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ELECTROSTATIC ION-CYCLOTRON PLASMA INSTABILITIES IN A TWO-FLUID HYDRODYNAMIC THEORY

Hermann K. Wimmel

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ABSTRACT

A two-fluid hydrodynamic theory is shown to describe certain electrostatic ion-cyclotron plasma instabilities of the Harris[1] type in a semiquantitative way. The ions (electrons) are described by the first three (two) moment equations of the collisionless Boltzmann equation, implying zero electron temperature. Only purely longitudinal ("electrostatic") oscillations are considered. If the electron plasma frequency ω_p is greater than the ion cyclotron frequency Ω_c and if the ion pressure is sufficiently anisotropic, unstable oscillations occur with $|\mathrm{Re}(\omega)| \approx \Omega_c$. Owing to the limitations of the model, only the cases in which the wavelength of the oscillation mode is large compared to the ion cyclotron radius can be treated. Instabilities at higher harmonics of the ion cyclotron frequency are also excluded. Quantitative agreement with the results of Harris' exact treatment of the infinite plasma is found only for $\underline{k}$ nearly parallel to $\underline{B}^o$, whereas otherwise our equations contain stabilizing extra terms not obtained from the rigorous theory. Estimates of the critical pressure anisotropy, at which instability sets in, are given. The critical anisotropy is smallest for the "resonance" case $\omega_p \approx \Omega_c$ with $\underline{k}$ lying nearly parallel to $\underline{B}^o$. Short-wavelength modes are less stable than long-wavelength modes. The two-fluid hydrodynamic model allows a pictorial understanding of the unstable oscillation modes and of the fact that geometric boundaries have little influence on the conditions for instability.

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1. E. G. Harris, Phys. Rev. Letters 2, 34 (1959); Oak Ridge National Laboratory Report ORNL-2728 (1959); Proc. Conf. Theor. Aspects Contr. Fusion Res., Gatlinburg, Tennessee [TID-7582 (ORNL-2805), 1959], p. 131; P. B. Burt and E. G. Harris, Bull. Am. Phys. Soc., Series II, 6, 300 (1961).

TABLE OF CONTENTS

		Page No.
	Abstract	iii
I.	Introduction	1
II.	Moment Equations	2
III.	Unstable Ion-Cyclotron Oscillations	12
IV.	Discussion	20

I. INTRODUCTION

It has been shown by E. G. Harris[1] that a static homogeneous plasma in a uniform external magnetic field $\underline{B}^o$ may exhibit unstable longitudinal plasma oscillations (electrostatic ion-cyclotron instabilities) with $|Re(\omega)|$ nearly equal to the ion cyclotron frequency Ω_c or a multiple thereof. These instabilities occur if the electron plasma frequency ω_p is greater than the pertinent multiple of Ω_c and if the ion pressure is sufficiently anisotropic. Harris uses the Vlasov equations, with the equilibrium distribution functions chosen as two-temperature Maxwellians with finite $T_\perp$, but $T_\parallel = 0$. (The denotations "perpendicular" and "parallel" refer to the direction of $\underline{B}^o$.) The case of $T_\perp$ and $T_\parallel$ both being finite has been studied numerically by Drummond, Rosenbluth, and Johnson.[2] The electrostatic ion-cyclotron instabilities in a cylindrical plasma shell have been investigated by Burt and Harris.[1] Remarkably, the results in this case are very similar to those obtained for the infinite plasma. Experimental evidence that such instabilities may exist in the Oak Ridge DCX mirror machine has been reported by Barnett.[3]

In the present paper we should like to approach the problem of these instabilities with the aid of a two-fluid hydrodynamic model. In place of the collisionless Boltzmann equation we use its first three, respectively two, moment equations, in order to describe the ions, respectively the electrons. The electron equation of motion and the ion pressure equation are decoupled from the higher moment equations by putting the electron pressure and the ion pressure diffusion term equal to zero. As Bernstein and Trehan[4] have indicated,

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2. W. E. Drummond, M. N. Rosenbluth, M. L. Johnson, Bull. Am. Phys. Soc., Series II, 6, 185 (1961).
3. C. F. Barnett, Bull. Am. Phys. Soc., Series II, 6, 196 (1961).
4. I. B. Bernstein and S. K. Trehan, Nucl. Fus. 1, 3 (1960).

such an approximation is obviously justified whenever the phase velocity of an oscillation mode is much greater than the average thermal particle velocity. It is, however, not evident that the approximation is innocuous, if it is applied to a stability problem. In cases of overstability, as are being considered here, very often $|\mathrm{Im}(\omega)| \ll |\mathrm{Re}(\omega)|$; hence, the use of approximate equations could very well cause appreciable errors in the results. We thought that this question might best be decided on the basis of a concrete calculation.

The hydrodynamic approach restricts us to frequencies near to or smaller than the ion cyclotron frequency, and to wavelengths large compared to the ion gyration radius. In spite of these restrictions, testing a hydrodynamic approach seemed interesting in view of the mathematical difficulties connected with Vlasov-type investigations of instabilities, especially in spatially inhomogeneous plasmas. One would hope that a hydrodynamic theory, if applicable, might simplify matters considerably.

II. MOMENT EQUATIONS

From the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \underline{w} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{\underline{K}}{M} \cdot \frac{\partial f}{\partial \underline{w}} = 0; \quad \underline{K} \equiv e(\underline{E} + \frac{\underline{w}}{c} \times \underline{B}) \tag{1}$$

the following moment equations are easily derived:

$$\frac{\partial N}{\partial t} + \mathrm{div}(N\underline{V}) = 0 \tag{2}$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \mathrm{Grad}\underline{V} = \frac{e}{M} (\underline{E} + \frac{\underline{V}}{c} \times \underline{B}) - \frac{1}{MN} \mathrm{Div}\underline{\underline{P}} \tag{3}$$

$$\frac{\partial}{\partial t}\left(N\,\underline{V}\,\underline{V} + \frac{\underline{\underline{P}}}{M} \right) = -\left\{ \nabla\cdot(N\,\underline{V}\,\underline{V}\,\underline{V}) + \frac{1}{M}\nabla\cdot(\underline{V}\,\underline{\underline{P}} + \underline{\underline{P}}\,\underline{V}) + \frac{1}{M}(\nabla\cdot\underline{\underline{P}}\,\underline{V})^T \right\} \tag{4}$$

$$+ \frac{Ne}{M}\left\{ (\underline{E}\,\underline{V} + \underline{V}\,\underline{E}) + \frac{1}{c}(\underline{V}\,\underline{V} \times \underline{B} - \underline{B} \times \underline{V}\,\underline{V}) \right\}$$

$$+ \frac{e}{M^2 c}(\underline{\underline{P}} \times \underline{B} - \underline{B} \times \underline{\underline{P}}) - \frac{1}{M}\nabla\cdot\underline{\underline{Q}}$$

