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STABILITY OF PLASMAS AGAINST ELECTROSTATIC PERTURBATIONS

T. K. Fowler

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Stability of Plasmas Against Electrostatic Perturbations

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ABSTRACT

With the notation $(\partial f/\partial t) + Vf = 0$ for the linearized self-consistent field (Vlasov) equation governing "collisionless" plasmas, it is shown that, if only the Coulomb force between particles is taken into account, a sufficient (and generally necessary) condition for stability of an equilibrium distribution $f_o(\vec{x},\vec{v})$ against small perturbations f is that the linear operator V, a function of f_o, be anti-Hermitian in some representation. Using this symmetry requirement, a class of stable functions is found, essentially by inspection, which includes spatially-confined distributions which might represent a plasma held together by gravitational forces. Also, some qualitative limits on the nature of unstable perturbations in thermonuclear "mirror" machines are determined. The stability criterion is shown to apply to other classical systems with central two-body forces.

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I. INTRODUCTION

A highly-ionized gas, or plasma, is susceptible to instabilities in which a small, localized imbalance in charge, once initiated, grows to macroscopic proportions. In this paper, we shall treat the stability against such disturbances of a "collisionless" plasma, that is, a plasma of such density and temperature that it may be regarded as a system of independent particles acted upon only by the average self-electric field of the plasma, together with any external forces.[1] (We neglect the plasma self-magnetic field.) In that most self-consistent field theories have similar mathematical structure, we shall also be able to point out the applicability of our results to other classical systems.

The particles of our system having been assumed independent, their distribution in space, velocity and time, $F(\vec{x},\vec{v},t)$, satisfies a "single-particle" Liouville equation made self-consistent, and non-linear, by including the self-field force. A time-independent solution, $f_o(\vec{x},\vec{v})$, of this non-linear equation is defined to be stable if every physically reasonable, small perturbation f remains suitably bounded in time. In practice, introducing $F = f_o + f$ into the equation and linearizing in f with f_o fixed, f_o is regarded as stable if no real solution f grows indefinitely in time. The linearized equation has the form

$$\frac{\partial f}{\partial t} + Vf = 0, \tag{1}$$

where V, a function of f_o, is a time-independent, real linear operator, to be

referred to as the Vlasov operator. If f_0 is stable, we shall also say that V is stable.

Each of the two treatments of (1) in the literature, on the one hand the Laplace transform method by Landau,[2] and on the other normal mode expansions by Van Kampen,[3] has generally been restricted in practice to equilibrium distributions spatially uniform and infinite.[4] We shall present here a stability criterion not restricted in this way, and, as an example of its use, we shall prove the stability of a class of distributions isotropic in velocity space which includes the well-known stable case of a spatially uniform and infinite plasma with a Maxwell velocity distribution but which also includes spatially-confined distributions which might represent stellar plasmas held together by gravitational forces. Of course, our method also is limited in scope. Our criterion, which calls for the existence of certain structural features in the operator V, is most useful when, as in our examples, the structure requisite for stability is evident essentially by inspection.

The paper is organized as follows. In Section 3 we develop a stability criterion in analogy with Lyapunov's well-known stability theorem for a system of first-order, ordinary, linear differential equations, discussed in Section 2. Then, after displaying the operator V in detail in Section 4, we apply the criterion to known stable cases in Section 5, and to new cases in Section 6. Implications for thermonuclear mirror machines are discussed in Section 7.

2. AN ANALOGOUS PROBLEM

For guidance, we consider first an equation analogous to (1) in which f is an n-dimensional vector function of t, V is an n x n matrix independent of t, and $\partial/\partial t \to d/dt$. The stability of solutions of such equations was given definitive treatment by Lyapunov.[5] We shall specialize to real matrices whose trace has zero real part,

$$\mathrm{Re}(\mathrm{tr}\ V) = 0, \tag{2}$$

which, as we shall show in Section 4, holds also for the Vlasov operator if we admit only electrostatic perturbations.

In consequence of (2), either V is imaginary, that is, all its eigenvalues are imaginary, or there occur eigenvalues with real parts having both signs. Certainly, then, a necessary condition for stability is that V be imaginary. Otherwise, for some eigenvalue ω_i, with corresponding eigenvector g_i, there exists a solution $f = g_i \exp(-\omega_i t)$ of (1) which blows up in time, and, because V is real so that f^* is also a solution, there is in fact an unbounded real solution, Re f.

If the eigenvectors of V were complete, so that every solution of (1) could be written in the form $f = \sum_i \alpha_i g_i \exp(-\omega_i t)$, α_i constant, the necessary stability condition that V be imaginary would also be sufficient, since then

$f(t) \leq \sum_i |\alpha_i g_i|$, all t. Now, it is well known that, if the eigenvectors are not complete, or, equivalently, if V is not diagonalizable,[6] no transformation separates (1) into n independent equations; some variables remain coupled, driving each other resonantly; and the result is that some solutions f of (1) contain terms with time dependence like $t^m e^{-\omega_i t}$, which, of course, blows up if V is imaginary so that ω_i is imaginary.[7] Thus, if V is imaginary, it is both necessary and sufficient for stability that V be diagonalizable. Since also it is necessary that stable V be imaginary, V is stable if and only if it is imaginary and diagonalizable.[8] Such a matrix can always be given an anti-Hermitian "representation," that is, for some non-singular matrix P the similarity transformation $P^{-1}VP$ is anti-Hermitian.[9] It is this symmetry criterion which we wish to exploit.

