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MILNE PROBLEM WITH CAPTURE

S. N. Purohit

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S. N. Purohit

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ABSTRACT

The general relation between the angular flux of the neutrons in a lattice measured with a neutron chopper and the total neutron flux is discussed. A new approximate method for calculating the spatial variation of neutron flux and the emergent angular distribution of neutrons in a Milne problem with capture is formulated. This method is based upon the representation of non-asymptotic part of neutron flux by a single decaying exponential. The coefficient and exponent of this exponential were determined by the iteration method. The emergent angular distribution and total flux of neutrons in the Milne problem were calculated for a large range of absorptions and compared with the results of LeCaine, obtained by the variational method. The agreement between the two results was found to be excellent in all cases.

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INTRODUCTION

In this paper we discuss two problems: first, the angular distribution of neutrons at a point inside a semi-infinite medium; second, the spatial flux and the angular distribution of emergent neutrons in the Milne problem with capture. The relation between the angular distribution of neutrons about a given direction and the total flux at the same point in space is needed to interpret the measurements on the lattices with the help of a neutron chopper. The experimentalist extracts a beam of neutrons from a point situated at a distance from the surface (Fig. 1). From these measurements the total neutron

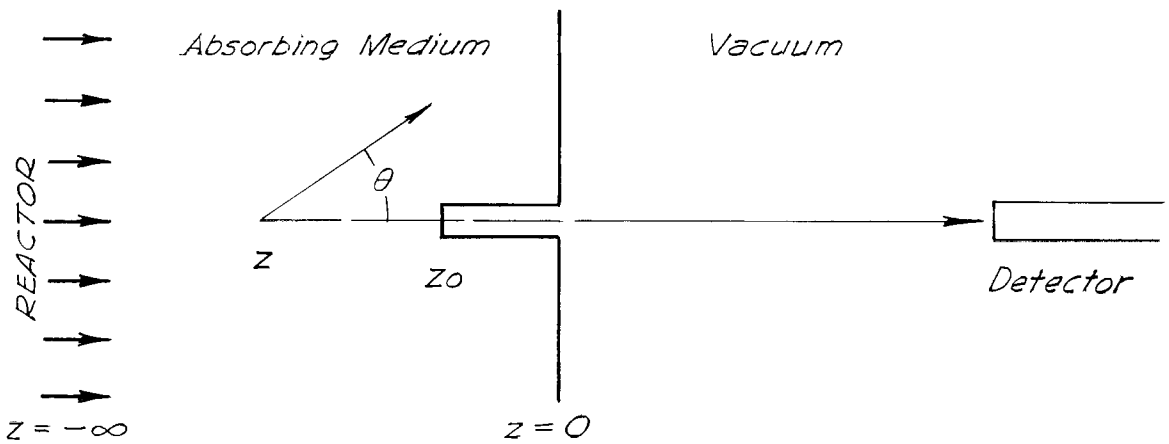

Fig. 1. Milne Geometry

flux at the point of extraction of the beam is determined. The theoretical expression for the angular distribution of neutrons about a given direction is given in terms of the total neutron flux. The physical approximations involved in obtaining the final result are discussed at each stage.

The Milne problem with capture was studied analytically by Adler and Mark,[1] using the Weiner-Hopf method. LeCaine[2] studied the same problem by

1. F. T. Adler and C. Mark, Milne's Problem with Capture, MT-66.
2. J. LeCaine, Canadian J. Res. A28, 242 (1950); see also J. LeCaine, Milne Problem with Capture, MT-119 (1945).

the variational method and carried out extensive numerical calculations for the neutron density and the emergent angular distribution. In a few cases LeCaine compared her results with that of Adler and Mark and found them in good agreement. Numerical calculations based upon the studies of Adler and Mark are not available. It must, however, be pointed out that LeCaine's formula for the non-asymptotic part of the neutron density involves a complicated function. We have proposed in this study a new approximate method for a Milne problem with capture. In this method the asymptotic part of neutron flux was assumed to be known. The non-asymptotic part of neutron flux was represented by a single exponential. The coefficient and exponent of this exponential were determined by the iteration method. Placzek[3,4] used a similar method to determine the neutron density in a Milne problem without capture. We have, however, extended the Placzek method to the capture case. Extensive numerical results are obtained and compared with LeCaine's results.

MATHEMATICAL FORMULATION

The transport behavior of neutrons in a plane geometry is given by the following Boltzmann transport equation.[5]

$$\mu \frac{\partial \phi}{\partial z}(z,\mu,E) + \Sigma_t(E)\phi(z,\mu,E) = \int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{+1}\int_{E'} \Sigma_s(E' \to E;\ \vec{\Omega}' \to \vec{\Omega})$$

$$\phi(E',z,\vec{\Omega}')\ dE'\ d\mu'\ d\varphi'\ d\varphi + S(z,E,\mu) \qquad (1)$$

---

3. G. Placzek, The Neutron Density Near a Plane Surface, MT-16 (1944).
4. V. Kourganoff, Basic Methods in Transfer Problems, Clarendon Press, Oxford (1952).
5. A. M. Weinberg and E. P. Wigner, The Physical Theory of Neutron Chain Reactors, The University of Chicago Press, Chicago (1958)

where

$\phi(z,\mu,E)$ = angular flux of neutrons,

z,μ,E = space, angle, and energy variables,

$S(z,E,\mu)$ = external source,

$\Sigma_t(E)$ = total macroscopic cross section,

$\Sigma_s(E' \to E;\ \vec{\Omega}' \to \vec{\Omega})$ = scattering frequency,

μ = cosine of the angle between angular flux and the z axis.

We shall make the following assumptions:

1. The medium is source free. The source is located at $z = -\infty$, and emits a constant beam of neutrons.
2. Medium extends from $z = 0$ to $z = -\infty$.
3. There is no flow of neutrons from vacuum to medium at the boundary $z = 0$.
$$\phi(0,\mu,E) = 0 \quad \text{for } (-1 \leq \mu < 0).$$
4. For z approaching $-\infty$
$$\phi(z,\mu,E)\ e^{\frac{z\Sigma_t(E)}{\mu}} \to 0.$$
This is true for a finite source of neutrons.

