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THE DISTORTED-WAVE THEORY OF DIRECT NUCLEAR REACTIONS

I: "Zero-Range" Formalism without Spin-Orbit Coupling, and the Code SALLY

R. H. Bassel
R. M. Drisko
G. R. Satchler

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ABSTRACT

The distorted-wave theory of direct nuclear reactions is presented in a unified manner in which the effects of assuming various reaction mechanisms and nuclear models appear only in certain radial form factors. The zero-range approximation is used, and spin-orbit coupling is neglected in the distorted waves. Formulae are given for transition amplitudes, cross-sections and polarizations. Then a description is given of the IBM-704 computer code SALLY which is based on these formulae.

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PART I: The Distorted-Wave Theory

1. Introduction

In recent years the broad class of reactions called "direct" has become of great importance as a source of information about nuclear structure. There are two reasons for this; firstly, since they do not pass through an intermediate or compound nucleus stage, they are dominated by a direct overlap between the initial and final nuclear states, and provide information in a way comparable to electromagnetic transitions. Secondly, they suffer no inhibition when feeding low-lying states of the residual system (in fact, a large energy loss is unfavorable). These states in general are least likely to be populated by more complicated processes such as compound nucleus formation and decay, yet they are the ones we most wish to study in terms of nuclear models.

One may say[1] that direct reactions only involve a few internal degrees of freedom of the system. Whether it is appropriate to describe these in terms of single particle or collective modes of motion, only one or two are involved in any particular reaction. Most of the current theories further assume weak-coupling; that is, they assume that elastic scattering is the most important process which occurs, and that inelastic or reaction events can be treated as perturbations. This leads for example to the so-called distorted-wave Born-approximation expression for the transition amplitude[1] in which the interaction responsible for the reaction appears once; that is, it describes a simple one-step process. The relative motion before and after the inelastic event is described by distorted-waves, which include the elastic scattering (usually calculated in an optical model approximation), so the

transition is one between elastic scattering states. The name Born-approximation, however, may be misleading. For example, the interaction one uses may be an "effective" interaction, in close analogy to the residual interactions one introduces into shell-model calculations. The theory is, perhaps, more closely akin to the impulse approximation. A great deal more has been done, in principle, than is usually implied by the term Born-approximation.

For some years theories have been used which are based upon plane waves instead of distorted waves.[2] That is, the elastic scattering is also assumed to be small. At the same time it was usually argued that one should exclude the nuclear interior from the integrals because of strong absorption. This basic inconsistency no longer appears in the distorted wave theory. Any strong absorption present is automatically taken care of by using the appropriate distorted waves. Distortion effects often considerably modify the shapes of differential cross-sections. In addition they give rise to polarization effects not allowed by the plane-wave theories (such as vector spin-polarizations) and modify others which are allowed (such as the angular distribution and correlation of γ-rays following the reaction). A further important feature of the distorted-wave theory is the possibility of predicting absolute cross-sections (apart from nuclear structure factors, and it is often the purpose of experiment to extract just these).

The plane-wave theories had one advantage that their results could often be expressed in simple analytical terms.[2] This is not possible with distorted-waves, and unless very dubious approximations are made it is impractical to carry out these calculations without the aid of a high-speed computer. For

this reason codes have been written for use on the IBM-704 or 7090 computers. The present report describes the simplest of these, called SALLY, which was originally written for the IBM-704 but also may be used on the 7090 with compatibility.

It is based on the theory using the so-called zero-range approximation, and neglecting the effects of spin-orbit coupling on the elastic scattering. It may be used for inelastic scattering, stripping (and pick-up) or knock-on reactions. It calculates differential cross-sections, polarizations, and, where appropriate, γ-ray correlation parameters, and in addition the elastic scattering cross-sections for incident and outgoing channels. The running time for this is typically 2 or 3 minutes on the 704 and less than 1/2 minute on the 7090. A variety of interaction models for the reaction, and a variety of optical models for the elastic scattering, are available as options, as described below. Up to 49 partial waves may be used.

Later reports will describe developments including (i) the effects of spin-orbit coupling in the distorted waves and (ii) the effects of finite range of the interactions.

2. The Distorted-Wave Theory

The distorted-wave theory of a direct nuclear reacton A(a,b)B is based upon a transition amplitude of the form

$$T = \int d\underset{\sim}{r}_{aA} \int d\underset{\sim}{r}_{bB}\, \phi^{(-)*}_{bB}(\underset{\sim}{k}_b, \underset{\sim}{r}_{bB}) \langle b,B | V | a,A \rangle\, \phi^{(+)}_{aA}(\underset{\sim}{r}_{aA}) \qquad \cdots (1)$$

for particle a incident with relative momentum $\underset{\sim}{k}_a$ and b emitted with relative momentum $\underset{\sim}{k}_b$. The ϕ are distorted waves (plane plus spherical scattered waves, outgoing or incoming denoted by the superscript $(\pm)$ respectively). We assume they are functions only of the separation of the centers of mass of the colliding pair, and independent of their spins. In practice they are generated from optical-model potentials such as describe the observed elastic scattering. They then satisfy Schrodinger equations of the form

$$\left[\nabla^2 + k^2 - \frac{2\mu}{\hbar^2} U(r) - \frac{2k\eta}{r}\right] \phi(\underset{\sim}{k}, \underset{\sim}{r}) = 0 \qquad \cdots (2)$$

where U(r) is the optical-model potential, μ the reduced mass of the pair, and η the Coulomb parameter, $\eta = Z_1 Z_2 e^2/\hbar v$. Asymptotically,

$$\phi^{(+)}(\underset{\sim}{k}, \underset{\sim}{r}) \longrightarrow e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + f(\theta, \phi)\, \frac{e^{ikr}}{r}$$

$$\phi^{(-)}(\underset{\sim}{k}, \underset{\sim}{r}) \longrightarrow e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + f^*(\pi-\theta, \pi+\phi)\, \frac{e^{-ikr}}{r},$$

which illustrates the general relation $\phi^{(-)*}(\underset{\sim}{k}, \underset{\sim}{r}) = \phi^{(+)}(-\underset{\sim}{k}, \underset{\sim}{r})$.

The remaining factor in (1) is the matrix element of the interaction taken between the <u>internal</u> states of the colliding pairs,

$$\langle b, B | V | a, A \rangle = \int \psi_b^* \psi_B^* V \psi_a \psi_A \, d\tau . \qquad \cdots (3)$$

It plays the role of an effective interaction for scattering from one elastic scattering state to another, and contains all the information on nuclear structure, angular momentum selection rules, and even the type of reaction considered (whether stripping, inelastic scattering, etc.). By having a number

of form factors available as options for this part of the matrix element, we may study a variety of nuclear models and reactions.

The effective interaction (3) is in general a function oı both $\underset{\sim}{r}_{aA}$ and $\underset{\sim}{r}_{bB}$; however we assume here (as in all current distorted wave calculations) that we may put these two vectors parallel, $\underset{\sim}{r}_{bB} = (\mathcal{M}_A/\mathcal{M}_B)\underset{\sim}{r}_{aA}$, where $\mathcal{M}_A$, $\mathcal{M}_B$ are the masses of A, B. We call this the "zero-range" approximation. It reduces the amplitude (1) to a single (3-dimensional) integral, which greatly facilitates its computation. When the reaction is inelastic scattering, so $\mathcal{M}_A = \mathcal{M}_B$, this approximation corresponds to assuming the interaction V is local, and neglecting exchange terms, for then the coordinates of the incoming and outgoing particles are the same. For a knock-on reaction, in which $\underline{a}$ knocks $\underline{b}$ out from the target $\underline{A}$ through an interaction V_{ab}, and is itself captured, we have to assume that V_{ab} is of zero-range (and local) in the separation of the centers of mass of a and b, $V_{ab} \propto \delta(\underset{\sim}{r}_a - \underset{\sim}{r}_b)$, which gives $\underset{\sim}{r}_{bB} = (\mathcal{M}_A/\mathcal{M}_B)\underset{\sim}{r}_{aA}$. In a stripping reaction, we have $\underline{a} = \underline{b} + \underline{x}$, where $\underline{x}$ is the particle transferred, so that $\underline{B} = \underline{A} + \underline{x}$ also. The interaction is generally assumed to be V_{bx}, and the factor (3) reduces to a "spectroscopic amplitude" (or nuclear overlap sometimes called a relative reduced width), times

$$\psi_x(\underset{\sim}{r}_{xA})\, D(\underset{\sim}{r}_{bx}) \qquad \cdots (4)$$

where ψ_x is the bound-state wave function for the captured particle $\underline{x}$ within $\underline{B}$, and D is the product of the interaction and the internal wave-function for the relative motion of $\underline{b}$ and $\underline{x}$ within $\underline{a}$,

$$D(\underset{\sim}{r}_{bx}) = V_{bx}(\underset{\sim}{r}_{bx})\, \psi_a(\underset{\sim}{r}_{bx}) \qquad \cdots (5)$$

The usual zero-range approximation consists of putting D equal to a delta-function, $D(r_{bx}) = D_o \delta(\underset{\sim}{r}_{bx})$, which leads to $\underset{\sim}{r}_{bB} = (m_A/m_B)\underset{\sim}{r}_{aA}$. Strictly this also implies the neglect of tensor terms in V_{bx} and of all but S-states in ψ_a. We note that $\underline{x}$ (and/or $\underline{b}$) may be a complex particle, such as in so-called "double-stripping" where $\underline{x}$ is a pair of nucleons. These cases are discussed in more detail in Section 5. Because of this zero-range approximation, this version of the theory is inadequate to deal with heavy-particle, or target, stripping in which the emitted particle is supposed to originate from the target nucleus (i.e. the target is stripped instead of the projectile). In this case it is never reasonable to replace the analog of (5) by a delta function.

3. The Transition Amplitude

We must now specify the amplitude (1) more carefully. If the spins of the particles are designated J_A, J_B, s_a, s_b and their corresponding z-components by M_A, M_B, m_a, m_b, the amplitude (1) becomes

$$T \equiv \langle J_B M_B, s_b m_b; \underset{\sim}{k}_b | V | J_A M_A, s_a m_a; \underset{\sim}{k}_a \rangle \quad \cdots (6)$$

$$= J \int d\underset{\sim}{r}_{aA} \int d\underset{\sim}{r}_{bB} \, \phi^{(-)*}_{bB}(\underset{\sim}{k}_b, \underset{\sim}{r}_{bB}) \langle J_B M_B, s_b m_b | V | J_A M_A, s_a m_a \rangle \phi^{(+)}_{aA}(\underset{\sim}{k}_a, \underset{\sim}{r}_{aA})$$

(J is the Jacobian of the transformation to the relative variables $\underset{\sim}{r}_{aA}$ and $\underset{\sim}{r}_{bB}$.) It is convenient to expand the effective interaction $\langle V \rangle$ into terms corresponding to transfer to the target nucleus of a total angular momentum $\underline{j}$ comprised of an orbital part $\underline{\ell}$ and spin part $\underline{s}$, according to the vector coupling

$$\underset{\sim}{j} = \underset{\sim}{J}_B - \underset{\sim}{J}_A \,, \qquad \underset{\sim}{s} = \underset{\sim}{s}_a - \underset{\sim}{s}_b \,, \qquad \underset{\sim}{j} = \underset{\sim}{\ell} + \underset{\sim}{s} \,. \qquad \cdots (7)$$

(Often only one value j, ℓ and s is important (or allowed) in a transition.) Then we may write, with Clebsch-Gordan coefficients[3] corresponding to eqs. (7).

$$J\langle J_B M_B, s_b m_b | V | J_A M_A, s_a m_a \rangle = \sum_{j\ell s} \langle J_A j M_A, M_B - M_A | J_B M_B \rangle$$

$$\times \langle \ell\, s\, m, m_a - m_b | j\, M_B - M_A \rangle \langle s_a\, s_b\, m_a, -m_b | s\, m_a - m_b \rangle$$

$$\times (-)^{s_b - m_b}\, i^{-\ell}\, A_{\ell s j}(B b, A a)\, f_{\ell s j, m}(\underset{\sim}{r}_{bB}, \underset{\sim}{r}_{aA}) \quad \cdots (8)$$

where of course $m = M_B + m_b - M_A - m_a$. The separation into the coefficient A and the form factor f is one of convenience, so that for example universal form factors with simple normalization may be used in computation. Then A will include both the strength of the interaction V and a spectroscopic amplitude (which depends upon the internal nuclear structure) arising from the overlap of the nuclear wave functions in eq. (8). Some examples are studied in section 5. By construction the factor $f_{\ell s j, m}(\underset{\sim}{r}_{bB}, \underset{\sim}{r}_{aA})$ behaves under rotation of coordinate axes like the spherical harmonic Y_ℓ^{m*}. We now define matrix elements for the transfer of definite angular momenta $\ell s j$,

$$\sqrt{2\ell+1}\, \beta_{sj}^{\ell m} = \frac{i^{-\ell} k_b^2}{\sqrt{4\pi}} \frac{m_A}{m_B} \int d\underset{\sim}{r}_{bB} \int d\underset{\sim}{r}_{aA}\, \phi_{bB}^{(-)*}(\underset{\sim}{k}_b, \underset{\sim}{r}_{bB})\, f_{\ell s j, m}(\underset{\sim}{r}_{bB}, \underset{\sim}{r}_{aA})\, \phi_{aA}^{(+)}(\underset{\sim}{k}_a, \underset{\sim}{r}_{aA}) \quad \cdots (9)$$

(The normalization adapted is for later convenience.[4]). We have not yet made the zero-range approximation. When we do, the form factor becomes

$$f_{\ell sj,m}(\underset{\sim}{r}_{bB},\underset{\sim}{r}_{aA}) \simeq F_{\ell sj}(r_{aA})\, Y_\ell^{m*}(\hat{r}_{aA})\, \delta\left(\underset{\sim}{r}_{bB} - \frac{m_A}{m_B}\underset{\sim}{r}_{aA}\right), \quad \cdots (10)$$

where $\hat{r}$ appearing as the argument of a spherical harmonic denotes the polar angles of $\underset{\sim}{r}$. We use (10) from now on, and drop the subscripts aA for simplicity. To evaluate (9) we need the partial-wave expansions of the distorted-waves,

$$\phi^{(+)}_{aA}(\underset{\sim}{k}_a,\underset{\sim}{r}_a) = \frac{4\pi}{k_a r_a}\sum_{L_a M_a} Y_{L_a}^{M_a}(\hat{r}_a)\, Y_{L_a}^{M_a *}(\hat{k}_a)\, i^{L_a}\, \chi^{(a)}_{L_a}(k_a r_a),$$

$$\phi^{(-)*}_{bB}(\underset{\sim}{k}_b,\underset{\sim}{r}_b) = \frac{4\pi}{k_b r_b}\sum_{L_b M_b} Y_{L_b}^{M_b}(\hat{r}_b)\, Y_{L_b}^{M_b *}(\hat{k}_b)\, i^{-L_b}\, \chi^{(b)}_{L_b}(k_b r_b). \quad \cdots (11)$$

Each partial-distorted-wave $\chi_L^{(i)}(kr)$ is a solution of a radial Schrödinger equation with a central potential $U_i(r)$ (which may be complex)

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2\eta_i k}{r} - \frac{2\mu_i}{\hbar^2}U_i(r) - \frac{L(L+1)}{r^2}\right]\chi_L^{(i)}(kr) = 0. \quad \cdots (12)$$

The CMS kinetic energy is $\hbar^2 k^2/2\mu_i$, where μ_i is the reduced mass of the pair, and η_i their Coulomb parameter $Z_1 Z_2 e^2/\hbar v$. At large radii where $U_i(r)$ is negligible, χ_L has the form

$$\chi_L(kr) = e^{i\sigma_L}\left[F_L(kr) + C_L H_L(kr)\right]$$
$$\sim \frac{1}{2i}\left[e^{-i(kr-\eta\ln 2kr-\ell\frac{\pi}{2})} - e^{2i(K_L+\sigma_L)}\, e^{+i(kr-\eta\ln 2kr-\ell\frac{\pi}{2})}\right]$$

where $\sigma_L = \arg\Gamma(1+L+i\eta)$ is the Coulomb phase-shift and where $H_L = G_L + iF_L$, and F_L, G_L are the regular and irregular Coulomb functions, respectively. The scattering amplitude $C_L = e^{iK_L}\sin K_L$; the phase shift K_L is introduced by the nuclear potential U(r), and in the absence of U we have $K_L = 0$. On the other hand, if the particle is uncharged, so $\eta = 0$, F_L and H_L become the spherical bessel and hankel functions, respectively,

$$F_L(kr) = kr\, j_L(kr)$$

and

$$H_L(kr) = ikr\, h_L^{(1)}(kr)$$

Thus in the absence of both Coulomb and nuclear interaction, $\eta = U = 0$, $\chi_L(kr) = (kr)\, j_L(kr)$, corresponding to plane waves, $\phi^{(\pm)}(\underset{\sim}{k},\underset{\sim}{r}) = \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r})$.

Insertion of (10) and (11) into (9) gives immediately

$$\beta_{sj}^{\ell m} = 4\pi \frac{m_A}{m_B}\frac{k_b}{k_a}\sum_{L_a L_b M_b}\langle L_b \ell M_b, -m | L_a M_b - m\rangle\langle L_b \ell 00 | L_a 0\rangle\, i^{L_a - L_b - \ell}\sqrt{\frac{2L_b+1}{2L_a+1}}$$
$$\times Y_{L_b}^{M_b}(\hat{k}_b)\, Y_{L_a}^{m-M_b *}(\hat{k}_a)\int dr\, \chi_{L_b}^{(b)}\left(\frac{m_A}{m_B}k_b r\right) F_{\ell sj}(r)\, \chi_{L_a}^{(a)}(k_a r) \quad \text{---(14)}$$

This expression simplifies if we choose the z-axis along $\underset{\sim}{k}_a$ and the y-axis along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$, and put θ as the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$. For $m \geqslant 0$ we can then write

$$\beta_{sj}^{\ell m}(\theta) = (-)^m \beta_{sj}^{\ell -m}(\theta) = \sum_{L_b L_a} \Gamma_{L_b L_a}^{\ell m}\, P_{L_b}^m(\theta)\, f_{L_b L_a}^{\ell}, \quad \text{---(15)}$$

where

$$\Gamma_{L_b L_a}^{\ell m} = i^{L_a - L_b - \ell}(2L_b+1)\sqrt{\frac{(L_b-m)!}{(L_b+m)!}}\langle L_b \ell 00 | L_a 0\rangle\langle L_b \ell m, -m | L_a 0\rangle \quad \text{---(16)}$$

the $P_L^m(\theta)$ are associated Legendre functions, and the radial integral is

$$f^{\ell}_{L_b L_a} = \frac{m_A k_b}{m_B k_a} \int \chi^{(b)}_{L_b}\left(\frac{m_A}{m_B} k_b r\right) F_{\ell s j}(r)\, \chi^{(a)}_{L_a}(k_a r)\, dr \qquad \ldots (17)$$

For low values of ℓ the coefficients $T^{\ell m}_{L_b L_a}$ have simple explicit forms; these are given in Appendix I, for ℓ = 0 through 4. The occurrence of the "parity conserving" Clebsch-Gordan coefficient $\langle L_b \ell 0 0 / L_a 0 \rangle$ ensures that only even values of $L_a + L_b + \ell$ contribute; that is to say, $(-)^{\ell}$ is the parity change in the transition. The explicit form of $\beta^{\ell m}$ also shows the origin of the orbital angular momentum transfer ; it comes from a change in the relative orbital momentum before and after the collision,

$$\underset{\sim}{\ell} = \underset{\sim}{L}_a - \underset{\sim}{L}_b$$

The code SALLY computes the matrix elements $\beta^{\ell m}$ defined by (15), (16) and (17). For this purpose a variety of optical potentials U(r) and form factors F(r) are available as options as described in Section 11 below. If required, the various $\beta^{\ell m}$ and integrals $f^{\ell}_{L_b L_a}$ may be printed out.

4. Cross-Sections and Polarizations

The transition amplitude (1) or (6) now has the form

$$T \equiv \langle J_B M_B, s_b m_b ; \underset{\sim}{k}_b | V | J_A M_A, s_a m_a ; \underset{\sim}{k}_a \rangle$$

$$= \frac{\sqrt{4\pi}}{k_b^2}\left(\frac{m_B}{m_A}\right)^2 \sum_{\ell s j} \langle J_A j M_A M_B - M_A | J_B M_B \rangle \langle s_a s_b m_a, -m_b | s\, m_a - m_b \rangle (-)^{s_b - m_b}$$

$$\times \langle \ell s m\, m_a - m_b | j\, M_B - M_A \rangle \sqrt{2\ell+1}\, A_{\ell s j}\, \beta^{\ell m}_{s j} \qquad \ldots (18)$$

(This form does not depend upon the zero-range assumption[10], so the results of this section are subject only to the neglect of spin-orbit coupling.)