The analogy between the Vlasov equation (1) and the finite dimensional vector differential equation we have discussed should be apparent. Such a vector-matrix equation arises if in (1) we retain explicitly differentiation with respect to t but give a numerical approximation to the $\vec{x}$ and $\vec{v}$ dependence of f and the linear operator V in which $\vec{x},\vec{v}$-space is represented by a grid of n points. If V has the symmetry requisite for stability by the above criterion, this can be retained in an n x n matrix approximating V and thereby assures stability of the numerical solutions. In Section 3 we shall draw the converse conclusion that the exact solutions of (1) are then also stable by direct proof rather than by examination of the limit $n \to \infty$. We shall only prove sufficiency, and simply infer necessity, at least with large probability, from the vector-matrix analogy together with, in Section 4, the direct proof that stable V are necessarily

imaginary. In practice, it is the sufficiency of the criterion which would be of most use, as it would be difficult to show instability by proving the non-existence of an anti-Hermitian representation of V. The necessity of the criterion is desirable only because it would imply a wide scope of application. Fortunately, in Sections 5 and 6, we are able to demonstrate by example, both for cases known stable and for new cases, that the sufficiency criterion is at least sometimes capable of satisfaction.

3. STABILITY CRITERION

We first must give some definitions. If S is a space of complex functions of $\vec{x}$ and $\vec{v}$ square integrable over a volume $\mathcal{V}$ in $\vec{x},\vec{v}$-space, we define as our basic Hermitian scalar product of functions f and g in S,

$$(f,g) = \int_{\mathcal{V}} d\vec{x}\, d\vec{v}\, f^*(\vec{x},\vec{v})\, g(\vec{x},\vec{v}). \tag{3}$$

The volume of integration $\mathcal{V}$ may include all $\vec{x},\vec{v}$-space or only a large but finite volume. In the latter case, functions belonging to S will be required to satisfy periodic boundary conditions at the surface of $\mathcal{V}$.

Unless otherwise stated, we define Hermitian symmetry always with respect to (3). $C^\dagger$ is the Hermitian conjugate of the linear operator C if for all f, g in S,

$$(f, C^\dagger g) = (g, Cf)^*. \tag{4}$$

Then, for example, $H = H^\dagger$ is Hermitian, as usual; a real integral operator is Hermitian if its kernel is symmetric in $\vec{x}$ and $\vec{v}$; $\partial/\partial\vec{x}$ is anti-Hermitian, and so forth.

With these definitions, we are in position to discuss the following stability criterion. If by A we mean "some anti-Hermitian operator," by H_+ "some positive definite Hermitian operator," and by the superscript (-1) the existence of an inverse, then it is sufficient for stability that V satisfy any one of the relations:

$$A = H_+V \text{ (or } VH_+^{-1}) \tag{5}$$

$$V = AH_+ \text{ (or } H_+^{-1}A) \tag{6}$$

$$H_+ = AV. \tag{7}$$

The content of each of these equations is the same as that of the stability criterion for the vector-matrix equation analogous to (1) discussed in Section 2; namely, V satisfying one of these relations has an anti-Hermitian representation. Returning to the matrix problem for a moment, if H_+ is a positive definite Hermitian matrix, there exists a non-singular matrix P such that $H_+ = P^\dagger P$. Then, for example, operating on $H_+V = A$ in (5) left by $(P^\dagger)^{-1}$ and right by P^{-1} shows that V satisfying this relation is similar to the anti-Hermitian matrix $(P^\dagger)^{-1}AP^{-1}$. In function space, we can more conveniently discuss "representations" in terms of scalar products rather than similarity transformations. The most general Hermitian scalar product over S is obtained from our basic product (3) by introducing into (3) an Hermitian positive definite operator as a weighting. If V satisfies one of the relations (5)-(7), which pair with V some Hermitian positive definite H_+, we define as a corresponding Hermitian scalar product [f,g] of functions f and g in S,

$$[f,g] = (f,H_+g), \tag{8}$$

where the weighting is the H_+ paired with V. Then any one of the relations (5)-(7) between V and an H_+ implies that V is anti-Hermitian in the corresponding [,] representation, which is to say V is anti-Hermitian by definition (4) if the product definition [,] in (8) is substituted for (,) in (4). We shall find, by the way, that V is never anti-Hermitian in our usual sense, that is, with respect to (3) itself.