The above four assumptions constitute the basis of the standard Milne problem. We further assume that the medium is isotropic, and the scattering frequency depends upon the angle between the initial and final directions.

$$\Sigma_s(E' \to E;\ \vec{\Omega}' \to \vec{\Omega}) = \Sigma_s(\vec{E}' \to E;\ \vec{\Omega}' \cdot \vec{\Omega}). \tag{2}$$

We expand the scattering frequency and the angular flux $\phi(E',z,\vec{\Omega}')$ by means of the spherical Harmonics method. Using these expansions and substituting in the scattering integral, we obtain the following final expression for the scattering integral I:

$$I = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{2} \int_{E'} \Sigma_{s\ell}(E' \to E)\ P_\ell(\mu)\ \phi_\ell(E',z)\ dE' \tag{3}$$

where

$$\phi_\ell(E',z) = \int_{-1}^{+1} \phi(E',z,\mu')\ P_\ell(\mu')\ d\mu' \tag{4}$$

$$\Sigma_{s\ell}(E' \to E) = \int_{-1}^{+1} \Sigma_s(E' \to E;\ \mu_o)\ P_\ell(\mu_o)\ d\mu_o \tag{5}$$

and

$\mu_o = \vec{\Omega}' \cdot \vec{\Omega}$

= cosine of the scattering angle between the initial and final directions of neutrons.

On substituting Eq. (3) into Eq. (1) and integrating with respect to z we get:

$$\phi(z,\mu,E)\ d\mu = e^{-\frac{z\Sigma_t(E)}{\mu}} \left[A + \frac{1}{\mu} \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{2} \int_0^z \left\{ \int_{E'} \Sigma_{s\ell}(E' \to E)\ P_\ell(\mu)\ \phi_\ell(E',z')\ dE' \right\} e^{\frac{z'\Sigma_t(E)}{\mu}}\ dz' \right] d\mu\ . \tag{6}$$

The constant of integration A is to be obtained from the boundary conditions. We have two cases:

A. <u>For positive μ: $0 < \mu \leq 1$</u>

For $z = -\infty$, Eq. (6) is given as follows:

$$\lim_{z \to -\infty} \phi(z,\mu,E)\, e^{\frac{z\Sigma_t(E)}{\mu}}\, d\mu = \Bigg[A + \frac{1}{\mu} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \int_0^{-\infty} \Bigg\{ \int_{E'} \Sigma_{s\ell}(E' \to E)$$

$$\phi_\ell(E',z') P_\ell(\mu)\, dE' \Bigg\} e^{\frac{z'\Sigma_t(E)}{\mu}}\, dz' \Bigg] d\mu \,. \qquad (7)$$

Since it is assumed that $\lim_{z \to -\infty} \phi(z,\mu,E)\, e^{\frac{z\Sigma_t(E)}{\mu}} = 0$ Eq. (7) gives A. The integral expression for the angular distribution is given as follows:

$$\phi(z,\mu,E)\, d\mu = \Bigg[\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \int_{-\infty}^{z} \Bigg\{ \int_{E'} \Sigma_{s\ell}(E' \to E)\, P_\ell(\mu)\, \phi_\ell(E',z')\, dE' \Bigg\}$$

$$e^{-(z-z')\frac{\Sigma_t(E)}{\mu}}\, dz' \Bigg] \frac{d\mu}{\mu} \,. \qquad (8)$$

B. <u>For Negative μ</u>: <u>$(-1 \leq \mu < 0)$</u>

$$\phi(0,\mu,E) = 0 \quad \text{for } (-1 \leq \mu < 0). \qquad (9)$$

Using the above boundary condition, we get:

$$\phi(z,\mu,E)\, d\mu = \Bigg[\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \int_0^{z} \Bigg\{ \int_{E'} \Sigma_{s\ell}(E' \to E)\, P_\ell(\mu) \phi_\ell(E',z')\, dE' \Bigg\}$$

$$e^{-(z-z')\frac{\Sigma_t(E)}{\mu}}\, dz' \Bigg] \frac{d\mu}{\mu} \,.$$

Equations (8) and (10) give the angular flux for neutrons of energy E at the point z in a Milne problem. We are, however, interested in the angular distribution in the positive μ interval only. It is given by Eq. (8). In the case of a finite medium of width "a" the appropriate boundary condition at $z = -a$ instead of $z = -\infty$ should be used. This condition would depend upon the source of neutrons.

ANGULAR AND TOTAL NEUTRON FLUX

In a neutron chopper experiment the neutron beam is extracted with the help of a <u>re-entrant</u> channel. The experimentalist wants to know the relation between the angular flux measured in a particular direction and the total flux. Let us suppose that the neutron beam is extracted from a point at a distance z equal to z_o from the boundary. The angular flux at this point is given by Eq. (8), putting z equal to z_o.

$$\phi(\mu,E,z_o)\ d\mu = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \left[\int_{-\infty}^{z_o} \left\{ \int_{E'} \Sigma_{s\ell}(E' \to E)\ P_\ell(\mu)\phi_\ell(E',z')\ dE' \right\} e^{-(z_o-z')\frac{\Sigma_t(E)}{\mu}}\ dz' \right] \frac{d\mu}{\mu} \ . \tag{11}$$

The scattering integrals corresponding to various values of "ℓ" are obtained by multiplying Eq. (1) by μ^ℓ and integrating over all angles. Zeroth and first angular moments are given by the following equations:

$$\frac{\partial J(E,z)}{\partial z} + \Sigma_t(E)\phi(E,z) = \int_{E'} \Sigma_{s_o}(E' \to E)\ \phi(E',z)\ dE' \tag{12}$$

$$\frac{\partial \overline{\mu^2\phi(E,z)}}{\partial z} + \Sigma_t(E)J(E,z) = \int_{E'} \Sigma_{s_1}(E' \to E)J(E',z)\ dE' \tag{13}$$

where:

$$\phi(E,z) = \int_{\mu} \phi(z,\mu,E)\ d\mu \qquad \text{Total Neutron Flux} \tag{14}$$

$$J(E,z) = \int_{\mu} \mu\phi(z,\mu,E)\ d\mu \qquad \text{Total Neutron Current} \tag{15}$$

$$\overline{\mu^2} = \frac{\int_{\mu} \mu^2\phi(z,\mu,E)\ d\mu}{\int_{\mu} \phi(z,\mu,E)\ d\mu} . \tag{16a}$$

Since $\mu^2 = \frac{1}{3} P_0(\mu) + \frac{2}{3} P_2(\mu)$

$$\overline{\mu^2} = \frac{1}{3}\left[1 + 2\,\frac{\phi_2(z,E)}{\phi_0(z,E)} \right] . \tag{16b}$$

ϕ_2 and ϕ_0 are given by Eq. (4). If ϕ_2 is zero, which is equivalent to the P_1-approximation for the angular flux, then $\overline{\mu^2}$ is equal to (1/3).

