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UC-20 - Controlled Thermonuclear Processes

EXACT RELATIVISTIC FOKKER-PLANCK COEFFICIENTS FOR PLASMA AND RADIATION: III

Albert Simon

OAK RIDGE NATIONAL LABORATORY • U.S. ATOMIC ENERGY COMMISSION

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Page 6 - Replace 8π by 4π everywhere in Eqs. (3.4), (3.5), (3.6), and (3.7).

Page 7 - Delete the factor of 2 on the third line and sixth line of the equation.

Page 8 - Change 8π to 4π on the third line of the equation, and delete the 2 in the last line.

Page 10 - Delete the factor 2 on the second line of Eq. (4.15).

Page 11 - Delete the factor 2 on the second line in Eqs. (4.16) and (4.17), and also on the last line of print before the $P^*(u')$.

Page 13 - Delete the 2 in $(u_i^2 - c^2)/2$ in Eq. (5.5).

Page 14 - Delete the 2 in $(u_i^2 - c^2)/2$ in Eq. (5.13).

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Exact Relativistic Fokker-Planck Coefficients
for Plasma and Radiation: III

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Abstract

Exact relativistic Fokker-Planck coefficients have been derived in a previous paper for the case of a plasma composed of electrons and infinite mass ions. In this paper, these results are generalized to the case of an arbitrary number of finite-mass ion species.

I. Introduction

In two previous papers[1,2] we have expanded the "Hierarchy" equations for plasma and radiation by a generalization of the Rostoker-Rosenbluth technique. To first order we obtained a pair of coupled integral equations for the particle-particle and particle-oscillator pair correlation functions. We then considered the case of a plasma which, in zero order, was static, uniformly distributed in space and with no external magnetic or electric fields, together with a zero-order oscillator distribution which was also static. We were then able to solve the particle-particle pair correlation equation exactly, by a generalization of the method of Lenard and Balescu, for the case of plasma composed of electrons and infinite mass ions. The resultant time-asymptotic solution immediately yields the Fokker-Planck coefficients for the electron distribution.

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In this paper, we will sketch the generalization of the above procedures for the case of a plasma composed of electrons and an arbitrary number of finite-mass ion species, and obtain the corresponding Fokker-Planck coefficients.

II. The Pair Correlation Equations for the Multi-Ion-Species Case

The Liouville equation for plasma and radiation has been written down in a very general form in I [see Eqs. (1) through (9)]. It was only in the perturbation expansion, going from Eq. (9) to Eq. (10), that we specialized to a plasma of electrons and infinite-mass ions. However, from Eq. (9) it is quite obvious how to write down Eq. (10) if this assumption is not made. The resultant zeroth-order equation still has a solution which is a product of one-particle and one-oscillator distribution functions and each of these obeys an equation which is an obvious generalization of Eqs. (13a) and (13b). Thus

$$\left\{ \frac{\partial}{\partial t} + \underset{\sim}{v}_j \cdot \frac{\partial}{\partial \underset{\sim}{q}_j} + \frac{e_j}{m_j} \left[\frac{1}{c} \sum_{\lambda} \int \left\{ q_\lambda \underset{\sim}{v}_j \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}_\lambda) - v_\lambda \underset{\sim}{A}_\lambda \right\} f_0^1(\lambda)\, dq_\lambda dv_\lambda \right.\right.$$

$$\left.\left. - \sum_{\ell} \frac{N_\ell e_\ell}{V} \int \frac{\partial}{\partial \underset{\sim}{q}_j} \left(\frac{1}{r_{j\ell}} \right) f_0^1(\ell)\, dq_\ell dv_\ell \right] \frac{\partial}{\partial \underset{\sim}{v}_j} \right\} f_0^1(j) = 0 \qquad (2.1)$$

and

$$\left\{ \frac{\partial}{\partial t} + v_\lambda \frac{\partial}{\partial q_\lambda} - \omega_\lambda^2 q_\lambda \frac{\partial}{\partial v_\lambda} + \left[\sum_{\ell} \frac{N_\ell e_\ell}{Vc} \int \underset{\sim}{v}_\ell \cdot \underset{\sim}{A}_\lambda f_0^1(\ell)\, d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell \right] \right.$$

$$\left. \cdot \frac{\partial}{\partial v_\lambda} \right\} f_0^1(\lambda) = 0 \qquad (2.2)$$

where the sums over ℓ denote a sum over all the species of particles in the plasma (electrons and ions) each having its own total particle number N_ℓ and electric charge e_ℓ.