The differential cross-section

$$\frac{d\sigma}{d\omega} = \frac{\mu_a \mu_b}{(2\pi\hbar^2)^2} \frac{k_b}{k_a} \frac{\sum_{M_B M_A m_b m_a} |T|^2}{(2J_A+1)(2s_a+1)} \quad \cdots (19)$$

then becomes

$$\frac{d\sigma}{d\omega} = \frac{\mu_a \mu_b}{\pi \hbar^4} \left(\frac{m_B}{m_A}\right)^4 \frac{2J_B+1}{(2s_a+1)(2J_A+1)} \frac{1}{k_b^3 k_a} \sum_{\ell s j} |A_{\ell s j}|^2 \sum_m |\beta_{sj}^{\ell m}|^2 \quad \cdots (20)$$

We note there is no interference between transitions with different $\ell s j$ or m.

SALLY computes $\sigma_{\ell s j}(\theta)$, for a given choice of form factor $F_{\ell s j}$, where

$$\sigma_{\ell s j}(\theta) = \frac{m_B^5 m_b m_a}{m_A^3 (m_a+m_A)(m_b+m_B)} \frac{9.268}{k_b^3 k_a} \sum_m |\beta_{sj}^{\ell m}|^2 \ \underline{mb/st.} \quad \cdots (21)$$

when energies are in Mev, lengths in fermis and masses in p.m.u. (proton mass) units). Then

$$\frac{d\sigma}{d\omega} = \frac{2J_B+1}{2J_A+1} \sum_{\ell s j} \left[\frac{|A_{\ell s j}|^2}{(2s_a+1)\, 5.093\times 10^3}\right] \sigma_{\ell s j}(\theta) \quad \underline{mb/st} \quad \cdots (22)$$

This normalization is so chosen that for deuteron stripping (where $s_a = 1$ and $s = \frac{1}{2}$) if we take $F_{\ell s j}(r)$ to be the radial wave function of the captured nucleon (and assume it independent of j) the quantity $\sum_j |A_{\ell s j}|^2 / [(2s_a+1)\, 5.093\times 10^3]$ is just the spectroscopic factor[5] $\mathcal{S}_\ell$ (or "reduced width in single particle units," although this notation is somewhat misleading); see Section 5 for details.

The vector spin-polarization of the emitted particles b, $\underset{\sim}{P} = \langle \underset{\sim}{s_b}/s_b \rangle$, only has a non-zero component along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$, of value

$$P(\theta) = 2\sqrt{\frac{(s_b+1)(2s_b+1)}{s_b}} \sum_{j\ell ss'} \sqrt{\frac{(2s+1)(2s'+1)(2\ell+1)}{\ell(\ell+1)}} \; (-)^{s_a - s_b - j - \ell}$$

$$\times W(ss'\ell\ell;1j)\, W(s_b s_b s s';1 s_a)\; A_{\ell sj}\, A^*_{\ell s'j} \qquad \text{---(23)}$$

$$\times \frac{\sum_{m=0}^{\ell-1} \sqrt{(\ell+m+1)(\ell-m)}\; \mathrm{Im}\left[\beta^{\ell m}_{sj}\, \beta^{\ell m+1\,*}_{s'j}\right]}{\sum_{\ell sj} |A_{\ell sj}|^2 \sum_m |\beta^{\ell m}_{sj}|^2}$$

The W are Racah coupling coefficients,[3] and ensure that P vanishes if $\ell = 0$ or $s_b = 0$ or $s + s' = 0$. There is now interference between different s values. In the special case $s = s' = \frac{1}{2}$, $s_a = 1$, and $s_b = \frac{1}{2}$, (e.g. deuteron stripping) with only one ℓ contributing,

$$P(\theta) = \frac{\frac{4}{3}\sum_j |A_{\ell\frac{1}{2}j}|^2 \frac{\ell(j-\ell)}{2j+1}\, \Pi_{\ell\frac{1}{2}j}}{\sum_j |A_{\ell\frac{1}{2}j}|^2} \qquad \text{---(24)}$$

where

$$\Pi_{\ell sj} = \frac{2}{\ell} \frac{\sum_{m\geq 0} \sqrt{(\ell+m+1)(\ell-m)}\; \mathrm{Im}\left[\beta^{\ell m}_{sj}\, \beta^{\ell m+1\,*}_{sj}\right]}{\sum_m |\beta^{\ell m}_{sj}|^2} \qquad \text{---(25)}$$

is the average projection of the vector $\underset{\sim}{\ell}$ along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$,

$$\Pi_{\ell sj} = \langle \ell_y/\ell \rangle, \qquad |\Pi| \leq 1 \qquad \text{---(26)}$$

If the form factor is independent of j, so is π, and (24) becomes

$$P(\theta) = \frac{1}{3}\frac{\ell}{\ell+1}\Pi, \qquad \text{if } j = \ell + \tfrac{1}{2},$$
$$= -\frac{1}{3}\Pi, \qquad \text{if } j = \ell - \tfrac{1}{2}. \qquad \text{---(27)}$$

Of course, P vanishes (but π does not) if $s_b = 0$, while both vanish if $\ell = 0$. SALLY also computes the quantity π as defined by (25), for a given form factor $F_{\ell s j}(t)$.

If the spin s_b is greater than $\frac{1}{2}$, other tensor polarization moments may be produced, but only of rank $\leq 2s_{max}$, where s_{max} is the maximum value of the spin transfer s. Thus they are not allowed in deuteron pick-up, (p,d) and (n,d), for even though $s_b = 1$ here, we have $s = \frac{1}{2}$ only. However they could appear in reactions which allow $s \geqslant 1$, such as (α,d) or (d,d') (and the (p,d) and (n,d) if viewed as knock-on rather than pick-up processes). If $s_b = 1$, the tensor polarization moments are the expectation values of the operators T_{2q}

$$T_{20} = \sqrt{\tfrac{1}{2}}\,(3 s_0^2 - 2)$$

$$T_{2\pm1} = \sqrt{\tfrac{3}{2}}\,(s_0 s_{\pm1} + s_{\pm1} s_0) \qquad \text{---}(28)$$

$$T_{2\pm2} = \sqrt{3}\,(s_{\pm1})^2$$

where $s_0 = s_z$, $s_{\pm1} = \mp (s_x \pm i s_y)/\sqrt{2}$. Then $t_{2q} = \langle T_{2q} \rangle$ is the same quantity that Lakin and Wolfenstein define.[6] With the amplitude (18) these become (with z-axis along $\underset{\sim}{k}_a$, y along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$)

$$t_{2q} = (-)^q t^*_{2-q} = \frac{\sum_{m_a m_b M_A M_B} \langle 1\,2\,m\,q \,|\, 1\,m+q \rangle \sqrt{5}\; T_{M_B m_b, M_A m_a}\; T^*_{M_B m_b+q, M_A m_a}}{\sum_{m_a m_b M_A M_B} \left| T_{M_B m_b, M_A m_a} \right|^2}$$

i.e.

$$t_{2q} = \sqrt{3} \sum_{j\ell\ell' ss'} \sqrt{(2\ell+1)(2\ell'+1)(2s+1)(2s'+1)} \quad A_{\ell s j} \; A_{\ell' s' j}^{*}$$

$$\times (-)^{j-\ell-s_a+1} W(11ss';2s_a) W(\ell\ell' ss';2j)$$

$$\times \frac{(-)^{q} \rho_{2,-q}(\ell s j, \ell' s' j)}{\sum_{\ell s j} |A_{\ell s j}|^2 \sum_m |\beta_{sj}^{\ell m}|^2} \qquad \text{---(29)}$$

where the tensors ρ_{kq} are given by

$$\rho_{kq}(\ell s j, \ell' s' j') = (-)^{\ell-\ell'+q} \rho_{k-q}(\ell' s' j', \ell s j)^{*}$$

$$= \sum_m (-)^{\ell'-m+q} \langle \ell\ell' m\, q-m | kq \rangle \beta_{sj}^{\ell m} \beta_{s'j'}^{\ell' m-q\,*} \qquad \text{...(30)}$$

[We shall require these tensors again later. They give a convenient way of describing the polarization properties of the reaction. The $\Pi_{\ell s j}$ of eq. (25) is proportional to ρ_{11}, while the differential cross-section is proportional to ρ_{00}]. It also follows from the symmetry $\beta^{\ell m} = (-)^m \beta^{\ell -m}$ (with our choice of axes) that

$$\rho_{kq}(\ell s j, \ell' s' j') = (-)^{k} \rho_{kq}(\ell s' j', \ell s j)^{*}, \qquad \text{--- (31)}$$

and hence from the symmetry of the Racah coefficients in eq. (29) that

the t_{2q} are real. From the definition (30) we see the ρ_{2q}, and hence t_{2q}, vanish if $\ell + \ell' < 2$, in particular for $\ell = \ell' = 0$. Further the Racah coefficients impose the selection rule $t_{2q} = 0$ if $s + s' < 2$, as remarked above. Just as with the vector polarization $\underset{\sim}{P}$, these t_{2q} may be detected by a second scattering.(6)

Using an incident polarized beam induces an asymmetry in the differential cross-section given by

$$\left(\frac{d\sigma}{d\omega}\right)_{pol.} = \left(\frac{d\sigma}{d\omega}\right)_{unpol} \left[1 + \frac{3 s_a}{s_a + 1} \underset{\sim}{p} \cdot \underset{\sim}{P}^{(in)} + \sum_q (-)^q \tau_{2-q} t_{2q}^{(in)} \right] \quad \cdots (32)$$

where $(d\sigma/d\omega)_{unpol.}$ is the differential cross-section, (19) or (20), with unpolarized particles incident. $\underset{\sim}{P}^{(in)}$ is the vector polarization of the incident beam, $\underset{\sim}{P}^{(in)} = \langle \underset{\sim}{S_a}/s_a \rangle_{in}$, while the $t_{2q}^{(in)}$ are the tensor polarization components, defined as in (28), $t_{2q}^{(in)} = \langle T_{2q} \rangle_{in}$. Of course the $t_{2q}^{(in)}$ vanish if $s_a = 0$ or $\frac{1}{2}$. The vector $\underset{\sim}{p}$ has a non-zero component along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ only, whose value is given by the expression (23) if s_a and s_b are interchanged, and $A_{\ell s j}$ is replaced by $\left[(-)^{j-s} A_{\ell s j} \right]$. Similarly the tensor coefficients τ_{2q} are given by eq. (29) (for $s_a = 1$) if s_a is there replaced by s_b and $A_{\ell s j}$ by $\left[(-)^{j-s} A_{\ell s j} \right]$. Hence the same selection rules operate for the $\underset{\sim}{p}$ and τ_{2q} as for $\underset{\sim}{P}$ and t_{2q}. (Note also this implies the choice of z-axis along k_a and y-axis along $k_a \times k_b$.) If the incident spin $s_a > 1$ (for example, Li^7 ions with $S_a = \frac{3}{2}$), additional terms could appear in (32).

An expression similar to (32) is obtained for the differential cross-section when the target nucleus is polarized[7] and the coefficients may again be expressed in terms of the ρ_{kq}.

In general the residual nucleus B is also polarized, and this may be studied by observing the angular distribution or correlation of any de-excitation γ-rays.[8] The orientation of B is conveniently described by statistical tensors[9] of rank $K \leqslant 2J_B$, with z-component $-K \leqslant Q \leqslant K$,

$$\rho_{KQ}(J_B J_B) = \sum_{M_B M_A m_b m_a} \langle J_B J_B M_B Q - M_B | KQ \rangle (-)^{J_B + Q - M_B} T_{M_B m_b, M_A m_a} T^*_{M_B - Q m_b, M_A m_a} \quad \cdots (33)$$

By construction ρ_{KQ} behaves under coordinate rotations like the spherical harmonic Y_K^{Q*} and has the symmetry $\rho_{KQ}^* = (-)^Q \rho_{K-Q}$. Then, for example, the angular distribution of γ-rays in coincidence with particles b emitted along $\underset{\sim}{k}_b$, is given by

$$W(\theta\phi; \underset{\sim}{k}_a \underset{\sim}{k}_b) = \sum_{KQ} \rho_{KQ}(J_B J_B) \, F_K \, (2K+1)^{-\frac{1}{2}} \, Y_K^Q(\theta\phi) \quad \cdots (34)$$

where $$F_K = \sum_{LL'} C_L C_{L'} F_K(LL' J_c J_B),$$

and C_L^2 is the relative probability of a 2^L-pole γ-ray in the transition $J_B \rightarrow J_C$. Then $F_0 = 1$ if $\sum_L C_L^2 = 1$. The γ-correlation coefficients $F_K(LL'J_C J_B)$ have been tabulated[9], and only even values of K enter. Tensors with odd K are needed to calculate, for example, the circular polarization of the γ-rays.[10]

Using the amplitude (18), (24) becomes

$$\rho_{KQ}(J_B J_B) = \sum_{jj'll's} \sqrt{(2l+1)(2l'+1)(2j+1)(2j'+1)}\,(2J_B+1)(-)^{J_A-J_B-s} W(jj'J_BJ_B;KJ_A)$$

$$\times\ W(ll'jj';Ks)\,(-)^{l'} A_{lsj} A^*_{l'sj'}\ \rho_{KQ}(lsj, l'sj') \qquad \cdots (35)$$

where the $\rho_{KQ}(lsj, l'sj')$ have been defined in eq. (30). The zero-rank tensors are proportional to the differential cross-section,

$$\rho_{00}(J_B J_B) = \sqrt{2J_B+1} \sum_{lsj} |A_{lsj}|^2 \sum_m |\beta^{lm}_{sj}|^2, \qquad \cdots (36)$$

and

$$\rho_{00}(lsj, l's'j') = \delta_{ll'} \frac{1}{\sqrt{2l+1}} \sum_m \beta^{lm}_{sj} \beta^{lm\,*}_{s'j'}. \qquad \cdots (37)$$

We notice that while there is interference between different l, j in (26), there is none between different spin-transfers s. If $l = l'$, and also j = j' and s = s', or the form factor is assumed independent of j and s, it is convenient to define[8]

$$\underline{K\text{even}}: \quad d_{KQ}(lsj) = \frac{\rho_{KQ}(lsj, lsj)}{(-)^{l}\langle ll00|K0\rangle\sqrt{2l+1}\ \rho_{00}(lsj, lsj)} \qquad \cdots (38)$$

when K is even, so $d_{00} = 1$. This normalization is not possible for odd K, for then $\langle ll00|K0\rangle$ vanishes. However, we may normalize the odd rank tensors as

$$\underline{K\text{odd}}: \quad d_{KQ}(lsj) = \frac{\rho_{KQ}(lsj, lsj)}{\sum_m |\beta^{lm}_{sj}|^2} \qquad \cdots (39)$$

With the re-definition (38) the angular correlation (34) may be rewritten for the special cases with $\ell = \ell'$ and either (i) only one value of j = j' and s = s' contributing, or (ii) the form factor (and hence d_{KQ}) independent of s and j,

$$W(\theta\phi, \underset{\sim}{k}_a \underset{\sim}{k}_b) = \sum_{K\,\mathrm{even},\,Q} g_K\, d_{KQ}\, C_{KQ}(\theta\phi) \qquad \cdots(40)$$

where $C_{KQ} = \sqrt{4\pi/(2K+1)}\; Y_K^{\,Q}(\theta\phi)$ is normalized so $C_{00} = 1$ and $C_{K0} = P_K(\cos\theta)$. The coefficients separate, $g_K = A_K F_K$, with F_K as before and

$$A_K = (2\ell+1)\sum_{sjj'} \sqrt{(2j+1)(2j'+1)(2J_B+1)}\,(-)^{J_A-J_B-s}\langle \ell\ell 00|K0\rangle$$

$$\times\, W(jj'J_BJ_B;KJ_A)\,W(\ell\ell'jj';Ks)\left[\frac{A_{\ell sj}\,A^*_{\ell sj'}}{\sum_j |A_{\ell sj}|^2}\right] \qquad \cdots(41)$$

with $A_0 = 1$

When s = 0, we have j = j' = ℓ and

$$A_K = (-)^{J_A-J_B}(2\ell+1)\sqrt{2J_B+1}\,\langle \ell\ell 00|K0\rangle\, W(\ell\ell J_BJ_B;KJ_A)$$
$$= (-)^{J_A-J_B}\, i^K\, Z(\ell J_B \ell J_B; J_A K)\Big/\sqrt{2J_B+1} \qquad \cdots(42)$$

in terms of the tabulated Z coefficients.[11] Again, for $s = \frac{1}{2}$ we can write A_K in terms of tabulated η_K,[12]

$$A_K = \sum_{jj'}\left[\frac{A_{\ell\frac{1}{2}j}\,A^*_{\ell\frac{1}{2}j'}}{\sum_j |A_{\ell\frac{1}{2}j}|^2}\right]\eta_K(jj'J_AJ_B). \qquad \cdots(43)$$

Because of the symmetry $\beta^{\ell m} = (-)^m \beta^{\ell -m}$, we have that d_{KQ} is (real/imaginary) as K is (even/odd), when we use a z-axis along $\underset{\sim}{k}_a$ and y-axis along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$. However, it is more convenient to have the tensors referred to $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ as $\hat{z}$-axis and $\underset{\sim}{k}_a$ as $\hat{x}$-axis.(5) The $\hat{\beta}^{\ell m}$ referred to these new axes are given in terms of the calculated $\beta^{\ell m}$ by the coordinate rotation transformation

$$\hat{\beta}^{\ell m} = \sum_{m'} \beta^{\ell m'} D^{\ell}_{m' m}\left(-\tfrac{\pi}{2}, -\tfrac{\pi}{2}, 0\right)^* \qquad \cdots (44)$$

where $(-\frac{\pi}{2}, \frac{\pi}{2}, 0)$ are the Euler angles of the rotation taking the old axes into the new. In this way we obtain expressions for $\hat{d}_{KQ}$ referred to the new axes. Detailed results are given in Appendix I and II. The symmetry $\beta^{\ell m} = (-)^m \beta^{\ell -m}$ in the other coordinate system leads to $\hat{\beta}^{\ell m} = 0$ if $(\ell - m)$ is odd, so that $\hat{d}_{KQ} = 0$ if Q is odd. The non-vanishing $\hat{d}_{KQ}$ are now complex, and may be written(8)

$$\hat{d}_{KQ} = |\hat{d}_{KQ}| \, e^{-iQ\alpha_{KQ}} \qquad \cdots (45)$$

Because the number of tensors one can define is greater than the number of independent $\beta^{\ell m}$ matrix elements, one is led to relations between them (and hence between experimental results) which depend only upon the general assumptions of the theory for their validity.(8) Then it is not necessary to compute all the tensors; again details are given in Appendices I and II. SALLY computes the tensors d_{KQ} for ℓ = 1 and 2 (there are none for ℓ = 0). Those for other ℓ could be obtained by a trivial extension of the program, or by hand from the computed $\beta^{\ell m}$.

Another quantity of interest is the orientation of the residual nucleus averaged over the angular distribution of the outgoing particles b. This

determines the angular distribution of any ensuing radiation when <u>not</u> measured in coincidence with particles b. For this we need the average values of the ρ_{KQ} of (33) and (35) (referred to $\underset{\sim}{k}_a$ as z-axis; the Q = 0 terms vanish when averaged in azimuth around $\underset{\sim}{k}_a$. Also the ρ_{KO} vanish for odd K),

$$\langle \rho_{KO} \rangle = \frac{\int \rho_{KO}(\theta) \sin\theta \, d\theta}{\int \rho_{OO}(\theta) \sin\theta \, d\theta} \qquad \text{---(46)}$$

so $\langle \rho_{OO} \rangle = 1$. Here θ is the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$. From the correlation formula (34) we find the angular distribution of γ-rays to be

$$w(\theta_\gamma) = \sum_{\text{even } K} F_K \langle \rho_{KO} \rangle P_K(\cos\theta_\gamma) \qquad \text{---(47)}$$

Of course w is symmetrical around $\underset{\sim}{k}_a$, and θ_γ is measured from $\underset{\sim}{k}_a$. The restriction to even K also ensures w is symmetrical about $\theta_\gamma = 90^\circ$. If we use the normalization (38), then from (40) we get the expression equivalent to (47),

$$w(\theta_\gamma) = \sum_{\text{even } K} g_K \langle d_{KO} \rangle P_K(\cos\theta_\gamma) \qquad \text{---(48)}$$

where

$$\langle d_{KO} \rangle = \frac{\int d_{KO}(\theta) \, \sigma(\theta) \sin\theta \, d\theta}{\int \sigma(\theta) \sin\theta \, d\theta}$$

and $\sigma(\theta)$ is given by (21). SALLY computes the $\langle d_{KO} \rangle$ for = 1 and 2 transfers.

SALLY also computes the coefficients for the correlation (40), in the scattering plane, for the special case $\ell = 2$ and $J_A = 0$, $J_B = j = 2$, s = 0 or 1, corresponding to inelastic excitation of the first state in even nuclei,

$$I(\phi) = (\alpha + y\beta) + y\gamma \sin^2(\phi - \phi_0) + (1 - \tfrac{2}{3}y)\sin^2 2(\phi - \phi_1) \quad ; \quad y = |A_{212}|^2 / |A_{202}|^2 .$$

For spin - $\frac{1}{2}$ particles, y is the ratio of spin-flip to spin-independent intensity. If y = 0 (as for alphas), this becomes the well-known

$$I(\phi) = A + B \sin^2 2(\phi - \phi_1) \quad , \quad A/B = \alpha .$$

The α, β, γ, ϕ_q are given in terms of the $\hat{d}_{kq}$ in Appendix II.