Restricting to linear anisotropic scattering $\ell = 0$ and $\ell = 1$ we obtain from Eq. (11)

$$\phi(z_0,\mu,E)\ d\mu = \frac{d\mu}{\mu}\left[\int_{-\infty}^{z_0} \left\{ \frac{1}{2}\left[\frac{\partial J(E,z')}{\partial z'} + \Sigma_t(E)\phi(E,z') \right] + \frac{3}{2}\left[\Sigma_t(E)J(E,z') + \frac{\partial\overline{\mu^2}}{\partial z'}\,\phi(E,z') \right] \right\} e^{-(z_0-z')\frac{\Sigma_t(E)}{\mu}}\ dz' \right] . \tag{17}$$

The above integral is the exact expression for the angular flux for the linear anisotropic scattering case. The neutron current is given by Eq. (13). In determining the neutron current we assume $J(E',z) \simeq J(E,z)$ in the scattering integral for the neutron current. This is the Selengut-Goertzel approximation for treating the slowing down in hydrogen. The expression for the neutron current is given as follows:

$$J(E,z) = -\frac{\partial \overline{\mu^2}\phi(E,z)}{\partial z}\left[\frac{1}{\Sigma_t(E) - \Sigma_s(E)\,\overline{\mu_L}}\right] \tag{18}$$

where

$\overline{\mu_L}$ = average cosine of the scattering angle in the laboratory system.

In evaluating neutron current from the above expression, we shall assume that $\overline{\mu^2}$ does not vary spatially, $\overline{\mu_L}$ is obtained for the isotropic scattering in the center of mass with the struck particle at rest. In this approximation $\overline{\mu_L}$ is equal to (2/3)A.

The final result for angular flux after substituting the above expression for the neutron current is:

$$\phi(z_o,\mu,E)\,d\mu = \frac{d\mu}{2\mu}\left[\int_{-\infty}^{z_o}\left\{\Sigma_t(E)\phi(E,z') + a\,\frac{\partial\phi(E,z')}{\partial z'} + b\,\frac{\partial^2\phi(E,z')}{\partial z'^2}\right\} e^{-(z_o-z')\frac{\Sigma_t(E)}{\mu}}\,dz'\right] \tag{19}$$

where

$$a = 3\overline{\mu^2}\left\{1 - \frac{\Sigma_t(E)}{\Sigma_t(E) - \Sigma_s(E)\overline{\mu_L}}\right\} \tag{20}$$

$$b = - \frac{\overline{\mu^2}}{\Sigma_t(E) - \Sigma_s(E)\overline{\mu_L}} \quad . \tag{21}$$

We expand $\phi(E,z)$, $\frac{\partial\phi(E,z)}{\partial z}$ and $\frac{\partial^2\phi(E,z)}{\partial z^2}$ by the Taylor series about $z = z_o$ if the point z_o is at a distance from the boundary. This expansion is not valid for z_o at the boundary. The neutron flux has a logarithmically infinite derivative at the boundary although flux is finite.

Using the Taylor series expansions the integral expression for the angular flux reduces to the following form:

$$\phi(z_o,\mu,E)\ d\mu = \left[\sum_{n=0}^{\infty} \int_{-\infty}^{z_o} \frac{(z' - z_o)^n}{n!} \left\{ \Sigma_t(E)\phi^n(z_o) + a\ \phi^{n+1}(z_o) + b\ \phi^{n+2}(z_o) \right\} e^{-(z_o - z')\frac{\Sigma_t(E)}{\mu}}\ dz' \right] \frac{d\mu}{2\mu} \tag{22}$$

where

$$\phi^n(z_o) = \left[\frac{\partial^n\phi(E,z')}{\partial z'^n} \right]_{z'=z_o} .$$

The above expression involves integrals of the following type:

$$I = \int_{-\infty}^{z_o} (z' - z_o)^n\ e^{-(z_o - z')\frac{\Sigma_t(E)}{\mu}}\ dz' . \tag{23a}$$

This integral can be reduced to the standard type by making the substitution $(z_o - z') = x$. The above integral reduces to the following value:

$$I = (-1)^n \, n! \left(\frac{\mu}{\Sigma_t(E)} \right)^{n+1} . \tag{23b}$$

The final expression for the angular flux is given in the following form:

$$\phi(z_o,\mu,E)\, d\mu = \frac{d\mu}{2\mu} \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{\mu}{\Sigma_t(E)} \right)^{n+1} \left\{ \Sigma_t(E)\phi^n(z_o) + a\, \phi^{n+1}(z_o) + b\, \phi^{n+2}(z_o) \right\} \right] . \tag{24}$$

We shall limit ourselves to n = 2 terms only.

$$\phi(z_o,\mu,E)\, d\mu = \frac{d\mu}{2} \left[\phi(E,z_o) + c \left\{ \frac{\partial\phi(E,z')}{\partial z'} \right\}_{z'=z_o} + d \left\{ \frac{\partial^2\phi(E,z')}{\partial z'^2} \right\}_{z'=z_o} \right] \tag{25}$$

where

$$c = \frac{1}{\Sigma_t(E)} \, [a - \mu] \tag{26}$$

$$d = \frac{1}{\Sigma_t^2(E)} \, [b\, \Sigma_t(E) - a\mu + \mu^2] . \tag{27}$$

Equation (25) gives the total flux $\phi(E,z_o)$ from the measured angular flux provided the gradient terms are known theoretically or experimentally. In a neutron chopper experiment the neutron beam is extracted along the

direction $\mu = 1$. For a special case of isotropic scattering in the laboratory system ($\overline{\mu_L} = 0$) the angular flux for $\mu = 1$ is given by the following expression:

$$\phi(z_o,1,E)\ d\mu = \frac{d\mu}{2}\left[\ \phi(E,z_o) - \frac{1}{\Sigma_t(E)}\left\{\frac{\partial\phi(E,z')}{\partial z'}\right\}_{z'=z_o}\right] \tag{28}$$

[Neglecting second derivative term in Eq. (25)].

The above expression was used by Poole[6] as well as Slovacek and Stone.[7] In order to determine the flux gradient terms at the point of extraction of beam the foils of resonance detectors are employed. Indium, gold, and manganese measure the neutron flux corresponding to energies 1.44 ev, 4.9 ev, and 300 ev, respectively. Thermal-neutron flux is measured by the cadmium-covered foils.

In Table 1 the values of $\overline{\mu^2}$ for different absorptions, as given by Case, Placzek, and deHoffmann,[8] are given. These were obtained for the monoenergetic neutrons suffering isotropic scattering.