It follows that each of the relations (5)-(7) between V and an H_+ implies that, for any real solution f(t) of (1), $[f(t),f(t)] \equiv (f(t),H_+f(t))$ is independent of time. To see this, take the bracket-type scalar product of f with (1), which gives $\frac{1}{2}\frac{\partial}{\partial t}[f,f] = -[f,Vf] = 0$ for real f. Therefore, if [f,f] constitutes a suitable measure of f whose boundedness in time indicates stability, our criteria (5)-(7) are, as claimed, sufficient conditions for stability.

The quantity [f,f] satisfies two of the three criteria for measures of stability suggested by Backus.[10] First, because of the required positive definiteness of H_+, this quantity is zero if and only if f is zero, which helps prevent the possibility that f develops wild oscillations whose plus and minus excursions cancel in taking the norm; and second, if [f(0),f(0)] is finite, f(0) should be a meaningful initial condition for a solution of (1). Whether [f,f] satisfies Backus' third requirement of possessing clear physical significance depends, of course, on the weighting H_+ in a specific case. We shall return to this point in treating examples.

We wish to point out the relation between [f,f] and a norm defined by $(f,f)^{1/2}$, which was suggested by Van Kampen[3] but which also lacks an evident physical interpretation. Since H_+ is definite, there exists a real constant ω such that, for all f in S,

$$\frac{(f,H_+f)}{(f,f)} \geqq \omega > 0. \tag{9}$$

If V satisfies (5)-(7), we have seen that the numerator is a constant, just $[f(0),f(0)] \equiv (f(0),H_+f(0))$, and in that case

$$(f(t),f(t)) \leqq \frac{1}{\omega}\,[f(0),f(0)]. \tag{10}$$

Thus $(f,f)^{1/2}$ is bounded if $[f,f]$ is and these norms are in fact equivalent if we could impose one other constraint that $[f(0),f(0)]$ be finite if $(f(0),f(0))$ is, which expresses a boundedness on H_+. We shall find for the Maxwell distribution and our other examples that H_+ is so bounded if we restrict ourselves to physically sensible perturbations.

We note in passing that a similarity transformation of some interest is one effected by a transformation of coordinates $\vec{x},\vec{v},t$. Transforming $\vec{x},\vec{v},t \to \vec{x}_o,\vec{v}_o,t$, the natural Hermitian product in terms of the new variables, $\int d\vec{x}_o\, d\vec{v}_o\, f^*g$, which is just (3) with the Jacobian of transformation, $\mathcal{J}$, omitted, could be obtained from (3) by (8) with $H_+ = \mathcal{J}^{-1}$, which is Hermitian diagonal because $\mathcal{J}$ is just a real function and positive definite by the definition of $\mathcal{J}$.

4. THE VLASOV OPERATOR

In a collisionless plasma composed of several species of charged particles, the distribution function, $F^{(i)}(\vec{x},\vec{v},t)$, one for each species, charge q_i and mass m_i, satisfies a self-consistent, single-particle Liouville equation, called in the plasma literature the Vlasov equation, also the collisionless Boltzmann equation, of the form[3]

$$\frac{\partial F^{(i)}}{\partial t} + \vec{v}\cdot \frac{\partial F^{(i)}}{\partial \vec{x}} + \frac{1}{m_i} \frac{\partial F^{(i)}}{\partial \vec{v}} \cdot \left[\vec{\mathcal{F}}^{(i)}_{ext} - \frac{\partial}{\partial \vec{x}} \sum_j \Phi_{ij} F^{(j)} \right] = 0, \tag{11}$$

where $\vec{\mathcal{F}}^{(i)}_{ext}$ is any external force and Φ_{ij}, the potential operator, is an integral operator with symmetric kernel independent of $\vec{v}$ and $\vec{v}'$,

$$\Phi_{ij}(\vec{x},\vec{v};\vec{x}',\vec{v}') = q_i q_j \frac{1}{|\vec{x} - \vec{x}'|} . \tag{12}$$

Writing $F^{(i)} = f_o^{(i)}(\vec{x},\vec{v}) + f^{(i)}(\vec{x},\vec{v},t)$ and linearizing in f yields

$$\frac{\partial f^{(i)}}{\partial t} + L_i f^{(i)} - \frac{q_i}{m_i} \frac{\partial f_o^{(i)}}{\partial \vec{v}} \cdot \frac{\partial}{\partial \vec{x}} \sum_j \Phi_{ij} f^{(j)} = 0. \tag{13}$$

The operator L_i, to be called a Liouville operator and defined by

$$L_i = \vec{v}\cdot \frac{\partial}{\partial \vec{x}} + \frac{1}{m_i} \left[\mathcal{F}^{(i)}_{ext} - \frac{\partial}{\partial \vec{x}} \sum_j \Phi_{ij} f_o^{(j)} \right] \cdot \frac{\partial}{\partial \vec{v}} , \tag{14}$$

governs equilibrium; e.g., $L_i f_o^{(i)} = 0$.