5. Form Factors for Various Reaction Models

The characteristics of the particular reaction under study are expressed entirely in the radial form factors $F_{\ell s j}(r)$ defined by eqs. (10) and (8), within our zero-range approximation. In this section we shall derive expressions for some typical types of reaction and nuclear models. Other reaction models may easily be dealt with along similar lines.

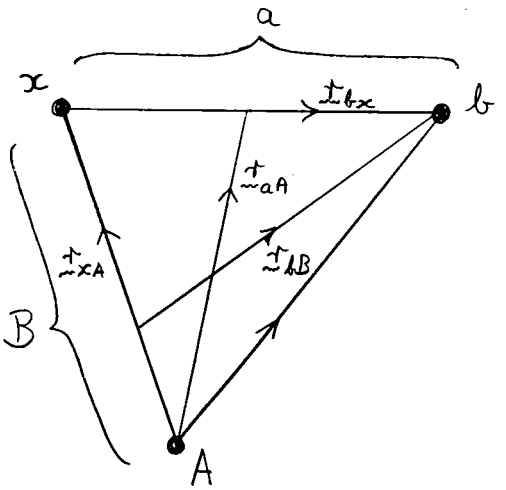
5.1 Stripping Reactions, $A(a,b)B$

Here the projectile a is assumed made up of the emitted particle b and another particle x which is captured by the target, so a = b + x and B = A + x. In deuteron stripping, (d,p) for example, a = d, b = p, and x = n. The interaction responsible is taken to be V_{bx}. For the present purpose we shall assume V_{bx} to be central,[13] $V_{bx} = V(r_{bx})$, and that b, x are in an s-state of relative motion within a. The matrix element (3) or (8) is then

$$\langle J_B M_B, s_b m_b | V | J_A M_A, s_a m_a \rangle \qquad \ldots (50)$$

$$= \int \psi^*_{J_B M_B}(\xi, \underline{r}_{xA}\, \sigma_x)\, \psi^*_{s_b m_b}(\sigma_b)\, V(r_{bx})\, \psi_{J_A M_A}(\xi)\, \psi_{s_a m_a}(r_{bx}, \sigma_b \sigma_x)\, d\xi\, d\sigma_b\, d\sigma_x$$

where ξ represents the internal coordinates of the target and σ_i the internal coordinates (e.g. spin) of particle i. From the position-vector diagram



$$-\underset{\sim}{r}_{bx} = \alpha\left(\delta\,\underset{\sim}{r}_{aA} - \underset{\sim}{r}_{bB}\right),$$

$$\underset{\sim}{r}_{xA} = \alpha\left(\underset{\sim}{r}_{aA} - \gamma\,\underset{\sim}{r}_{bB}\right),$$

$$\alpha = \frac{aB}{x\,(A+a)},$$

$$\delta = \frac{A}{B},$$

$$\gamma = \frac{b}{a},$$

where a, A, --- represent the masses of the corresponding particles. The Jacobian of the transformation from the "natural" variables $\underset{\sim}{r}_{xA}$ and $\underset{\sim}{r}_{bA}$ (or $\underset{\sim}{r}_{xA}$ and $\underset{\sim}{r}_{bx}$) is just

$$J = \alpha^3 = \left[\frac{aB}{x\,(a+A)}\right]^3 \qquad \text{---}(51)$$

The integral over ξ in (50) may be carried out formally, and expressed in terms of angular momentum states of the extra particle x.[5] (For simplicity we omit explicit reference to isotopic spin -- see ref. (5))

$$\int \psi^*_{J_B M_B}(\xi, \underset{\sim}{r}_{xA}\,\sigma_x)\,\psi_{J_A M_A}(\xi)\,d\xi$$

$$= \sum_{jlsm\mu} \langle J_A\, j\, M_A M_B - M_A | J_B M_B\rangle \langle l s m \mu | j\, M_B - M_A\rangle \qquad \text{---}(52)$$

$$\times\; \mathcal{J}_{BA}(lsj)\; u_l(r_{xA})\left[i^l\, Y_l^m(\hat{r}_{xA})\right]^* \psi_{s\mu}(\sigma_x)^*$$

The factor i^{ℓ} is included to ensure that $\mathcal{J}(\ell s j)$ is real.[9] σ_x represents any internal variables for particle x (e.g. intrinsic spin). The particle x moves within B with total angular momentum j comprised of an orbital part ℓ and a spin s. When x is a composite particle or cluster (such as in so-called "double-stripping" where x is a pair of nucleons[14]), the "spin" or internal angular momentum s of the cluster need not have a unique value in general.[14]

The parentage coefficient $\mathcal{J}(\ell s j)$ in the expansion (52) selects out a particular particle x. If we consider n identical particles x within B, the cross section is proportional to the spectroscopic factor[5]

$$\mathcal{S}(\ell s j) = n\left[\mathcal{J}(\ell s j)\right]^2 \qquad \cdots (53)$$

$\sqrt{\mathcal{S}}$ may be called a spectroscopic amplitude. $\mathcal{S}$ is analogous to a reduced width measured in "single particle" units. A similar expansion can be carried out for particle a. If we assume the pair b, x are in an s-state of relative motion, we may integrate over their internal variables

$$\int \psi^*_{s_b m_b}(\sigma_b)\, \psi^*_{s\mu}(\sigma_x)\, \psi_{s_a m_a}(\underline{r}_{bx}, \sigma_b \sigma_x)\, d\sigma_x\, d\sigma_b$$

$$= a(s)\, \phi_a(r_{bx})\, \langle s_b\, s\, m_b \mu \,|\, s_a m_a \rangle \qquad \cdots (54)$$

where $\int \phi_a(r)\, r^2\, dr = 1$. The constant $a(s)$ includes the overlap of the internal states of <u>b</u> and <u>x</u> when initially forming <u>a</u>, and when finally free (<u>b</u>) or within B(<u>x</u>). If <u>x</u> and/or <u>b</u> are composite, or clusters, these overlaps

may be less than unity (e.g. a cluster x may be smaller when part of a than when part of B), but of course for single nucleons the overlaps are unity (as for deuteron stripping). The remainder of $a(s)$ represents the probability amplitude of finding a made up of b and x in this way.

Again, if there are ν identical particles x within a, the cross-section is proportional to ν and the effective overlap becomes $\sqrt{\nu}\, a(s)$. In deuteron stripping, for example, this factor becomes unity.

Inserting (52) and (54) in (50) we obtain exactly the form of (8) with

$$A_{lsj}(Bb, Aa)\, f_{lsj,m}(\underline{r}_{bB}, \underline{r}_{aA}) = J \sqrt{\frac{2s_a+1}{2s+1}} \sqrt{n\nu}\, a(s)\, \mathcal{J}(lsj) \times V(r_{bx})\, \phi_a(r_{bx})\, u_l(r_{xA})\, Y_l^m(\hat{r}_{xA})^* \qquad \cdots(55)$$

where J is the Jacobian (51) and the weights $\sqrt{n}\sqrt{\nu}$ have been included. The zero-range approximation consists here of putting

$$V(r_{bx})\, \phi_a(r_{bx}) \approx D_0\, \delta(\underline{r}_{bx}) \qquad \cdots(56)$$

Since $-\underline{r}_{bx} = \alpha(\delta \underline{r}_{aA} - \underline{r}_{bB})$, we have

$$\delta(\underline{r}_{bx}) = \frac{1}{\alpha^3}\, \delta(\delta \underline{r}_{aA} - \underline{r}_{bB}) = \frac{1}{J}\, \delta(\delta \underline{r}_{aA} - \underline{r}_{bB}) \qquad \cdots(57)$$

Then the Jacobian J cancels out from the amplitude (55). Comparing these equations with eq. (10) we may identify the form factor (noting $\underline{r}_{xA} = \underline{r}_{aA}$ when $\underline{r}_{bx} = 0$)

$$F_{lsj}(r) = u_l(r) \qquad \cdots(58)$$

and the coefficient A,

$$A_{\ell s j} = \sqrt{\frac{2s_a+1}{2s+1}}\ \sqrt{n\nu}\ \ a(s)\ \ \mathcal{J}(\ell s j)\ D_0 \qquad \cdots (59)$$

The radial function $u_\ell(r)$ may in general be expanded in terms of the radial eigenfunctions for the particle x moving in some central potential (i.e., shell model orbitals),

$$u_\ell(r) = \Sigma_N\ c_N\ u_{N\ell}(r)$$

where N is the principal quantum number.[15] Of course in practical applications it is assumed only one such orbital $u_{N\ell}(r)$ contributes, (with $C_N = 1$). In SALLY these are computed as harmonic oscillator functions for $r < R_N$,

$$F_\ell^{(0)}(r) \equiv u_{N\ell}(r) \propto r^\ell e^{-\frac{1}{2}\beta r^2} F\left(\frac{\ell+1-n}{2}; \ell+\tfrac{3}{2}; \beta r^2\right),$$

where β is adjusted so that $u_{N\ell}$ matches smoothly on to a negative-energy Coulomb function (or hankel function if x is uncharged) of given binding energy, for $r \geqslant R_N$. Then the function is normalized so that

$$\int_0^\infty u_{N\ell}(r)^2\ r^2\, dr = 1.$$

Further, if V is really the potential binding b and x to form a (rather than some effective interaction), some estimates of the magnitude of D_0 may be made. ϕ_a then satisfies the Schrodinger equation (with the reduced mass $\mu = bx/a$)

$$\left(\nabla^2 - \alpha^2\right)\phi(r) = \frac{2\mu}{\hbar^2}\ V(r)\ \phi(r) \qquad \cdots (60)$$

Two procedures are possible. First we may use

$$D_0 = \int d\underset{\sim}{r}\; V(r)\; \phi(r) \qquad \text{— from (56)}$$

$$= \frac{\hbar^2}{2\mu} \int d\underline{r}\, \left(\nabla^2 - \alpha^2\right) \phi(r) \qquad \text{— " (60)}$$

$$= -\varepsilon \int d\underline{r}\; \phi(r), \qquad \text{--- (61)}$$

where $\varepsilon = \alpha^2\hbar^2/2\mu$ is the binding energy of $\underline{a} \rightarrow \underline{b} + \underline{x}$. (The term in ∇^2 vanishes upon applying Green's theorem). If we choose the form for ϕ which is correct asymptotically, $\phi(r) = \sqrt{\frac{\alpha}{2\pi}}\; e^{-\alpha r}/r$, we get

$$D_0^2 = \frac{8\pi\varepsilon^2}{\alpha^3} = 8\pi \left(\frac{\hbar^2}{2\mu}\right)^{3/2} \varepsilon^{1/2}$$

$$= 2.38 \times 10^3 \; \frac{\varepsilon^{\frac{1}{2}}(\text{Mev})}{\mu^{3/2}(\text{proton masses})} \qquad \text{--- (62)}$$

(This form is equivalent to assuming V(r) itself has zero-range, $V = V_0\,\delta(\underline{r})$). Alternatively we may improve ϕ for small separations r by taking the Hulthen form (which also changes the magnitude, but not the form, at large r),

$$\phi(r) = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}}\; \frac{e^{-\alpha r} - e^{-\beta r}}{r}$$

which gives

$$D_0^2 = \frac{8\pi\varepsilon^2}{\alpha^3} \left(\frac{\alpha+\beta}{\beta}\right)^3 \qquad \text{--- (63)}$$

and reduces to (62) when $\beta \gg \alpha$. An alternative point of view is to consider the fourier transform of $V\phi$,

$$G(K) = \int d\underset{\sim}{r} \; e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}} \; V(r)\,\phi(r)$$

$$= -\frac{\hbar^2}{2\mu}\left(K^2+\alpha^2\right)\int d\underset{\sim}{r} \; e^{-i\underset{\sim}{K}\cdot\underset{\sim}{r}} \; \phi(r), \qquad \ldots (64)$$

the last step following from (60) and the Hermitian property of ∇^2. Clearly $D_o = G(0)$ and corresponds to taking only the zero momentum components of $V\phi$. The Hulthén wave function leads to

$$G(K) = -\frac{\varepsilon\sqrt{8\pi}}{\alpha^{3/2}}\left(\frac{\alpha+\beta}{\beta}\right)^{3/2}\frac{\beta^2}{K^2+\beta^2} \qquad \cdots (65)$$

so the correction for finite momentum components K is $\beta^2/(K^2+\beta^2)$. In the plane wave theory there is a unique value of K at each scattering angle, namely

$$K_o = \left|\frac{b}{a}\underset{\sim}{k}_a - \underset{\sim}{k}_b\right|,$$

and the "zero-range" amplitude should be multiplied by $G(K_o)/D_o$ to correct for this. With distorted waves many other components K are introduced, although the plane wave value of K may indicate roughly a lower limit on the correction factor. It is worth noting, however, that if only the very low momentum components of $V\phi$ contribute then the so-called zero-range approximation becomes exact. This would be the case for low energy, low Q reactions (so both $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$ were small in magnitude) if the captured particle was loosely bound (so the reaction proceeds at large distances from the nucleus where the distortion does not introduce momenta much different from $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$).

For deuteron stripping, (d,p) or (d,n), a binding energy of $\mathcal{E}$ = 2.23 Mev with $\mu = \frac{1}{2}$ leads to α = 0.23 fermi^{-1}. For the single exponential eq. (62) then gives $D_o^2 \approx 1.0 \times 10^4$ Mev^2 fermi^3, which is the value used in deriving eqs. (21) and (22) for the amplitudes computed by SALLY. For the Hulthén function with $\beta \approx 7\alpha$, eq. (63) gives* $D_o^2 \approx 1.5 \times 10^4$. This and other values of D_o^2 may be incorporated very simply, since the cross-section is proportional to D_o^2. Then for the general stripping reaction we have from (53) and (59)

$$\frac{d\sigma}{d\omega} = \frac{2J_B+1}{2J_A+1} \sum_{lsj} \mathcal{S}(lsj) \frac{2\nu a^2(s)}{2s+1} \left[\frac{D_o^2(b,x)}{1.018 \times 10^4}\right] \sigma_{lsj}(\theta) \quad \underline{mb/st} \quad \cdots (66)$$

which for (d,p) or (d,n) reactions, with our choice of D_o^2(n,p), reduces to

$$\frac{d\sigma}{d\omega} = \frac{2J_B+1}{2J_A+1} \sum_{lsj} \mathcal{S}(lsj)\, \sigma_{lsj}(\theta) \quad \underline{mb/st} \quad \cdots (67)$$

If the Hulthén function is preferred*, multiply (67) by 1.5.

5.2 Pick-Up Reactions

These, of course, are just the inverse of the stripping processes discussed in the previous section. The analysis of Appendix I shows that for the pick-up reaction B(b,a)A we should use the spectroscopic amplitude $A_{lsj}(b,a)$,

*Application of effective range theory[1] gives the same value for D_o^2.

$$A_{\ell s j}(b,a) = \sqrt{\frac{2J_B+1}{2J_A+1}}\,(-)^{J_B-J_A+s_b-s_a}(-)^{j-s}A_{\ell s j}(a,b) \quad \cdots(68)$$

where $A_{\ell s j}(a,b)$ is the stripping amplitude for which expressions were given in the previous section. Then for example the differential cross-section for B(b,a)A is

$$\frac{d\sigma}{d\omega} = \frac{2s_a+1}{2s_b+1}\sum_{\ell s j} \mathcal{S}(\ell s j)\,\frac{2\nu a^2(s)}{2s+1}\left[\frac{D_0^2(b,x)}{1.018\times 10^4}\right]\sigma_{\ell s j}(\theta)\ \underline{mb/st} \quad \cdots(69)$$

which for (p,d) or (n,d) becomes (with our choice of $D_0^2(n,p)$),

$$\frac{d\sigma}{d\omega} = \frac{3}{2}\sum_{\ell j}\mathcal{S}(\ell\tfrac{1}{2}j)\ \sigma_{\ell\frac{1}{2}j}(\theta)\ \underline{mb/st.} \quad \cdots(70)$$

5.3 Inelastic Scattering, A(a,a')A*

In this section we consider inelastic scattering, but neglecting (i) exchange of the scattered particle with one in the target nucleus, and (ii) non-local interactions. (Exchange may be regarded as a knock-out process,[16] as discussed in Section 5.4). We have b = a', and B = A*, so that $M_B = M_A$ and $\underset{\sim}{r}_{bB} = \underset{\sim}{r}_{aA}$ and the "zero-range" assumption (10) is automatically satisfied. (Also, the Jacobian of the transformation to the variables $\underset{\sim}{r}_{aA} = \underset{\sim}{r}_{bB}$ is unity.)

The general interaction which induces the scattering may be expanded in multipoles

$$V(\underset{\sim}{r},\sigma_a,\xi) = \sum_{\ell s j}(-)^{j-\mu}V_{\ell s j;\mu}(\xi,r)\,T_{\ell s j;-\mu}(\hat{r},\sigma_a) \quad \cdots(71)$$

where ξ represents the internal coordinates of the target, and σ_a those of the scattered particle $\underline{a}$. The spin-angle tensor is constructed as

$$T_{\ell s j,\mu}(\hat{r},\sigma_a) = \sum_m \langle \ell s\, m\, \mu{-}m | j\mu \rangle\, i^\ell\, Y_\ell^{\,m}(\hat{r})\, S_{s,\mu-m}(\sigma_a) \quad \cdots (72)$$

where $S_{00} = 1$, and for $S_{1,\lambda}$ we use the λth. component of the spin operator for a (or the Pauli spin operator if $s_a = \frac{1}{2}$), etc. Of course, we can only have $s \leqslant 2 s_a$. By construction, both $V_{\ell s j,\mu}$ and $T_{\ell s j,\mu}$ rotate like the spherical harmonic $Y_j^{\,\mu}$, and (71) has the correct scalar-product form.(3) For the special case $s = 0$, eq. (72) gives

$$T_{\ell 0 \ell, m}(\hat{r}) = i^\ell\, Y_\ell^{\,m}(\hat{r}) \quad \cdots (73)$$

so if we only have s = 0 (e.g. if $s_a = 0$, as for α-particles) we may abbreviate (71) to

$$V(\underline{r},\xi) = \sum_{\ell m} V_{\ell m}(r,\xi)\left[i^\ell\, Y_\ell^{\,m}(\hat{r})\right]^* \quad \cdots (74)$$

where, of course,

$$V_{\ell m}(r,\xi) = i^\ell \iint V(\underline{r},\xi)\, Y_\ell^{\,m}(\theta\phi)\, \sin\theta\, d\theta\, d\phi .$$

The terms with $s \neq 0$ we may call "spin-flip," although this notation is not literally correct because of the uncertainty principle. The transfer of spin angular momentum $s \neq 0$ includes some transitions in which the z-component of the spin of the scattered particle does <u>not</u> change, $m_a = m_b$, as may be seen from the general amplitude (18).

Taking the matrix-element of the interaction (71) between the internal states of the colliding pair immediately gives the form (8), (10) with

$$A_{\ell s j}\, F_{\ell s j}(r) = (-)^s \sqrt{\frac{2s_a+1}{2s+1}}\, \langle s_a \| S_s \| s_a \rangle \langle J_B \| V_{\ell s j}(r) \| J_A \rangle \quad \cdots (75)$$

where we use the Wigner-Eckart theorem[3] in the form

$$\langle J_B M_B | V_{j\mu} | J_A M_A \rangle = \langle J_A\, j\, M_A \mu | J_B M_B \rangle \langle J_B \| V_j \| J_A \rangle \quad \cdots (76)$$

Then $\langle s_a \| S_0 \| s_a \rangle = 1$, and $\langle s_a \| S_1 \| s_a \rangle = \sqrt{s_a(s_a+1)}$ if $\underset{\sim}{S}_1 = \underset{\sim}{s}_a$ or $\langle \frac{1}{2} \| S_1 \cdot \| \frac{1}{2} \rangle = \sqrt{3}$ if $\underset{\sim}{S}_1 = \underset{\sim}{\sigma}$ and $s_a = \frac{1}{2}$. Thus for the special case (74) in which s = 0,

$$A_\ell F_\ell(r) = \sqrt{2 s_a + 1} \langle J_B \| V_\ell(r) \| J_A \rangle \quad \cdots (77)$$

We now proceed to some special nuclear models in order to obtain explicit expressions for (75) or (77).