For large values of absorption $\overline{\mu^2}$ is very large compared to the zero-absorption value of (1/3). For example, the value of α corresponding to thermal neutrons for lithium hydride is equal to 0.673. The corresponding value of $\overline{\mu^2}$ is equal to 0.684. Equation (25) may be employed to obtain total neutron flux from the measurement of the angular flux.

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6. M. J. Poole, "Measurement of Neutron Spectra in Reactors," Proceedings of Neutron Thermalization Conference, ORNL-2739 (1958).
7. R. E. Slovacek and R. S. Stone, "Low Energy Spectra Measurements," Ibid.
8. K. M. Case, F. deHoffmann, and G. Placzek, Introduction to the Theory of Neutron Diffusion, U.S. Government Printing Office, Washington 25, D. C. (1953).

Table 1. Values of μ^2

$\alpha = \frac{\Sigma_a(E)}{\Sigma_t'(E)}$	$\overline{\mu^2}$
0.0	0.3333
0.1	0.3622
0.2	0.3963
0.3	0.4369
0.4	0.4859
0.5	0.5454
0.6	0.6176
0.7	0.7036
0.8	0.8001
0.9	0.9
1.0	1.0

NEUTRON FLUX

We discuss in brief the approximate method developed to determine neutron flux for the monoenergetic neutrons scattered isotropically in the laboratory system - $\phi_\ell(E,z) = 0$ for $\ell \geq 1$. We divide the neutron flux into asymptotic and non-asymptotic parts and express it as follows:

$$\phi(z) = \phi_{asy}(z) + \Delta\phi(z). \tag{29}$$

In the above equation $\Delta\phi(z)$ is the non-asymptotic part of the flux. Adler and Mark,[1,2] using the Wiener-Hopf method, gave the following expressions for the asymptotic flux as a function of distance z and at the boundary.

$$\phi_{asy}(z) = J(0) \sqrt{\frac{2\nu^2}{\alpha} \frac{(1 - \nu^2)}{(\nu^2 - \alpha)}} \sinh \nu(z + z_o) \tag{30}$$

$$\phi_{asy}(0) = J(0) \sqrt{\frac{\nu^2}{\alpha}} \quad \text{at the boundary} \tag{31}$$

where

α = ratio of absorption to total cross sections,

z_o = extrapolated distance measured in total mean free path,

z = distance measured in total mean free path,

ν = real root of the following transcendal equation

$$(1 - \alpha) \frac{\tanh^{-1} \nu}{\nu} = 1 \tag{32}$$

$J(0)$ = total neutron current at the boundary.

In order to obtain the non-asymptotic part of neutron flux, we need the total neutron flux $\phi(z)$. It is given by the following integral equation:[8]

$$\phi(z) = \frac{(1 - \alpha)}{2} \int_0^\infty \phi(z') \, E_1(|z - z'|) \, dz'. \tag{33}$$

Let us assume that the non-asymptotic part can be represented by a single exponential term in the following manner:

$$\phi(z) = J(0) \sqrt{\frac{2\nu^2}{\alpha} \frac{(1 - \nu^2)}{(\nu^2 - \alpha)}} \left[\sinh \nu(z + z_o) + Ae^{-Bz} \right]. \tag{34}$$

The parameters A and B have to be determined. We consider the expression for $\phi(z)$, as given by Eq. (34), as the zeroth iteration expression. Using this expression we carry out the first iteration with the help of the integral Eq. (33). To these two expressions we apply the integral condition at the boundary as given by Eq. (31). This leads to two equations which enable the determination of A and B. These two equations are as follows:

$$A = \sqrt{\frac{\nu^2 - \alpha}{2(1 - \nu^2)}} - \sinh \nu z_o \tag{35}$$

$$\sqrt{\frac{(\nu^2 - \alpha)}{2(1 - \nu^2)}} - \frac{\sinh \nu z_o}{2} + \frac{\cosh \nu z_o}{2} \left\{ \frac{(1 - \alpha) \log(1 - \nu^2)}{2\nu} \right\}$$

$$= \frac{1 - \alpha}{2} A \log \frac{(1 + B)}{B} . \tag{36}$$

The values of ν and z_o for different values of α were given by Case et al.[8] and LeCaine.[2] The calculated values of A and B along with ν and z_o for different values of α are given in Table 2.

Table 2. Values of ν,[8] z_o,[2] -A, and B for Various Values of α

α	ν	z_o	-A	B
0.1	0.5254	0.7896	0.0784	3.4259
0.2	0.7104	0.8891	0.1194	3.2692
0.3	0.8286	1.0181	0.1621	3.0616
0.4	0.9073	1.1923	0.2115	2.8860
0.5	0.9575	1.4408	0.2778	2.7934
0.6	0.9856	1.8249	0.3893	2.4035
0.7	0.9974	2.4947	0.6491	1.6261
0.8	0.999909	3.9238	1.8360	1.3939

EMERGENT ANGULAR DISTRIBUTION

A general formula for the determination of directed angular neutron flux at a point z along the direction $\cos^{-1}\mu$ was given by Eq. (5). We shall apply that formula to obtain the emergent angular distribution of neutrons at the boundary. The monoenergetic neutrons which are scattered isotropically in the laboratory system have the following angular distribution at the boundary ($z = 0$).

$$\phi(0,\mu)\ d\mu = \frac{1}{2\mu}\left[\int_0^\infty \Sigma_{s_o}\phi(z')\ e^{-\frac{z'\Sigma_t}{\mu}}\ dz'\ d\mu\right]$$

$$\text{for } (\infty \leqslant z' \leqslant 0). \tag{37}$$

In the above equation z' increases from zero to infinity. Expressing z' in terms of total mean free path and substituting for $\phi(z')$, according to Eq. (32) the emergent angular distribution is given as follows:

$$\phi(0,\mu)\ d\mu = d\mu\ J(0)\ \frac{(1-\alpha)}{2}\sqrt{\frac{2\nu^2(1-\nu^2)}{\alpha(\nu^2-\alpha)}}$$

$$\left[\frac{e^{\nu z_o}}{2(1-\mu\nu)} - \frac{e^{-\nu z_o}}{2(1+\mu\nu)} + \frac{A}{1+B\mu}\right]. \tag{38}$$

NUMERICAL RESULTS

In the following Tables 3 and 4 extensive numerical results for the neutron flux as a function of distance and the emergent angular distribution of neutrons are presented. Some of these results are plotted in Figs. 2 and 3. The values of α range from 0.1 to 0.8. The normalization condition assumed

in these calculations is of the unit neutron density at the boundary. Mathematically, it is expressed as follows:

$$\phi(0) = J(0)\sqrt{\frac{\nu^2}{\alpha}} = 1. \tag{39}$$

For the sake of comparison, the numerical results given by LeCaine[2] using the variational method are also listed. The agreement is excellent. The symbols used in Tables 3 and 4 represent the various physical quantities in the following manner.