In addition to plasmas, Eqs. (11)-(14) apply directly to a system with gravitational interaction with the replacement $q_i \to (-G)^{1/2} m_i$, G being the gravitational constant, or to a combination of gravitational and Coulomb forces with $q_j q_i \to q_i q_j - G m_i m_j$. With only a redefinition of Φ_{ij}, the equations apply also to self-consistent field approximations to other classical systems with central two-body forces if, as in the plasma case, a self-consistent potential linearly related to the source distribution is adequate. The latter constraint means that only single interactions contribute to the effective field, so to speak a classical "Born approximation." Our results will not hold in general if Φ_{ij} depends on velocity, which eliminates magnetic interactions.

For a more concise notation, if there are n charge species we define vectors and matrices

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(n)} \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & & & 0 \\ & L_2 & & \\ & & \ddots & \\ 0 & & & L_n \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{pmatrix} \tag{15}$$

with

$$K_i = -\frac{1}{m_i} \frac{\partial f_o^{(i)}}{\partial \vec{v}} \cdot \frac{\partial}{\partial \vec{x}}, \tag{16}$$

and we define the complete potential operator, Φ, to be an n x n matrix with integral operator components Φ_{ij} given by (12). Then (13) assumes the form (1) with the Vlasov operator V given by

$$V = L + K\Phi. \tag{17}$$

Employing the property that for conservative forces, dq/dt is independent of q if q is any component of $\vec{x}$ or $\vec{v}$, which holds even if $d\vec{v}/dt$ contains velocity-dependent terms like $\vec{v} \times \vec{B}(\vec{x})$, integration of $(f, L_i g)$ by parts yields the anti-Hermiticity condition,

$$(f, L_i g) = -(g, L_i f)^*, \tag{18}$$

if f,g fall off fast enough so that the integrated term, an integral over the surface bounding $\mathcal{V}$, vanishes. If $\mathcal{V}$ is a finite but large volume, (18) again holds with periodic boundary conditions at the surface.

We shall, in fact, adopt (18) as a boundary condition on functions in S, that is, as a constraint on admissible perturbations, and thus hereafter L_i, hence L, will be regarded as an anti-Hermitian operator. We have thereby

omitted certain integrable but not square integrable perturbations with respect to which, according to Backus,[10] even the Maxwell distribution is unstable. We have also restricted somewhat the behavior of f at the surface of $\mathcal{V}$, that is, at "infinity," but we are guided by the fact that these restrictions are by and large just those that promote the total conservation of particles under the action of equilibrium forces. Also, we have shown elsewhere the compatibility of (18) with the customary restrictions on perturbations in that reducing (13) to an integral equation by finding a Green's function $(-\lambda + L_i)^{-1}$ satisfying (18) as a boundary condition leads to dispersion relations agreeing with those in the literature.[11] Actually, there is little physical interest in the exact requirements on f at infinite $\vec{x}$ and $\vec{v}$ in order that (18) be rigorously correct, as the behavior at infinity reflects back at most weakly to the bounded region of $\vec{x},\vec{v}$-space, call it $\mathcal{V}_1$, where f_o, hence $K\Phi$, has appreciable magnitude, this being the source region of any instability. Outside $\mathcal{V}_1$, the Vlasov equation (13) reduces to $[(\partial/\partial t) + L]\, f = 0$, a true Liouville equation which conserves particles and will not of itself cause f to grow in time.

In a similar vein, we would assume that it is not important physically whether, say, f_o vanishes exactly outside $\mathcal{V}_1$ or whether this is only approximately so. Thus, later, when the question arises whether such an f_o has an inverse throughout $\mathcal{V}$, we shall take the liberty to insure that this is so, and also to smooth f_o in other ways which relieve non-essential mathematical difficulties. We summarize this practical point of view in the following definition:

I. <u>It is sufficient for the stability of f_o that there exist for arbitrary $\epsilon > 0$ a stable equilibrium distribution $g(\vec{x},\vec{v})$ such that $|f_o - g| < \epsilon$, all $\vec{x},\vec{v}$.</u>

We would assume that, if there also exists a not-too-pathological, unstable g arbitrarily close to f_o, the rate of growth of instability approaches zero as $g \to f_o$.

Having established (18) as a boundary condition, we are now in a position to prove Eq. (2) implying that stable V are necessarily imaginary. We note also the fact, mentioned previously, that V is never anti-Hermitian in (3), which follows from the anti-Hermiticity of L and the asymmetry of $K\Phi$ with respect to $\vec{v}$ and $\vec{v}'$. Equation (2) has been verified in the special cases in which dispersion relations have been found.[1] Our general proof will take two forms. First, defining $Q = K\Phi$ and denoting the Hermitian part of an operator by superscript h, we obtain from the theorem that the trace of a sum is the sum of the traces and the fact that L has imaginary eigenvalues,

$$\mathrm{Re}\ (\mathrm{tr}\ V) = \mathrm{tr}\ V^h = \mathrm{tr}\ Q^h = \sum_i \mathrm{tr}\ Q^h_{ii}. \tag{19}$$