5.3.1 Collective Rotations

We derive the interaction from a deformed, or non-spherical, potential well which has a definite shape in a body-fixed coordinate system,[17] $V = V(r\ \theta'\ \phi')$. There are various ways of defining the shape of such a potential; we mention two only. The first assumes the potential depends only upon the distance from the nuclear "surface," which is allowed to be non-spherical and defined by $r = R(\theta'\phi')$ where

$$R(\theta'\phi') = R_0 \left[1 + \sum_{kq} a_{kq} Y_k^q(\theta'\phi') \right] \quad \cdots (78)$$

For example, for a quadrupole deformation k = 2 we have

$$a_{20} = \beta \cos\gamma , \quad a_{21} = a_{2-1} = 0 , \quad a_{22} = a_{2-2} = \frac{\beta \sin\gamma}{\sqrt{2}} \quad \cdots (79)$$

in terms of the familiar deformation β and asymmetry γ. Then we take $V = V[r - R(\theta'\phi')]$. The much-used Woods-Saxon potential is a spherical form of this type,

$$V(r-R) = \frac{-V_0}{e^x+1}, \qquad x = \frac{r-R}{a} . \qquad \cdots (80)$$

Inserting (78) and expanding in a Taylor series about $R = R_0$,

$$V = V(r-R_0) - \delta R \frac{d}{dr} V(r-R_0) + \frac{1}{2} \delta R^2 \frac{d}{dr^2} V(r-R_0) - \cdots$$

where

$$\delta R = R_0 \sum_{kq} a_{kq} Y_k^q(\theta'\phi') \qquad \cdots (81)$$

We note that, on this model, a given multipole deformation k of the surface $R(\theta'\phi')$ contributes, in second and higher orders, to other multipole terms, ℓ, in the interaction. In second order, for example, it contributes to even values of ℓ in the range $0 \leq \ell \leq 2k$. However, in first order $\ell = k$, and we have in body-fixed axes

$$V^{(1)} = -R_0 \frac{dV(r-R_0)}{dr} \sum_{\ell q} a_{\ell q} Y_\ell^q(\theta'\phi') .$$

We may identify $V(r - R_0)$ as the spherical optical potential which gives rise to elastic scattering. The other terms we rotate into space-fixed axes

$$Y_\ell^q(\theta'\phi') = \sum_m Y_\ell^m(\theta\phi) D^\ell_{mq}(\mathcal{R}^{-1}) ,$$

where $\mathcal{R}$ is the rotation taking the body-fixed axes into the space-fixed axes. We can then identify

$$V^{(1)}_{\ell m}(r,\xi) = -i^\ell R_0 \frac{dV}{dr} \sum_q a^*_{\ell q} D^\ell_{qm}(\mathcal{R}) . \qquad \cdots (82)$$

The spherical-harmonic addition theorem may be applied to the second-order terms in the expansion (81), to give

$$\delta R^2 = R_0^2 \sum_{\ell\mu k k' q} a_{kq}\, a_{k'\mu-q} \langle k k' q\, \mu-q | \ell\mu \rangle \langle k k' 00 | \ell 0 \rangle$$

$$\times \sqrt{\frac{(2k+1)(2k'+1)}{4\pi\,(2\ell+1)}}\; Y_\ell^{\mu}(\theta'\phi') \qquad \cdots (83)$$

which for k = k' = 2 and an axially symmetric deformation, $a_{20} = \beta$, $a_{2\pm1} = a_{2\pm2} = 0$, becomes

$$\delta R^2 = \frac{\beta^2 R_0^2}{4\pi}\left[1 + \frac{4}{7}\sqrt{5\pi}\, Y_2^0(\theta'\phi') + \frac{12}{7}\sqrt{\pi}\; Y_4^0(\theta'\phi')\right] \cdots (84)$$

Rotating (83) into space-fixed axes now enables us to identify the second-order interaction terms

$$V_{\ell m}^{(2)}(r,\xi) = \frac{i^\ell R_0^2}{4\sqrt{\pi}} \cdot \frac{d^2V}{dr^2} \sum_\mu D_{\mu m}^{\ell}(R) \left\{ \sum_{kk'q} \left(a_{kq}\, a_{k'\mu-q}\right)^* \right.$$

$$\left. \times \langle k k' q\, \mu-q | \ell\mu \rangle \langle k k' 00 | \ell 0 \rangle \sqrt{\frac{(2k+1)(2k'+1)}{2\ell+1}} \right\} \cdots (85)$$

which for the axially-symmetric quadrupole deformation k = k' = 2, as in (84), becomes

$$V_{\ell m}^{(2)}(r,\xi) = \frac{5}{4\sqrt{\pi}}\, i^\ell \beta^2 R_0^2 \frac{d^2V}{dr^2}\; D_{0m}^{\ell}(R)\; \frac{\langle 2200 | \ell 0 \rangle^2}{\sqrt{2\ell+1}} \cdots (86)$$

for even ℓ, and zero otherwise.

If we take the Wood-Saxon form (80) for the potential $V(r - R_0)$, then

$$\frac{dV}{dr} = \frac{1}{a}\frac{dV}{dx} = \frac{V_0}{a}\frac{e^x}{(e^x+1)^2} = -\frac{V_0}{a}F^{(2)}(r),$$

$$\frac{d^2V}{dr^2} = \frac{1}{a^2}\frac{d^2V}{dx^2} = \frac{V_0}{a^2}\frac{e^x}{(e^x+1)^2}\left[1-\frac{2e^x}{e^x+1}\right] = \frac{V_0}{a}F^{(9)}(r). \quad \text{--- (87)}$$

(The form factors $F^{(2)}$ and $F^{(9)}$ are options available in SALLY.)

An alternative way to define the potential $V(r\theta'\phi')$ is in terms of the volume-conserving equipotential surfaces

$$r' = r\left[1 - \frac{1}{4\pi}\sum_{kq} a_{kq}^2 + \sum_{kq} a_{kq} Y_k^q(\theta'\phi')\right]^{-1},$$

so that

$$V \equiv V(r') = V(r) + \delta r\frac{dV}{dr} + \frac{1}{2}\delta r^2\frac{d^2V}{dr^2} \cdots$$

where

$$\delta r = -r\sum_{kq} a_{kq} Y_k^q(\theta'\phi') \quad \text{+ higher order terms.} \quad \text{--- (88)}$$

Thus to first order the only effect is to replace the R_0 of eq. (82) by r, which tends to give the interaction a longer "tail." When V is taken to be Woods-Saxon, the corresponding form factor, $F^{(1)}(r) = \frac{r}{R_0}\frac{d}{dx}\frac{1}{e^x+1} = \frac{r}{R_0}F^{(2)}(r)$, is available as an option in SALLY. Contributions from (88) which are second-order in the deformation are associated with both $r\,dV/dr$ and $r^2\,d^2V/dr^2$; the form factor $F^{(10)}(r) = \left(\frac{r}{R_0}\right)^2\frac{d^2}{dx^2}\frac{1}{e^x+1}$ is also available in SALLY.

It now remains to calculate the matrix element $\langle J_B \| V_\ell \| J_A\rangle$. Here we shall only consider transitions induced in even-even nuclei considered as asymmetric rotators.[17] The extension to odd nuclei is straightforward.

The wave function for any asymmetric rotator may be written[18]

$$\psi_{\alpha JM} = \sum_K A_K^\alpha \, \psi_{KJM} ,$$
$$\psi_{KJM} = \sqrt{\frac{2J+1}{16\pi^2(1+\delta_{K,0})}} \left[D^J_{KM}(R) + (-)^J D^J_{-KM}(R) \right]. \quad \cdots (89)$$

The (real) mixing coefficients A_K are given in references[18]. For axially symmetric nuclei, only $K = 0$ enters, and $A_0 = 1$. The interactions (82) and (85) can be written

$$V_{\ell m} = \sum_\mu V_{\ell\mu} \, D^\ell_{\mu m}(R), \qquad V_{\ell,-\mu} = V_{\ell\mu} , \qquad \cdots (90)$$

which has matrix elements

$$\langle \beta J_B \| V_\ell \| \alpha J_A \rangle = \sum_{K_A K_B} A^\alpha_{K_A} A^\beta_{K_B} \langle K_B J_B \| V_\ell \| K_A J_A \rangle ,$$

where

$$\langle K_B J_B \| V_\ell \| K_A J_A \rangle = \sqrt{\frac{2J_A+1}{2J_B+1}} \frac{1}{\sqrt{(1+\delta_{K_A,0})(1+\delta_{K_B,0})}}$$
$$\times \left\{ \langle J_A \ell K_A K_B - K_A | J_B K_B \rangle V_{\ell, K_B - K_A} + (-)^{J_A} \langle J_A \ell, -K_A, K_B + K_A | J_B K_B \rangle V_{\ell, K_B + K_A} \right\}$$

For even nuclei, $J_A = K_A = 0$, so we must have $J_B = \ell$, and

$$\langle \beta \ell \| V_\ell \| \alpha 0 \rangle = \sqrt{\frac{2}{2\ell+1}} \sum_K \frac{A^\beta_K V_{\ell K}}{\sqrt{1+\delta_{K,0}}} \qquad \cdots (91)$$

In the axially symmetric case, $K_B = 0$ only, so (91) becomes

$$\langle \beta \ell \| V_\ell \| \alpha 0 \rangle = \frac{V_{\ell 0}}{\sqrt{2\ell+1}} \qquad \cdots (92)$$

According to eq. (90) the $V_{\ell\mu}$ are obtained from eq. (82) to first-order in the deformation, and from eq. (85) to second-order. For convenience, we summarize here the results for scattering from even nuclei with an axially-symmetric quadrupole deformation, and thus K = 0, using the Woods-Saxon potential. If we identify the form factor $F_\ell(r)$ in eq. (77) with the options $F^{(i)}(r)$ available in SALLY and defined by eqs. (87), we have for the spectroscopic amplitudes A_ℓ (remembering s = 0 in this model)

$$\left.\begin{aligned} \frac{A_2^{(1)}}{\sqrt{2s_a+1}} &= -\sqrt{\tfrac{1}{5}}\left(\frac{\beta R_0 V_0}{a}\right) && \text{to first order} \\ \frac{A_2^{(2)}}{\sqrt{2s_a+1}} &= +\frac{1}{14\sqrt{\pi}}\left(\frac{\beta^2 R_0^2 V_0}{a^2}\right) && \text{to second order} \end{aligned}\right\} \begin{array}{l} 0^+ \to 2^+ \\ \text{i.e.} \\ J_A = 0,\ J_B = 2 \end{array} \qquad \text{---(93)}$$

$$\left.\begin{aligned} \frac{A_4^{(1)}}{\sqrt{2s_a+1}} &= 0 && \text{to first order} \\ \frac{A_4^{(2)}}{\sqrt{2s_a+1}} &= -\frac{1}{14\sqrt{\pi}}\left(\frac{\beta^2 R_0^2 V_0}{a^2}\right) && \text{to second order} \end{aligned}\right\} \begin{array}{l} 0^+ \to 4^+ \\ \text{i.e.} \\ J_A = 0,\ J_B = 4 \end{array}$$

5.3.2 Collective Vibrations

The interaction model adopted here is the same as in the previous section, namely a non-spherical potential well, except that the deformation parameters a_{kq} are now treated as dynamical variables, creating and annihilating phonons.[19]

So we now write $a_{kq} \rightarrow \alpha_{kq}$, where

$$\alpha_{kq} = \sqrt{\frac{\hbar\omega_k}{2C_k}}\left[b_{kq} + (-)^q b^*_{k,-q}\right] = (-)^q \alpha^*_{k,-q}$$

and b_{kq}, b_{kq}^* are the usual boson annihilation and creation operators for a 2^k-pole oscillation with angular momentum k and z-component q.

Since the deformation is no longer static, the expressions (78) and (81) may now be regarded as referring to space-fixed coordinates, and to first order we have

$$V^{(1)}_{\ell m} = -i^\ell R_0 \frac{dV(r-R_0)}{dr} \alpha^*_{\ell m}, \qquad \text{---}(94)$$

while to second-order, the analog of (85) is

$$V^{(2)}_{\ell m} = \frac{i^\ell}{4\sqrt{\pi}} R_0^2 \frac{d^2V(r-R_0)}{dr^2} \left\{ \sum_{kk'q} (\alpha_{kq}\alpha_{k'm-q})^* \right.$$
$$\left. \times \langle kk'q\, m-q | \ell m\rangle \langle kk'00|\ell 0\rangle \sqrt{\frac{(2k+1)(2k'+1)}{2\ell+1}} \right\} \qquad \text{---}(95)$$

Again, we shall only discuss matrix-elements for even nuclei, but again extension to odd nuclei is straightforward. We denote the states by $|N,JM\rangle$, where N is the number of phonons present. The ground state has zero-spin and no phonons, $J_A = 0$ and $N_A = 0$, and the first order interaction (94) can only excite one 2^ℓ-pole phonon, so to first order we must have $N_B = 1$, $J_B = \ell$. Since

$$\langle 1,\ell \| \alpha^*_\ell \| 0,0\rangle = \sqrt{\frac{\hbar\omega_\ell}{2C_\ell}},$$

eq. (77) then gives for single phonon excitation

$$\frac{A_\ell^{(1)} F_\ell(r)}{\sqrt{2s_a+1}} = -i^\ell R_0 \frac{dV(r-R_0)}{dr} \sqrt{\frac{\hbar\omega_\ell}{2C_\ell}} \qquad \text{---(96)}$$

Using the Saxon form for V, and taking $F_\ell(r) = F^{(2)}(r)$ of eq. (87),

$$\frac{A_\ell^{(1)}}{\sqrt{2s_a+1}} = i^\ell \frac{V_0 R_0}{a} \sqrt{\frac{\hbar\omega_\ell}{2C_\ell}} \qquad \text{---(97)}$$

If we compare (97) for a quadrupole transition, $\ell = 2$, with the analogous rotational excitation, $0^+ \rightarrow 2^+$, of eq. (93), we see the phonon transition strength is the same as for an equivalent deformation β where

$$\beta = \sqrt{5} \sqrt{\frac{\hbar\omega_2}{2C_2}} \qquad \text{---(98)}$$

For the excitation of two 2^k-pole phonons by the second-order interaction (95), we soon find

$$\frac{A_\ell^{(2)} F_\ell(r)}{\sqrt{2s_a+1}} \equiv \langle 2,\ell \| V_\ell^{(2)} \| 0,0\rangle = \frac{i^\ell R_0^2}{4\sqrt{\pi}} \frac{d^2V(r-R_0)}{dr} \frac{2k+1}{\sqrt{2\ell+1}}$$

$$\times \langle kk00|\ell 0\rangle \langle 2,\ell \| \alpha_k^* \| 1,k\rangle \langle 1,k \| \alpha_k^* \| 0,0\rangle, \qquad \text{---(99)}$$

if ℓ is even, and zero otherwise. For quadrupole phonons, k = 2, we can have $\ell = 0, 2, 4$; also

$$\langle 2,\ell \| \alpha_2^* \| 1,2\rangle = \sqrt{2} \sqrt{\frac{\hbar\omega_2}{2C_2}}$$

Again taking the Saxon form for V and putting $F_\ell(r) = F^{(9)}(r)$ of eq. (87), the excitation of two quadrupole phonons leads to

$$\frac{A_\ell^{(2)}}{\sqrt{2s_a+1}} = -\sqrt{\frac{5}{2\pi}}\ \frac{V_0 R_0^2}{a^2}\ \frac{\hbar\omega_2}{2C_2} \times \begin{cases} \frac{1}{2} & \text{if } \ell=0, \\ \sqrt{\frac{1}{7}} & \text{if } \ell=2,4. \end{cases} \qquad \text{---(100)}$$

Since the second-order interaction is quadratic in the α_{kq}, it cannot contribute to the excitation of a single phonon state. It does, however, have diagonal elements (excitation and subsequent annihilation of a phonon) which will affect the elastic scattering from the state. For example, the scalar part of (95) contributes a potential to the N-phonon states of

$$\Delta U = \frac{1}{\sqrt{4\pi}}\langle N,J \| V_0^{(2)} \| N,J\rangle = \frac{2N+5}{8\pi}\ R_0^2\ \frac{d^2V(r-R_0)}{dr^2}\ \frac{\hbar\omega_k}{2C_k} \qquad \text{---(101)}$$

It is easy to see this has the effect of making more diffuse the surface of the central potentials seen in the higher excited states, as would be expected on physical grounds.

5.3.3 Excitation by Two-Body Forces

The interaction here is taken to be a sum of interactions between the incident particle and the particles in the target nucleus.

$$V(\underset{\sim}{r}, \sigma_a, \xi) = \sum_i v_{ia}(\underset{\sim}{r} - \underset{\sim}{r}_i, \sigma_a, \sigma_i) \qquad \text{---(102)}$$

In general, isotopic-spin variables will enter also.[16] An example of such an interaction is the nucleon-nucleon scattering amplitude used in the impulse approximation description of the inelastic scattering of high-energy nucleons.[20] The spatial dependence of such interactions may be expanded in multipoles,

$$g(|\underset{\sim}{r}-\underset{\sim}{r}_i|) = \sum_{\ell m} g_\ell(r, r_i)\; Y_\ell^m(\hat{r}_i)\; Y_\ell^m(\hat{r})^*, \qquad \text{---(103)}$$

and the Y_ℓ^m taken with the various spin-dependent terms to form the tensors (72). For example, for nucleons, an interaction of the form

$$v_{ia} = g(|\underset{\sim}{r}-\underset{\sim}{r}_i|)\left[A + B\,\underset{\sim}{\sigma}_a\cdot\underset{\sim}{\sigma}_i\right] \qquad \text{---(104)}$$

(where A, B may be i-spin dependent, $A = A_\alpha + A_\beta\,\underset{\sim}{\tau}_a\cdot\underset{\sim}{\tau}_i$, etc.) becomes

$$v_{ia} = \sum_{\ell s j,\mu} (-)^s a_s\, g_\ell(r, r_i)\,(-)^{j-\mu}\, T_{\ell s j,\mu}(\hat{r}_i,\sigma_i)\, T_{\ell s j,-\mu}(\hat{r},\sigma_a), \qquad \text{---(105)}$$

if $a_o = A$, $a_1 = B$, and $T_{\ell 0 \ell,m} = i^\ell\, Y_\ell^m$. Then V has the form (71) if

$$V_{\ell s j,\mu}(r,\xi) = \sum_i (-)^s a_s\, g_\ell(r, r_i)\, T_{\ell s j,\mu}(\hat{r}_i,\sigma_i) \qquad \text{---(106)}$$

so that from (75), remembering $s_a = \frac{1}{2}$ here,

$$A_{\ell s j}\, F_{\ell s j}(r) = a_s\sqrt{2}\,\left\langle J_B \,\middle\|\, \sum_i g_\ell(r, r_i)\, T_{\ell s j}(\hat{r}_i\sigma_i) \,\middle\|\, J_A\right\rangle. \qquad \text{---(107)}$$

Further development requires a specific model for the nuclear wave-functions, for example in terms of a single-particle excitation[21], or a rotational excitation with deformed-nucleus orbitals for the intrinsic structure.[22] The corresponding radial form factors will have the same general features as those discussed for the collective model, namely $F_\ell(r)$ zero at the origin, $r = 0$, and peaking inside or near the nuclear surface, $r = R_o$. Thus the same form factors may often be used, with suitable adjustment of parameters, to represent these more detailed models.

5.3.4 Coulomb Excitation

The inelastic scattering of charged particles is also accompanied by coulomb excitation[23]; indeed at energies well below the Coulomb barrier this is the main mode of excitation. When both nuclear and coulomb forces contribute, their amplitudes are coherent and interfere. We here indicate how the coulomb excitation may be included in distorted-wave approximation. Since the calculation is carried out by partial-wave expansions, the coulomb contribution is only accurately assessed for scattering angles $\theta > \theta_c = 2\eta/L_{max}$, the classical deflection angle (when $\eta \ll L_{max}$) for the maximum orbital angular momentum L_{max} which is used.[23] For $\theta \lesssim \theta_c$, the coulomb excitation is underestimated. An associated restriction is imposed by the upper cut-off, R_{max}, on the radial integrals. This corresponds to neglecting impact distances $> R_{max}$, or, classically, neglecting angular momenta $L > k R_{max}$, and so underestimates the cross-section for $\theta \lesssim 2\eta/kR_{max}$.

The non-relativistic Coulomb interaction may be regarded as a special case of the two-body forces of the previous section.

$$V_c = Z_a e \sum_i \frac{e_i}{|\underset{\sim}{r} - \underset{\sim}{r}_i|}$$

$$= 4\pi Z_a e \sum_i e_i \sum_{\ell m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_\ell^m(\hat{r}_i) Y_\ell^{m*}(\hat{r}) \qquad \ldots(108)$$

where $Z_a e$ is the charge on the projectile, and e_i the charge of the ith nucleon in the target, while $r_>$, $r_<$ are the greater and the lesser of r_i, r. Thus, for $r > r_i$,

$$V_{\ell m}^{(C)} = \frac{i^{\ell}\, 4\pi Z_a e}{(2\ell+1)\; r^{\ell+1}} \sum_i e_i r_i^{\ell}\, Y_{\ell}^{m}(\hat{r}_i) \qquad \text{---(109)}$$

$$= \frac{i^{\ell}\, 4\pi Z_a e}{(2\ell+1)\; r^{\ell+1}} \; \mathcal{M}_e(\ell, m)$$

where $\mathcal{M}_e(\ell m) = \sum_i e_i r_i^{\ell} Y_{\ell}^{m}(\hat{r}_i)$ is the multipole operator for electric transitions.[19] We note that in this approximation $V^{(c)}$ only contributes to s = 0 transitions. Then denoting the purely nuclear s = 0 interaction by $V_{\ell m}^{(N)}$, we have

$$A_{\ell o\ell}\, F_{\ell o\ell}(r) = \sqrt{2s_a+1}\, \langle J_B \| V_{\ell}^{(N)} + V_{\ell}^{(C)} \| J_A \rangle$$

$$= A_{\ell}^{(N)} \left[F_{\ell}^{(N)}(r) - \frac{b}{r^{\ell+1}} \right], \qquad \text{--- (110)}$$

where the latter form is valid if we neglect contributions from $r < r_i$. The nuclear amplitude is defined (as in the preceding sections) by

$$A_{\ell}^{(N)}\, F_{\ell}^{(N)}(r) = \sqrt{2s_a+1}\, \langle J_B \| V_{\ell}^{(N)} \| J_A \rangle \qquad \text{---(111)}$$

and the coulomb excitation parameter by

$$b_{\ell} = - \frac{i^{\ell}\, 4\pi Z_a e}{2\ell+1} \; \frac{\langle J_B \| \mathcal{M}_e(\ell) \| J_A \rangle}{A_{\ell}^{(N)}} \qquad \text{---(112)}$$

The well-known reduced transition probability for de-excitation is just[19][23]

$$B_{B\to A}(E\ell) = \left|\langle J_B \,\|\, m_e(\ell) \,\|\, J_A\rangle\right|^2 . \qquad \text{---(113)}$$

The radial form factor in (110),

$$F_\ell(r) = F_\ell^{(N)}(r) - \frac{b_\ell}{r^{\ell+1}} , \qquad \text{---(114)}$$

is available in SALLY for $\ell = 2$ and 3 with the shape (87), $F_\ell^{(N)}(r) = \frac{d}{dx}\frac{1}{e^x+1}$, $x = \frac{r-R_0}{a_0}$. The chief feature of the coulomb term is its long tail, so its effects should be seen mainly at small scattering angles. Since the Coulomb potential is repulsive, and the nuclear potential attractive, the interference between them is mainly destructive.