Similarly, the first-order equation still has a solution which is a product of a pair correlation function and one-particle and one-oscillator distribution functions, summed over all possible pair correlations (excluding two oscillator correlations since these are absent in first order in a classical problem). Thus the generalization of Eqs. (25), (26), (27), and (28) of I are obvious; and we may then go through precisely the same procedure as in I to obtain a set of four equations in four unknowns. Once again, these may be reduced to two equations in two unknowns, the particle-particle pair correlation function $g(i,j)$ and the particle-oscillator pair correlation function $g(i,\lambda)$. The generalization of Eq. (38) of I is easily shown to be:

$$\begin{aligned}
&\Bigg\{ \frac{\partial}{\partial t} + \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{q}_i} + \underset{\sim}{F}_o(i)\cdot \frac{\partial}{\partial \underset{\sim}{v}_i} + v_\lambda \frac{\partial}{\partial q_\lambda} - \omega_\lambda^2 q_\lambda \frac{\partial}{\partial v_\lambda} \\
&\qquad + \Bigg[\sum_\ell \frac{N_\ell e_\ell}{Vc} \int \underset{\sim}{v}_\ell \cdot \underset{\sim}{A}_\lambda \, f_o^1(\ell)\, d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell \Bigg] \frac{\partial}{\partial v_\lambda} \Bigg\} g(i,\lambda) \\
&= \frac{e_i}{m_i V} \frac{\partial}{\partial \underset{\sim}{v}_i} f_o^1(i) \cdot \sum_\ell N_\ell e_\ell \int \frac{\partial}{\partial \underset{\sim}{q}_i} \left(\frac{1}{r_{i\ell}} \right) g(\ell,\lambda)\, d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell \\
&\quad - \frac{1}{Vc} \frac{\partial}{\partial v_\lambda} f_o^1(\lambda) \sum_\ell N_\ell e_\ell \int \underset{\sim}{v}_\ell \cdot \underset{\sim}{A}_\lambda \, g(i,\ell)\, d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell - \frac{e_i}{m_i c} f_o^1(\lambda) \frac{\partial}{\partial \underset{\sim}{v}_i} f_o^1(i) \\
&\quad \cdot \Big\{ q_\lambda [\underset{\sim}{v}_i \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}_\lambda)] - v_\lambda \underset{\sim}{A}_\lambda \Big\} - \frac{e_i}{c} f_o^1(i) \frac{\partial}{\partial v_\lambda} f_o^1(\lambda)\, \underset{\sim}{v}_i \cdot \underset{\sim}{A}_\lambda, \qquad (2.3)
\end{aligned}$$

where

$$\underset{\sim}{F}_o(i) = \frac{e_i}{m_i}\left\{ \frac{1}{c} \sum_{\nu} \int \left\{ q_\nu [\underset{\sim}{v}_i \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}_\nu)] - v_\nu \underset{\sim}{A}_\nu \right\} f_o^1(\nu)\ dq_\nu dv_\nu \right.$$

$$\left. - \sum_{\ell} \frac{N_\ell e_\ell}{V} \int \frac{\partial}{\partial \underset{\sim}{q}_i} \left(\frac{1}{r_{i\ell}} \right) f_o^1(\ell)\ d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell + \frac{1}{c} [\underset{\sim}{v}_i \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}^e) - \dot{\underset{\sim}{A}}^e] - \frac{\partial}{\partial \underset{\sim}{q}_i} \phi^e \right\} \tag{2.4}$$

and the generalization of Eq. (40) of I is also easily found to be:

$$\left\{ \frac{\partial}{\partial t} + \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{q}_i} + \underset{\sim}{v}_j \cdot \frac{\partial}{\partial \underset{\sim}{q}_j} + \underset{\sim}{F}_o(i) \cdot \frac{\partial}{\partial \underset{\sim}{v}_i} + \underset{\sim}{F}_o(j) \cdot \frac{\partial}{\partial \underset{\sim}{v}_j} \right\} g(i,j)$$

$$= \frac{e_i}{m_i V} \frac{\partial f_o^1(i)}{\partial \underset{\sim}{v}_i} \cdot \sum_{\ell} N_\ell e_\ell \int \frac{\partial}{\partial \underset{\sim}{q}_i} \left(\frac{1}{r_{i\ell}} \right) g(j,\ell)\ d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell$$