Q^h_{ii} is an integral operator whose kernel is obtained by symmetrizing the kernel belonging to Q_{ii},

$$Q^h_{ii}(\vec{x},\vec{v};\vec{x}',\vec{v}') = \frac{q_i^2}{2m_i}\left(\frac{\partial f_o^{(i)}(\vec{x},\vec{v})}{\partial\vec{v}} - \frac{\partial f_o^{(i)}(\vec{x}',\vec{v}')}{\partial\vec{v}'}\right)\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{20}$$

Q^h_{ii}, a product of antisymmetric factors, has zero diagonal elements, hence zero trace, and (2) follows. Difficulties arising from the singularity at $\vec{x} = \vec{x}'$ are avoided if the second factor, just $-(\partial/\partial\vec{x})\ \Phi$, is regarded as the electric field at $\vec{x}$ due to a unit charge at $\vec{x}'$ distributed over a sphere with small radius r and the limit $\vec{x} \to \vec{x}'$ is allowed to precede the limit $r \to 0$.

We shall also verify the above conclusion by examining the eigenvalues of Q^h_{ii} directly. Introducing (20) into the following homogeneous integral equation,

$$\lambda\, \psi_\lambda(\vec{x},\vec{v}) = \int d\vec{x}'\, d\vec{v}'\, Q^h_{ii}(\vec{x},\vec{v};\vec{x}',\vec{v}')\, \psi_\lambda(\vec{x}',\vec{v}'), \tag{21}$$

we see that, for $\lambda \neq 0$, ψ_λ must have the form

$$\psi_\lambda(\vec{x},\vec{v}) = \frac{1}{\lambda}\left[\frac{\partial f_o^{(i)}}{\partial \vec{v}} \cdot \vec{a}_\lambda(\vec{x}) - b_\lambda(\vec{x})\right], \tag{22}$$

where

$$\vec{a}_\lambda(\vec{x}) = \frac{q_i^2}{2m_i}\int d\vec{x}'\, d\vec{v}'\, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, \psi_\lambda(\vec{x}',\vec{v}') \tag{23}$$

$$b_\lambda(\vec{x}) = \frac{q_i^2}{2m_i}\int d\vec{x}'\, d\vec{v}'\, \frac{\partial f_o^{(i)}}{\partial \vec{v}'} \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, \psi_\lambda(\vec{x}',\vec{v}'). \tag{24}$$

Since the second term of (22) is independent of $\vec{v}$, it is necessary to confine our system to a large volume $\mathcal{V}$ in velocity space if ψ is to be normalizable and (23) is to converge. Even so, as we have noted, with the boundary condition that functions be periodic on the surface bounding $\mathcal{V}$, L remains anti-Hermitian with respect to (3), and it remains appropriate to examine only Q^h_{ii} in order to establish (2) at each stage of taking the limit $\mathcal{V} \to \infty$. (We simply assume that this limit approaches the problem of physical interest.)

Introducing (22) into (23), the contribution from the first term of ψ_λ is seen to vanish if $f_o^{(i)}$ is periodic in $\mathcal{V}$, so that, in terms of b_λ, $\vec{a}_\lambda$ is,

$$\vec{a}_\lambda(\vec{x}) = -\frac{1}{\lambda}\,\frac{q_i^2}{2m_i}\int d\vec{x}'\, d\vec{v}'\, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, b_\lambda(\vec{x}'). \tag{25}$$

Similarly, introducing (25) into (22) and the result into (24) yields

$$\lambda^2 b_\lambda(\vec{x}) = \mathcal{V}\int d\vec{x}''\, b_\lambda(\vec{x}'') \left\{\int d\vec{x}'\, d\vec{v}' \right.$$

$$\left. \left(\frac{q_i^2}{2m_i}\, \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \cdot \frac{\partial f_o^{(i)}}{\partial \vec{v}'}\right)\left(\frac{q_i^2}{2m_i}\, \frac{\vec{x}' - \vec{x}''}{|\vec{x}' - \vec{x}''|^3} \cdot \frac{\partial f_o^{(i)}}{\partial \vec{v}'}\right)\right\}. \qquad (26)$$

The kernel in curly brackets, which could be written $Q_{ii}^{\dagger}\, Q_{ii}$, is by its form Hermitian positive semi-definite, which is to say eigenvalues satisfy $\lambda^2 \geqq 0$.[12] Thus, λ is real, as it should be if it is to be an eigenvalue of the Hermitian operator Q^h. Again (2) follows, since to each solution b_λ of (25) with eigenvalue λ^2, there correspond two eigenfunctions of Q_{ii}^h given by (22) with eigenvalues $\pm\sqrt{\lambda^2}$.

Our proof of (2) for the inverse square law force goes through for other central two-body forces as well. In the second proof, for example, except for properties promoting convergence of integrals, the only property of the potential operator necessary is its lack of dependence on $\vec{v}$ and $\vec{v}'$.