Explicit expressions for b may be obtained if the collective model is used. Equating the deformation variables to the mass-moments[19]

$$\alpha_{\ell m} = \frac{4\pi}{3A}\sum_i \left(\frac{r_i}{R_c}\right)^\ell Y_\ell^m(\hat{r}_i)^* ,$$

and assuming the charge and mass distributions coincide, gives

$$M_e(\ell m) = \frac{3Ze}{4\pi} R_c^\ell \alpha_{\ell m}^* . \qquad \text{---(115)}$$

Using the vibrational model of Section 5.3.2 to first order, eq. (94), with the Saxon form factor (87), $F_\ell^{(N)} = \frac{d}{dx}\frac{1}{e^x+1}$, and hence the nuclear amplitude (97), $A_\ell^{(N)} \equiv A_\ell^{(1)} = \sqrt{2S_a+1}\, i^\ell \frac{V_0 R_0}{a}\sqrt{\frac{\hbar\omega_\ell}{2C_\ell}}$, we soon find for the Coulomb parameter

$$b_\ell = -\frac{3}{2\ell+1} Z_a Z_A e^2 \frac{a R_c^\ell}{V_0 R_0} \qquad \text{---(116)}$$

$$= -\left(\frac{5}{2\ell+1}\right) 0.864 \frac{Z_a Z_A a R_c^\ell}{V_0 R_0}$$

if R_0, a in fermis, V_0 in Mev. Typically, $R_c \approx 1.25 \times A^{\frac{1}{3}}$ fermis

An analogous derivation for the rotational model of section 5.3.1 leads to the same expression (116) for b .

5.4 Knock-On Reactions, A(a,b)B

These are visualized as processes in which the target is comprised of a core plus the emitted particle, A = C + b, the latter being ejected by the incident particle, which is itself captured so that B = c + a. The interaction is taken to be that between a and b, V_{ab}. Exchange between an incident nucleon and the struck target nucleon in inelastic scattering is a special case of this type.[16][21] These reactions may be included in the "zero-range" formalism if the interaction V_{ab} is itself taken to be of zero-range in the separation of the centers-of-mass of particles a and b.

The parentage expansions of the nuclear wave functions corresponding to A = C + b, and B = C + a are of the same type as already discussed for stripping reactions, eq. (52).

$$\psi_{J_A M_A}(\xi_c, \hat{r}_{bc}\, \sigma_b) = \sum_{J_c M_c j_b l_b \mu_b m'_b} \langle J_c\, j_b\, M_c\, M_A - M_c | J_A\, M_A \rangle \langle l_b\, s_b\, \mu_b\, m'_b | j_b\, M_A - M_c \rangle \qquad \text{--(117)}$$

$$\times\ \mathcal{J}_{AC}(l_b s_b j_b)\ u_{l_b}(r_{bc})\, i^{l_b}\, Y_{l_b}^{\mu_b}(\hat{r}_{bc})\ \psi_{J_c M_c}(\xi_c),$$

and an analogous expression for B = C + a. The internal coordinates of the core are denoted ξ_c, those of b by σ_b. This expansion selects out a particular particle; if there are n_a identical particles a within B, and n_b identical particles b within A, the cross section is proportional to $n_a n_b$. We shall account for this by including the factor $\sqrt{n_a n_b}$ in the transition amplitude,

which then includes two "spectroscopic amplitudes" $\sqrt{S} = \sqrt{n}\,\mathcal{J}$, each similar to the one (53) appearing in a stripping amplitude in Section 5.1. If these expansions are introduced into the matrix element (8) of V, and if we assume V is scalar in spin-space (i.e. contains no spin-orbit or tensor terms), some straightforward Racah algebra[3] leads to the following expression

$$A_{lsj}\, f_{lsj,m}(\underset{\sim}{r}_{aA}, \underset{\sim}{r}_{bB}) = \frac{J}{\sqrt{4\pi}} \sum i^{l-l_b-l_a} \sqrt{n_a}\, \mathcal{J}_{BC}(l_a s_a j_a) \sqrt{n_b}\, \mathcal{J}_{AC}(l_b s_b j_b)$$

$$\times\ (2\Sigma+1)\, V^{\Sigma}_{ab}\ u_{l_a}(r_{ac})\, u_{l_b}(r_{bc})\ \mathcal{Y}_l^{m\,*}(\hat{r}_{ac}, \hat{r}_{bc})$$

$$\times\ (-)^{J_c - J_A + j - j_a} \langle l_a l_b\, 00 | l 0 \rangle \sqrt{\frac{(2s+1)(2J_A+1)}{2J_c+1}} \sqrt{(2j_a+1)(2j_b+1)(2l_a+1)(2l_b+1)}$$

$$\times\ W(J_A J_B j_b j_a; j J_c)\, W(s_a s_b s_b s_a; \Sigma s) \begin{pmatrix} j & j_a & j_b \\ l & l_a & l_b \\ s & s_a & s_b \end{pmatrix} \qquad \cdots (118)$$

summed over $J_c, j_b, l_b, j_a, l_a, \Sigma$, and where

$$\langle l_a l_b\, 00 | l 0 \rangle \sqrt{\frac{(2l_a+1)(2l_b+1)}{4\pi\,(2l+1)}}\ \mathcal{Y}_l^m(\hat{r}_a, \hat{r}_b) = \sum_{\mu} \langle l_a l_b \mu\, m-\mu | l m \rangle\ Y_{l_a}^{\mu}(\hat{r}_a)\, Y_{l_b}^{m-\mu}(\hat{r}_b)\,.$$

When $\hat{r}_a = \hat{r}_b = \hat{r}$, $\mathcal{Y}_l^m(\hat{r}_a, \hat{r}_b)$ becomes just $Y_l^m(\hat{r})$. We have also put for the value of V_{ab} in the state with total spin Σ,

$$V^{\Sigma}_{ab} = \langle (s_a s_b)\, \Sigma, M_\Sigma \,|\, V_{ab} \,|\, (s_a s_b)\, \Sigma, M_\Sigma \rangle .$$

(Since we assumed V scalar in spin-space, it is diagonal in Σ). For example, for spin - $\frac{1}{2}$ particles, $s_a = s_b = \frac{1}{2}$, the general interaction of this type is $V = A + B\, \underset{\sim}{\sigma}_a \cdot \underset{\sim}{\sigma}_b$, and we have

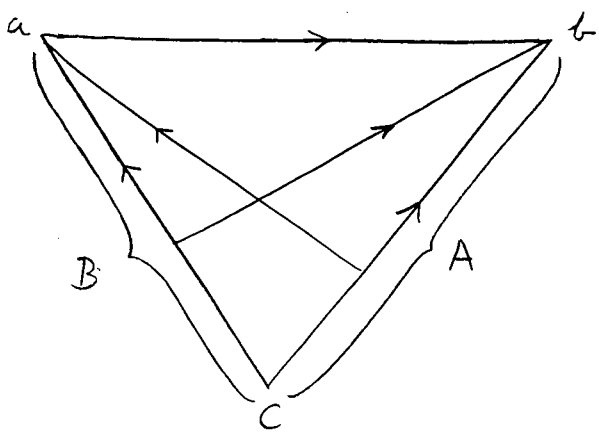
$$V^{\Sigma} = A + [2\Sigma(\Sigma+1) - 3]\, B = \begin{cases} A - 3B & \text{if } \Sigma = 0, \text{ singlet,} \\ A + B & \text{if } \Sigma = 1, \text{ triplet.} \end{cases} \qquad \cdots (119)$$

Because V_{ab} does not involve the core coordinates it is diagonal in the core states, so that the target A and residual B nuclear states must have at least one parent core state $\psi_{J_c M_c}$ in common for the transition amplitude not to vanish.

To introduce the zero-range approximation we put

$$V^{\Sigma}_{ab} = -G_{\Sigma}\,\delta(\underset{\sim}{r}_b - \underset{\sim}{r}_a) \qquad \cdots (120)$$

For the position-vector diagram we have



$$\underset{\sim}{r}_{bC} - \underset{\sim}{r}_{aC} = \frac{B}{B+b}\left(\underset{\sim}{r}_{bB} - \frac{A}{B}\underset{\sim}{r}_{aA}\right)$$

so that $\delta(\underset{\sim}{r}_{bc} - \underset{\sim}{r}_{aC}) = \left(\frac{B+b}{B}\right)^3 \delta\left(\underset{\sim}{r}_{bB} - \frac{A}{B}\underset{\sim}{r}_{aA}\right)$. Also, the Jacobian J of the transformation from $\underset{\sim}{r}_{aC}$, $\underset{\sim}{r}_{bC}$ to $\underset{\sim}{r}_{aA}$, $\underset{\sim}{r}_{bB}$ is $[AB/C(B+b)]^3$. The product of these factors gives $(A/C)^3$, which also follows immediately from $\underset{\sim}{r}_{aC} = \frac{A}{C}\underset{\sim}{r}_{aA}$ when $\underset{\sim}{r}_{ba} = 0$. From eq. (10) we then have for the zero-range amplitude and form-factor,

$$A_{\ell s j} F_{\ell s j}(r) = \sqrt{\frac{1}{4\pi}}\left(\frac{m_A}{m_c}\right)^3 \sum_{\Sigma J_c j_a j_b \ell_a \ell_b} \sqrt{n_a}\, \mathcal{J}_{BC}(\ell_a s_a j_a) \sqrt{n_b}\, \mathcal{J}_{AC}(\ell_b s_b j_b)\,(2\Sigma+1)\,G_{\Sigma}$$

$$\times\, u_{\ell_a}\left(\frac{m_A}{m_c} r\right) u_{\ell_b}\left(\frac{m_B}{m_c} r\right) i^{\ell-\ell_b-\ell_a} (-)^{J_c-J_A+j-j_a-1} \langle \ell_a \ell_b 00 | \ell 0\rangle \sqrt{\frac{2s+1,\, 2J_A+1}{2J_c+1}} \qquad \cdots (121)$$

$$\times \sqrt{(2j_a+1)(2j_b+1)(2\ell_a+1)(2\ell_b+1)}\, W(J_A J_B j_b j_a; j J_c)\, W(s_a s_b s_b s_a; \Sigma s) \begin{pmatrix} j & j_a & j_b \\ \ell & \ell_a & \ell_b \\ s & s_a & s_b \end{pmatrix}$$

The complexity of this expression arises entirely from the recoupling of the various angular momenta involved, from the coupling scheme appropriate to the nuclear states,

$$\underset{\sim}{J}_A = \underset{\sim}{J}_c + \underset{\sim}{j}_b \;, \qquad \underset{\sim}{j}_b = \underset{\sim}{\ell}_b + \underset{\sim}{s}_b \;,$$

$$\underset{\sim}{J}_B = \underset{\sim}{J}_c + \underset{\sim}{j}_a \;, \qquad \underset{\sim}{j}_a = \underset{\sim}{\ell}_a + \underset{\sim}{s}_a \;,$$

to that exhibiting the angular momentum transfers in the transition between them, (which also displays the selection rules for j, ℓ and s),

$$\underset{\sim}{j} = \underset{\sim}{J}_B - \underset{\sim}{J}_A = \underset{\sim}{j}_a - \underset{\sim}{j}_b = \underset{\sim}{\ell} + \underset{\sim}{s}$$

$$\underset{\sim}{\ell} = \underset{\sim}{\ell}_a - \underset{\sim}{\ell}_b \;, \qquad \underset{\sim}{s} = \underset{\sim}{s}_a - \underset{\sim}{s}_b \;.$$

Some simplification is obtained by taking special cases; for example, if the interaction is spin-independent so that $G_\Sigma = G$, independent of Σ, the sum over Σ in (121) may be carried out,

$$\sum_\Sigma (2\Sigma+1)\, W(s_a s_b s_b s_a ; \Sigma s) = 1. \qquad \cdots(122)$$

Again, if only one core state, of zero spin, contributes, $J_c = 0$, we have $J_A = j_b$, $J_B = j_a$, and

$$\frac{W(J_A J_B j_b j_a ; j J_c)}{\sqrt{2J_c+1}} \xrightarrow{J_c=0} \frac{(-)^{J_A+J_B-j}}{\sqrt{(2J_A+1)(2J_B+1)}}$$

In many cases it will be possible to assume only one orbital ℓ_a and only one orbital ℓ_b contribute, so the radial form factor is basically $u_{\ell_a}(r)\, u_{\ell_b}(r)$ (where the $u_\ell(r)$ may be taken as shell model orbital wave-functions), and the remainder of (121) may be taken as the amplitude $A_{\ell s j}$:

$$F_{\ell s j}(r) = u_{\ell_a}\left(\frac{m_A}{m_c} r\right) u_{\ell_b}\left(\frac{m_B}{m_c} r\right), \qquad (\text{for all allowed } \ell, s, j)$$

$$A_{\ell s j} = \sqrt{\frac{1}{4\pi}}\left(\frac{m_A}{m_c}\right)^3 \sum \sqrt{n_a}\, J_{BC}(\ell_a s_a j_a)\sqrt{n_b}\, J_{AC}(\ell_b s_b j_b)(2\Sigma+1)\, G_\Sigma$$

$$\times\, i^{\ell-\ell_b-\ell_a}(-)^{J_c-J_A+j-j_a-1}\langle \ell_a \ell_b 00|\ell 0\rangle\sqrt{(2j_a+1)(2j_b+1)(2\ell_a+1)(2\ell_b+1)} \qquad \text{---(123)}$$

$$\times\sqrt{\frac{(2s+1)(2J_A+1)}{(2J_c+1)}}\; W(J_A J_B j_b j_a; j J_c)\, W(s_a s_b s_b s_a; \Sigma s)\begin{pmatrix} j & j_a & j_b \\ \ell & \ell_a & \ell_b \\ s & s_a & s_b \end{pmatrix}$$

It will often be possible to represent this $F_{\ell s j}(r)$ to a good approximation by one of the options available in SALLY. (When doing this, it must be remembered that (123) implies the normalization $\int u_\ell(r)^2 r^2 dr = 1$).

PART II: The Code SALLY

6. Method of Calculation

The amplitudes and cross-sections described in Part I are computed by the code called SALLY. It was written for the IBM-704, (with 8000-word fast memory; a 16,000 or 32,000-word machine would require revision of the program) but may also be used on the IBM-709 or 7090 with compatibility. The running time for a typical case, with about 15 partial waves in both entrance and exit channels, is about 2 or 3 minutes on the 704 and less than 1/2 minute on the 7090, with the minimal output. Of course, these figures depend upon the number of partial waves used, details of the integration intervals and step-lengths and the amount of output demanded.

The code calculates the amplitudes described in Section 3, eq. (15) and (16), using the Γ coefficients tabulated in Appendix III.

$$\beta^{\ell m}(\theta) = \sum_{L=0}^{L_b} \overline{\beta_L^m} \; P_L^m(\theta), \qquad m \geqslant 0$$

$$= (-)^m \beta^{\ell -m}(\theta),$$

$$\overline{\beta_L^m} = \sum_{L'} \Gamma_{LL'}^{\ell m} \; f_{LL'}^{\ell}, \quad \begin{cases} L-\ell \leqslant L' \leqslant L+\ell \\ (L+L'+\ell) \text{ even} \\ L' \leqslant L_a \end{cases} \qquad \text{---}(124)$$

$$f_{LL'}^{\ell} = \frac{m_A}{m_B} \frac{k_b}{k_a} \int \chi_L^{(b)}\left(\frac{m_A}{m_B} k_b r\right) F_\ell(r) \, \chi_{L'}^{(a)}(k_a r) \, dr$$

From these the various quantities specified in Section 11 and Appendix II are calculated, including the cross-section (21)

$$\sigma(\theta) = N \sum_{m=-\ell}^{\ell} |\beta^{\ell m}(\theta)|^2$$

where ...(125)

$$N = \frac{9.268}{k_a k_b^3} \frac{m_B^5 m_b m_a}{m_A^3 (m_A+m_a)(m_B+m_b)},$$

with the masses in p.m.u., the wave numbers in (fermi)$^{-1}$.

Because of the limited storage of the 704, SALLY is conveniently broken into three parts. In part A the form factor $F_\ell(r)$ is computed at specified radii and stored on a drum (in the 7090 $F_\ell(r)$ is stored in core). Program B is then called from tape and is written over A. B first calculates the optical model wave functions for the incident channel and stores them on tape as a function of radius, in blocks $\left(\chi_0^{(a)}(r_i), \chi_1^{(a)}(r_i) \text{ ------ } \chi_L^{(a)}(r_i)\right)$ for each r_i. On completion of the integration, the normalizations of the various partial waves are computed, (by matching onto the asymptotic coulomb functions) and stored on tape. This program then returns to compute the optical wave functions for the outgoing channel at r_1, calls in the incident channel wave functions and the form factor at this radius to calculate the integrands, normalizes by the incident channel normalizations and accumulates the integrals by the trapezoidal rule. This calculation proceeds until a cutoff (upper or lower) radius is reached, at which point the integrals are stored on tape. The calculation then continues, outputting integrals at every specified cutoff, if any, until R_{max} is reached. The normalizations of the outgoing wave functions are then computed and stored.

Program C is then called, obtains the integrals from tape, applies the normalizations from the second optical model pass and proceeds to calculate the amplitudes and other quantities of physical interest specified (see Section 11).

If lower cutoffs are called for, Program C differences the integrals at R_{max} and R_{cutoff} to obtain the integrals needed. Throughout the calculation, precautions have been taken to insure that underflow will not introduce errors. Overflow is, of course, monitored but no remedial action can be taken. In this case, the calculation is stopped and a message printed, identifying the stop and the routine in which it stopped.

The code makes use of a Kutta-Simpson integration method to solve the radial Schrodinger equation. This has the advantage of using a variable radial mesh, so that sensitive regions of the potential, etc., can be investigated carefully. In addition this method is relatively insensitive to the starting guess made for the function. The solutions are scaled to prevent overflow of the un-normalized wave functions. This feature also guarantees that the higher partial waves will not be started until they make a finite contribution to the total wave function.

The coulomb functions F_o, G_o, F_1 and G_1 are computed using asymptotic formulae.[24] The solutions are of high accuracy if $\rho_{max} = k R_{max}$ is chosen as

$$\begin{aligned} \rho_{max} &\geqslant \tfrac{5}{12}(\eta^2 + 4\eta + 3) \qquad \text{if } \eta > 3, \\ &\geqslant 10 \qquad\qquad\qquad\qquad \text{if } \eta < 3 \end{aligned} \qquad \ldots(126)$$

where η is the Coulomb parameter. The Coulomb functions for the higher angular momenta are found by recursion relations;[25] (upward recursion for G_L, downward for F_L). Here the program assumes values for $F_{L'}$ and $F_{L'+1}$, with $L' > L_{max}$. On completion of this calculation the program assumes values for $F_{L''}$ and $F_{L''+1}$, with $L'' > L'$, and recalculates. If the two results compare to sufficient accuracy the calculation passes on to the next phase. If not,

the program trys 5 times to compute these functions. If still unsuccessful, the program increments ρ_{max} and trys again. SALLY is pre-set to try to find Coulomb functions 25 times. If still unsuccessful, the program stops. These failures occur if ρ_{max} is not large enough or if the function has a zero in this range of ρ_{max}. The remedial action suggested is to try a larger ρ_{max}, (but one within the bounds of the physics).

The most common remarks (messages) output by SALLY are the following:

"Remark # 11 in subroutine 1

Increase $\rho_{max} = \rho_{max} + \delta\rho_{max}$"

This remark occurs because of the failure to find Coulomb functions of sufficient accuracy (see above).

"Remark # 1 in subroutine 11"

This routine checks whether L_{max} is sufficiently large to include all partial waves affected by the potential. It also checks to see whether ρ_{max} is large enough so that the nuclear potentials are negligible. According to which failure, the program will increment L_{max} by 1 or ρ_{max} by $\delta\rho$, and try again. Again SALLY is pre-set to make only 25 such increments.