$$+ \frac{e_j}{m_j V} \frac{\partial f_o^1(j)}{\partial \underset{\sim}{v}_j} \cdot \sum_{\ell} N_\ell e_\ell \int \frac{\partial}{\partial \underset{\sim}{q}_j} \left(\frac{1}{r_{j\ell}} \right) g(i,\ell)\ d\underset{\sim}{q}_\ell d\underset{\sim}{v}_\ell$$

$$- \frac{e_i}{m_i c} \frac{\partial f_o^1(i)}{\partial \underset{\sim}{v}_i} \cdot \sum_{\nu} \int \left\{ q_\nu [\underset{\sim}{v}_i \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}_\nu)] - v_\nu \underset{\sim}{A}_\nu \right\} g(j,\nu)\ dq_\nu dv_\nu$$

$$- \frac{e_j}{m_j c} \frac{\partial f_o^1(j)}{\partial \underset{\sim}{v}_j} \cdot \sum_{\nu} \int \left\{ q_\nu [\underset{\sim}{v}_j \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}_\nu)] - v_\nu \underset{\sim}{A}_\nu \right\} g(i,\nu)\ dq_\nu dv_\nu$$

$$+ \frac{e_i e_j}{m_i} f_o^1(j) \frac{\partial f_o^1(i)}{\partial \underset{\sim}{v}_i} \cdot \frac{\partial}{\partial \underset{\sim}{q}_i} \left(\frac{1}{r_{ij}} \right) + \frac{e_j e_i}{m_j} f_o^1(i) \frac{\partial f_o^1(j)}{\partial \underset{\sim}{v}_j} \cdot \frac{\partial}{\partial \underset{\sim}{q}_j} \left(\frac{1}{r_{ij}} \right) \tag{2.5}$$

We shall not write down the multi-species generalization of Eqs. (41), (42), and (43) of I since these should be obvious from the form of the results obtained above.

III. The Fourier-Laplace Transform of the Particle-Particle Pair Correlation Equation

As in II, we now specialize to the case of a plasma which in zero order is static, spatially uniform and with no external electric or magnetic fields. The zero-order equations are then satisfied by

$$f_o^1(i) = f_o(\underset{\sim}{v}_i) \tag{3.1}$$

$$f_o^1(\lambda) = f_o(v_\lambda^2 + \omega_\lambda^2 q_\lambda^2) \tag{3.2}$$

where the f_o are completely arbitrary functions of the indicated arguments for each particle species and each oscillator, and are normalized to unity for integration over the argument. Once again we may then take the Fourier transform in space and the Laplace transform in time of Eqs. (2.3) and (2.5). Proceeding as in II, we obtain

$$\begin{aligned}
[ip + \underset{\sim}{k}\cdot(\underset{\sim}{v}_i - \underset{\sim}{v}_j)]\, g(\underset{\sim}{v}_i\underset{\sim}{v}_j) &= \frac{4\pi}{k^2}\underset{\sim}{k}\cdot\Bigg[\frac{e_i}{m_i}\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\sum_\ell n_\ell e_\ell \int g^*(\underset{\sim}{v}_j\underset{\sim}{v}_\ell)\, d^3v_\ell \\
&\quad - \frac{e_j}{m_j}\frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j}\sum_\ell n_\ell e_\ell \int g(\underset{\sim}{v}_i\underset{\sim}{v}_\ell)\, d^3v_\ell\Bigg] - \frac{e_i}{m_i}\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \\
&\quad \cdot [\underset{\sim}{v}_i \times (\underset{\sim}{k} \times \underset{\sim}{E}_j^*(\underset{\sim}{v}_j)) + \underset{\sim}{F}_j^*(\underset{\sim}{v}_j)] \\
&\quad + \frac{e_j}{m_j}\frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j}\cdot[\underset{\sim}{v}_j \times (\underset{\sim}{k} \times \underset{\sim}{E}_i(\underset{\sim}{v}_i)) + \underset{\sim}{F}_i(\underset{\sim}{v}_i)] + \frac{4\pi}{pk^2}\vec{k} \\
&\quad \cdot\left[\frac{e_i e_j}{m_i} f_o(\underset{\sim}{v}_j)\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} - \frac{e_j e_i}{m_j} f_o(\underset{\sim}{v}_i)\frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j}\right]
\end{aligned} \tag{3.3}$$