5. STABILITY OF THE MAXWELL DISTRIBUTION

Consider a one-species plasma of electrons neutralized by a uniform positive background with no external forces, $L = \vec{v}\cdot(\partial/\partial\vec{x})$, and in equilibrium let the electron distribution at every point $\vec{x}$ be $f_o = N \exp(-\alpha H_o)$, α and N being positive constants and $H_o = (1/2)mv^2$. Then, the operator K appearing in V, Eq. (16), can be written $K = -f_o'L = -Lf_o'$, where $f_o' = (\partial f_o/\partial H_o)$ and to obtain the factor L we have used $(\partial H_o/\partial\vec{v}) = m\vec{v}$. Since $(-f_o') = \alpha f_o > 0$ for all $\vec{x},\vec{v}$ within an arbitrarily large volume $\mathcal{V}$ of $\vec{x},\vec{v}$-space, the real function $(-f_o')$ possesses an inverse in $\mathcal{V}$ and may be regarded as a diagonal, positive definite Hermitian operator. Factoring out of V first L, then $(-f_o')$, we obtain

$$V = K\,[(-f_o')^{-1} + \Phi\,]. \qquad (27)$$

As it is the product of commuting Hermitian and anti-Hermitian factors, K is anti-Hermitian. The factor in brackets is Hermitian in that both terms are. $(-f_0')^{-1}$ is positive definite, because $(-f_0')$ is, and we shall show that Φ, defined by (12), is positive semi-definite (zero admitted), so that the bracket is definite. Thus, V has the form (6), and f_0 is stable.

For our electron plasma, Φ reduces to $e^2 \, |\vec{x} - \vec{x}'|^{-1}$, which, if it is semi-definite, satisfies for all f in S,

$$(f,\Phi f) \equiv e^2 \int d\vec{x}\, d\vec{v} \int d\vec{x}'\, d\vec{v}'\; f^*(\vec{x},\vec{v}) \frac{1}{|\vec{x} - \vec{x}'|} f(\vec{x}',\vec{v}') \geq 0. \tag{28}$$

For a real perturbation f, Φf is e times the perturbed electrostatic potential, and (28), just the total potential energy, is positive. The conclusion remains valid if f is complex. Defining $R(\vec{k},\vec{p};\vec{x}'\vec{v}') = g(\vec{p}) \frac{e}{k} \exp(-i\vec{k}\cdot\vec{x}')$ with g any function of $\vec{p}$ such that $\int d\vec{p}\, |g|^2 = 4\pi(2\pi)^{-3}$, then Φ is Hermitian positive semi-definite by virtue of the form,[12]

$$\Phi = R^\dagger R = \int d\vec{k}\, d\vec{p} \left(g^*(\vec{p}) \frac{e}{k} e^{i\vec{k}\cdot\vec{x}} \right) \left(g(\vec{p}) \frac{e}{k} e^{-i\vec{k}\cdot\vec{x}'} \right), \tag{29}$$

which, after integration on $\vec{p}$, is just the Fourier transform in space of $e^2 \, |\vec{x} - \vec{x}'|^{-1}$.

Now, recalling Section 3, what we have proved is that, for any real solution f(t) of the linearized Vlasov Eq. (1) corresponding to a Maxwell distribution in equilibrium, the quantity $(f(t), \; [(-f_0')^{-1} + \Phi]\, f(t))$, if finite at t = 0, remains finite and is in fact constant in time. As was pointed out above, the Φ-term is just the total potential energy in the distribution f, and should be required, by choice of f(0), to be finite at t = 0. The physical significance of

the other term is more obscure. We note, however, that for any square integrable $f(t)$ with (f,f) normalized to unity, $(f,(-f_o')^{-1}f)$ has the same bounds as $(-f_o')^{-1}$ itself, and hence is finite unless $f(0)$ falls off at infinite velocity more slowly than $f_o^{1/2}$. Even in the exceptional case, which could be avoided by keeping $\mathcal{V}$ large but finite, we would take the point of view, as in Section 4, that in the linearized stability theory there is no physical significance in such an infinity due only to the nature of $f(0)$ in the region of $\vec{x},\vec{v}$-space outside the bounded region in velocity space where f_o has appreciable magnitude. With this limitation on allowed perturbations, the bracket in (27) is not only definite but bounded, in which case we have also proved stability with respect to the norm $(f,f)^{1/2}$.