"Error # 2 in subroutine 10"

This routine calculates η, k, ka, ka', $2\eta/kR_c$. If E, a, a', or r_c are input as zero, in the course of calculation there is a division by zero. Check the input parameters!

7. Input for SALLY

The New York input routine NY-INP1 is used for SALLY. Each row of the input sheet corresponds to an 80-column card. The input is scanned so that one may enter any integer either with or without a decimal point. Negative

parameters are simply preceded by a minus sign. Each entry is followed by a comma, except the last in each row. Any entry left blank is read as zero, e.g. ---a,,b--- is equivalent to ---a, 0, b--- so that it is never necessary to fill in zeros. (A comma at the end of a row will be interpreted as being followed by a zero: ---a, = ---a, 0). There is no specified number of entries per row (card), but certain sections of data must start on a fresh row (card). Each row of input numbers should start in the 12th column, and the 11th column is left blank. The preceding columns 8, 9 and 10 are used for the instructions BCD (binary coded decimal) or DEC (decimal).

The first two rows (cards) are:

BCD 3XXXXXX

DEC 0

The X's may be replaced by 6 other symbols (e.g. for identification of the coder) but these are not printed on the output. They may be followed by up to 11 symbols (for designation of the run) which are printed on the output sheets. No commas are necessary on these two cards.

The next three sections of input contain parameters for the (i) form factor and integration intervals, (ii) optical model for the entrance channel, and (iii) optical model for the exit channel. Each section must start on a new row (card), but each may occupy more than one row (card).

(i) Form factor and integration controls: F1 through F5 are parameters for the form factor, and F6 controls the choice of form factor (see Section 9). ℓ is the orbital angular momentum transfer (in units of $\hbar$). The last nine entries control the integration intervals (see Section 8).

Total: 27 entries (including zeros).

(ii) Optical model for entrance channel: E_a is the laboratory energy (Mev) of the incident particle a, $\mathcal{M}_a$ its mass (amu) and $\mathcal{M}_A$ the mass of the target nucleus (pmu). Z_aZ_A is the product of the charges of incident and target nuclei (in units of e^2). Next follow the parameters for the optical model potential for the pair a, A (see Section 10). (Note: even if the particle is uncharged, $Z_aZ_A = 0$, a non-zero number must be entered for the charge radius r_c; for example, put $r_c = r_o$). L_a is the maximum orbital angular momentum (in units of $\hbar$) considered for the pair a, A, and so controls the number of partial waves used in the entrance channel. L_a should be chosen on physical grounds; a rough guide is that L_a should be of the order of $2k_aR$, where k_a is the wave number and R the nuclear radius. Values up to $L_a = 49$ may be used, provided $L_a \geqslant 3$.

KA1 controls the choice of optical model (see Section 10).

Total: 24 entries (including zeros).

(iii) Optical model for exit channel: as for last section, except the first entry is the Q-value of the reaction (Mev) and the last four entries are used as option controls. L_b, the maximum angular momentum in the exit channel, is also chosen on physical grounds and should be of order $2k_bR$. Again, values up to $L_b = 49$ may be used provided $L_b \geqslant 3$.

KB1 controls the choice of optical model for the exit channel (see Section 10).

KB2 controls the choice of lower or upper cutoffs on the radial integrals; enter 0 for upper, 1 for lower. (Upper cutoff R means the integrals are carried out for $0 \leqslant r \leqslant R$, while lower cutoff R means $R \leqslant r \leqslant R_{max}$). If no cutoffs are required (see below), enter 0.

KB3 controls the output; entry 0 gives the cross-section, polarization and tensors (where applicable), while entry 1 gives in addition the matrix elements $\beta^{\ell m}$, $\overline{\beta_L^{\ m}}$ and the radial integrals $f^{\ell}_{L\,L'}$ (see Section 11). When cutoffs are employed, this data is printed out for each value of the cutoff as well. The form factor, and the elastic cross-sections and partial-wave scattering amplitudes are always printed out.

KB4 controls the choice of quantization axis for the matrix elements β^m and $\overline{\beta_L^m}$ (and the tensor phase angles ϕ_i) but for $\ell = 1$ and 2 only; entry 0 gives them referred to the incident beam as axis, entry 1 refers them to the recoil direction $\underset{\sim}{Q} = \underset{\sim}{k}_a - \frac{m_A}{m_B}\underset{\sim}{k}_b$ as axis (see Section 5) and also prints out the recoil angles. For other ℓ values, enter 0 here.

KB5 controls the angular intervals at which the reaction cross sections, etc., are evaluated; entry 0 outputs them at every 5° from 0° to 180°, entry 1 at every 2.5^o for the same range. With these two options, enter zeros for Θ_o and $\Delta\Theta$ in the places preceding L_b. More control over the angular intervals is provided by entering Θ_o, the smallest angle at which output is required, and $\Delta\Theta$ the interval, in these positions, and entering the number of angles required (up to 75) in KB5.

If cutoffs are to be employed, up to 29 of these (in fermis) may be entered on the next row or rows (cards). If no cutoffs are required, these rows(cards) may be omitted. Finally a transfer card is required: TRA 3,4

Thus the input requires a <u>minimum</u> of 6 cards (if no cutoffs) or 7 cards (with cutoffs). This input is summarized in columns on the following page, and on a key input sheet. A sample input for a typical case is also shown.

IBM **Share 709 Symbolic Coding Form**

KEY TO INPUT SHEET FOR "SALLY"

Problem: SALLY

Coder | Date | Page of

*	Location	Operation	Address, Tag, Decrement/Count	Comments	Identification
1	2–6 7	8 9 10	11–12	72	73–80
	First card →	B C D	3 X X X X X X	plus up to 11 symbols to label the run	
	Second card →	D E C	0		
	Third card →	D E C	F1, F2, F3, F4, F5	form factor parameters	
		D E C	,,,ℓ,,F6,,,,,,	angular momentum, and model	
		D E C	a, b, c, d, e, f, g, h, i	integration controls	
	Start new card →	D E C	$E_a, \mathfrak{M}_a, \mathfrak{M}_A, Z_aZ_A$	incident data	
		D E C	V, W, r_o, r_c, a,,,r_o', a', W'	a + A optical model	
		D E C	,,,L_a,,KA1,,,,		
	Start new card →	D E C	Q, $\mathfrak{M}_b, \mathfrak{M}_B$, Z_bZ_B	outgoing data	
		D E C	V, W, r_o, r_c, a,,, r_o', a', W'	b + B optical model	
		D E C	, θ_o, $\Delta\theta$, L_b,,KB1, KB2, KB3, KB4, KB5		
	Start new card - may be omitted →	D E C		up to 29 cut-off radii if required - may be omitted	
	Last card →	T R A	3, 4		

UCN-363
(123 12-58)

IBM **Share 709 Symbolic Cod...g Form**

SAMPLE INPUT SHEET

Problem: SALLY # 1828

Coder: John Doe | Date: 1 / 10 / 61 | Page of

* 1	Location 2–6	7	Operation 8–10	Address, Tag, Decrement/Count 12–72	Comments	Identification 73–80
			BCD	3XXXXXX NA/DP/0/2A		
			DEC	0		
			DEC	6.96,,.96,,3		
			DEC	,,,2,,12,,,,,,,		
			DEC	.06,.5,5,20,,7,30,60,		
			DEC	7.8,2,23,11		
			DEC	83,28,1.5,1.5,.55,,,,,		
			DEC	,,,10,,2,,,,		
			DEC	4.731,1,24,11		
			DEC	48,7.5,1.33,1.3,.5,,,1.33,.98,		
			DEC	,,,10,,1,,1,,		
			TRA	3,4		

– for the reaction $Na^{23}(d,p)Na^{24}$
with $E_d = 7.8$, $Q = 4.731$ Mev, $\ell = 2$

UCN-363
(123 12-58)

"SALLY" INPUT : SUMMARY

1. Form factor and integration controls		2. Entrance Channel option model		3. Exit Channel optical model	
F1	form factor parameters	E_a (Mev)	lab. energy	Q (Mev)	Q-value
F2		m_a (pmu)	incident mass	m_f (pmu)	outgoing mass
F3		m_A (pmu)	target mass	m_B (pmu)	residual mass
F4		Z_aZ_A (e^2)	charge product	Z_bZ_B (e^2)	charge product
F5		V (Mev)	optical potential for a + A	V (Mev)	optical potential for b + B
0		W (Mev)		W (Mev)	
0		r_o (f)		r_o (f)	
0		r_c (f)		r_c (f)	
ℓ (ħ)	angular momentum transfer	a (f)		a	
0		0		0	
F6	form factor option	0		0	
0		r'_o (f)		r'_o (f)	
0		a' (f)		a' (f)	
0		W' (Mev)		W' (Mev)	
0		0		0	
0		0		θ_o (deg)	initial angle
0		0		$\Delta\theta$ (deg)	angle interval
0		L_a (ħ)	maximum angular momentum	L_b (ħ)	max. ang. mom.
a	integration controls: radii (f)	0		0	
b		KA1	model option	KB1	model option
c		0		KB2	cutoffs
d		0		KB3	output
e		0		KB4	quant. axis
f	number of intervals	0		KB5	angle control
g					
h					
i					
27 entries		24 entries		24 entries	

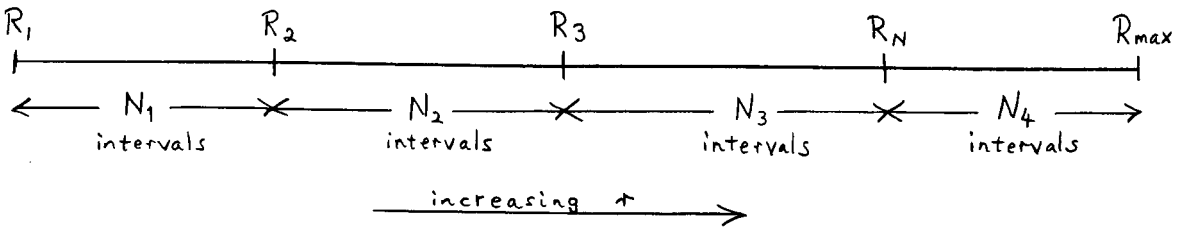
(f = fermi = 10^{-13} cm; pmu = proton mass unit)

8. Integration Controls for SALLY

The last 9 entries, a through i, in the first section of input parameters afford full control over the radial integration intervals. The complete input (radii in fermis)

$$\boxed{R_1,\ R_2,\ R_3,\ R_N,\ R_{max},\ N_1,\ N_2,\ N_3,\ N_4}$$

corresponds to four integration regions:



Not all of these have to be used (see simplified input below).

The starting value for the Lth partial-wave radial function in each channel is taken to be $\rho_i^{L+1}/2^L(L+1)!$ at the first ρ_i for which this quantity is $\geqslant 2^{-127}$ (where $\rho_i = kR_i$). Thus $u_0(\rho_1) = \rho_1$; $u_1(\rho_1) = \frac{1}{4}\rho_1^2$, etc. The quantity ρ_1 must be greater than zero and small enough to allow the waves to start. Normally $\rho_1 = 0.0625$ is satisfactory, only introducing an error in the real part of the s-wave phase shift of order 10^{-4} radian.

For charged particles, the value of $_{max} = kR_{max}$ in each channel should satisfy the inequalities (126),

$$65. \gtrsim \rho_{max} \gtrsim \frac{5}{12}\left[\eta^2 + 4\eta + 3\right],$$ or 10, whichever is greater. (Values of η up to approximately 11 can be handled by SALLY.)

The other criteria for R_{max} are physical. It must be chosen sufficiently large that the optical potentials and the form factor have fallen to negligible values. The form factor is usually the over-riding consideration here (remember

it is weighted by r^2 in the radial integrals). In most cases of physical interest R_{max} should be chosen at least twice the nuclear radius, and preferably three times it; in some circumstances larger values may be needed.

The N_1 intervals between R_1 and R_2 are chosen to be small to ensure accuracy in starting the integration of the radial differential equations; they should be approximately $\delta\rho \approx 0.0625$ (or $\delta r \approx 0.0625/k$) or less. Also, it is suggested that the value of $\rho_2 = kR_2$ should be at least 0.5 in each channel. At low energies, when k is small, the more important parameter is the local wave number K inside the optical potential, which may be considerably larger than k. Then a starting radius of $R_1 = 0.0625/K$ and an interval size of $\delta r = 0.0625/K$ may be advisable.

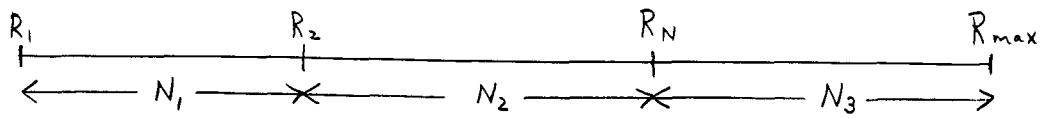
The remaining intervals must be chosen so that they are small compared to (i) the wavelength in either channel, and (ii) distances over which the optical potentials or form factor vary appreciably. Condition (i) appears to be satisfied if $\delta\rho \lesssim 0.25$ (or $\delta r \lesssim 0.25/k$), while for Saxon potentials with a typical surface diffuseness $\underline{a} \approx 0.5$, a step length as large as $\delta r \approx 0.35$ (provided $\delta\rho \lesssim 0.25$ simultaneously) does not seem to impair the accuracy greatly, although a smaller interval would be advisable.

(Note also that the radius R_N is part of the form factor input for options 0, 7, 12 and 13, although with other options it is a free parameter.)

Not all these parameters have to be used. A useful combination is effectively to omit R_3, entering

$$\boxed{R_1,\ R_2,\ R_N,\ R_{max},\ 0,\ N_1,\ N_2,\ N_3,\ 0} \equiv \boxed{R_1,\ R_2,\ R_N,\ R_{max},,\ N_1,\ N_2,\ N_3,}$$

corresponding to:

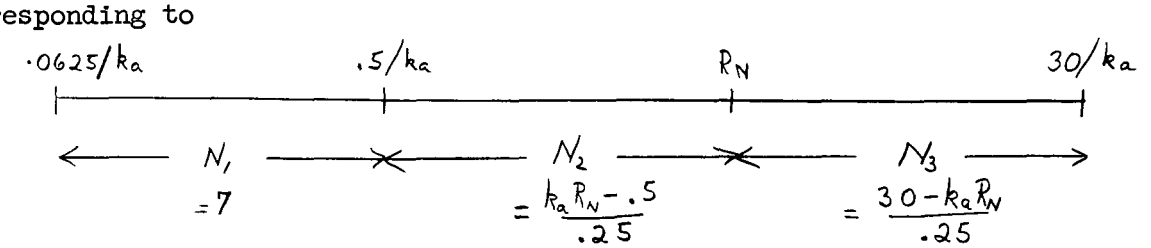


These three regions are sufficient to allow the use of smaller intervals in the range R_1 to R_N, where the form factor, optical potentials and distorted waves may be varying rapidly ("inside the nucleus"), than in the "external" region R_N to R_{max}. Then R_N should be chosen (except with options 0, 7, 12, 13) to be somewhat larger than the nuclear radius.

A flexible, simplified, system of preset values for the radii and the intervals is also available. This is based on the three integration regions above with $R_1 = 0.0625/k_a$, $R_2 = 0.5/k_a$, $R_{max} = 30./k_a$, and using $N_1 = 7$ and N_2, N_3 as the smallest integers for which δr_2, $\delta r_3 \leq 0.25/k_a$ respectively. (Note that for charged particles, $\rho_{max} = 30$ is too <u>small</u> if the Coulomb parameter $\eta > 6.5$. Also note that $R_{max} = 30/k_a$ leads to $\rho_{max} = 30\ k_b/k_a$ in the exit channel.) Only R_N needs to be entered as input, and any value between R_2 and R_{max} may be used for it (except that for options 0, 7, 12, 13, R_N has the usual significance (Section 8)). The input is now written:

$$\boxed{R_N,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0} \equiv \boxed{R_N,,,,,,,,}$$

corresponding to



with N_2, N_3 rounded upward. More control may be gained by entering a value

(i) for N_3: $\boxed{R_N, 0, 0, 0, 0, N_3, 0, 0, 0} \equiv \boxed{R_N,,,,, N_3,,,}$

(ii) for R_{max}: $\boxed{R_N, R_{max}, 0, 0, 0, N_3, 0, 0, 0} \equiv \boxed{R_N, R_{max},,,, N_3,,,}$

(iii) for R_2: $\boxed{R_2, R_N, R_{max}, 0, 0, N_2, N_3, 0, 0} \equiv \boxed{R_2, R_N, R_{max},,, N_2, N_3,,}$

In either of these, or the more complete input, any of the N_i can be entered as 0, when it will be taken as $k_a(R_{i+1} - R_i)/0.25$ (rounded upward) for $i \geqslant 2$, or $k_a(R_2 - R_1)/0.0625$ (rounded upward) for $i = 1$.

<u>Example 1</u>: <u>(d,p) deuteron stripping, incident energy 8 Mev</u>

For this energy, $k_a \approx 0.8$, so the preset value of $R_{max} = 30/k_a \approx 37.5$. The preset interval size of $0.25/k_a \approx 0.3$ may be adequate. For the exit channel consider two Q values:

<u>(i)</u> $Q = 0$, so $k_b \approx 0.6$, and the preset value of R_{max} leads to a $\rho_{max} \approx 22.5$,

(ii) $Q = 8$ Mev, so $k_b \approx 0.8$ and the preset $\rho_{max} \approx 30$. The Coulomb parameter η_a for the deuterons is approximately $Z_A/12$, so is well below the limiting value of 11 even for the heaviest nuclei. For the protons, η_b is even smaller.

We could then use the input:

$$R_N, 0, 0, 0, 0, 0, 0, 0, 0 \equiv R_N,,,,,,,,$$

While these preset integration controls are adequate, the R_{max} is much larger than is usually necessary, and the running time could be reduced by about 40% by using $R_{max} \approx 20$, say. This leads to $\rho_{max} \approx 16$ for the deuterons and

$\rho_{max} \approx 12$, (for Q = 0) or $\rho_{max} \approx 16$, (for Q = 8) for the protons. The input is then

$$R_N,\ 20,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \quad \equiv \quad R_N,\ 20,,,,,,,$$

If, in addition, the interval size of 0.3 is inadequate for the nuclear interior, a smaller interval may be used by explicitly writing input for R_2 and N_2. Suppose $R_N = 5.3$ and we require $\delta r = 0.15$. The preset R_2 is $0.5/k_a \approx 0.63$. We may take $R_2 = 0.65$, then $R_N - R_2 = 4.65$ and $N_2 = 31$ gives the required interval. The input is then

$$.65,\ 5.3,\ 20,\ 0,\ 0,\ 31,\ 0,\ 0,\ 0 \quad \equiv \quad .65,\ 5.3,\ 20,,,\ 31,,,$$

Leaving $N_3 = 0$ retains the interval $\delta r \approx 0.3$ for $R_N \leq r \leq R_{max}$.

<u>Example 2</u>: <u>Inelastic α-scattering, incident energy 40 Mev</u>

Here $k_a \approx 2.7$, so the preset value of $R_{max} = 30/k_a \approx 11$ is rather small, although experience has shown it adequate for lighter nuclei (say $M_A \lesssim 50$). For heavier nuclei a larger R_{max} should be used. The limitation $\rho_{max} \lesssim 65$ allows $R_{max} \lesssim 24$. For small energy losses, $k_a \approx k_b$, so ρ_{max} for the exit channel has a value similar to the entrance channel. The preset interval of $\delta r = 0.25/k_a \approx 0.1$ is certainly adequate. Since options 0, 7, 12, 13 will not be used generally for this reaction, the value assigned to R_N is arbitrary (between $0.5/k_a$ and R_{max}) and the simplified input may be used:

$$R_N, R_{max},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \quad = \quad R_N, R_{max},,,,,,,$$

9. Form Factor Options in SALLY

Ten interaction form factors F(r) are available, choice being controlled by entry F6 as follows:

F6 =	F(r) =	
0, or 12	Bound - state wave - function $u_{N\ell}(r)$	
1,	$\frac{r}{R_0} f(x)$, $r < R_{CE}$ $\frac{r}{R_0} f(x) - \frac{b}{r^3}$, $r \geqslant R_{CE}$	$f(x) = \frac{d}{dx} \frac{1}{e^x+1} = \frac{-e^x}{(e^x+1)^2}$, $x = \frac{r-R_0}{a_0}$
2,	$f(x)$, $r < R_{CE}$ $f(x) - \frac{b}{r^3}$, $r \geqslant R_{CE}$	
3,	$f(x)$, $r < R_{CE}$ $f(x) - \frac{b}{r^4}$, $r \geqslant R_{CE}$	
4,	$\frac{V}{e^x+1}$ Saxon	$x = \frac{r-R_0}{a_0}$
5,	$V e^{-x^2}$ Gauss	
7, or 13,	$e^{-x^2} u_{N\ell}(r)$, $r \leqslant R_N$ $u_{N\ell}(r)$, $r \geqslant R_N$	Damped bound - state $x = \frac{r-R_N}{a_N}$
8,	$-f\left(\frac{r-R_0}{a_I}\right)$, $r \leqslant R_0$ $-f\left(\frac{r-R_0}{a_E}\right)$, $r \geqslant R_0$	$f(x) = \frac{d}{dx} \frac{1}{e^x+1}$, $x = \frac{r-R_0}{a_0}$
9,	$\frac{d}{dx} f(x)$	
10,	$\left(\frac{r}{R_0}\right)^2 \frac{d}{dx} f(x)$	

The parameters for these options are entered in the locations F1 through F5 according to the following table:

F1 =	k_N	R_o	R_o	R_o	R_o	R_o	k_N	R_o	R_o	R_o	B_N	B_N
F2 =	η_N	a_o	a_o	a_o	a_o	a_o	η_N	a_E	a_o	a_o	Z_1Z_2	Z_1Z_2
F3 =	μ_N	b	b	b	V	V	μ_N	a_I	0	0	μ_N	μ_N
F4 =	0	R_{CE}	R_{CE}	R_{CE}	0	0	a_N	0	0	0	0	a_N
F5 =	N	0	0	0	0	0	N	0	0	0	N	N
F6 =	0	1	2	3	4	5	7	8	9	10	12	13

<u>Option 0 or 12</u> is designed for stripping (or pick-up) reactions and represents the bound-state wave-function for the captured particle of reduced mass μ(amu), Coulomb charge parameter η_N and binding energy $B_N = h^2k^2_N/2\mu_N$ (Mev). The units of k_N are inverse fermis, so

$$k_N \simeq 0.2195 \sqrt{\mu_N(\text{pmu})\, B_N(\text{Mev})}$$

$$\eta_N \simeq 0.1581 \; Z_1 Z_2 \sqrt{\frac{\mu_N(\text{pmu})}{B_N(\text{Mev})}} ,$$

where Z_1e is the charge on the captured particle and Z_2e the charge of the nucleus to which it is bound. In option 0 one enters k_N and μ_N; in option 12, B_N and Z_1Z_2 are entered.