$$(ip + \underset{\sim}{k}\cdot\underset{\sim}{v}_i)\ \underset{\sim}{E}_i(\underset{\sim}{v}_i) - \underset{\sim}{F}_i(\underset{\sim}{v}_i) = \frac{4\pi e_i}{k^2 m_i}\,\underset{\sim}{k}\cdot\ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\ \sum_{\ell} n_\ell e_\ell \int \underset{\sim}{E}_\ell(\underset{\sim}{v}_\ell)\ d^3v_\ell$$

$$- \frac{4\pi e_i}{p}\ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\cdot\ [\underset{\sim}{v}_i \times (\underset{\sim}{k} \times Q_o)] \qquad (3.4)$$

$$(ip + \underset{\sim}{k}\cdot\underset{\sim}{v}_i)\ \underset{\sim}{F}_i(\underset{\sim}{v}_i) - \omega_k^2 \underset{\sim}{E}_i(\underset{\sim}{v}_i) = \frac{4\pi e_i}{k^2 m_i}\,\underset{\sim}{k}\cdot\ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\ \sum_{\ell} n_\ell e_\ell \int \underset{\sim}{F}_\ell(\underset{\sim}{v}_\ell)\ d^3v_\ell$$

$$- 4\pi \sum_{\ell} n_\ell e_\ell \int (I\cdot\underset{\sim}{v}_\ell)\, g(\underset{\sim}{v}_i \underset{\sim}{v}_\ell)\ d^3v_\ell - \frac{4\pi e_i}{p}\ \omega_k^2\ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\cdot Q_o - \frac{4\pi e_i}{p}\ (\underset{\sim}{v}_i\cdot I)\ f_o(\underset{\sim}{v}_i)$$

$$(3.5)$$

In the equations above, $\underset{\sim}{E}$ and $\underset{\sim}{F}$ have been defined slightly differently than in II. Thus

$$\underset{\sim}{E}(\underset{\sim}{v}) = (4\pi V)^{\frac{1}{2}} \sum_{\kappa} \underset{\sim}{u}_\kappa \int q_\kappa\ g(\underset{\sim}{v}\ q_\kappa v_\kappa u_\kappa)\ dq_\kappa dv_\kappa \qquad (3.6)$$

$$\underset{\sim}{F}(\underset{\sim}{v}) = (4\pi V)^{\frac{1}{2}} \sum_{\kappa} i\underset{\sim}{u}_\kappa \int v_\kappa\ g(\underset{\sim}{v}\ q_\kappa v_\kappa u_\kappa)\ dq_\kappa dv_\kappa \qquad (3.7)$$

Equations (3.3), (3.4), and (3.5) above are the generalization of Eqs. (2.11), (2.12), and (2.13) of II. We now define the constant vectors

$$\sum_{\ell} n_\ell e_\ell \int \underset{\sim}{E}_\ell(\underset{\sim}{v}_\ell)\ d^3v_\ell = \underset{\sim}{C}_1 \qquad (3.8)$$

$$\sum_{\ell} n_\ell e_\ell \int \underset{\sim}{F}_\ell(\underset{\sim}{v}_\ell)\ d^3v_\ell = \underset{\sim}{C}_2 \qquad (3.9)$$

and solve Eqs. (3.4) and (3.5) for $\mathbf{E}$ and $\mathbf{F}$ in terms of $\mathbf{C}_1$ and $\mathbf{C}_2$ just as before. Then these results are substituted in Eq. (3.3) to give a single integral equation for $g(\mathbf{v}_i\mathbf{v}_j)$. As before we also consider only the time-asymptotic value of $g(\mathbf{v}_i\mathbf{v}_j)$. As a result we obtain the following generalization of Eq. (2.21) of II:

