86.1	43.1	7.1
60,000	98.4	89.7	65.8	33.1	6.0
80,000	81.2	74.0	54.3	27.3	5.2
100,000	70.1	63.9	46.9	23.6	4.6
150,000	54.3	49.5	36.3	18.3	3.6

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 10 FEET CONTAINING ONE-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURE (^{o}F)				
	r=0 ft	r=5 ft	r=10 ft	r=20 ft	r=100 ft
500	1,276.6	1,101.7	565.4	33.6	0.0
1,000	1,597.4	1,393.9	794.2	108.4	0.0
1,500	1,704.6	1,500.3	902.3	172.9	0.0
2,000	1,733.7	1,535.2	954.8	221.6	0.0
3,000	1,701.4	1,518.4	983.8	283.4	0.0
4,000	1,625.8	1,458.2	986.6	315.3	0.0
5,000	1,540.0	1,386.1	936.4	330.3	0.0
7,000	1,375.6	1,243.9	858.9	335.1	0.3
10,000	1,175.7	1,067.5	750.8	318.1	1.3
15,000	948.3	864.1	617.1	279.8	4.2
20,000	801.1	731.2	526.2	246.8	7.3
25,000	698.2	638.0	461.2	220.7	9.9
30,000	622.1	568.8	412.4	200.0	11.8
40,000	516.6	472.7	343.8	169.3	14.2
60,000	395.4	362.1	264.2	132.0	15.6
80,000	326.5	299.1	218.5	109.8	15.4
100,000	282.0	258.3	188.7	95.1	14.7
150,000	218.5	200.1	146.2	73.7	12.8

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 5 FEET CONTAINING SIX-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURE (^{o}F)				
	r=0 ft	r=2.5 ft	r=5 ft	r=10 ft	r=50 ft
500	83.1	72.5	45.1	9.9	0.0
1,000	92.9	82.0	54.0	16.2	0.0
1,500	96.9	86.1	58.0	19.6	0.0
2,000	99.1	88.3	60.3	21.8	0.0
3,000	101.1	90.4	62.7	24.5	0.1
4,000	101.7	91.2	63.8	26.0	0.3
5,000	101.7	91.3	64.3	26.9	0.5
7,000	100.8	90.7	64.4	27.9	0.9
10,000	98.6	88.8	63.5	28.4	1.4
15,000	94.3	85.1	61.3	28.1	2.0
20,000	90.1	81.4	58.8	27.5	2.4
25,000	86.2	77.9	56.5	26.6	2.7
30,000	82.6	74.7	54.3	25.8	2.9
40,000	76.4	69.2	50.3	24.2	3.1
60,000	66.9	60.6	44.3	21.5	3.1
80,000	60.1	54.5	39.8	19.5	3.1
100,000	54.9	49.8	36.5	17.9	2.9
150,000	45.8	41.6	30.5	15.1	2.7

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 10 FEET CONTAINING SIX-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURES (^{o}F)				
	r=0 ft	r=5 ft	r=10 ft	r=20 ft	r=100 ft
500	212.5	183.0	93.9	5.5	0.0
1,000	277.0	241.2	136.9	18.2	0.0
1,500	308.3	270.6	161.6	29.8	0.0
2,000	327.0	288.6	177.7	39.3	0.0
3,000	347.8	309.1	197.4	53.4	0.0
4,000	358.6	320.0	208.7	63.2	0.0
5,000	364.4	326.2	215.8	70.2	0.0
7,000	368.9	331.5	223.4	79.5	0.0
10,000	367.4	331.3	227.0	87.2	0.2
15,000	357.6	323.5	225.0	92.4	0.9
20,000	345.1	312.8	219.5	93.7	1.8
25,000	332.4	301.7	213.0	93.2	2.7
30,000	320.2	290.9	206.2	91.9	3.5
40,000	298.1	271.2	193.4	88.4	4.8
60,000	263.0	239.6	172.1	80.8	6.6
80,000	237.2	216.3	155.9	74.3	7.5
100,000	217.4	198.4	143.3	68.9	7.9
150,000	182.2	166.3	120.6	58.8	8.1