The function is calculated as an harmonic oscillator radial function of principal quantum number N, orbital angular momentum ℓ, matched smoothly on to a hankel function (or its Coulomb analog) at $r = R_N$ (entered in the integration controls section) with the required binding energy B_N (i.e. asymptotically it falls off as e^{-kr}/r).

$$u_{N\ell}(r) \propto r^{\ell} e^{-\beta r^2} F\left(\frac{\ell+1-N}{2}, \ell+\frac{3}{2}, \beta r^2\right), \quad r \leq R_N,$$

$$\propto r^{-1} W\left(-\eta_N, \ell+\tfrac{1}{2}, 2k_N r\right), \quad r \geq R_N$$

F is a confluent hypergeometric function, and W the real Whittaker function; if $\eta = 0$, $W = (2kr/\pi)^{1/2} K_{\ell+1/2}(kr)$. β is chosen by the program so the log derivatives match at $r = R_N$ and the function normalized so that $\int_0^\infty \mu_N^2(r)\, r^2 dr = 1$. N is the number of oscillator quanta plus one; the number of radial nodes (excluding $r = 0$ and $r = \infty$) in $\mu_{N\ell}$ is $\frac{1}{2}(N-\ell-1)$. The relation to a more conventional notation is as follows

	N,ℓ	Conventional
———	6p, 6f	3p, 2f
———	5s, 5d, 5g	3s, 2d, 1g
———	4p, 4f	2p, 1f
———	3s, 3d	2s, 1d
———	2p	1p
———	1s	1s

This option in SALLY will compute $u_{N\ell}$ for $N \leq 6$, $\ell \leq 4$. On the first page of output, below the values of $u_{N\ell}(r)$, are also printed out the values of

$$\int_{R_N}^{\infty} u^2 r^2 dr, \quad \int_0^{R_N} u^2 r^2 dr, \quad u(R_N), \quad \hbar^2 R_N u^2(R_N)/2\mu, \quad \beta$$

and $R_N \, d(\ln ru)/dr \big|_{r=R_N}$.

No simple rule can be given for choosing values of the matching radius R_N, although it will be of the same order as a nuclear radius. For the capture of a neutron or proton, it has been found empirically that in order to reproduce approximately the wave function of a nucleon bound by a few Mev in a Saxon potential of radius $1.3\ A^{1/3}$, the values of R_N needed are approximately

$R_N/A^{1/3}$	$\approx$	1.75	1.7	1.65	1.6	1.55
when A	$\sim$	10	25	60	100	200.

In detail, however, it is possible for R_N to depend upon N, ℓ, and B_N, as well as A.

Options 7 and 13 are the same as options 0 and 7 except that $u_{N\ell}(r)$ is multiplied by the damping factor: $\exp - \left(\frac{r - R_N}{a_N}\right)^2$, for $r \leqslant R_N$. This allows contributions from the nuclear interior to be reduced smoothly.

Options 1, 2 and 3 were designed especially as form factors for inelastic scattering (see Section 5.3), but may of course be used for other purposes. The terms in b take account of Coulomb excitation to first order, but excluding contributions from $r < R_{CE}$ (see Section 5.3.4). For interaction between particles of like charge, b is negative. The form b/r^3 in options 1 and 2 is appropriate to an $\ell = 2$ transition, the b/r^4 in option 3 to an $\ell = 3$ transition.

b may be put zero, but R_{CE} must remain finite (for example, put $R_{CE} = R_o$). These options may be used with any $\ell \leqslant 4$.

Option 8 is similar to options 2 and 3, but with b = 0, and different values for a_o inside and outside the peak $r = R_o$, allowing more flexibility in the choice of shape.

Option 9 is the derivative of the form factor 2 and 3 (with b = 0), and again was designed for inelastic scattering (Section 5.3). It is zero at $r = R_o$, changing sign there.

Option 10 is just $(r/R_o)^2$ times option 9 (see Section 5.3).

Options 4 and 5 are Saxon and Gaussian respectively.

10. Optical Model Options in SALLY

There are three choices of optical model potential U(r) for each channel, chosen by entries KA1 and KB1. Each has a Saxon-Woods form for the real part:

$$\text{Real } U(r) = -\frac{V}{e^x + 1}, \quad x = \frac{r - R}{a}, \quad R = r_0 (M_i)^{\frac{1}{3}} \qquad (i = A \text{ or } B)$$

For the imaginary (absorptive) part we have the choices

Ki1 =	Imag. U(r) =
1,	$-W e^{-(x')^2}, \quad x = \frac{r - R'}{a'}, \quad R' = r_0' (M_i)^{\frac{1}{3}}$ Put $W' = 0$. (surface Gaussian)
2,	$-\frac{W}{e^x + 1}, \quad x = \frac{r - R}{a}$, R and a as for real part Put $r_0' = a' = W' = 0$. (Saxon)
4,	$-\frac{W}{e^{x'} + 1}, \quad x' = \frac{r - R'}{a'}, \quad R' = r_0' (M_i)^{\frac{1}{3}}$ Put $W' = 0$. (independent Saxon)
7,	$-\left[W - W' \frac{d}{dx'}\right] \frac{1}{e^{x'} + 1}, \quad x' = \frac{r - R'}{a'}, \quad R' = r_0' (M_i)^{\frac{1}{3}}$ (independent Saxon plus derivative)

These give adequate control over the distribution of surface or volume absorption. Of course, Model 2 is a special case of Model 7 (with $W' = 0$ and $r_o' = r_o$, $a' = a$), but its use avoids having to enter both r_o, a and r_o', a' when these are equal. Similarly, Model 4 is a special case of Model 7 (with $W' = 0$). The signs are chosen so that a positive entry for V or W gives an attractive potential. In Model 7 we may put $W = 0$ (pure surface absorption) or $W' = 0$ (pure volume absorption). A purely real potential is given by any model with $W = W' = 0$ (however, a' must not be entered zero unless Model 2 is used). Similarly, V may be made zero, but a must remain some (arbitrary) finite value. When considering values of W and W' to enter for Model 7 it should be remembered that, while $(e^x + 1)^{-1}$ has the value $\frac{1}{2}$ at $r = R$, $d/dx\ (e^x + 1)^{-1}$ has the value $\frac{1}{4}$.

In addition to this potential, there is in the entrance channel a Coulomb potential due to a uniform sphere of radius $R_c = r_c \mathcal{M}_A^{1/3}$ and total charge $(Z_a Z_A e^2)$, that is

$$V_{Coulomb} = \frac{Z_a Z_A e^2}{r}, \qquad r \geqslant R_c$$

$$= \frac{Z_a Z_A e^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right), \qquad r \leqslant R_c,$$

and similarly for the exit channel. For neutral particles $Z_a Z_A = 0$, but a finite r_c must be entered; it is suggested to put $r_c = r_o$.

Putting both V and W (and W') equal to zero is equivalent to using just Coulomb distorted-waves. If $Z_a Z_A = 0$ also, it is equivalent to using plane waves. (Again, a, a' and r_c must be non-zero, except for Model 2 where $a' = 0$ always.)

11. Output from SALLY

SALLY always outputs the form factor used, the elastic scattering in entrance and exit channels, and the reaction differential cross-section (and polarizations were applicable). In addition under option, the amplitudes and radial integrals may be printed out.

On the first page are printed the form factor controls F1 through F6, and values of the form factor F(r) at each integration step, starting at $r = R_1$ and finishing at $r = R_{max}$ (see Section 8). These numbers appear in the form: $a\ E\,y = a \times 10^{y}$ (with the radii in fermis),

e.g. 1.23 --- E-02 = 1.23 --- $\times 10^{-2}$

-4.56 --- E 09 = -4.56 --- $\times 10^{9}$

When the form factor options F6 = 0, 7, 12 or 13 (the bound-state wave function) are used, $R(r) = u_{N\ell}(r)$ is followed by some additional data

$$\text{EXT AREA} = C \int_{R_N}^{\infty} u_{N\ell}^{2}\, r^{2}\, dr$$

$$\text{INT AREA} = C \int_{0}^{R_N} u_{N\ell}^{2}\, r^{2}\, dr$$

(where the normalization constant C = EXT AREA + INT AREA)

RN = R_N, matching radius

ZETA (RN)/RN = $\sqrt{C}\ u_{N\ell}(R_N)$

GAMMA = $C\,(\hbar^2 R_N / 2\mu_N)\, u_{N\ell}^{2}(R_N)$

BETA = β, oscillator parameter

RN*LOGDER = $R_N \frac{d}{dr} \ln(r u_{N\ell})\Big|_{r=R_N}$

Below are printed the (dimensionless) integration intervals, and step-lengths $\delta\rho_1$, $\rho_2 - \delta\rho_1$, $\delta\rho_2$, $\rho_N - \delta\rho_2$, $\delta\rho_3$, $\rho_{Max} - \delta\rho_3$, where $\rho_i = k_a R_i$ (see Section 8), to be used in the entrance channel.

The next page carries data for the entrance channel elastic scattering. These are expressed in the form: $y\ a = .a \times 10^{y}$

e.g. -3 123 --- = .123 --- $\times 10^{-3}$

2 -456 --- = -.456 --- $\times 10^{2}$

At the top of the left-hand column is the reaction or absorption cross-section = SIGMAR (in f^2 = 10 mb units) predicted by the optical model. The remainder of the column contains input-, reduced-, and integration-data in the following way

OPTION x: the optical model (see Section 10) $(x = KA1)$

ELAB = energy of incident particle in lab. system (Mev)

MSUBI = $\mathfrak{M}_a$, mass of incident particle (pmu)

MSUBB = $\mathfrak{M}_A$, mass of target nucleus (pmu)

ZZP = $Z_a Z_A$, product of charges of projectile and target (e^2)

V = V (Mev); A = a(f)

W = W (Mev); RG = r_o'(f)

RO = r_o(f); B = a'(f)

RC = r_c(f); TAU1 = W'(Mev)

parameters for optical model for a + A (See Section 10)

ECM = energy in CMS (Mev)

ETA = η, Coulomb parameter = $0.15805086 \sqrt{\frac{\mathfrak{M}_a(pmu)}{E_{lab}(Mev)}}$

K = k_a, wave number (f^{-1}) = $0.2195376 \frac{\mathfrak{M}_A}{\mathfrak{M}_A + \mathfrak{M}_a} \sqrt{\mathfrak{M}_a(pmu) E_{lab}(Mev)}$

KA = $k_a a$

KB = $k_a a'$

RHOBN = $k_a r_o (\mathfrak{M}_A)^{1/3}$

RHOBC = $k_a r_c (\mathfrak{M}_A)^{1/3}$

RHOBG = $k_a r_o' (\mathfrak{M}_A)^{1/3}$

RHOIN = $k_a R_1$, lower limit of radial integration - followed by the other $k_a R_i$ (unlabelled) - (see Section 8)

RHOMAX = $k_a R_{max}$

DRHOIN = $k_a \delta r_1$, the initial step length - followed by the other $k_a \delta r_i$ (unlabelled) - (see Section 8)

The next three columns contain the CMS scattering angle THETA = Θ (at CMS intervals of 4°), the elastic differential cross-section (= SIGMATH) in f^2 = 10 mb. units, and, for charged particles ($Z_a Z_A \neq 0$), its ratio to the Rutherford scattering cross-section (= SIG - SIGC). Finally are printed the real and imaginary parts of the partial-wave elastic-scattering-amplitudes $C_L = e^{iK_L} \sin K_L$, and the normalizing factors obtained when the integrated partial waves are matched on to the exterior wave functions. At the bottom of this page are then printed the (dimensionless) integration intervals and step lengths to be used in the exit channel, $\delta\rho_1'$, $\rho_2' - \delta\rho_1'$, etc., where $\rho_i' = k_b R_i = (k_b/k_a)\rho_i$

The third page carried equivalent information for the exit channel, except that ELAB is to be interpreted as the lab. energy which would give the same CMS energy if the residual nucleus were stationary in the laboratory, (i.e. the lab. energy for the inverse reaction). In the exit channel, ECM is the ECM of the entrance channel, plus the Q-value of the reaction.

The fourth page carries the results of the calculation of the reaction, for an upper cut-off at R_{max}, i.e. the radial integrals are over the range $0 < r < R_{max}$. If other cut-offs (upper or lower) are employed also, similar results of calculation with these are printed on the following pages. At the top appears the angle-integrated cross-section for the reaction

TOTAL SIGMA = $\sigma_T = 2\pi \int \sigma(\Theta) \sin\Theta \, d\Theta$,

also, for $\ell = 1$ and 2, the tensors averaged over the differential cross-section (and referred to $\underset{\sim}{k}_a$ as z-axis, see Section 4 and Appendix II)

AV D20 = $\langle d_{20} \rangle = 2\pi \int \sigma(\Theta)\, d_{20} \sin\Theta \, d\Theta/\sigma_T$ for $\ell = 1$ and 2,

AV D40 = $\langle d_{40} \rangle = 2\pi \int \sigma(\Theta)\, d_{40} \sin\Theta \, d\Theta/\sigma_T$ for $\ell = 2$ only.

Then arranged in columns are the CMS scattering angle, THETA = Θ, the differential cross-section SIGMA = $\sigma(\Theta)$ defined by eqs. (21) and ( ), and (except for $\ell = 0$) the "polarization" POL = $\pi(\Theta)$ defined by eq. (25). For $\ell = 1$ we also have LAMBDA = $\lambda(\Theta)$ and 2PHIO = $2\phi_o$, while for $\ell = 2$, there is D20 = $\hat{d}_{20}$, D22 = $|\hat{d}_{22}|$, D44 = $|\hat{d}_{44}|$, D32 = $|\hat{d}_{32}|$, 2PHIO = $2\phi_o$, 4PHI1 = $4\phi_1$, and 2PHI2 = $2\phi_2$ (see Appendix II). The phase angles $q\phi_i$ are given in degrees, and are physically unchanged by the addition or subtraction of 360° (that is, ϕ_o and $\phi_o \pm 180^o$, or ϕ_1 and $\phi_1 \pm 90^o$, are physically equivalent). When control KB4 is set as zero, the ϕ_i are measured from the incident direction $\underset{\sim}{k}_a$, a right-hand rotation about $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ being positive. When KB4 is set to 1, the ϕ_i are measured from the direction of the recoil momentum $\underset{\sim}{q}(\Theta) = \underset{\sim}{k}_a - (m_A/m_B)\underset{\sim}{k}_b$; also the recoil angle between $\underset{\sim}{q}$ and $\underset{\sim}{k}_a$ is printed out on a separate page preceding the reaction results at the same values of Θ (see Appendix II). When plane waves are used (that is, both optical potentials and the charges are set zero), we have $\phi_o = \phi_1 = -$(angle of recoil) for KB4 = 0, or $\phi_o = \phi_1 = 0$ for KB4 = 1 (see Appendix IV).

For $\ell = 0$ and $\ell = 1$ the amplitudes (real and imaginary parts) $\beta^{\ell m}(\Theta)$ are also printed out on the same page.

For $\ell = 2$, the additional angular correlation parameters α, β and γ defined on page 20 (see Appendix II) are also printed out.

When the control KB3 is set as zero, this is the only output. If KB3 is set as 1, the real and imaginary parts of all the amplitudes $\beta^{\ell m}(\theta)$ and $\overline{\beta^{m}_{L}}$, and the radial integrals $f^{\ell}_{LL'}$, defined in eqs. (15), (16) and (17), or ( ), are also printed out. This is repeated for each value of the cut-off radii if these are employed. When KB4 is set zero, then m is the component of ℓ along $\underset{\sim}{k}_a$, but when KB4 = 1, m is the component of ℓ along the recoil direction $\underset{\sim}{q}(\Theta)$. (In both cases the y-axis is taken along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$.)

APPENDIX I

Some General Properties

We consider here some general properties of the amplitudes $\beta^{\ell m}(\theta)$, defined by eq. (14), and the tensors ρ_{KQ} defined by eq. (30).

(i) Special coordinate systems. By construction, both $\beta^{\ell m}$ and ρ_{KQ} behave like the complex conjugate of a spherical harmonic under rotations of coordinate axes,

$$\beta^{\ell m}(2) = \sum_{m'} \beta^{\ell m'}(1) \, D^{\ell}_{m'm}{}^{*}(\alpha\beta\gamma) ,$$

$$\rho_{KQ}(2) = \sum_{Q'} \rho_{KQ'}(1) \, D^{K}_{Q'Q}{}^{*}(\alpha\beta\gamma) , \qquad \text{---(A.1)}$$

where $(\alpha\ \beta\ \gamma)$ are the Euler angles of the rotation which has to be applied to set # 1 of coordinate axes to make them coincide with set # 2. Two coordinate systems are particularly useful; the first has x and z-axes in the scattering plane (defined by $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$) and y along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$. The other puts $\hat{z}$ along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ and leaves $\hat{x}$, $\hat{y}$ in the scattering plane. Quantities referred to the latter axes we denote by a circumflex, $\hat{}$, those referred to the former we leave unmarked.

In the former coordinate system we then have

$$\beta^{\ell m} = (-)^{m} \beta^{\ell -m} \qquad \text{---(A.2)}$$

and it follows for the tensors which are diagonal in ℓ, s, j (or diagonal in ℓ and independent of s, j),

$$\rho_{KQ} = (-)^{K-Q} \rho_{K,-Q} = (-)^{Q} \rho^{*}_{K,-Q} \qquad \text{---(A.3)}$$

so that ρ_{KQ} is real if K is even, imaginary if K is odd, and $\rho_{KO} = 0$ if K is odd.

In the latter corrdinate system we have instead

$$\hat{\beta}^{\ell m} = 0, \quad \text{if } (\ell - m) \text{ is odd}, \qquad \text{---(A.4)}$$

and hence

$$\hat{\rho}_{KQ} = 0, \quad \text{if } Q \text{ is odd}, \qquad \text{---(A.5)}$$

but the $\hat{\rho}_{KQ}$ are now complex in general. If the two sets of axes are further restricted so that $\hat{x}$ is parallel to z, the rotation from (xyz) to $(\hat{x}\hat{y}\hat{z})$ has Euler angles $(-\frac{\pi}{2}, -\frac{\pi}{2}, 0)$, and for example A.1 gives

$$\begin{aligned}
\hat{\rho}_{10} &= -i\sqrt{2}\,\rho_{11}, \\
\hat{\rho}_{20} &= -\sqrt{\tfrac{3}{2}}\,\rho_{22} - \tfrac{1}{2}\rho_{20}, \qquad \text{---(A.6)} \\
\hat{\rho}_{22} &= -\tfrac{1}{2}\rho_{22} + \sqrt{\tfrac{3}{8}}\,\rho_{20} + i\,\rho_{21}, \qquad \underline{\text{etc}}.
\end{aligned}$$

Of course, ρ_{00}, which is proportional to the cross-section, is scalar and invariant under rotations, $\hat{\rho}_{00} = \rho_{00}$, etc.