$$
\begin{aligned}
g(\mathbf{v}_i\mathbf{v}_j) = \lim_{p\to 0} \frac{4\pi}{ip + \mathbf{k}\cdot(\mathbf{v}_i - \mathbf{v}_j)} \Bigg\{ & \frac{\mathbf{k}}{k^2}\cdot\left[\frac{e_i}{m_i}\frac{\partial f_0(\mathbf{v}_i)}{\partial \mathbf{v}_i} \sum_\ell n_\ell e_\ell \int g^*(\mathbf{v}_j\mathbf{v}_\ell)\, d^3v_\ell \right.\\
& \left. - \frac{e_j}{m_j}\frac{\partial f_0(\mathbf{v}_j)}{\partial \mathbf{v}_j}\cdot \sum_\ell n_\ell e_\ell \int g(\mathbf{v}_i\mathbf{v}_\ell)\, d^3v_\ell \right] + \frac{\mathbf{k}}{k^2}\cdot\left[\frac{e_ie_j}{m_i} f_0(\mathbf{v}_j)\frac{\partial f_0(\mathbf{v}_i)}{\partial \mathbf{v}_i}\right.\\
\ast\quad & \left. - \frac{e_je_i}{m_j} f_0(\mathbf{v}_i)\frac{\partial f_0(\mathbf{v}_j)}{\partial \mathbf{v}_j}\right] + 2\mathbf{k}\cdot\left[\frac{e_i}{m_i}\frac{\partial f_0(\mathbf{v}_i)}{\partial \mathbf{v}_i}\frac{\mathbf{v}_i}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2}\cdot\sum_\ell n_\ell e_\ell \right.\\
& \int (\mathrm{I}\cdot\mathbf{v}_\ell)\, g^*(\mathbf{v}_j\mathbf{v}_\ell)\, d^3v_\ell - \frac{e_j}{m_j}\frac{\partial f_0(\mathbf{v}_j)}{\partial \mathbf{v}_j}\frac{\mathbf{v}_j}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2}\cdot\\
& \left. \qquad\qquad \cdot \sum_\ell n_\ell e_\ell \int (\mathrm{I}\cdot\mathbf{v}_\ell)\, g(\mathbf{v}_i\mathbf{v}_\ell)\, d^3v_\ell \right]\\
& + 2\,\mathbf{v}_i\cdot\mathrm{I}\cdot\mathbf{v}_j\, \mathbf{k}\cdot\left[\frac{e_ie_j}{m_i}\frac{f_0(\mathbf{v}_j)}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2}\frac{\partial f_0(\mathbf{v}_i)}{\partial \mathbf{v}_i} - \frac{e_je_i}{m_j}\frac{f_0(\mathbf{v}_i)}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2}\frac{\partial f_0(\mathbf{v}_j)}{\partial \mathbf{v}_j}\right]\\
& - \frac{e_ie_j}{m_im_jk^2}\,\mathbf{k}\cdot\frac{\partial f_0(\mathbf{v}_i)}{\partial \mathbf{v}_i}\,\mathbf{k}\cdot\frac{\partial f_0(\mathbf{v}_j)}{\partial \mathbf{v}_j}\left[\frac{\mathbf{v}_i\cdot[\mathbf{C}_2^* + (\mathbf{k}\cdot\mathbf{v}_j)\mathbf{C}_1^*]}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} - \frac{\mathbf{v}_j\cdot[\mathbf{C}_2 + (\mathbf{k}\cdot\mathbf{v}_i)\mathbf{C}_1]}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2}\right]
\end{aligned}
$$

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$$+ \frac{e_i e_j}{m_i m_j k^2} [(\mathbf{k}\cdot\mathbf{v}_i)(\mathbf{k}\cdot\mathbf{v}_j) - k^2c^2]\mathbf{k}\cdot \left[\frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \frac{\frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i}\cdot \mathbf{C}_1^*}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} \right.$$

$$\left. - \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \frac{\frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j}\cdot \mathbf{C}_1}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \right] \Bigg\}$$

$$- 4\pi \Bigg\{ \frac{e_i}{m_i} \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot \frac{\sum_\ell n_\ell e_\ell \int (\mathrm{I}\cdot\mathbf{v}_\ell)\; g^*(\mathbf{v}_j\mathbf{v}_\ell)\; d^3v_\ell}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} + \frac{e_j}{m_j} \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \cdot$$

$$\frac{\sum_\ell n_\ell e_\ell \int (\mathrm{I}\cdot\mathbf{v}_\ell)\; g(\mathbf{v}_i\mathbf{v}_\ell)\; d^3v_\ell}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} + \frac{e_i e_j}{m_i} \frac{f_o(\mathbf{v}_j)}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot \mathrm{I}\cdot\mathbf{v}_j$$

$$+ \frac{e_j e_i}{m_j} \frac{f_o(\mathbf{v}_i)}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \cdot \mathrm{I}\cdot\mathbf{v}_i - \frac{e_i e_j}{m_i m_j} \frac{[(\mathbf{k}\cdot\mathbf{v}_i)(\mathbf{k}\cdot\mathbf{v}_j) + k^2c^2]}{[(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2][(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2]}$$