$\phi_{asy}(z)$ = asymptotic neutron flux,

$-\Delta\phi(z)$ = non-asymptotic neutron flux,

$a\%$ = percentage deviation of neutron flux

$$= -\left[\frac{\Delta\phi(z)}{\phi_{asy}(z)}\right]\%,$$

$\phi(0,\mu)\,d\mu$ = emergent angular distribution of neutrons at the boundary,

$$b = \frac{\phi(0,\mu)}{\phi(0,1)}$$

= ratio between emergent angular distribution of neutrons in directions μ and $\mu = 1$,

$$c\% = -\left[\frac{\Delta\phi(0,\mu)}{\phi_{asy}(0,\mu)}\right]\%$$

= percentage deviation of the emergent angular distribution of neutrons from the angular distribution corresponding to the asymptotic neutron flux,

* = LeCaine's results by the variational method.

Table 3. Neutron Flux for Various Values of z and α

z	$\phi_{asy}(z)$	$-\Delta\phi(z)$	$-a\%$	$-a^{*}\%$
		$\alpha = 0.1$		
0.00	1.2250	0.22499	18.37	18.29
0.10	1.3905	0.15972	11.49	10.38
0.20	1.5599	0.11339	7.27	6.92
0.30	1.7339	0.08050	4.64	4.85
0.40	1.9124	0.05715	2.99	3.51
0.50	2.0966	0.04057	1.94	2.60
1.00	3.1136	0.00732	0.24	0.70
1.50	4.3469	0.00018	0.04	0.23
2.00	5.8817	0.00012	0.003	0.08
		$\alpha = 0.2$		
0.00	1.2163	0.21634	17.79	17.76
0.10	1.3741	0.15602	11.35	10.35
0.20	1.5387	0.11251	7.31	6.995
0.30	1.7111	0.08113	4.74	4.95
0.40	1.8923	0.05851	3.09	3.60
0.50	2.0828	0.04219	2.03	2.68
1.00	3.2149	0.00822	0.26	0.71
1.50	4.7569	0.0016	0.034	0.22
2.00	6.9053	0.0003	0.004	0.07
		$\alpha = 0.3$		
0.00	1.2065	0.20643	17.11	17.07
0.10	1.3562	0.15200	11.21	10.20
0.20	1.5151	0.11190	7.39	6.999
0.30	1.6846	0.08239	4.89	5.01
0.40	1.8654	0.06656	3.57	3.67
0.50	2.0593	0.04469	2.17	2.74
1.00	3.2706	0.00967	0.30	0.74
1.50	5.0513	0.00268	0.053	0.23
2.00	7.7117	0.00045	0.006	0.07

Table 3 (continued)

z	$\phi_{asy}(z)$	$-\Delta\phi(z)$	$-a\%$	$-a^{*}\%$
		$\alpha = 0.4$		
0.00	1.1934	0.19333	16.20	16.18
0.10	1.3348	0.14486	10.85	9.91
0.20	1.4872	0.10855	7.30	6.90
0.30	1.6521	0.08134	4.92	4.99
0.40	1.8304	0.06037	3.30	3.69
0.50	2.0237	0.04567	2.26	2.77
1.00	3.2780	0.00797	0.24	0.75
1.50	5.2181	0.00188	0.04	0.23
2.00	8.2514	0.00060	0.01	0.07
		$\alpha = 0.5$		
0.00	1.1755	0.17550	14.93	14.94
0.10	1.3088	0.13273	10.14	9.36
0.20	1.4542	0.10038	6.90	6.61
0.30	1.6149	0.07516	4.65	4.84
0.40	1.7864	0.05684	3.18	3.61
0.50	1.9762	0.04342	2.20	2.73
1.00	3.2389	0.01075	0.33	0.75
1.50	5.2578	0.00266	0.05	0.23
2.00	8.5057	0.00066	0.01	0.07
		$\alpha = 0.6$		
0.00	1.1528	0.1528	13.25	13.18
0.10	1.2787	0.12012	9.39	8.41
0.20	1.4168	0.09447	6.67	6.01
0.30	1.5689	0.07428	4.73	4.44
0.40	1.7363	0.05841	3.36	3.34
0.50	1.9203	0.04593	2.39	2.54
1.00	3.1637	0.01381	0.44	0.72
1.50	5.1917	0.00415	0.08	0.22
2.00	8.5004	0.00125	0.02	0.07
		$\alpha = 0.7$		
0.00	1.1218	0.12180	10.86	10.63
0.10	1.2411	0.10352	8.34	6.84
0.20	1.3726	0.08799	6.41	4.91
0.30	1.5178	0.07478	4.93	3.65

Table 3 (continued)

z	$\phi_{asy}(z)$	$-\Delta\phi(z)$	$-a\%$	$-a^{*}\%$
		$\alpha = 0.7$		
0.40	1.6783	0.06356	3.79	2.75
0.50	1.8553	0.05402	2.91	2.10
1.00	3.0592	0.02396	0.78	0.60
1.50	5.0411	0.01063	0.21	0.18
2.00	8.3024	0.00471	0.06	0.05
		$\alpha = 0.8$		
0.00	1.0788	0.07836	7.44	7.03
0.10	1.1923	0.07686	6.60	4.48
0.20	1.3178	0.06686	5.20	3.20
0.30	1.4648	0.05816	4.07	2.37
0.40	1.6097	0.05059	3.22	1.78
0.50	1.7791	0.04401	2.53	1.36
1.00	2.9332	0.02192	0.77	0.40
1.50	4.8361	0.01092	0.23	0.12
2.00	7.9735	0.00544	0.07	0.04

Table 4. Emergent Angular Distribution $\phi(0,\mu)$ for Various Values of μ and α