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 5 FEET CONTAINING ELEVEN-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURE (^{o}F)				
	r=0 ft	r=2.5 ft	r=5 ft	r=10 ft	r=50 ft
500	52.4	45.8	28.5	6.3	0.0
1,000	58.8	51.9	34.1	10.2	0.0
1,500	61.6	54.7	36.8	12.4	0.0
2,000	63.1	56.3	38.4	13.9	0.0
3,000	64.8	58.0	40.2	15.6	0.1
4,000	65.7	58.9	41.2	16.7	0.2
5,000	66.1	59.3	41.7	17.4	0.3
7,000	66.3	59.7	42.3	18.2	0.6
10,000	66.0	59.5	42.5	18.9	0.9
15,000	64.9	58.5	42.1	19.2	1.3
20,000	63.5	57.3	41.4	19.2	1.6
25,000	62.1	56.1	40.6	19.0	1.8
30,000	60.7	54.8	39.7	18.7	2.0
40,000	58.0	52.5	38.1	18.2	2.2
60,000	53.5	48.4	35.3	17.0	2.4
80,000	49.7	45.0	32.9	16.0	2.4
100,000	46.4	42.1	30.8	15.0	2.4
150,000	39.8	36.1	26.4	13.0	2.2

TEMPERATURE HISTORY OF A SPHERICAL CAVITY OF RADIUS 10 FEET CONTAINING ELEVEN-YEAR-OLD FISSION PRODUCTS AND SURROUNDED BY AN INFINITE MEDIUM. TEMPERATURES GIVEN AT FIVE RADIAL DISTANCES r.

TIME (hours)	TEMPERATURE (^{o}F)				
	r=0 ft	r=5 ft	r=10 ft	r=20 ft	r=100 ft
500	134.0	115.4	59.2	3.5	0.0
1,000	175.2	152.5	86.5	11.5	0.0
1,500	195.6	171.7	102.4	18.8	0.0
2,000	208.1	183.6	112.9	24.9	0.0
3,000	222.7	197.9	126.2	34.0	0.0
4,000	231.0	206.1	134.2	40.4	0.0
5,000	236.2	211.4	139.6	45.1	0.0
7,000	242.0	217.3	146.2	51.6	0.0
10,000	245.2	221.0	151.0	57.4	0.2
15,000	244.9	221.4	153.5	62.3	0.6
20,000	242.0	219.2	153.3	64.6	1.2
25,000	238.1	215.9	151.9	65.6	1.8
30,000	233.8	212.3	149.9	65.9	2.3
40,000	223.2	204.6	145.4	65.5	3.3
60,000	209.1	190.3	136.1	63.0	4.7
80,000	195.2	177.9	127.7	60.1	5.6
100,000	183.1	166.9	120.2	57.2	6.1
150,000	157.9	144.1	104.2	50.4	6.6

APPENDIX V

The Steady State

Some worthwhile information pertaining to storage of radioactive wastes in salt can be obtained by considering a constant-heat-generation, steady-state situation. Solutions for the steady state are not complicated and can be solved backwards for heat generation as a function of temperature, i.e., if it is desirable not to exceed a certain temperature, then a heat generator can be found which will produce the limiting temperature but not exceed it.

For the sphere, the steady-state boundary-value problem is

$$T_1'' + \frac{2}{r}T_1' = -A/K_1 \quad , \quad 0 \leq r < a \quad ,$$

$$T_2'' + \frac{2}{r}T_2' = 0 \quad , \quad r > a \quad ,$$

where primes are written for derivatives with respect to r, K is thermal conductivity, and A is a constant heat generation per unit time per unit volume.