$$\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i}\, \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j}\, \mathbf{v}_i\cdot Q_o\cdot \mathbf{v}_j - \frac{e_i e_j}{m_i m_j} \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot Q_o \cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j}$$

$$+ \frac{e_i e_j}{m_i m_j} \left[\frac{(\mathbf{k}\cdot\mathbf{v}_i)\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i}}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \mathbf{v}_i\cdot Q_o\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} + \frac{(\mathbf{k}\cdot\mathbf{v}_j)\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j}}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} \mathbf{v}_j\cdot Q_o\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \right]$$

$$- \frac{e_i e_j \mathbf{k}}{m_i m_j k^2} \cdot \left[\frac{\frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j}}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot \mathbf{C}_2^* + \frac{\frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i}}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \cdot \mathbf{C}_2 \right] \Bigg\}$$

(3.10)

IV. Generalization of the Lenard Balescu Method

We can solve Eq. (3.10) in the same manner as in II if we generalize our previous definitions as follows:

$$h(\underset{\sim}{v}_i) = \sum_\ell n_\ell e_\ell \int g(\underset{\sim}{v}_i, \underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.1}$$

$$H(u) = k \sum_\ell n_\ell e_\ell \int h(\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.2}$$

$$\underset{\sim}{p}(\underset{\sim}{v}_i) = \sum_\ell n_\ell e_\ell \int (I\cdot\underset{\sim}{v}_\ell)\ g(\underset{\sim}{v}_i, \underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.3}$$

$$\underset{\sim}{P}(u) = k \sum_\ell n_\ell e_\ell \int \underset{\sim}{p}(\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.4}$$

$$\underset{\sim}{S}(u) = k \sum_\ell n_\ell e_\ell \int (I\cdot\underset{\sim}{v}_\ell)\ h(\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.5}$$

$$T(u) = k \sum_\ell n_\ell e_\ell \int \underset{\sim}{p}(\underset{\sim}{v}_\ell)(I\cdot\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.6}$$

as well as those for the modified zero-order functions:

$$F_o(u) = k \sum_\ell n_\ell e_\ell^2 \int f_o(\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.7}$$

$$F_o'(u) = \underset{\sim}{k} \cdot \sum_\ell \frac{n_\ell e_\ell^2}{m_\ell} \int \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.8}$$

$$\underset{\sim}{J}_o'(u) = \underset{\sim}{k} \cdot \sum_\ell \frac{n_\ell e_\ell^2}{m_\ell} \int \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\ (I\cdot\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.9}$$

$$\underset{\sim}{J}_o(u) = k \sum_\ell n_\ell e_\ell^2 \int (I\cdot\underset{\sim}{v}_\ell)\ f_o(\underset{\sim}{v}_\ell)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\ d^3v_\ell \tag{4.10}$$

$$\underset{\sim}{K}_o(u) = k\sum_{\ell} \frac{n_\ell e_\ell^2}{m_\ell} \int I\cdot \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\, \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\; d^3v_\ell \tag{4.11}$$

$$M_o'(u) = \underset{\sim}{k}\cdot\sum_{\ell} \frac{n_\ell e_\ell^2}{m_\ell} \int \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\, (I\cdot\underset{\sim}{v}_\ell)(\underset{\sim}{v}_\ell\cdot I)\; \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\; d^3v_\ell \tag{4.12}$$

$$M_o(u) = k\sum_{\ell} n_\ell e_\ell^2 \int f_o(\underset{\sim}{v}_\ell)(I\cdot\underset{\sim}{v}_\ell)(\underset{\sim}{v}_\ell\cdot I)\; \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\; d^3v_\ell \tag{4.13}$$

$$L_o(u) = k\sum_{\ell} \frac{n_\ell e_\ell^2}{m_\ell} \int (I\cdot\underset{\sim}{v}_\ell) I\cdot \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\; \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v}_\ell)\; d^3v_\ell \tag{4.14}$$