μ	$\phi(0,\mu)$	$[\phi(0,\mu)]^*$	b	b^*	c%
			$\alpha = 0.1$		
0.00	0.4500	0.4502	2.551×10^{-1}	2.566×10^{-1}	18.37
0.01	0.4607	0.4642	2.612×10^{-1}	2.645×10^{-1}	17.52
0.05	0.5022	0.5077	2.848×10^{-1}	2.893×10^{-1}	14.68
0.10	0.5513	0.5562	3.126×10^{-1}	3.170×10^{-1}	12.96
0.50	0.9510	0.9497	5.391×10^{-1}	5.412×10^{-1}	3.77
0.60	1.0702	1.0686	6.067×10^{-1}	6.090×10^{-1}	3.00
0.90	1.5404	1.5378	8.733×10^{-1}	8.763×10^{-1}	1.58
0.94	1.6222	1.6196	9.197×10^{-1}	9.230×10^{-1}	1.46
0.96	1.6714	1.6629	9.475×10^{-1}	9.476×10^{-1}	1.39
0.98	1.7106	1.7079	9.698×10^{-1}	9.733×10^{-1}	1.33
1.00	1.7639	1.7548	10.000×10^{-1}	10.000×10^{-1}	1.28
			$\alpha = 0.2$		
0.00	0.4000	0.4001	1.800×10^{-1}	1.810×10^{-1}	17.78
0.01	0.4090	0.4117	1.842×10^{-1}	1.864×10^{-1}	17.00
0.05	0.4436	0.4483	1.998×10^{-1}	2.030×10^{-1}	14.36
0.10	0.4859	0.4900	2.188×10^{-1}	2.219×10^{-1}	11.83
0.50	0.8776	0.8774	3.952×10^{-1}	3.974×10^{-1}	3.61
0.60	1.0184	1.0172	4.587×10^{-1}	4.607×10^{-1}	2.02
0.90	1.7422	1.7402	7.846×10^{-1}	7.880×10^{-1}	1.24
0.94	1.9056	1.9037	8.582×10^{-1}	8.621×10^{-1}	1.10
0.96	1.9981	1.9963	8.999×10^{-1}	9.041×10^{-1}	1.04
0.98	2.0683	2.0973	9.315×10^{-1}	9.498×10^{-1}	0.98
1.00	2.2204	2.2081	10.000×10^{-1}	10.000×10^{-1}	0.91
			$\alpha = 0.3$		
0.00	0.3500	0.3501	1.186×10^{-1}	1.186×10^{-1}	17.12
0.01	0.3573	0.3595	1.210×10^{-1}	1.218×10^{-1}	16.40
0.05	0.3858	0.3896	1.307×10^{-1}	1.320×10^{-1}	13.98
0.10	0.4212	0.4245	1.426×10^{-1}	1.438×10^{-1}	11.60
0.50	0.7883	0.7875	2.671×10^{-1}	2.668×10^{-1}	3.49
0.60	0.9410	0.9398	3.188×10^{-1}	3.184×10^{-1}	2.63
0.90	1.9634	1.9621	6.651×10^{-1}	6.648×10^{-1}	0.97
0.94	2.2710	2.2692	7.693×10^{-1}	7.689×10^{-1}	0.81
0.96	2.4611	2.4599	8.337×10^{-1}	8.335×10^{-1}	0.74
0.98	2.6852	2.6840	9.097×10^{-1}	9.094×10^{-1}	0.67
1.00	2.9537	2.9514	10.000×10^{-1}	10.000×10^{-1}	0.60

Table 4 (continued)

μ	$\phi(0,\mu)$	$[\phi(0,\mu)]^*$	b	b^*	c%
			$\alpha = 0.4$		
0.00	0.3000	0.3000	6.94×10^{-2}	6.94×10^{-2}	16.20
0.01	0.3057	0.3074	7.07×10^{-2}	7.11×10^{-2}	15.70
0.05	0.3286	0.3315	7.60×10^{-2}	7.67×10^{-2}	13.36
0.10	0.3572	0.3599	8.26×10^{-2}	8.32×10^{-2}	7.13
0.50	0.6847	0.6841	15.83×10^{-2}	15.82×10^{-2}	3.35
0.60	0.8365	0.8354	19.34×10^{-2}	19.32×10^{-2}	2.48
0.90	2.1638	2.1622	50.04×10^{-2}	50.01×10^{-2}	0.74
0.94	2.7090	2.7073	62.65×10^{-2}	62.62×10^{-2}	0.57
0.96	3.0953	3.0946	71.58×10^{-2}	71.57×10^{-2}	0.49
0.98	3.6109	3.6087	83.51×10^{-2}	83.46×10^{-2}	0.42
1.00	4.3241	4.3237	100.00×10^{-2}	100.00×10^{-2}	0.34
			$\alpha = 0.5$		
0.00	0.2500	0.2500	3.40×10^{-2}	3.40×10^{-2}	14.93
0.01	0.2544	0.2556	3.46×10^{-2}	3.47×10^{-2}	14.36
0.05	0.2721	0.2776	3.70×10^{-2}	3.77×10^{-2}	12.40
0.10	0.2944	0.2964	4.00×10^{-2}	4.03×10^{-2}	10.44
0.50	0.5702	0.5696	7.75×10^{-2}	7.74×10^{-2}	5.41
0.60	0.7084	0.7076	9.62×10^{-2}	9.61×10^{-2}	2.26
0.90	2.2471	2.2456	30.53×10^{-2}	30.50×10^{-2}	0.55
0.94	3.1181	3.1160	42.63×10^{-2}	42.33×10^{-2}	0.39
0.96	3.8608	3.8605	52.54×10^{-2}	52.44×10^{-2}	0.31
0.98	5.0714	5.0673	68.90×10^{-2}	68.83×10^{-2}	0.23
1.00	7.3606	7.3616	100.00×10^{-2}	100.00×10^{-2}	0.16
			$\alpha = 0.6$		
0.00	0.2000	0.2000	1.22×10^{-2}	1.22×10^{-2}	13.25
0.01	0.2032	0.2040	1.24×10^{-2}	1.24×10^{-2}	12.81
0.05	0.2159	0.2175	1.31×10^{-2}	1.32×10^{-2}	11.22
0.10	0.2324	0.2337	1.41×10^{-2}	1.42×10^{-2}	9.58
0.50	0.4491	0.4492	2.73×10^{-2}	2.73×10^{-2}	5.28
0.60	0.5636	0.5633	3.43×10^{-2}	3.42×10^{-2}	2.17
0.90	2.0847	2.0840	12.67×10^{-2}	12.62×10^{-2}	0.46
0.94	3.2125	3.2091	19.53×10^{-2}	19.49×10^{-2}	0.29
0.96	4.3937	4.3896	26.71×10^{-2}	26.66×10^{-2}	0.21
0.98	6.9394	6.9361	42.18×10^{-2}	42.13×10^{-2}	0.13
1.00	16.4500	11.6464	100.00×10^{-2}	100.00×10^{-2}	0.05