We conclude this section with another application to a known stable case, the class of "single-humped" functions of velocity with no external force.[13] We shall treat just the one-species, one-dimensional problem, which generalizes to infinite plasmas in three dimensions if f_o is isotropic in $\vec{v}$.[1,13] In one dimension the appropriate potential operator, $\Phi = 2\pi\, e^2\, |x - x'|$, is Hermitian positive semi-definite by (29) with the replacement $\vec{k}\cdot\vec{x} \to kx$. Assuming first that $f_o(v)$ has a single maximum at $v = 0$, we find

$$K = -\frac{1}{m}\frac{\partial f_o}{\partial v}\frac{\partial}{\partial x} = \left(\frac{1}{m}\frac{1}{|v|}\left|\frac{\partial f_o}{\partial v}\right|\right)\left(v\frac{\partial}{\partial x}\right). \tag{30}$$

On the right, the second factor is L for the problem. The first factor, corresponding to $-f_o'$ above, commutes with L, so that K is anti-Hermitian; and it is real, positive definite, since $\frac{1}{|v|}\left|\frac{\partial f_o}{\partial v}\right|$ cannot be zero at $v = 0$ if f has a maximum there (then $\frac{\partial^2 f}{\partial v^2} \neq 0$), and any zero for $v \neq 0$ can be handled by definition

(I), Section 4. Again, V factors as in (27), and f_o is stable by (6). Further, this proof carries over to any distribution obtained from our case by Galilean transformation, so that in the new variables f_o is maximum at $v \neq 0$, since such a change of reference frames is a similarity transformation.

6. OTHER STABLE DISTRIBUTIONS

Consider a one-species plasma, like that of the previous example, but now we include in L a force in equilibrium, $\vec{F}_o = -(\partial/\partial\vec{x})\,\pi(\vec{x}) + (q/c)\,\vec{v} \times \vec{B}_o(\vec{x})$, where $\pi(\vec{x})$ and the external magnetic field $\vec{B}_o$ may have any spatial dependence and the scalar potential π may represent either an external interaction or a self-consistent field, or both. Then, the equilibrium Hamiltonian is

$$H_o = \frac{1}{2}\,mv^2 + \pi(\vec{x}), \tag{31}$$

which depends on π in $\vec{F}_o$, but not $\vec{B}_o$.

Let f_o be a function of H_o only with the property that, for all H_o such that $(1/2)\,mv^2 \geqq 0$,

$$f_o' \equiv \frac{\partial f_o}{\partial H_o} < 0. \tag{32}$$

Then, $-f_o'$ is an Hermitian positive definite operator, and, in exact analogy with the previous proof for the Maxwell distribution, we shall show that f_o is stable.

Because K in V, Eq. (17), acts on Φ, ultimately the potential Φf, which is independent of $\vec{v}$, we are free to add to K a term in $\partial/\partial\vec{v}$. Defining

$$K_1 = -\frac{1}{m}\,\frac{\partial f_o}{\partial\vec{v}} \cdot \frac{\partial}{\partial\vec{x}} + (-f_o')\,\frac{1}{m}\,\vec{F}_o \cdot \frac{\partial}{\partial\vec{v}}, \tag{33}$$

We may redefine $V = L + K_1\Phi$. For f_o a function of H_o only,

$$K_1 = (-f_o')L = L(-f_o'), \tag{34}$$

where commutation follows from the fact that H_o, hence f_o and $\partial f_o/\partial H_o$, is an equilibrium constant of motion. K_1 is anti-Hermitian by (34). If f_o satisfies (32), $(-f_o')^{-1}$ exists and K_1 factors left out of V to give (27) with K replaced by K_1, and f_o is stable by (6).

There are many f_o satisfying (32) which are also, properly, everywhere positive and normalizable. Equation (32) need only be satisfied over the range $\mathcal{R}$ of H_o covered by $H_o(\vec{x},\vec{v})$ in (31) as $\vec{x}$ and $\vec{v}$ take on all values. If $\pi(\vec{x})$ is an attractive potential, $\pi < 0$ and $\to 0$ as $\vec{x} \to \infty$, the range $\mathcal{R}$ has a finite negative lower limit at the minimum of $\pi(\vec{x})$, if there is a minimum. In any case, any positive, monotonic decreasing function of H_o in $\mathcal{R}$ satisfies (32). If π is an attractive well of finite depth and if we cut off f_o at a negative upper limit on H_o, all particles have negative energy and f_o is spatially confined and normalizable in $\vec{x}$ and $\vec{v}$. By definition (I), Section 4, in regions of $\mathcal{R}$ where f_o is constant or zero, we would approximate f_o by a function satisfying (32). If the true f_o had a dip or a hump, this could not be done and our criterion would not insure stability.

The above results apply to a plasma composed of electrons and any number of species of ions. In that K_1 and L are diagonal, the factorization of V goes through, yielding

$$V = K_1 \; [T + \Phi], \tag{35}$$

where T also is diagonal with components $T_i = \left(- \dfrac{\partial f_o^{(i)}}{\partial H_o^{(i)}} \right)^{-1}$, and T is surely Hermitian positive definite if each T_i is (sufficient, not necessary). Finally,

the general Φ remains semi-definite, for, if f is a vector function with one component for each charge species, $(f,\Phi f) = \int d\vec{x}\ \rho^*(\vec{x})\ |\vec{x} - \vec{x}'|^{-1}\ \rho(\vec{x}')$, where $\rho(\vec{x}) = \sum_j q_j \int d\vec{v}\ f^{(j)}$ is just the net charge when f is real. In (29) we have already shown $|\vec{x} - \vec{x}'|^{-1}$ to be definite over complex functions of $\vec{x}$, in that $g(\vec{p})$ played no essential role.