The boundary conditions are

$$T_1(a) = T_2(a),$$

$$K_1T_1'(a) = K_2T_2'(a),$$

and

$$T_1 \text{ finite as } r \to 0 \quad , \text{ and } T_2 \text{ finite as } r \to \infty.$$

The solutions to this problem are found to be

$$T_1 = A/3\left[a^2/K_2 + a^2/2K_1 - r^2/2K_1\right] \quad ,$$

and

$$T_2 = Aa^3/3K_2r \quad .$$

If it is desirable to keep the temperature at the edge of the sphere below a certain temperature T_{max}, then the heat generation in the sphere must not exceed A_{max}, where

$$A_{max} = \frac{3K_1T_{max}}{a^2} \quad .$$

It is interesting to notice the variation in T_1 as the relationship between K_1 and K_2 is varied. Consider the temperature at $r = 0$. When $K_1 = K_2$,

$$T_1 = Aa^2/2K_2 \quad . \tag{i}$$

When $K_1 = 5K_2$,

$$T_1 = 11\ Aa^2/30\ K_2 \quad , \tag{ii}$$

and when $K_1 \rightarrow \infty$,

$$T_1 = Aa^2/3K_2 \quad . \tag{iii}$$

It is seen from comparing (i), (ii), and (iii) that the cases $K_1 = K_2$ and $K_1 \rightarrow \infty$ are upper and lower temperature bounds respectively for the case where $K_1 = 5K_2$.

APPENDIX VI

Point-Source Solutions

It is of interest to survey briefly another method for obtaining solutions to heat conduction boundary value problems. The conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

is satisfied by the solution[1]

$$T(r,r',t,t') = \frac{Q}{8C\ \pi\alpha(t-t')^{3/2}}\, e^{-\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]/4\alpha(t-t')},$$

where Q is a finite quantity of heat instantaneously liberated at (x',y',z',t'). If heat is generated at a rate $Q = Q(t)$ per unit time, the above solution can be integrated with respect to time to obtain solution for a continuous point source. If, in addition, the solution is integrated with respect to the proper space variables, a solution is obtained for a continuous volume source.

As an illustration, the temperature at the center of a sphere at time t due to liberation of an amount of heat

$$\sum_{i=1}^{N} A_i e^{-\lambda_i t'}\, dV'$$

at the point (r',θ,Φ') at time t' is

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[1] H. S. Carslaw, Introduction to the Mathematical Theory of the Conduction of Heat in Solids, (London: MacMillan and Company, Ltd., 1921) pp. 149-51.

$$T = \sum_{i=1}^{N} A_i e^{-\lambda_i t} \frac{e^{-r'^2/4\alpha(t-t')}}{8C\ \pi\alpha(t-t')^{3/2}}\, dV'\ dt' \quad .$$

In spherical co-ordinates, $dV' = r'^2 \sin\theta'\, dr'\, d\theta'\, d\Phi'$, so the temperature at the center of a sphere of radius a due to the totality of point sources in that sphere is

$$T = \sum_{i=1}^{N} \frac{A_i}{8C(\pi\alpha)^{3/2}} \int_0^t \frac{e^{-\lambda_i t'}\, dt'}{(t-t')^{3/2}} \int_0^a e^{-r'^2/4\alpha(t-t')}\, r'^2\, dr' \ \times$$

$$\times \int_0^{\pi} \sin\theta'\, d\theta' \int_0^{2\pi} d\Phi'$$

If the substitution $\tau = t-t'$ is made and the above equation is integrated, it is seen that

$$T = \sum_{i=1}^{N} \frac{A_i e^{-\lambda_i t}}{C} \int_0^t e^{\lambda_i \tau} \left\{ \operatorname{erf}\left[\frac{a}{2\sqrt{\alpha\tau}}\right] - \frac{a}{\sqrt{\alpha\pi\tau}}\, e^{-a^2/4\alpha\tau} \right\} d\tau \quad ,$$

which is the same as the expression (22) given on page 11, if the substitution $Q(t) = \sum_{i=1}^{N} A_i e^{-\lambda_i t}$ is made.

In the case of a cylinder occupying the region $0 < r < a$, $-b < z < b$, where $dV' = r'\, dr'\, dz'\, d\theta'$, the temperature at $r = z = 0$ is found to be

$$T = \sum_{i=1}^{N} \frac{A_i e^{-\lambda_i t}}{C} \int_0^t e^{\lambda_i \tau} \left\{ 1 - e^{-a^2/4\alpha\tau} \right\} \operatorname{erf}\left[\frac{b}{2\sqrt{\alpha\tau}} \right] d\tau \quad .$$

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