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On the Reality of the Eigen-Values for a One-group N-region Diffusion Problem

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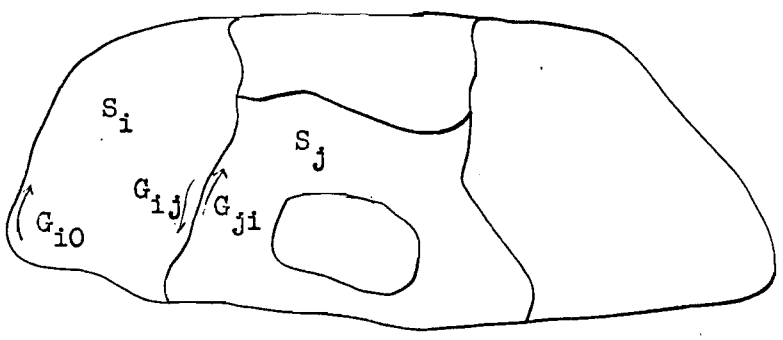
1. In the computation of the eigen-values for multiregion diffusion problems, some assurance is required that the eigen-values, and, in particular, the eigen-values of largest or smallest modulus, are real. This note provides a <u>sufficient</u> condition for the reality of the eigen-values for a fairly general one-group, multiregion two-dimensional diffusion problem. The condition is also sufficient to insure the orthogonality of eigen-functions corresponding to distinct eigen-values.

2. Consider the bounded, connected plane region S, which is subdivided into N subregions S_i, $i = 1, \ldots, N$, by the Jordan arcs G_{ij}, $i = 1, \ldots, N$; $j = 0, 1, \ldots, N$.



The arc G_{ij} is to be taken with a positive sense when viewed from the interior of the region S_i. Thus $G_{ji} = - G_{ij}$. The arc G_{i0} is the segment of the outer boundary of S which is common to the boundary of S_i.

In each region S_i, the linear differential equation with real coefficients

$$\text{(1a)} \qquad \nabla^2 u_i + D_i u_i = \lambda \theta_i u_i \; ; \quad \theta_i \geqslant 0, \; i = 1, \ldots, N; \; \sum_{i=1}^{N} \theta_i > 0.$$

(θ_i constant in each S_i ; D_i may be a function of position only)

is to be satisfied, with the eigen-value λ to be constant over those regions S_i where $\theta_i > 0$.

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The following interface conditions are to be satisfied on each G_{ij} , $i = 1, \ldots, N$
$j = 1, \ldots, N$.

$$(1b) \quad \begin{cases} u_i = \alpha_{ij} u_j + \beta_{ij} \dfrac{\partial u_j}{\partial n} \\ \dfrac{\partial u_i}{\partial n} = \gamma_{ij} u_j + \delta_{ij} \dfrac{\partial u_j}{\partial n} \end{cases} ; \quad \varphi_{ij} = (\alpha_{ij} \delta_{ij} - \beta_{ij} \gamma_{ij}) \neq 0 . \quad \alpha_{ij}, \ldots, \delta_{ij} \text{ real.}$$

On the outer boundary G_{i0} , $i = 1, \ldots, N$, we may specify either $u_0 = 0$ or $\dfrac{\partial u_0}{\partial n} = 0$, but not both, or, more generally, the determinant $\varphi_{i0} = 0$.

Note that where two subregions S_i and S_j have no common interface, a nonsingular $(\varphi_{ij} \neq 0)$ condition is satisfied vacuously, since $G_{ij} = 0$; and also that $\varphi_{ii} = 1$, $G_{ii} = 0$, which is to say that the solution is to be continuous within any given subregion S_i.

Note also that $\varphi_{ij} \neq 0$ implies

$$(2) \qquad \varphi_{ji} = \frac{1}{\varphi_{ij}} .$$

3. Assume that the problem has a complex eigen-value λ , and a corresponding eigen-function u. Then, because of the reality of the coefficients, the conjugates $\bar{\lambda}$ and $\bar{u}$ must also satisfy the problem:

$$\left.\begin{aligned} &(3a) \quad \nabla^2 u_i + D_i u_i = \lambda \theta_i u_i \\ &(3b) \quad \nabla^2 \bar{u}_i + D_i \bar{u}_i = \bar{\lambda} \theta_i \bar{u}_i \end{aligned}\right\} \qquad i = 1, \ldots, N$$

Multiply (3a) by $\bar{u}_i$, and (3b) by u_i, and take the (surface) integral of the difference.

$$(\lambda - \bar{\lambda})\, \theta_i \iint_{S_i} u_i \bar{u}_i \, d\sigma = \iint_{S_i} (\bar{u}_i \nabla^2 u_i - u_i \nabla^2 \bar{u}_i)\, d\sigma; \; i = 1, \ldots, N.$$

The application of Green's theorem in each subregion leads to

$$(\lambda - \bar{\lambda})\theta_i \iint_{S_i} u_i \bar{u}_i \, d\sigma = \sum_{j=0}^{N} \int_{G_{ij}} \left(\bar{u}_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial \bar{u}_i}{\partial n}\right) ds; \; i = 1, \ldots, N.$$