Here N, $\underline{V}$, $\underline{\underline{P}}$ are the ion density, average velocity, and pressure, defined in the usual way, while the pressure diffusion tensor $\underline{\underline{Q}}$ is defined as

$$\underline{\underline{Q}} \equiv MN\,\langle(\underline{w} - \underline{V})(\underline{w} - \underline{V})(\underline{w} - \underline{V})\rangle.$$

The symbol T, used as an upper index, stands for the transposition of a tensor. A similar set of equations holds for the electrons.

We consider the case that a small time-dependent perturbation is imposed on a homogeneous static plasma in a uniform external magnetic field $\underline{B}^o$. By linearizing our equations accordingly, and neglecting the spatial variation of $\underline{\underline{Q}}$, we obtain, with some slight change in notation, for the ions:

$$\frac{\partial N}{\partial t} + N^o \operatorname{div} \underline{V} = 0 \tag{5}$$

$$\frac{\partial \underline{V}}{\partial t} = \frac{e}{M}\left(\underline{E} + \frac{1}{c}\,\underline{V} \times \underline{B}^o\right) - \operatorname{Div} \underline{\underline{\pi}} \tag{6}$$

$$\frac{\partial \underline{\underline{\pi}}}{\partial t} = -\nabla\cdot(\underline{V}\,\underline{\underline{\pi}}^o + \underline{\underline{\pi}}^o\,\underline{V}) - (\nabla\cdot\underline{\underline{\pi}}^o\,\underline{V})^T \tag{7}$$

$$+ \frac{e}{Mc}\left\{ (\pi^o \times B - B \times \pi^o) + (\underline{\underline{\pi}} \times \underline{B}^o - \underline{B}^o \times \underline{\underline{\pi}}) \right\}$$

If $\underline{B}^o$ is a parallel to the z-direction of a cartesian coordinate system,

$$\underline{\underline{\pi}}^o = \begin{pmatrix} \pi_\perp^o & 0 & 0 \\ 0 & \pi_\perp^o & 0 \\ 0 & 0 & \pi_z^o \end{pmatrix}. \tag{8}$$

The symbols without an upper index now indicate perturbation increments, and

$$\underline{\underline{\pi}}^o \equiv \frac{1}{MN^o}\,\underline{\underline{P}}^o; \quad \underline{\underline{\pi}} \equiv \frac{1}{MN^o}\,\underline{\underline{P}}.$$

Again a similar set of equations describes the electrons. We now assume that all perturbed quantities vary as

$$e^{i\underline{k}\cdot\underline{r}-i\omega t}, \quad [\mathrm{Re}(\omega) \gtrless 0],$$

in time and space. We further simplify the problem by considering longitudinal perturbations only:

$$\underline{B} \equiv 0, \qquad \mathrm{curl}\ \underline{E} \equiv 0 \tag{9}$$

$$\underline{E} = -\ \mathrm{grad}\ \underline{\Phi}; \qquad \Delta\underline{\Phi} = -\ 4\pi e(N - n). \tag{10}$$

According to Harris[1] and Bernstein[5] this approximation is justified for $|\omega| \ll kc$. On assuming zero electron pressure, and using the symbols n and $\underline{v}$ for the electron density and average velocity, the electron equations read:

$$n = \frac{n_o}{\omega}\,(\underline{k}\cdot\underline{v}) \tag{11}$$

$$i\omega\underline{v} = \frac{e}{m}\,\underline{E} + \omega_c(\underline{v} \times \underline{e}_z), \tag{12}$$

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5. I. B. Bernstein, Phys. Rev. 109, 10 (1958).

where $\underline{e}_z$ is the unit vector in z-direction. The equations for the ions are ($N^o = n_o$ assumed):

$$N = \frac{n_o}{\omega} (\underline{k}\cdot\underline{V}) \tag{13}$$

$$- i\omega\underline{V} = \frac{e}{M} \underline{E} + \Omega_c(\underline{V} \times \underline{e}_z) - i(\underline{k}\cdot\underline{\underline{\pi}}) \tag{14}$$

$$\omega\underline{\underline{\pi}} = \underline{k}\cdot(\underline{V}\,\underline{\underline{\pi}}^o + \underline{\underline{\pi}}^o\underline{V}) + (\underline{k}\cdot\underline{\underline{\pi}}^o\underline{V})^T + i\Omega_c(\underline{\underline{\pi}} \times \underline{e}_z - \underline{e}_z \times \underline{\underline{\pi}}). \tag{15}$$

Here, as throughout the paper, e symbolizes the ion charge (singly charged ions are assumed for convenience), and ω_c and Ω_c designate the angular electron and ion cyclotron frequency:

$$\omega_c \equiv \frac{eB^o}{mc}\ ,\ \Omega_c \equiv \frac{eB^o}{Mc}\ .$$

Furthermore

$$k^2\phi = 4\pi e(N - n) \tag{16}$$

By eliminating n and $\underline{v}$ from Eqs. (11), (12), and (16), the electric potential can be expressed in terms of ion variables only:

$$\phi = \frac{M\omega}{ek^2} G(\omega)(k\cdot V) \tag{17}$$

with

$$G(\omega) \equiv \Omega_p^2 \left[\omega^2 - \frac{\alpha_z^2 - \gamma^2}{1 - \gamma^2} \omega_p^2\right]^{-1} \tag{18}$$

where Ω_p and ω_p are the ion and electron plasma frequencies (angular frequencies):

$$\Omega_p^2 = \frac{4\pi e^2 n_o}{M} , \quad \omega_p^2 = \frac{4\pi e^2 n_o}{m} ,$$

and $\alpha_z = \underline{e}_k \cdot \underline{e}_z$ [$\underline{e}_k$ is the unit vector in $\underline{k}$-direction]; the quantity $\gamma \equiv \omega/\omega_c$ will be put equal to zero in the remainder of this paper, since only $\omega \approx \Omega_c$ will be considered, and $\Omega_c/\omega_c \ll 1$.