We now proceed exactly as in II. The first equation is obtained by integrating Eq. (3.10) over d^3v_j, multiplying by n_je_j and summing over all j. We obtain

$$h(\underset{\sim}{v}_i) = \lim_{\epsilon\to 0} \int \frac{du'}{i\epsilon + u_i - u'}\; \frac{4\pi}{k^2}\left\{ \frac{e_i}{m_i}\, \frac{\underset{\sim}{k}}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\, [H^*(u') + F_o(u')] \right.$$

$$- F_o'(u')\; [h(\underset{\sim}{v}_i) + e_i f_o(\underset{\sim}{v}_i)] + 2\left[\frac{e_i}{m_i}\, \frac{\underset{\sim}{k}}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\; \frac{\underset{\sim}{v}_i\cdot\underset{\sim}{P}^*(u')}{(u')^2 - c^2}\right.$$

✗ $$\left.\left. - \underset{\sim}{J}_o'(u')\cdot \frac{\underset{\sim}{p}(\underset{\sim}{v}_i)}{u_i^2 - c^2} \right]\right\} + \;\ldots\ldots\ldots \tag{4.15}$$

where $ku_i = \underset{\sim}{k}\cdot\underset{\sim}{v}_i$ and where we have written only the first few terms in the equation since these are all that are necessary to illustrate the method of solution and since the remaining terms may be readily obtained from Eq. (3.10).

Similarly, the first few terms of the equation for $H(u_i)$ are:*

*This definition of H(u) and its use in obtaining a solution for many-species in the pure Coulomb case was first pointed out to me by N. Rostoker (private communication).

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$$H(u_i) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u_i - u'} \frac{4\pi}{k^2} \Big\{ F_0'(u_i) \; [H^*(u') + F_0(u')] - F_0'(u')$$

$$\cdot \; [H(u_i) + F_0(u_i)] + 2 \left[\underset{\sim}{J}_0'(u_i) \cdot \frac{\underset{\sim}{P}^*(u')}{(u')^2 - c^2} - \underset{\sim}{J}_0'(u') \cdot \frac{\underset{\sim}{P}(u_i)}{u_i^2 - c^2} \right] + \ldots \qquad (4.16)$$

and for $\underset{\sim}{S}(u_i)$, we have

$$\underset{\sim}{S}(u_i) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u_i - u'} \frac{4\pi}{k^2} \Big\{ \underset{\sim}{J}_0'(u_i) \; [H^*(u') + F_0(u')] - F_0'(u')$$

$$\cdot \; [\underset{\sim}{S}(u_i) + \underset{\sim}{J}_0(u_i)] + 2 \left[M_0'(u_i) \cdot \frac{\underset{\sim}{P}^*(u')}{(u')^2 - c^2} - \underset{\sim}{J}_0'(u') \cdot \frac{T(u_i)}{u_i^2 - c^2} \right] + \ldots \qquad (4.17)$$

Note that Eqs. (4.16) and (4.17) are formally identical to Eqs. (3.16) and (3.17) of II except for small differences in n and m due to altered dimensions in the definitions of $F_0(u_i)$, etc. Of course, a similar set of equations exists for $\underset{\sim}{p}(\underset{\sim}{v}_i)$, $\underset{\sim}{P}(u_i)$, and $T(u_i)$.

Once again, as in II, we reduce Eqs. (4.15), (4.16), and (4.17) to a single equation which no longer has an integral term in it by multiplying each equation by a 3 by 3 determinant and adding. It is clear from the form of the coefficients of $H^*(u')$ and $2\,\underset{\sim}{P}^*(u')/[(u')^2 - c^2]$ that these determinants are:

$$\mathcal{A} = \begin{vmatrix} F_0' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix} \qquad (4.18)$$

* SEE ERRATA

$$\beta = -\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \begin{vmatrix} 1 & v_{i1} & v_{i2} \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix} \tag{4.19}$$

$$\mathcal{C}_1 = \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \begin{vmatrix} 1 & v_{i1} & v_{i2} \\ F_o' & J_1' & J_2' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix} \tag{4.20}$$

$$\mathcal{C}_2 = -\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \begin{vmatrix} 1 & v_{i1} & v_{i2} \\ F_o' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \end{vmatrix} \tag{4.21}$$

which are identical to those used in II except that v_i and u_i have subscripts on them, and now F_o', $\underset{\sim}{J}_o'$, and M_o' have their generalized definitions. It is clear now that the entire procedure of solution goes through as before.