Table 4 (continued)

μ	$\phi(0,\mu)$	$[\phi(0,\mu)]^*$	b	b^*	c%
		$\alpha = 0.7$			
0.00	0.1500	0.1500	2.30×10^{-3}	2.30×10^{-3}	10.86
0.01	0.1520	0.1526	2.33×10^{-3}	2.34×10^{-3}	10.58
0.05	0.1603	0.1618	2.46×10^{-3}	2.48×10^{-3}	9.53
0.10	0.1714	0.1733	2.63×10^{-3}	2.65×10^{-3}	8.40
0.50	0.3272	0.3284	5.02×10^{-3}	5.03×10^{-3}	2.99
0.60	0.4119	0.4129	6.32×10^{-3}	6.32×10^{-3}	2.19
0.90	1.6483	1.6455	25.30×10^{-3}	25.18×10^{-3}	0.45
0.94	2.7076	2.7013	41.55×10^{-3}	41.33×10^{-3}	0.27
0.96	3.9791	3.9728	61.06×10^{-3}	60.93×10^{-3}	0.18
0.98	7.5231	7.4954	115.50×10^{-3}	114.70×10^{-3}	0.09
1.00	65.1626	65.355	1000.00×10^{-3}	1000.00×10^{-3}	0.01
		$\alpha = 0.8$			
0.00	0.1000	0.1000	8.44×10^{-5}	8.43×10^{-5}	7.27
0.01	0.1012	0.1023	8.54×10^{-5}	8.62×10^{-5}	7.10
0.05	0.1062	0.1073	8.96×10^{-5}	9.04×10^{-5}	6.45
0.10	0.1130	0.1143	9.53×10^{-5}	9.63×10^{-5}	5.74
0.50	0.2111	0.2114	17.81×10^{-5}	17.90×10^{-5}	2.14
0.60	0.2654	0.2666	22.38×10^{-5}	22.47×10^{-5}	1.57
0.90	1.0744	1.0751	90.63×10^{-5}	90.61×10^{-5}	0.32
0.94	1.7920	1.7925	151.20×10^{-5}	151.10×10^{-5}	0.19
0.96	2.6877	2.6879	226.70×10^{-5}	226.50×10^{-5}	0.12
0.98	5.3667	5.3659	452.70×10^{-5}	452.20×10^{-5}	0.06
1.00	11.8548×10^{2}	11.865×10^{2}	1.00	1.0	0.00

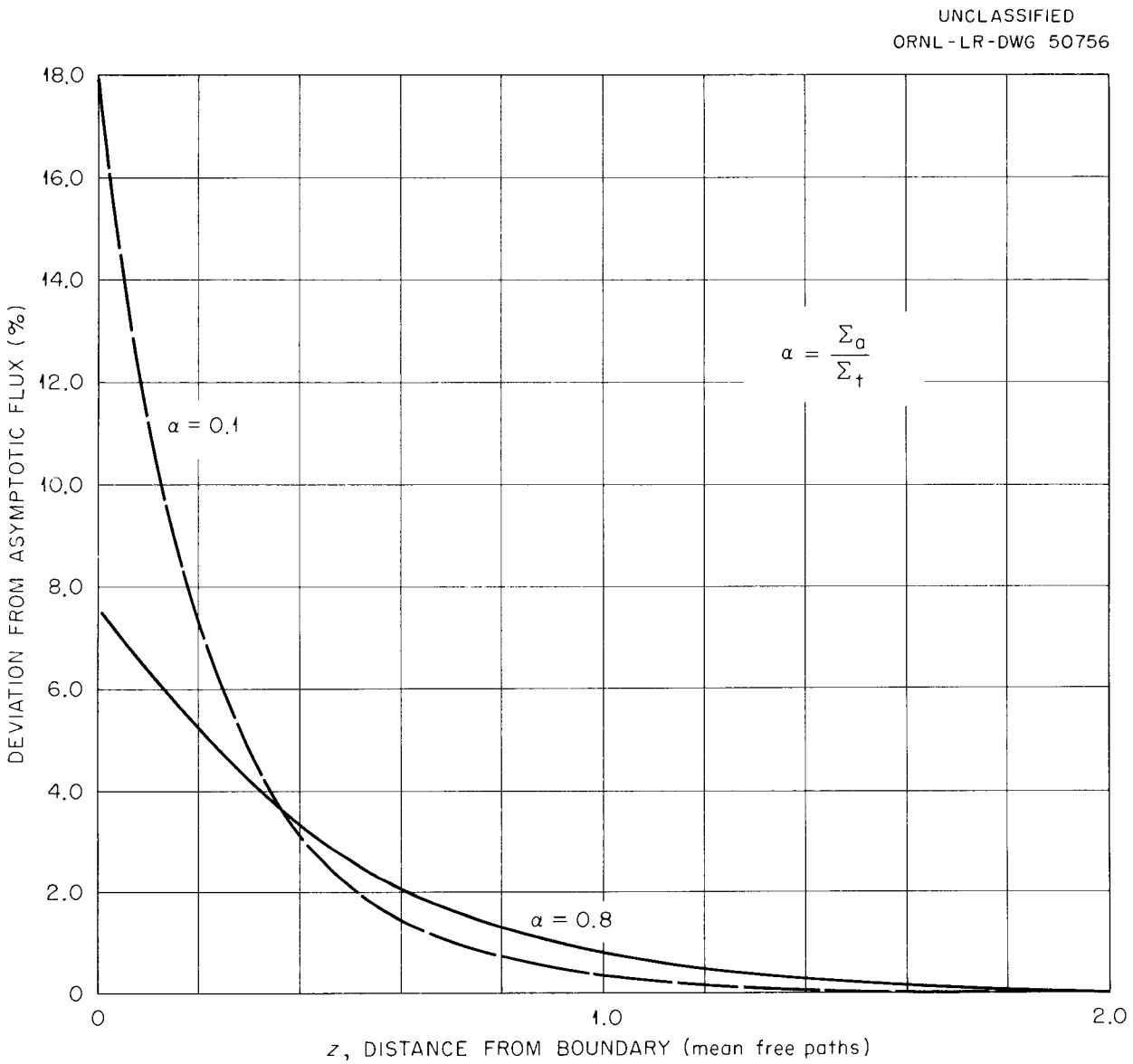

Fig. 2. Percentage Deviation from Asymptotic Flux as a Function of Distance from Boundary.