After insertion of Eq. (17) into Eq. (14), the equations for $\underline{V}$ and $\underline{\underline{\pi}}$ form a closed set. On using the following abbreviations:

$$u \equiv \frac{\omega}{k} , \quad u_c \equiv \frac{\omega_c}{k} , \quad U_c \equiv \frac{\Omega_c}{k} ,$$

$$u_p \equiv \frac{\omega_p}{k} , \quad U_p \equiv \frac{\Omega_p}{k} ,$$

and measuring all velocities in units of U_c (i.e., putting $U_c = 1$), we obtain

$$u\underline{V} - u\,\tilde{G}(u)(\underline{V}\cdot\underline{e}_k)\underline{e}_k - i(\underline{V} \times \underline{e}_z) - \underline{e}_k\cdot\underline{\underline{\pi}} = 0 \tag{19}$$

$$u\underline{\underline{\pi}} - \underline{e}_k\cdot(\underline{V}\,\underline{\underline{\pi}}^o + \underline{\underline{\pi}}^o\underline{V}) - (\underline{e}_k\cdot\underline{\underline{\pi}}^o\underline{V})^T - i(\underline{\underline{\pi}} \times \underline{e}_z - \underline{e}_z \times \underline{\underline{\pi}}) = 0 \tag{20}$$

with

$$\tilde{G}(u) = \frac{U_p^2}{u^2 - \dfrac{\alpha_z^2 - \gamma^2}{1 - \gamma^2}\,u_p^2} \approx \frac{U_p^2}{u^2 - \alpha_z^2 u_p^2} \quad [\gamma \ll 1], \tag{21}$$

and $\underline{\underline{\pi}}^o$ given by Eq. (8).

It is convenient to assume that $\underline{k}$ lie in the x-z plane; consequently

$$\underline{e}_k \cdot \underline{e}_y = 0, \quad \underline{e}_k \cdot \underline{e}_x = \alpha_x, \quad \alpha_x^2 + \alpha_z^2 = 1.$$

Equations (19) and (20), when written in components, assume now the following form:

$$\left.\begin{aligned} uV_x - uG(u)(\alpha_x^2 V_x + \alpha_x \alpha_z V_z) - iV_y - \alpha_x \pi_{xx} - \alpha_z \pi_{xz} &= 0 \\ uV_y \qquad\qquad\qquad\qquad + iV_x - \alpha_x \pi_{xy} - \alpha_z \pi_{yz} &= 0 \\ uV_z - uG(u)(\alpha_x \alpha_z V_x + \alpha_z^2 V_z) \qquad - \alpha_x \pi_{xz} - \alpha_z \pi_{zz} &= 0 \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} u\pi_{xx} - \pi_\perp^o(3\alpha_x V_x + \alpha_z V_z) - 2i\pi_{xy} &= 0 \\ u\pi_{yy} - \pi_\perp^o(\alpha_x V_x + \alpha_z V_z) + 2i\pi_{xy} &= 0 \\ u\pi_{zz} - \pi_z^o(\alpha_x V_x + 3\alpha_z V_z) \qquad\qquad &= 0 \\ u\pi_{xy} - \pi_\perp^o \alpha_x V_y \qquad + i(\pi_{xx} - \pi_{yy}) &= 0 \\ u\pi_{xz} - (\pi_\perp^o \alpha_x V_z + \pi_z^o \alpha_z V_x) - i\pi_{yz} &= 0 \\ u\pi_{yz} - \pi_z^o \alpha_z V_y \qquad\qquad + i\pi_{xz} &= 0 \end{aligned}\right\} \tag{23}$$

Solving Eqs. (23) for the π_{ik} and inserting the solutions in Eq. (22) finally leaves us with the following set of three equations only:

$$\left\{ s - \alpha_x^2 sg(s) - 3\alpha_x^2 \pi_\perp^o + \frac{4\alpha_x^2 \pi_\perp^o}{4 - s} - \alpha_z^2 \pi_z^o \frac{s}{s - 1} \right\} V_x \tag{24}$$

$$+ \left\{ 1 - \frac{2\alpha_x^2 \pi_\perp^o}{4 - s} + \frac{\alpha_z^2 \pi_z^o}{s - 1} \right\} y + \alpha_x \alpha_z \left\{ - sg(s) - \pi_\perp^o - \frac{s}{s - 1} \pi_\perp^o \right\} V_z = 0,$$

$$\left\{ 1 - \frac{2\alpha_x^2 \pi_\perp^o}{4 - s} + \frac{\alpha_z^2 \pi_z^o}{s - 1} \right\} V_x + \left\{ 1 + \frac{\alpha_x^2 \pi_\perp^o}{4 - s} - \frac{\alpha_z^2 \pi_z^o}{s - 1} \right\} y \tag{25}$$

$$+ \alpha_x \alpha_z \frac{\pi_\perp^o}{s - 1} V_z = 0,$$

$$\alpha_x \alpha_z \left\{ - sg(s) - \frac{\pi_z^o s}{s - 1} - \pi_z^o \right\} V_x + \alpha_x \alpha_z \frac{\pi_z^o}{s - 1} y \tag{26}$$

$$+ \left\{ s - \alpha_z^2 sg(s) - \alpha_x^2 \pi_\perp^o \frac{s}{s - 1} - 3\alpha_z^2 \pi_z^o \right\} V_z = 0$$

with $s \equiv u^2$, $y \equiv - iuV_y$, and

$$g(s) \equiv \frac{\xi s_p}{s - \alpha_z^2 s_p}, \quad s_p \equiv u_p^2, \quad \xi \equiv \frac{m}{M}. \tag{27}$$