V. The Fokker-Planck Coefficients

Thus we obtain the final result for the Fokker-Planck equation for the distribution of species i:

$$\text{r.h.s.} = -\frac{\partial}{\partial \underset{\sim}{v}_i} \cdot \left\{ \underset{\sim}{A}_i \, f_o(\underset{\sim}{v}_i) + B_i \cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \right\} \tag{5.1}$$

where

$$\underset{\sim}{A}_i = \frac{\omega_{pi}^2}{(2\pi)^3\ n_i} \int \frac{\underset{\sim}{k}}{k^2}\ \mathrm{IM}(\underset{\sim}{\alpha}_i \cdot \frac{1}{Y} \cdot \underset{\sim}{\alpha}_i)\ d^3k \tag{5.2}$$

$$B_i = -\frac{\omega_{pi}^2}{(2\pi)^3\ n_i m_i} \int \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^3}\ \mathrm{IM}(\underset{\sim}{\alpha}_i \cdot \frac{1}{Y} \cdot \frac{G}{G'} \cdot \underset{\sim}{\alpha}_i)\ d^3k \tag{5.3}$$

where

$$\underset{\sim}{\alpha}_i = (1,\ v_{1i},\ v_{2i}) \tag{5.4}$$

$$\star \quad Y(u_i) = \begin{pmatrix} 1+\psi & \chi_1 & \chi_2 \\ \chi_1 & \frac{u_i^2-c^2}{2} + (\theta+\tilde{B}_o)_{11} & (\theta+\tilde{B}_o)_{12} \\ \chi_2 & (\theta+\tilde{B}_o)_{21} & \frac{u_i^2-c^2}{2} + (\theta+\tilde{B}_o)_{22} \end{pmatrix} \tag{5.5}$$

$$G(u_i) = \begin{pmatrix} F_o & J_1 & J_2 \\ J_1 & M_{11} & M_{12} \\ J_2 & M_{21} & M_{22} \end{pmatrix} \tag{5.6}$$

$$G'(u_i) = \begin{pmatrix} F_o' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{pmatrix} \tag{5.7}$$

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and

$$\psi(u_i) = \lim_{\epsilon \to o} \frac{4\pi}{k^2} \int \frac{F_o'(u')\, du'}{i\epsilon + u_i - u'} \tag{5.8}$$

$$\underset{\sim}{\chi}(u_i) = \lim_{\epsilon \to o} \frac{4\pi}{k^2} \int \frac{\underset{\sim}{J}_o'(u')\, du'}{i\epsilon + u_i - u'} \tag{5.9}$$

$$\Theta\,(u_i) = \lim_{\epsilon \to o} \frac{4\pi}{k^2} \int \frac{M_o'(u')\, du'}{i\epsilon + u_i - u'} \tag{5.10}$$

$$\begin{aligned} \tilde{B}_o &= \frac{4\pi}{k^2} \int L_o(u)\, du \\ &= \sum_{\ell} \frac{\omega_{p\ell}^2}{k^2} \int (I\cdot\underset{\sim}{v}_\ell)\; I\cdot \frac{\partial f_o(\underset{\sim}{v}_\ell)}{\partial \underset{\sim}{v}_\ell}\, d^3v_\ell \end{aligned} \tag{5.11}$$

and F_o, F_o', etc. are defined in Eqs. (4.7) to (4.14).

The reduction to the isotropic case is as direct as before and we obtain

$$\underset{\sim}{A}_i = \frac{\omega_{pi}^2}{2\pi m_i} \int \frac{\underset{\sim}{k}}{k^4} \left\{ \frac{F_o'(u')}{\left| 1 + \psi(u_i) \right|^2} + \frac{(v_i^2 - u_i^2)\, M_{11}'(u_i)}{\left| \dfrac{u_i^2 - c^2}{2} + (\Theta + \tilde{B}_o)_{11} \right|^2} \right\} d^3k \tag{5.12}$$

* $$B_i = - \frac{\omega_{pi}^2}{2\pi m_i m_i} \int \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^5} \left\{ \frac{F_o(u_i)}{\left| 1 + \psi(u_i) \right|^2} + \frac{(v_i^2 - u_i^2)\, M_{11}(u_i)}{\left| \dfrac{u_i^2 - c^2}{2} + (\Theta + \tilde{B}_o)_{11} \right|^2} \right\} d^3k. \tag{5.13}$$

Of course, the entire right hand side of Eq. (5.1) vanishes when all species are in a Maxwell-Boltzmann distribution with the same temperature.

* SEE ERATA

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2. A. Simon (submitted to Phys. Fluids). This paper is referred to hereafter as II.

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