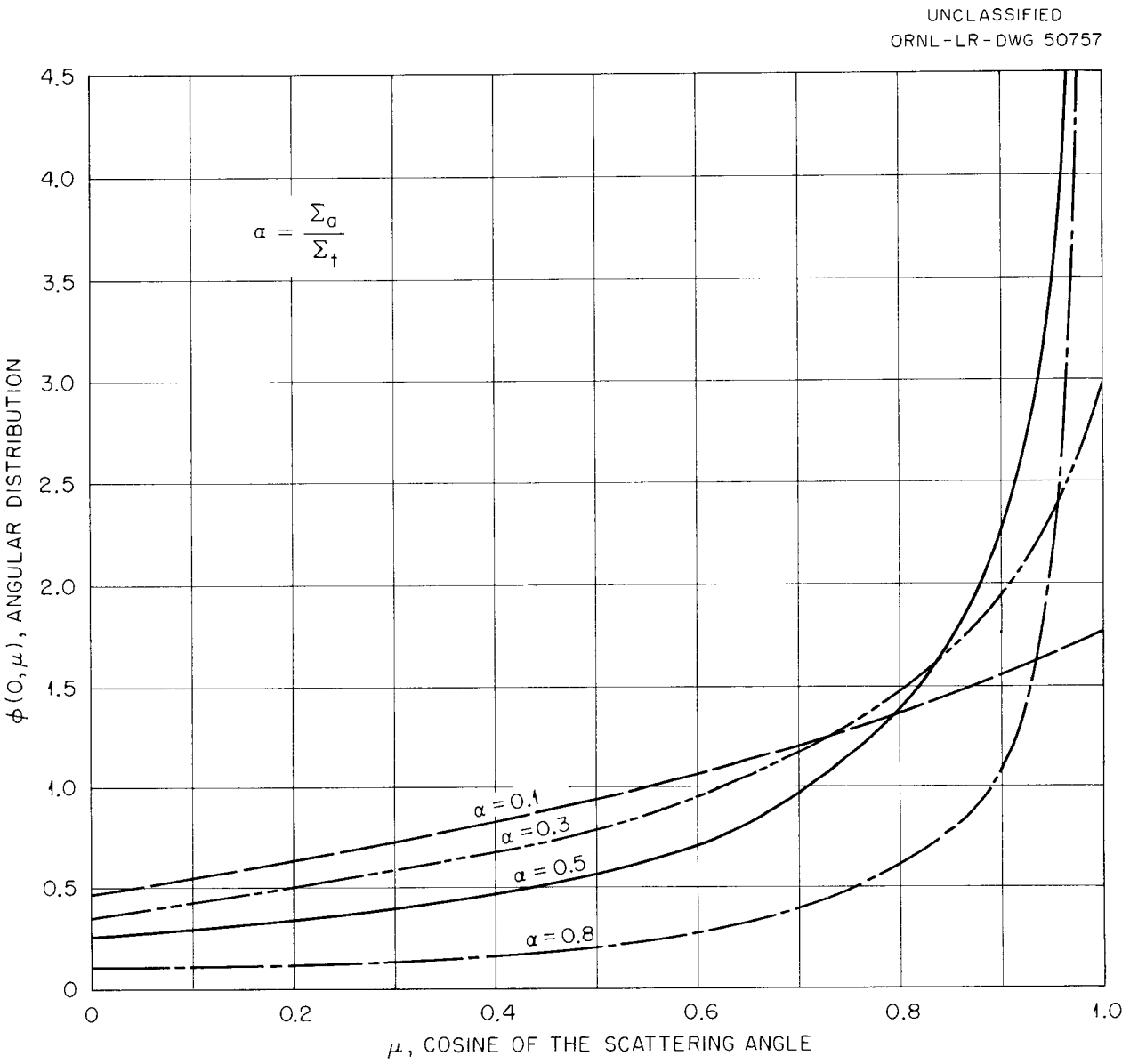

Fig. 3. Angular Distribution of Emergent Neutrons at the Boundary.

DISCUSSION

The general expression relating the angular neutron flux and the total neutron flux has been formulated. In a neutron chopper experiment the neutron beam is extracted from a point away from the boundary. The angular neutron flux in the direction $\mu = 1$ is measured. In order to obtain the exact total neutron flux from the measured angular flux the inverse Laplace transform of the measured quantity is required, which cannot be obtained easily. According to the formulas developed in this study it is possible to determine the total neutron flux provided the flux gradient terms can be estimated. If the point of extraction of beam is more than a distance of one mean free path from the boundary then the neutron flux may be represented by an asymptotic expression. But if this point is near the boundary then the non-asymptotic part of the flux cannot be ignored. The exact determination of the non-asymptotic flux, however, requires the exact solution of the transport equation which is in itself a formidable task. We have, however, estimated the non-asymptotic flux by an approximate method.

The total neutron flux is assumed to consist of two parts - asymptotic and non-asymptotic. The asymptotic expression for flux is known exactly and is given by the studies of Adler and Mark[1,2,8] for the absorbing medium. The non-asymptotic flux is represented by a single exponential. We have calculated the deviations of total flux from the asymptotic flux for different values of absorption (α) and distance (z). We note that the deviation decreases rapidly as we go away from the boundary. At a distance of one mean free path the deviation is less than 1%.

We also calculated the emergent angular distribution. According to Table 4 we note that for a fixed absorption the percentage deviation of the emergent angular distribution from the angular distribution corresponding to the asymptotic neutron flux decreases as the value of μ increases from zero to one. In other words, it means that the distribution of neutrons moving in the forward direction is given by the asymptotic flux. These neutrons come from the interior or the medium. On the other hand, the distribution of

neutrons along the direction $(0 \leq \mu < 1)$ is governed by the non-asymptotic flux. We further observe that for very large absorptions the emergent angular distribution is peaked in the forward direction. The ratio between the distributions along directions $\mu = 0$ and $\mu = 1$ decreases rapidly as the absorption is increased. For large absorption those neutrons contribute the most which come from the interior in the forward direction.

The agreement between the results reported in this study and those given by LeCaine using the variational method are good in all cases. This agreement is understandable as the asymptotic fluxes are given by the same expressions in both the cases. At the boundary the same normalization condition was used in both the cases. Thus at the boundary and at a distance far from the boundary both the methods give identical results. In between two extreme cases the above agreement indicates that a simple trial function will give the non-asymptotic part of the flux. The representation of non-asymptotic or transient flux by a single exponential merits serious consideration and further investigation.

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