In order to obtain the condition for instability, we must put the determinant D of the above equations equal to zero and solve for s:

$$D \equiv \| D_{ik} \| = 0.$$

It is convenient to consider instead the equation

$$\phi \equiv \| \phi_{ik} \| = 0, \quad \phi_{ik} \equiv F\, D_{ik} \tag{28}$$

with

$$F(s) \equiv \frac{\sigma_2 (s - 1)(4 - s)}{g(s)}, \quad \sigma_2 \equiv \xi s_p \equiv U_p^2. \tag{29}$$

We shall write ϕ, respectively ϕ_{ik}, as polynomials in $x \equiv s - 1$, e.g.,

$$\phi(x) = \sum_n \frac{\phi^{(n)}(0)}{n!} x^n. \tag{30}$$

For $\omega \approx \Omega_c$ we have $|x| \ll 1$ and may find an approximate solution of $\phi = 0$ by neglecting higher powers of x. With the further abbreviations

$$\sigma_1 \equiv 1 - \alpha_z^2 s_p; \quad \sigma_\perp \equiv \sigma_1 - \alpha_x^2 \sigma_2; \quad \sigma_z \equiv \sigma_1 - \alpha_z^2 \sigma_2, \tag{31}$$

the elements ϕ_{ik} assume the form:

$$\begin{aligned}\phi_{11}(x) = {} & 3\sigma_\perp x + (3 + 2\sigma_\perp)x^2 + (2 - \sigma_\perp)x^3 - x^4 \\ & + \alpha_x^2 \pi_\perp^0 \, [- 5\sigma_1 x + (-5 + 3\sigma_1)x^2 + 3x^3] \\ & + \alpha_z^2 \pi_z^0 \, [- 3\sigma_1 + (-3 - 2\sigma_1)x + (-2 + \sigma_1)x^2 + x^3]\end{aligned}$$

$$\begin{aligned}\phi_{12}(x) = {} & 3\sigma_1 x + (3 - \sigma_1)x^2 - x^3 + \alpha_x^2 \pi_\perp^0 \, [- 2\sigma_1 x - 2x^2] \\ & + \alpha_z^2 \pi_z^0 \, [3\sigma_1 + (3 - \sigma_1)x - x^2]\end{aligned}$$

$$\begin{aligned}\phi_{13}(x) = {} & \alpha_x \alpha_z \sigma_2 \, [- 3x - 2x^2 + x^3] \\ & + \alpha_x \alpha_z \pi_\perp^0 \, [- 3\sigma_1 + (-3 - 5\sigma_1)x + (-5 + 2\sigma_1)x^2 + 2x^3]\end{aligned}$$

$$\phi_{21}(x) = \phi_{12}(x)$$

$$\phi_{22}(x) = 3\sigma_1 x + (3 - \sigma_1)x^2 - x^3 + \alpha_x^2\pi^0\,[\sigma_1 x + x^2]$$

$$+ \alpha_z^2\pi_z^0\,[-3\sigma_1 + (-3 + \sigma_1)x + x^2]$$

$$\phi_{23}(x) = \alpha_x\alpha_z\pi^0\,[3\sigma_1 + (3 - \sigma_1)x - x^2]$$

$$\phi_{31}(x) = \alpha_x\alpha_z\sigma_2\,[-3x - 2x^2 + x^3]$$

$$+ \alpha_x\alpha_z\pi_z^0\,[-3\sigma_1 + (-3 - 5\sigma_1)x + (-5 + 2\sigma_1)x^2 + 2x^3]$$

$$\phi_{32}(x) = \alpha_x\alpha_z\pi_z^0\,[3\sigma_1 + (3 - \sigma_1)x - x^2]$$

$$\phi_{33}(x) = 3\sigma_z x + (3 + 2\sigma_z)x^2 + (2 - \sigma_z)x^3 - x^4$$

$$+ \alpha_x^2\pi^0\,[-3\sigma_1 + (-3 - 2\sigma_1)x + (-2 + \sigma_1)x^2 + x^3]$$

$$+ \alpha_z^2\pi_z^0\,[-9\sigma_1 x + (-9 + 3\sigma_1)x^2 + 3x^3]$$

From these expressions for the ϕ_{ik} the polynomial form of $\phi(x)$ can be obtained by straightforward, if somewhat tedious, calculations. A certain simplification arises from the fact that the quantities σ_2, $\pi_\perp^0$, π_z^0 are all small compared to 1 in the cases of interest and/or within the expected range of usefulness of the hydrodynamic approach. To keep σ_2 small we must restrict ω_p such that

$$\frac{m}{M}\,\omega_p^2 = \Omega_p^2 \ll \Omega_c^2 \,.$$

Since $\pi_\perp^o$, measured in units of U_c, is of the magnitude $(k\,R_c)^2$, R_c being the ion cyclotron radius, $\pi_\perp^o \ll 1$ restricts us to wavelengths large compared to R_c. For $u \approx U_c$ we have $(k\,R_c)^2 \approx (W_{therm}/u)^2$, W_{therm} being the thermal ion velocity. Since in the hydrodynamic approximation we assume $W_{therm} \ll u$, the smallness of $\pi_\perp^o$ is no new restriction. For further simplification we shall also assume

$$\frac{\sigma_2}{|\sigma_1|} \ll 1; \quad \frac{\pi_i^o}{|\sigma_1|} \ll 1;$$

this implies that $|\Omega_c^2 - \alpha_z^2\omega_p^2|$ should not become too small, i.e., in ordinary units:

$$|\Omega_c^2 - \alpha_z^2\omega_p^2| \gg \max\left\{\Omega_p^2,\ k^2\pi_\perp^o\right\}.$$

This excludes the "resonance" case $\omega_p = \Omega_c$ from our consideration.