Since $\pi(\vec{x})$ could represent a self-consistent gravitational field, stable distributions of the sort found here are of possible application to the stability of stars. The distribution would have the reasonable property of approaching colder "temperatures" nearest the surface of the star; any number of constituents could be handled, each with a different equilibrium distribution; and a realistic background galactic magnetic field could be assumed to be present, as it would have no effect on the calculation.

7. APPLICATION TO MIRROR MACHINES

While by our method we have not yet found stable distributions confined in space by a magnetic field, the stability of solutions discussed in the previous section has some bearing on thermonuclear devices, particularly high-energy injection mirror machines, such as the Oak Ridge DCX and Russian OGRA, in which hot ions are injected as an organized beam. In most such machines, ion motion can initially be well described by two constants of motion, the energy, $H_o^{(i)}$, and the canonical angular momentum, $p_\theta^{(i)} = m_i r^2\dot{\theta} + (q_i/c)\ r\ A_\theta(r,z)$, in cylindrical coordinates with z along the field axis, A_θ being the only non-zero component of the vector potential. Because collisions are infrequent, the development in time toward the steady-state distribution f_o is essentially "adiabatic" with respect to these constants, and thus, at least for ions, a function of $H_o^{(i)}$

and $p_\theta^{(i)}$ only is generally adequate to describe the plasma obtained.[14] Introducing besides the confining magnetic field a self-consistent equilibrium plasma potential, also cylindrically symmetric, to limit the escape of electrons at the ends of magnetic field lines, the electrons also are described by a function just of the analogous quantities, $H_o^{(e)}$ and $p_\theta^{(e)}$.

As before, because K_1 is diagonal, we can discuss ions and electrons separately. Then, dropping the species subscript, the dependence of f_o on p_θ introduces into K_1 in (34) just one other term,

$$- \frac{\partial f_o}{\partial p_\theta} \frac{1}{m} \frac{\partial p_\theta}{\partial \vec{v}} \cdot \frac{\partial}{\partial \vec{v}} = \left(- \frac{\partial f_o}{\partial p_\theta} \right) \frac{\partial}{\partial \theta} . \tag{36}$$

Acting on any perturbation f such that the potential Φf is independent of θ, this term vanishes. Thus, f_o would be stable against such perturbations, since its dependence on H_o is almost sure not to cause instability. In v_z, f_o is "single-humped" about $v_z = 0$. Further, a possible "two-stream" instability[15] between circulating ions and the electrons is restrained by the fact that electrons follow magnetic field lines. The known electrostatic instabilities in such systems, derived by Burt and Harris for a cylindrical shell of plasma in a uniform magnetic field,[16] concern ion cyclotron resonance with electron plasma oscillations in the z-direction, and, in agreement with (36), these are not excited by perturbations with no θ-dependence. We would conclude that, even in realistic fields, any electrostatic instabilities in mirror machines will almost surely arise from θ-dependent perturbations.

Similarly, in a linear "pinch," if f_o were a function only of H_o and $p_z = mv_z + (q/c)\, A_z(r,\theta)$, the p_z term in K_1 is proportional to $(\partial/\partial z)$. Again, even for realistic spatial dependence of the self-magnetic field, one would conclude that electrostatic instabilities would usually result from z-dependent perturbations.

8. CONCLUSIONS

We have investigated the applicability of Lyapunov's stability criterion to the stability of plasmas against small electrostatic perturbations. Denoting the linearized Vlasov equation governing perturbations f of the distribution in space and velocity by $(\partial/\partial t + V)f = 0$, Eq. (1), we showed (Section 3) that it is sufficient for stability (and likely necessary) that the linear operator V be anti-Hermitian with respect to some Hermitian scalar product defined on the space of allowed perturbations. This is so and V is stable if, for example, V can be factored into a product of an anti-Hermitian operator and an Hermitian positive definite operator [Eq. (6)]. After showing that the method gives correct results for certain known stable cases (Section 5), we presented (Section 6) a new class of stable distributions which includes f_o independent of $\vec{x}$ with a Maxwell distribution of velocities, but also certain spatially-dependent functions. In brief, this is the class of all plasmas composed of any number of species of charged particles such that for each species, i, the equilibrium distribution $f_o^{(i)}$ is a monotonic decreasing function of the equilibrium energy integral $H_o^{(i)}$ for that species, and $f_o^{(i)}$ depends on $H_o^{(i)}$ only. Such functions are, of course, isotropic in velocity space. They may represent plasmas confined in space by a scalar potential, such as that due to a self-consistent gravitational field, but not those <u>confined</u> by a magnetic field, though a magnetic field with any spatial dependence can be present. The solutions have possible application to the stability of stars. In Section 7 implications for thermonuclear mirror machines were discussed, with the conclusion that any electrostatic instabilities must be excited by perturbations with azimuthal variation.

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