Multiply each equation by a constant $K_i > 0$, to be determined, and sum:

$$(4) \quad (\lambda - \bar{\lambda}) \sum_{i=1}^{N} K_i \theta_i \iint_{S_i} u_i \bar{u}_i \, d\sigma = \sum_{i=1}^{N} \sum_{j=0}^{N} K_i \int_{G_{ij}} \left(\bar{u}_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial \bar{u}_i}{\partial n}\right) ds \quad .$$

Now for $i < j$, and for $j = 0$, we make the substitution indicated by the interface conditions (1b), so that

$$K_i \int_{G_{ij}} \left(\bar{u}_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial \bar{u}_i}{\partial n}\right) = K_i \int_{G_{ij}} (\alpha_{ij} \delta_{ij} - \beta_{ij} \gamma_{ij}) \left(\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}\right) ds$$

$$= K_i \varphi_{ij} \int_{G_{ij}} \left(\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}\right) ds.$$

Since $\varphi_{i0} = 0$, all terms containing $j = 0$ vanish. The right-hand side of (4) then becomes

$$\sum_{j=1}^{N} \sum_{i=1}^{j-1} K_i \varphi_{ij} \int_{G_{ij}} (\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}) ds + \sum_{j=1}^{N} \sum_{i=j}^{N} K_i \int_{G_{ij}} (\bar{u}_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial \bar{u}_i}{\partial n}) ds$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{j-1} K_i \varphi_{ij} \int_{G_{ij}} (\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}) ds + \sum_{i=1}^{N} \sum_{j=1}^{i-1} K_i \int_{G_{ij}} (\bar{u}_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial \bar{u}_i}{\partial n}) ds$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{j-1} \left[K_i \varphi_{ij} \int_{G_{ij}} + K_j \int_{G_{ji}} \right] (\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}) ds$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{j-1} (K_i \varphi_{ij} - K_j) \int_{G_{ij}} (\bar{u}_j \frac{\partial u_j}{\partial n} - u_j \frac{\partial \bar{u}_j}{\partial n}) ds .$$

The first equality results from interchanging the order of summation in the second term; the second by "renaming" the indices in the second term; the last from recalling that $G_{ji} = - G_{ij}$.

From the last expression, it is obvious that if

$$(5) \qquad K_i \varphi_{ij} - K_j = 0 \; ; \quad 0 < i < j \leqslant N ,$$

then the right-hand side of (4) vanishes; since $u_i \bar{u}_i = |u_i|^2 > 0$, and $K_i \Theta_i > 0$ for at least one i , the left-hand side of (4) vanishes

only if $\lambda = \bar{\lambda}$ in those S_i where $\theta_i \neq 0$. Equation (5) therefore furnishes a sufficient condition that the eigen-values for the problem be real: that there exist a set of N positive constants K_i such that $K_j = K_i \varphi_{ij}$ for all $0 < i < j \leqslant N$. Since $K_i > 0$, $K_j > 0$, $\varphi_{ij} > 0$.

4. Assuming that equation (5) holds, it is possible to develop a more succinct statement of the condition. Note that

$$K_j = K_i \varphi_{ij} = K_h \varphi_{hj} = K_i \varphi_{ih} \varphi_{hj} , \quad i < h < j .$$

Therefore

$$\varphi_{ij} = \varphi_{ih} \varphi_{hj} ,$$

and in view of (2),

$$(6) \qquad \begin{cases} \varphi_{ih} \varphi_{hj} \varphi_{ji} = 1 , & \text{for all } 0 < i, j, h \leqslant N \\ \varphi_{ij} > 0 & \text{all } 0 < i, j \leqslant N . \end{cases}$$

<u>Theorem</u>. A sufficient condition that the one-group, N-region, diffusion problem (1a, b) have real eigen-values is that the determinants φ_{ij} of the interface conditions (1b) satisfy (6).

5. If λ, $\bar{\lambda}$, u, $\bar{u}$ are interpreted now to represent distinct (real) eigen-values and their corresponding eigen-functions, the identical analysis, bearing in mind that now $(\lambda - \bar{\lambda}) \neq 0$, leads to the conclusion:

Corollary. Condition (6) is sufficient to insure that the eigen-functions corresponding to distinct (real) eigen-values be orthogonal over the regions S_i where $\theta_i > 0$.

6. In certain frequently occurring special cases, the condition (6) is automatically satisfied.

(a) Continuity across the interface:

$$\begin{cases} u_i = u_j \\ K_i \dfrac{\partial u_i}{\partial n} = K_j \dfrac{\partial u_j}{\partial n} \qquad K_i > 0, \quad \text{all } 0 < i \leqslant N \end{cases}$$

$$\varphi_{ij} = K_j / K_i$$

(b) Contact resistance

$$K_i \frac{\partial u_i}{\partial n} = K_j \frac{\partial u_j}{\partial n} = h_{ij}(u_i - u_j) \; ; \qquad \begin{matrix} K_i > 0 \\ h_{ij} \neq 0 \end{matrix} \quad \text{all } i, j$$

implies

$$\begin{cases} u_i = u_j + \dfrac{K_j}{h_{ij}} \dfrac{\partial u_j}{\partial n} \\ \dfrac{\partial u_i}{\partial n} = \dfrac{K_j}{K_i} \dfrac{\partial u_j}{\partial n} \end{cases}$$

$$\varphi_{ij} = K_j / K_i \; .$$

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