We now obtain the following Taylor coefficients for ϕ: $\phi(0) = \phi'(0) = 0$

$$a_o \equiv \frac{1}{2}\,\phi''(0) = \sigma_1^2\sigma_2\alpha_x^2\pi_\perp^o(27\alpha_x^2 - 54\alpha_z^2) - 108\sigma_1^3\alpha_z^2\pi_z^o + \sigma_1^2\sigma_2\alpha_z^2\pi_z^o(-27\alpha_x^2 \tag{32}$$

$$+ 108\alpha_z^2) - 9\sigma_1^3\alpha_x^2\pi_\perp^o\alpha_z^2\pi_z^o + 351\sigma_1^3\alpha_z^4(\pi_z^o)^2 + O[(\pi_i^o)^3,\ (\pi_i^o)^2\,\sigma_2],$$

$$a_1 \equiv \frac{\phi^{(3)}(0)}{3!} = -\,27\alpha_x^2\sigma_1^2\sigma_2 - 27\sigma_1^3\alpha_x^2\pi_\perp^o - 324\sigma_1^2\alpha_z^2\pi_z^o - 54\sigma_1^3\alpha_z^2\pi_z^o \tag{33}$$

$$+ O\,[\sigma_2^2, \sigma_2\pi_i^o,\ (\pi_i^o)^2]$$

$$a_2 \equiv \frac{\phi^{(4)}(0)}{4!} = 27\sigma_1^3 + O(\sigma_2, \pi_i^o) \tag{34}$$

$$a_3 \equiv \frac{\phi^{(5)}(0)}{5!} = 81\sigma_1^2 + O(\sigma_2, \pi_i^o) \tag{35}$$

$$a_4 \equiv \frac{\phi^{(6)}(0)}{6!} = -12\sigma_1^3 - 9\sigma_1^2 + 36\sigma_1 + O(\sigma_2, \pi_1^o). \tag{36}$$

The coefficients of the terms of higher order have not been calculated. The determinant D can now be written as:

$$D \equiv x^{-1}(3 - x)^{-3}(x + \sigma_1)^{-3} \sum_{n=0}^{N} a_n x^n. \tag{37}$$

In the next section we shall discuss the occurrence of complex-valued roots of the equation D = 0 and the unstable oscillations associated with them.

III. UNSTABLE ION-CYCLOTRON OSCILLATIONS

Unstable plasma oscillations occur, whenever the equation D = 0 has complex roots x (and vice versa). In the following we restrict ourselves to small values of x, i.e., to $\omega \approx \Omega_c$. Let us look for the zeros of the polynomials

$$P_N(x) \equiv \sum_{n=0}^{N} a_n x^n, \tag{38}$$

with N = 2 or 3. It will be shown that this approximation is reasonable under a fairly wide range of conditions.

<u>We start by discussing the case $\pi_z^o = 0$.</u> The roots of the equation

$$P_2(x) \equiv a_o + a_1 x + a_2 x^2 = 0 \tag{39}$$

are complex for negative discriminant,

$$\Delta_2 \equiv a_1^2 - 4a_0a_2 < 0$$

i.e., for

$$\alpha_x^2(\sigma_1\pi_\perp^0 - \sigma_2)^2 + 8\sigma_1\sigma_2\alpha_z^2\pi_\perp^0 < 0. \tag{40}$$

It follows immediately that instability is possible only for negative σ_1, i.e., for $\alpha_z^2\omega_p^2 > \Omega_c^2$. In the special case $\alpha_x^2 \to 0$ this is also sufficient for instability. The propagation vectors $\underline{k}$ of the unstable modes lie inside a double-cone about the direction of $\underline{B}^0$, the maximum inclination angle being given by the implicit equation:

$$tg^2\theta \equiv \left(\frac{\alpha_x}{\alpha_z}\right)^2 = \frac{-8\sigma_1\sigma_2\pi_\perp^0}{(\sigma_1\pi_\perp^0 - \sigma_2)^2} .$$

The right-hand side is always smaller than 2; this corresponds to an angle $\theta \approx 55^\circ$.

It must be shown yet that it is sufficient to use only $P_2(x)$. Consider, then, the equation

$$P_3(x) \equiv a_0 + a_1x + a_2x^2 + a_3x^3 = 0. \tag{41}$$

Two of its roots are complex, if the discriminant is negative:

$$\Delta_3 \equiv a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 + 18\, a_0a_1a_2a_3 - 27a_0^2a_3^2 < 0. \tag{42}$$

The discriminant can also be written in the form

$$\Delta_3 \equiv (a_2^2 - 4a_1a_3)(a_1^2 - 4a_0a_2) + a_0a_3(2a_1a_2 - 27a_0a_3)$$

$$= (a_2^2 - 4a_1a_3)\,\Delta_2 + a_0a_3(2a_1a_2 - 27a_0a_3)$$

By substituting the explicit expressions for the coefficients, one sees that approximately

$$\Delta_3 \approx a_2^2\,\Delta_2\ ,$$

the other terms being of higher order in either σ_2 or $\pi_\perp^0$. Hence, as long as σ_1 is not too small, the use of $P_2(x)$ is justified.

The solutions of $P_2(x) = 0$ turn out to be

$$x = \frac{\alpha_x^2(\sigma_1\pi_\perp^0 + \sigma_2)}{2\sigma_1} \pm \frac{\alpha_x\sqrt{\alpha_x^2(\sigma_1\pi_\perp^0 - \sigma_2)^2 + 8\sigma_1\sigma_2\alpha_z^2\pi_\perp^0}}{2\sigma_1}$$

In order for this to be a good approximation for the solution of the original determinantal equation, $D(x) = 0$, x must be small compared to 1. The case $\alpha_x \to 0$ is of special interest. If $\sigma_1 < 0$ and $\alpha_x \to 0$:

$$x \to \pm\, i\alpha_x\sqrt{\frac{2\sigma_2\pi_\perp^0}{|\sigma_1|}} \tag{44}$$

In ordinary units this reads

$$x \to \pm\, i\,\frac{\alpha_x k\omega_p}{\Omega_c}\sqrt{\frac{2\xi\pi_\perp^0}{\omega_p^2 - \Omega_c^2}}$$

or

$$\omega \approx \pm \Omega_c \pm i \cdot \alpha_x k \omega_p \sqrt{\frac{\xi \pi_\perp^o}{2(\omega_p^2 - \Omega_c^2)}} \quad . \tag{45}$$

We want to compare our results with those of E. G. Harris.[1] He obtains, in our notation (ordinary units):

$$\omega = \pm \Omega_c \pm i\, \alpha_z \omega_p \Omega_c \sqrt{\frac{\xi e^{-q_\perp} I\,(q_\perp)}{A\alpha_z^2 \omega_p^2 - \Omega_c^2}} \tag{46}$$

with $A \approx 1$ and

$$q_\perp \equiv \frac{\alpha_x^2 k^2 \pi_\perp^o}{\Omega_c^2} \tag{47}$$

We are only interested in the case where $q_\perp \ll 1$, whence Eq. (46) reduces to

$$\omega \approx \pm \Omega_c \pm i\, \alpha_x \alpha_z k \omega_p \sqrt{\frac{\xi \pi_\perp^o}{2(\alpha_z^2 \omega_p^2 - \Omega_c^2)}} \tag{48}$$

It is seen that our results, as given in Eqs. (40), (43), and (45), agree with Harris' formula for $\alpha_x \to 0$ ($\alpha_z \to 1$). For nonvanishing α_x our equations contain stabilizing terms not found in Harris' formulae, such that for $\alpha_x^2 \geq \frac{2}{3}$ we obtain always stability in contradiction to Harris' result. We must infer from this that the neglect of pressure diffusion, which was used in order to obtain a closed set of hydrodynamic equations, becomes less satisfactory, as the angle between the propagation vector $\underline{k}$ and the magnetic field $\underline{B}^o$ increases. This effect persists for non-zero π_z^o;however, the possibility exists that this peculiar stability for $\alpha_x^2 \geq \frac{2}{3}$ may not follow, if one solves for the unabridged determinant without use of the above approximations.

Finally, we present a discussion of the case of nonvanishing π_z^o, again in second-order approximation with respect to x. The discriminant associated with Eq. (39) now assumes the form

$$\Delta_2 \approx 27^2\sigma_1^4 \left\{ \alpha_x^4(\sigma_1\pi_\perp^o - \sigma_2)^2 + 8\sigma_1\sigma_2\alpha_x^2\alpha_z^2\pi_\perp^o + 16\sigma_1\alpha_z^2\pi_z^o \right. \tag{49}$$

$$\left. + (12 + 2\sigma_1)\ [2\alpha_x^2(\sigma_1\pi_\perp^o + \sigma_2) + (12 + 2\sigma_1)\alpha_z^2\pi_z^o]\ \alpha_z^2\pi_z^o \right\},$$

or

$$\Delta_2 \approx 27^2\sigma_1^4 \left\{ \alpha_x^4(\sigma_1\pi_\perp^o - \sigma_2)^2 + 8\sigma_1\sigma_2\alpha_x^2\pi_\perp^o + 16\sigma_1\alpha_z^2\pi_z^o \right\}, \tag{50}$$

where terms of higher order in $\pi_\perp^o$, π_z^o, and σ_2 have been dropped, since these quantities are assumed to be small. It is seen that a non-zero π_z^o has a stabilizing effect and that $\sigma_1 < 0$ is a necessary condition for instability, as before.

Written as a condition for π_z^o, the instability condition $\Delta_2 < 0$ reads:

$$\frac{\pi_z^o}{\pi^o} < \xi\ F(A,p,z) \tag{51}$$

with

$$F(A,p,z) \equiv \frac{1}{2}\ \frac{1 - z}{z - p} - \frac{1}{16}\ \frac{(1 - z)^2}{zA}\left(1 + \frac{A}{z - p}\right)^2, \tag{52}$$

where

$$A \equiv \frac{\xi}{\pi^o}, \quad p \equiv \frac{1}{s_p}, \quad z \equiv \alpha_z^2, \tag{53}$$

and only the range $p \leq 1$, $z \geq \max\ (p, \frac{1}{3})$ is of interest.

We are interested in the maximum of F, since this determines the onset of instability. Notice that p has a fixed value for a given homogeneous plasma and for specified magnetic field $\underline{B}^o$, whereas z and A are variables. (A depends on the wavelength through $\pi_\perp^o$.) We obtain the following results:

a. With respect to A the function F has a single maximum (A > 0) at A = z - p:

$$F(A_{max}) = \frac{3(1 - z)(z - \frac{1}{3})}{4z(z - p)} \tag{54}$$

b. With respect to z, the function $F(A_{max})$ has a single maximum in the range $\frac{1}{3} \leq z \leq 1$ at

$$z_{max} = \frac{1}{4 - 3p} \left[1 + \sqrt{1 - p(4 - 3p)}\ \right] \tag{55}$$

provided $p \leq \frac{1}{3}$; for $p > \frac{1}{3}$ the z-derivative of $F(A_{max})$ is always negative, and $F(A_{max}) \to +\infty$ at $z = p + 0$.

c. For $p \leq \frac{1}{3}$ we obtain the following maximum of F:

$$F(A_{max}, z_{max}) = \frac{1}{(2z_{max})^2} - \frac{3}{4}\ ; \quad [p \leq \frac{1}{3}\ ]; \tag{56}$$

with z_{max} given by Eq. (55). For $p = z_{max} = \frac{1}{3}$, F assumes the largest value within this range ($p \leq \frac{1}{3}$), i.e., $F = \frac{3}{2}$. Correspondingly

$$\frac{\pi_z^o}{\pi_\perp^o} > \frac{3}{2}\,\xi \approx 0.817 \times 10^{-3} \text{ (for protons)} \tag{57}$$

provides for stability in the range $p \leq \frac{1}{3}$, i.e., $(\omega_p/\Omega_c) \geq \sqrt{3}$. The result,

of course, refers only to oscillation modes with $\omega \approx \Omega_c$, not to higher harmonics of the ion cyclotron frequency.

d. The divergence of F for $p > \frac{1}{3}$, $z \to p + 0$, $A = z - p \to 0$, is unphysical, since this limit contradicts our assumptions

$$\pi_\perp^0 \ll 1; \quad \frac{\xi}{z - p} \ll 1; \quad \frac{p\pi_\perp^0}{z - p} \ll 1, \tag{*}$$

the latter two conditions being equivalent to

$$\frac{\sigma_2}{|\sigma_1|} \ll 1; \quad \frac{\pi_\perp^0}{|\sigma_1|} \ll 1.$$

It is easy to see that Eq. (54) is useful only as long as

$$z - p \gtrsim \sqrt{10\xi} = 0.0735 \text{ (for protons)},$$

and particularly:

$$p \lesssim 0.9265.$$

Hence we ought to choose, instead of $z_{max} = p + 0$,

$$z_{max} \approx p + \sqrt{10\xi}, \tag{58}$$

or something of similar magnitude. From this we obtain

$$F(A_{max}, z_{max}) \approx 10.2 \times p^{-1} (1 - p)(p - \tfrac{1}{3}), \tag{59}$$

with a maximum $F \approx 1.82$ at $p = 1/\sqrt{3}$. Consequently

$$\frac{\pi_z^o}{\pi_\perp^o} > 1.82\ \xi \tag{60}$$

will provide for stability in the range $\frac{1}{3} < p \leq 0.9265$ $[1.0389 \leq (\omega_p/\Omega_c) < \sqrt{3}]$, and $z \geq p + 0.0735$.

e. Another estimate of the critical pressure anisotropy can be obtained in the following way: For fixed values of z and A the function $F(A,p,z)$ has a single maximum with respect to p at $p = p_{max}$, with

$$\frac{1}{z - p_{max}} = \frac{1}{A}\ \frac{5z - 1}{1 - z}\ , \tag{61}$$

and

$$F(p_{max}) = \frac{1}{2A}\ (3z - 1). \tag{62}$$

The maximum with respect to z of the latter expression occurs at $z_{max} = 1$ $(p_{max} = 1)$, with $F = A^{-1}$. Consequently

$$\frac{\pi_z^o}{\pi_\perp^o} > \pi_\perp^o \tag{63}$$

provides for stability in all cases in which the assumptions (*) are valid. Since $\pi_\perp^o$ stands, in ordinary units, for $(k^2\pi_\perp^o/\Omega_c^2) \approx k^2R_c^2$, we see that for smaller wavelength a greater value of $(\pi_z^o/\pi_\perp^o)$ is required for stabilization.

f. The above results are limited in two respects. First, the case $\omega_p \to \Omega_c$, which probabily is most unstable, has not been treated, since in evaluating the coefficients a_i use was made of the assumptions (*). Secondly, the case of large k (wavelengths comparable with or smaller than the ion gyration radius) had to be left out of consideration as a consequence of the hydrodynamic approximation.

IV. DISCUSSION

The two-fluid hydrodynamic theory presented above has been shown to describe certain unstable modes of the Harris type in an approximate fashion. Generally, greater stability is predicted than on the basis of the Vlasov equations. Harris' necessary and sufficient condition for instability, viz., $n^2\Omega_c^2 < \alpha_z^2\omega_p^2$, with n = an entire number, occurs in our model merely as a necessary condition, with the restriction $n = \pm 1$. Harris' result saying that $\omega \approx n\Omega_c$, whenever $(\alpha_z^2\omega_p^2 - n^2\Omega_c^2)$ is sufficiently large, also follows from the present model (again with the restriction n = 1). Somewhat disturbingly, a fixed range of propagation directions permitting stable oscillation modes only, exists in our model. This might actually cast some doubt upon the usefulness for stability problems of the two-fluid hydrodynamic model. Still, the model reproduces the main features of Harris' results.

Several rough estimates of the critical pressure anisotropy required for instability have been derived in this paper. These critical values are in general too small as compared to Drummond, Rosenbluth, and Johnson's results.[2] According to their report, values of $(\pi_\perp^0/\pi_z^0)$ greater than 8 or 9 cause instability. The only estimate of ours that approaches their result, when evaluated for wavelengths somewhat larger than the ion cyclotron radius, is given in Eq. (63).

For the convenience of the reader the approximations that have been used in our paper are listed once more:

1. Hydrodynamic approximation, neglect of ion pressure diffusion variation $\underline{\underline{Q}}$. Consequences: Validity expected only for

$$k^2 \ll \frac{|\omega|^2}{\pi_\perp^o} \quad , \quad k^2 \ll \frac{|\omega|^2}{\pi_z^o} \; ;$$

which for $\omega \approx \Omega_c$, $\pi_z^o < \pi_\perp^o$ implies $k^2 R_c^2 \ll 1$ (in reduced units: $\pi_\perp^o \ll 1$, $\pi_z^o \ll 1$). Probably there exists also an upper frequency limit of validity of the order Ω_c or $2\Omega_c$ (see end of this Section IV). Cold electrons assumed for convenience.

2. Linearization, Fourier decomposition in space, time variation like $\exp(-i\omega t)$ assumed.

3. Purely longitudinal oscillations assumed, i.e., $\underline{B} = 0$, $|\omega| \ll kc$. The assumption has been used by Harris[1] and Bernstein.[5]

4. $(\omega/\omega_c) \to 0$, i.e., electrons tied to magnetic field lines. Unproblematic for the case considered ($\omega \approx \Omega_c$).

5. Evaluation of dispersion relation to second order in

$$x = \frac{\omega^2 - \Omega_c^2}{\Omega_c^2}$$

Hence limitation to small growth constants.

6. Small ion plasma frequency:

$$\Omega_p^2 = \frac{m}{M}\,\omega_p^2 \ll \Omega_c^2 \; ;$$

this provides an upper limit for the electron plasma frequency ω_p. In reduced units: $\sigma_2 \ll 1$.

7. Exclusion of the case $\alpha_z \omega_p \approx \Omega_c$. Specifically

$$|\Omega_c^2 - \alpha_z^2 \omega_p^2| \gg \max(\Omega_p^2,\ k^2\pi_\perp^o,\ k^2\pi_z^o)$$

or, in reduced units,

$$|\sigma_1| \gg \max(\sigma_2, \pi_\perp^o, \pi_z^o).$$

This excludes the case of strong resonance between ion cyclotron motion and electron plasma oscillations with its possible consequence of strong instability.

Use of the assumptions 1, 6, and 7 is made in evaluating the dispersion relation. These assumptions provide for validity of approximation 5 and allow the use of simplified coefficients in the polynomials $P_N(x)$.

The two-fluid hydrodynamic theory contributes to our qualitative understanding of the unstable oscillation modes. We consider first the motion of the electrons. In the limit used, i,e., for $\omega_c \to \infty$, electron density and velocity are given by:

$$\left.\begin{aligned} n &\approx \frac{n_0}{u}\,\alpha_z v_z \\ v_z &\approx -\frac{ie}{m\omega}\,E_z = -\frac{e}{mu}\,\alpha_z\phi; \quad v_x \approx v_y \approx 0. \end{aligned}\right\} \tag{64}$$

As one expects, the electrons (more properly: their guiding centers) are tied to the magnetic field lines and respond only to the z-component of the electric field. One obtains equally easily the following important relation between ion density N and electron density n (again for $\omega_c \to \infty$):

$$\frac{n}{N} = \frac{1}{1 - \dfrac{\omega^2}{\alpha_z^2\omega_p^2}} \tag{65}$$

If ω is real, electron and ion density are either in equal or in opposite phase, depending on whether $\omega^2 < \alpha_z^2\omega_p^2$ or $\omega^2 > \alpha_z^2\omega_p^2$. In the first instance, the electron density generally overcompensates for the ion charge density, while in the second instance the ion charge density is added to by the electrons. When ω is not purely real, but possesses a small imaginary part, i,e.,

$$\omega = \omega' + i\omega''; \quad |\omega''| \ll |\omega'|, \tag{66}$$

with ω', ω'' real, and

$$\frac{n}{N} = \left(1 - \frac{\omega'^2 - \omega''^2}{\alpha_z^2 \omega_p^2} - \frac{2i\omega'\omega''}{\alpha_z^2 \omega_p^2} \right)^{-1} \tag{67}$$

then n and N are still nearly in equal or opposite phase as before, as long as

$$\left| \frac{2\omega'\omega''}{\alpha_z^2 \omega_p^2 - \omega'^2} \right| \ll 1;$$

while otherwise appreciable phase-shifts will occur. The transition from effective charge overcompensation to charge addition occurs for $\mathrm{Re}(n/N) = \infty$, or

$$\alpha_z^2 \omega_p^2 - \omega'^2 + \omega''^2 \approx \alpha_z^2 \omega_p^2 - \omega'^2 = 0. \tag{68}$$

We note that this coincides with Harris' instability condition

$$\alpha_z^2 \omega_p^2 - n^2 \Omega_c^2 > 0, \tag{69}$$

as $\omega' = \pm n\Omega_c$. Apparently

$$\mathrm{Re}\left(\frac{n}{N}\right) > 0, \tag{70}$$

i.e., effective charge (over) compensation, is necessary for the Harris instability to occur.

We now consider the motion of the ions. On using the same approximations as in Chapters III and IV, especially $\omega \approx \Omega_c$, we obtain the following result for the macroscopic ion velocity $\underline{V}$ (velocities in units of U_c):

$$\left.\begin{aligned}
\lambda V_x &= 1 - \frac{1}{6}\alpha_x^2\pi_\perp^o + x\cdot\left[\left(\frac{2}{\sigma_1} + \frac{1}{3}\right) + \frac{1}{2}\,\frac{\sigma_2}{\sigma_1\pi_\perp^o}\right] + O(x^2, x\pi_\perp^o, \text{ etc.}) \\
\lambda V_y &= i\ \mathrm{sgn}(\omega)(1 + x)^{-1/2}\left\{-1 + \alpha_x^2\left(\frac{1}{2}\,\frac{\sigma_2}{\sigma_1} + \frac{7}{6}\,\pi_\perp^o\right)\right. \\
&\qquad\qquad \left. - x\cdot\left[\left(\frac{1}{\sigma_1} + \frac{2}{3}\right) + \frac{1}{2}\,\frac{\sigma_2}{\sigma_1\pi_\perp^o}\right] + O(x^2, x\pi_\perp^o, \text{ etc.})\right\} \\
\lambda V_z &= -x\cdot\left[\frac{\alpha_x}{2\alpha_z}\,\frac{\sigma_2}{\sigma_1\pi_\perp^o} + \frac{\alpha_x^3}{\alpha_z}\left(\frac{1}{2}\,\pi_\perp^o + \frac{1}{6}\,\frac{\sigma_2}{\sigma_1}\right)\right] + O(x^2, x\pi_\perp^o, \text{ etc.}),
\end{aligned}\right\} \tag{74}$$

where λ is an arbitrary constant and x is given by Eq. (43). The inverse of the square root, $(1 + x)^{-1/2}$, should be taken with positive real part. We see that a rough approximation for $\underline{V}$ is

$$V_x : V_y : v_z \approx 1 : \ [i\ \mathrm{sgn}(\omega)] : 0, \tag{75}$$

i.e., $\underline{V}$ is nearly perpendicular to the magnetic field $\underline{B}^o$. This is just what one would obtain by dropping the electric field and pressure terms in the ion equation of motion [Eq. (14)], in agreement with our exclusion of strong resonance between ion and electron motion (Approximation 7 on p. 21). Hence

it follows that in the limit of infinite wavelength and zero ion-electron coupling there exist collective ion modes with $\omega = \pm \Omega_c$.

It is plain, from our assumption of cold electrons, that the Harris instabilities are associated with the transformation of kinetic energy of the random ion cyclotron motion into electric field energy and macroscopic kinetic energy of ion and electron fluids. Our equations fail to reveal the "cause," or "mechanism," for this process, but lend themselves to the following tentative interpretation.

A. Under the right conditions (pressure anisotropy, ion-electron coupling), kinetic energy of random ion motion can be converted to "macroscopic" energy at frequencies $\omega \approx n\Omega_c$. As a frequency (resonance) condition, this would be insensitive to plasma geometry.

B. Proper ion-electron coupling, such as to cause instability, can be provided only if charge compensation obtains (in the sense explained above). The equation (69), expressing this condition, depends only on the propagation direction, not on the wavelength; hence, again, plasma geometry seems to be unimportant. The expectation that the stability conditions for Harris instability will be insensitive to plasma geometry is borne out by a study by Burt and Harris[1] of Harris instabilities in a cylindrical plasma shell.

A final remark concerns Harris instabilities at higher harmonics of Ω_c. The equations (24) to (26) contain singularities at $\omega = \pm \Omega_c$ and $\omega = \pm 2\Omega_c$. Further singularities at higher harmonics of Ω_c would arise, if we included higher-order moment equations. Since this would significantly alter our equations for $\omega \approx n\Omega_c$, $n \geq 3$, it is not possible to study Harris instabilities at the higher harmonics of Ω_c by our simple set of hydrodynamic equations.

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