<u>(ii) Transition amplitude for inverse reaction</u>. If the wave functions used behave under time-reversal in the conventional way,[9] $K_t \psi_{JM} = (-)^{J-M} \psi_{J,-M}$, the transition amplitudes (6) have the symmetry[27]

$$\langle J_B M_B, s_b m_b; \underset{\sim}{k}_b | V | J_A M_A, s_a m_a; \underset{\sim}{k}_a \rangle$$

$$= (-)^{J_B - J_A + s_b - s_a} (-)^{M_A - M_B + m_a - m_b} \langle J_A, -M_A, s_a, -m_a; -\underset{\sim}{k}_a | V | J_B, -M_B, s_b, -m_b; -\underset{\sim}{k}_b \rangle \qquad \text{---(A.7)}$$

relating the transition A(a,b)B, $\underset{\sim}{k}_a \rightarrow \underset{\sim}{k}_b$, to the inverse B(b,a)A, $-\underset{\sim}{k}_b \rightarrow -\underset{\sim}{k}_a$, with reversed spins (i.e. M $\rightarrow$ -M), but with the same quantization axes used to define z-components in both amplitudes. With the expansion (8) this leads to

$$\sqrt{2J_B+1}\; A_{\ell sj}\, f_{\ell sj,m}(\underset{\sim}{t}_b,\underset{\sim}{t}_a) = (-)^{J_B-J_A+s_b-s_a}\,(-)^{j-s}\sqrt{2J_A+1}\;\tilde{A}_{\ell sj}\,\tilde{f}_{\ell sj,m}(\underset{\sim}{t}_a,\underset{\sim}{t}_b) \quad \cdots(A.8)$$

where we use $\sim$ to denote quantities from the inverse reaction, namely the right-hand-side of (A.7). Using $\phi^{(-)*}(\underset{\sim}{k},\underset{\sim}{t}) = \phi^{(+)}(-\underset{\sim}{k},\underset{\sim}{t})$ for the distorted waves, this becomes with the normalization (9)

$$\sqrt{2J_B+1}\; A_{\ell sj}\;\beta^{\ell m}_{sj}\; m_B^4\, k_a^2 = (-)^{J_B-J_A+s_b-s_a}\,(-)^{j-s}\sqrt{2J_A+1}\;\tilde{A}_{\ell sj}\,\tilde{\beta}^{\ell m}_{sj}\; m_A^4\, k_b^2 \quad \cdots(A.9)$$

The form (radial and angular) of $f_{\ell sj,m}$ and $\tilde{f}_{\ell sj,m}$ must be the same, so if we choose them to be identical (in "zero-range", we choose $F_{\ell sj}(r) = \tilde{F}_{\ell sj}(r)$), we have

$$m_A^4\, k_b^2\;\tilde{\beta}^{\ell m}_{sj} = m_B^4\, k_a^2\;\beta^{\ell m}_{sj} \quad \cdots(A.10a)$$

and

$$\tilde{A}_{\ell sj} = \sqrt{\frac{2J_B+1}{2J_A+1}}\;(-)^{J_B-J_A+s_b-s_a}\,(-)^{j-s}\,A_{\ell sj} \quad \cdots(A.10b)$$

The property (A.10b) was used on page 15 in relating expressions for the polarization of emitted particles to those for the asymmetry induced by an incident polarized beam. Eq. (A.10a) also gives $k_b^2\,\tilde{\sigma}(\theta) = k_a^2\,\sigma(\theta)$.

The equality $(m_B^4 k_a^2)\beta^{\ell m} = (m_A^4 k_b^2)\tilde{\beta}^{\ell m}$ holds if we use the same quantization axes for both amplitudes, for example with z along $\underset{\sim}{k}_a$, y along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$. But the corresponding choice for the inverse reaction would have been $\tilde{z}$ along $-\underset{\sim}{k}_b$ (now the incident direction) and $\tilde{y}$ along $-\underset{\sim}{k}_b \times -\underset{\sim}{k}_a = \underset{\sim}{k}_b \times \underset{\sim}{k}_a$.

The Euler angles for the rotation into these axes are $(0, \pi + \theta, \pi)$ where θ is the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{k}_b$, so the $\tilde{\beta}^{lm}$ referred to them become, using (A.1)

$$\tilde{\beta}^{lm}(\tilde{z}) = \sum_{m'} \tilde{\beta}^{lm'}(z)\, D^{l\,*}_{m'm}(0, \pi+\theta, \pi)$$

$$= \left(\frac{m_B}{m_A}\right)^4 \left(\frac{k_a}{k_b}\right)^2 \sum_{m'} \beta^{lm'}(z)\, D^{l\,*}_{m'm}(0, \pi+\theta, \pi) .$$

The tensors ρ_{KQ} and other quantities are related in a similar way. The (dimensionless) $\hat{d}_{KQ}$ calculated by SALLY are referred to $\hat{z}$ along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$, $\hat{x}$ along $\underset{\sim}{k}_a$; the corresponding choice for the inverse would have been $\tilde{\hat{z}}$ along $\underset{\sim}{k}_b \times \underset{\sim}{k}_a$, $\tilde{\hat{x}}$ along $-\underset{\sim}{k}_b$, requiring a rotation $(0, \pi, -\theta)$. The rotation has a simple explicit form then, changing only the phase

$$\tilde{\hat{d}}_{KQ}(\tilde{\hat{z}}) = (-)^{K-Q}\, e^{-iQ\theta}\, \hat{d}_{K-Q}(\hat{z}) = (-)^{K}\, e^{-iQ\theta}\, \hat{d}^{*}_{KQ}(\hat{z}) .$$

<u>(iii) The "adiabatic" symmetry</u>. A special symmetry is displayed when the distorted waves for the incident and emitted particles are similar.[28] This requires $k_a = \frac{m_A}{m_B} k_b$, that is an energy release of

$$Q = E_a(lab) \frac{m_A}{m_A + m_a} \left[\frac{m_B m_a}{m_A m_b} - 1 \right] ;$$

and optical potentials related by

$$U_a(r) = \frac{m_A m_b}{m_B m_a} U_b\left(\frac{m_A}{m_B} r\right)$$

When the particles are charged there is the additional condition on the Coulomb interaction

$$\frac{Z_a Z_A}{Z_b Z_B} = \frac{m_b}{m_a} .$$

In practice this last is only satisfied in inelastic scattering, where $Z_a = Z_b$, $Z_A = Z_B$, $m_a = m_b$, otherwise the symmetry will only apply when Coulomb effects are negligible. The other two conditions are approximately satisfied in inelastic scattering if the energy loss is small, $|Q| \ll E_a$ (the adiabatic condition).

Under these circumstances the transition amplitudes, and hence observables from the reaction, are invariant under a rotation of 180° about the recoil direction $\underset{\sim}{q} = \underset{\sim}{k}_a - {}^{m_A}/_{m_B}\, \underset{\sim}{k}_b$. Thus the ρ_{KQ} with odd Q vanish (with z in the scattering plane and y along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$); or $\hat{\rho}_{KQ} = (-)^K \hat{\rho}_{K-Q}$ (with $\hat{z}$ along $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$, $\hat{x}$ along $\underset{\sim}{q}$). The consequences of this are discussed in ref. (28); in particular the polarization π vanishes and the phase angles ϕ_i of the $\hat{d}_{KQ}$ become just the recoil angle (the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{q}$).

Appendix II

Formulae for $\ell = 0$ through 4

(i) Transfer of $\ell = 0$

There is only one amplitude, $m = 0$,

$$\beta(\theta) = \sum_L \bar{\beta}_L P_L(\theta),$$

$$\bar{\beta}_L = (2L+1) f^0_{LL}.$$

The computed cross-section is, from eq. (125),

$$\sigma(\theta) = N \, |\beta(\theta)|^2,$$

and the polarization vanishes, $\pi = 0$.

(ii) Transfer of $\ell = 1$

There are now two amplitudes, $m = 0, 1$,

$$\left.\begin{aligned} \beta^0 &= \sum_L \overline{\beta^0_L} P_L(\theta), \quad \overline{\beta^0_L} = (L+1) f^1_{L,L+1} - L f^1_{L,L-1} \\ \beta^1 &= \sum_L \overline{\beta^1_L} P^1_L(\theta), \quad \overline{\beta^1_L} = \sqrt{\tfrac{1}{2}}\left[f^1_{L,L+1} + f^1_{L,L-1} \right] \end{aligned}\right\} \begin{aligned} &\underset{\sim}{k}_a \text{ as z-axis} \\ &\underset{\sim}{k}_a \times \underset{\sim}{k}_b \text{ as y-axis} \end{aligned}$$

and

$$\sigma = N \left[|\beta^0|^2 + 2|\beta^1|^2 \right]$$

$$\pi = \frac{N}{\sigma} \, 2\sqrt{2} \, \mathrm{Im}(\beta^0 \beta^{1*}), \quad = 0 \text{ at } \theta = 0^\circ \text{ or } 180^\circ.$$

If we write the tensors

$$d_{2Q} = D_{2Q} / D_{00}, \quad D_{2Q} = -\sqrt{\tfrac{3}{2}} \, \rho_{2Q},$$

$$\left.\begin{aligned} D_{00} &= 2|\beta^1|^2 + |\beta^0|^2 = \sigma/N \\ D_{20} &= |\beta^0|^2 - |\beta^1|^2 \\ D_{21} &= \sqrt{3} \, \mathrm{Re}(\beta^0 \beta^{1*}) \\ D_{22} &= \sqrt{\tfrac{3}{2}} \, |\beta^1|^2 \end{aligned}\right\} \begin{aligned} &z \text{ in } \underset{\sim}{k}_a, \underset{\sim}{k}_b \text{ plane} \\ &y \text{ along } \underset{\sim}{k}_a \times \underset{\sim}{k}_b \end{aligned}$$

so that: $\quad d_{20} + \sqrt{6} \, d_{22} = 1.$

Referred to the other axes,

$$\left.\begin{aligned} \hat{D}_{00} &= D_{00} = |\hat{\beta}^{1}|^2 + |\hat{\beta}^{-1}|^2 \\ \hat{D}_{20} &= -\tfrac{1}{2}\hat{D}_{00}, \quad \hat{D}_{21} = 0 \\ \hat{D}_{22} &= -\sqrt{\tfrac{3}{2}}\ \hat{\beta}^{1}\hat{\beta}^{-1*} \end{aligned}\right\} \quad \begin{aligned} &\hat{z} \text{ along } \underset{\sim}{k}_a \times \underset{\sim}{k}_b, \\ &\hat{x} \text{ in } \underset{\sim}{k}_a, \underset{\sim}{k}_b \text{ plane.} \end{aligned}$$

If we have $\hat{x}$ parallel to z, $\hat{z}$ parallel to y,

$$\hat{\beta}^{1} = -\sqrt{\tfrac{1}{2}}\,\beta^{0} - i\beta^{1} \quad ; \quad \hat{\beta}^{0} = 0,$$
$$\hat{\beta}^{-1} = +\sqrt{\tfrac{1}{2}}\,\beta^{0} - i\beta^{1}.$$

It is convenient[8] to compute parameters λ and ϕ_0, where

$$\hat{d}_{22} = \frac{N}{\sigma}\sqrt{\tfrac{3}{2}}\left\{\left[\tfrac{1}{2}|\beta^{0}|^2 - |\beta^{1}|^2\right] + i\sqrt{2}\,\mathrm{Re}(\beta^{0}\beta^{1*})\right\} = |\hat{d}_{22}|\,e^{-2i\phi_0},$$

$$\lambda = \sqrt{\tfrac{8}{3}}\,|\hat{d}_{22}| = \sqrt{1-\pi^2}; \quad 0 \leq \lambda \leq 1; \quad \lambda = 1 \text{ at } \theta = 0^\circ \text{ or } 180^\circ.$$

The latter also shows the relation between λ and the polarization π. When referred to the recoil $\underset{\sim}{q} = \underset{\sim}{k}_a - \frac{M_A}{M_B}\underset{\sim}{k}_f$ as z-axis, the amplitudes β_R^m are related to those computed with $\underset{\sim}{k}_a$ as z-axis by

$$\left.\begin{aligned} \beta_R^{0} &= \cos\theta_R\,\beta^{0} + \sqrt{2}\sin\theta_R\,\beta^{1} \\ \beta_R^{1} &= -\sqrt{\tfrac{1}{2}}\sin\theta_R\,\beta^{0} + \cos\theta_R\,\beta^{1} \end{aligned}\right\} \quad \begin{aligned} &z' \text{ along } \underset{\sim}{q}, \\ &y' \text{ along } \underset{\sim}{k}_a \times \underset{\sim}{k}_f. \end{aligned}$$

where θ_R is the angle between $\underset{\sim}{q}$ and $\underset{\sim}{k}_a$.

<u>(iii) Transfer of $\ell = 2$</u>

There are now three amplitudes, and

$$\sigma = N\left[|\beta^{0}|^2 + 2|\beta^{1}|^2 + 2|\beta^{2}|^2\right]$$

$$\pi = \frac{N}{\sigma}\left[\sqrt{6}\,\mathrm{Im}(\beta^{0}\beta^{1*}) + 2\,\mathrm{Im}(\beta^{1}\beta^{2*})\right]$$

For the tensors $d_{KQ} = D_{KQ}/D_{00}$, $D_{2Q} = -\sqrt{\tfrac{7}{2}}\,\rho_{2Q}$, $D_{4Q} = \sqrt{\tfrac{35}{18}}\,\rho_{4Q}$, we have

$$D_{00} = \sigma / N$$

$$\left.\begin{aligned}
D_{20} &= |\beta^0|^2 + |\beta^1|^2 - 2|\beta^2|^2 \\
D_{21} &= \mathcal{R}e\left[\sqrt{6}\,\beta^1\beta^{2*} + \beta^0\beta^{1*}\right] \\
D_{22} &= \sqrt{\tfrac{3}{2}}\,|\beta^1|^2 - 2\,\mathcal{R}e\,(\beta^0\beta^{2*}) \\
D_{40} &= \tfrac{1}{3}\,|\beta^2|^2 - \tfrac{4}{3}\,|\beta^1|^2 + |\beta^0|^2 \\
D_{41} &= \mathcal{R}e\left[-\sqrt{\tfrac{5}{9}}\,\beta^1\beta^{2*} + \sqrt{\tfrac{10}{3}}\,\beta^0\beta^{1*}\right] \\
D_{42} &= \sqrt{\tfrac{5}{3}}\,\mathcal{R}e(\beta^0\beta^{2*}) + \sqrt{\tfrac{10}{9}}\,|\beta^1|^2 \\
D_{43} &= \sqrt{\tfrac{35}{9}}\,\mathcal{R}e\,(\beta^1\beta^{2*}) \\
D_{44} &= \sqrt{\tfrac{35}{18}}\,|\beta^2|^2
\end{aligned}\right\}\quad \begin{aligned}&z \text{ in } \underset{\sim}{k}_a, \underset{\sim}{k}_b \text{ plane,}\\ &y \text{ along } \underset{\sim}{k}_a \times \underset{\sim}{k}_b.\end{aligned}$$

which gives the relations

$$\sqrt{10}\;d_{21} - \sqrt{3}\;d_{41} = \sqrt{21}\;d_{43}\,,$$

$$\sqrt{\tfrac{8}{5}}\;d_{42} + \sqrt{\tfrac{2}{3}}\;d_{22} - \sqrt{\tfrac{14}{5}}\;d_{44} = d_{20} - d_{40}\,,$$

$$d_{40} - \tfrac{10}{3}\;d_{20} + \tfrac{7}{3} = \sqrt{70}\;d_{44}\,.$$

Referred to the other axes,

$$\left.\begin{aligned}
\hat{D}_{00} &= |\hat{\beta}^2|^2 + |\hat{\beta}^0|^2 + |\hat{\beta}^{-2}|^2 = D_{00} \\
\hat{D}_{20} &= |\hat{\beta}^0|^2 - |\hat{\beta}^2|^2 - |\hat{\beta}^{-2}|^2 \\
\hat{D}_{22} &= -\left[\hat{\beta}^0\hat{\beta}^{-2*} + \hat{\beta}^2\hat{\beta}^{0*}\right] \\
\hat{D}_{40} &= \tfrac{7}{12}\,\hat{D}_{00} + \tfrac{5}{12}\,\hat{D}_{20} \\
\hat{D}_{42} &= -\sqrt{\tfrac{5}{12}}\,\hat{D}_{22} \\
\hat{D}_{44} &= \sqrt{\tfrac{35}{18}}\,\hat{\beta}^2\hat{\beta}^{-2*}
\end{aligned}\right\}\quad \begin{aligned}&\hat{z} \text{ along } \underset{\sim}{k}_a \times \underset{\sim}{k}_b,\\ &\hat{x} \text{ in } \underset{\sim}{k}_a, \underset{\sim}{k}_b \text{ plane}\end{aligned}$$

If we have $\hat{x}$ parallel to z, $\hat{z}$ parallel to y,

$$\hat{\beta}^{2} = -\tfrac{1}{2}\beta^{2} + \sqrt{\tfrac{3}{8}}\,\beta^{0} + i\beta^{1}$$

$$\hat{\beta}^{-2} = -\tfrac{1}{2}\beta^{2} + \sqrt{\tfrac{3}{8}}\,\beta^{0} - i\beta^{1}$$

$$\hat{\beta}^{1} = \hat{\beta}^{-1} = 0$$

$$\hat{\beta}^{0} = -\sqrt{\tfrac{3}{2}}\,\beta^{2} - \tfrac{1}{2}\beta^{0}$$

Hence the tensor $\hat{d}_{KQ}$ referred to the "hatted" axes can be expressed in terms of the amplitude β^{m} referred to the other axes, and are computed in this way by SALLY,

$$\hat{d}_{20} = \frac{N}{\sigma}\left[|\beta^{2}|^{2} - 2|\beta^{1}|^{2} - \tfrac{1}{2}|\beta^{0}|^{2} + \sqrt{6}\,\mathrm{Re}\,(\beta^{0}\beta^{2*})\right],$$

$$-1 \leqslant \hat{d}_{20} \leqslant 1,$$

$$\hat{d}_{22} = \frac{N}{\sigma}\left\{\left[\sqrt{\tfrac{3}{8}}\,|\beta^{0}|^{2} - \sqrt{\tfrac{3}{2}}\,|\beta^{2}|^{2} + \mathrm{Re}(\beta^{0}\beta^{2*})\right] + i\,\mathrm{Re}\left[\sqrt{6}\,\beta^{1}\beta^{2*} + \beta^{0}\beta^{1*}\right]\right\}$$

$$= |\hat{d}_{22}|\, e^{-2i\phi_{0}},$$

$$\hat{d}_{44} = \frac{N}{\sigma}\sqrt{\tfrac{35}{18}}\left\{\left[\tfrac{1}{4}|\beta^{2}|^{2} - |\beta^{1}|^{2} + \tfrac{3}{8}|\beta^{0}|^{2} - \sqrt{\tfrac{3}{8}}\,\mathrm{Re}(\beta^{0}\beta^{2*})\right] + i\,\mathrm{Re}\left[\sqrt{\tfrac{3}{2}}\,\beta^{0}\beta^{1*} - \beta^{1}\beta^{2*}\right]\right\}$$

$$= |\hat{d}_{44}|\, e^{-4i\phi_{1}}$$

These are related to the polarization π by

$$4\pi^{2} = (1 - \hat{d}_{20})^{2} - \frac{288}{35}\,|d_{44}|^{2}$$

At $\theta = 0^{\circ}$ or 180° they always take their plane wave values,

$$\hat{d}_{20} = -\tfrac{1}{2}, \quad \hat{d}_{22} = \sqrt{\tfrac{3}{8}}, \quad \hat{d}_{44} = \sqrt{\tfrac{35}{128}}.$$

For the odd-rank tensors,[11] normalized as in eq. (39),

$\hat{d}_{30} = \frac{1}{2}\hat{d}_{10} = \sqrt{\frac{1}{10}}\,\pi \qquad \hat{d}_{31} = \hat{d}_{33} = \hat{d}_{11} = 0$, and SALLY computes

$$\hat{d}_{32} = \frac{N}{\sigma}\sqrt{\frac{1}{2}}\left(\hat{\beta}^{2}\hat{\beta}^{0*} - \hat{\beta}^{0}\hat{\beta}^{-2*}\right)$$

$$= \frac{N}{\sigma}\left\{\mathrm{Im}\left[\sqrt{3}\,\beta^{1}\beta^{2*} - \beta^{0}\beta^{1*}\right] + i\sqrt{2}\,\mathrm{Im}\left(\beta^{2}\beta^{0*}\right)\right\}$$

$$= \left|\hat{d}_{32}\right| e^{-2i\phi_2}, \qquad (= 0 \text{ at } \theta = 0^\circ \text{ or } 180^\circ).$$

The angular correlation for an $\ell = 2$ transition with zero-spin target and s = 0 and/or 1, followed by a quadrupole γ-ray,[21] involves the coefficients α, β, γ (and phase angles ϕ_0, ϕ_1) as described on page 20. These are given by

$$\alpha = \frac{1}{2}\left[\frac{\sqrt{35}\,(1-\hat{d}_{20})}{12\sqrt{2}\,|\hat{d}_{44}|} - 1\right],$$

$$\beta = \frac{1}{3}\left[\frac{\sqrt{35}\,(1+\sqrt{\frac{2}{3}}\,|\hat{d}_{22}|)}{4\sqrt{2}\,|\hat{d}_{44}|} + 1\right],$$

$$\gamma = -\frac{1}{6}\sqrt{\frac{35}{3}}\,\frac{|\hat{d}_{22}|}{|\hat{d}_{44}|}.$$

At $\theta = 0^\circ$ or 180°, these take their plane-wave values, $\alpha = 0$, $\beta = \frac{4}{3}$, and $\gamma = -\frac{2}{3}$.

When referred to the recoil $\underset{\sim}{q} = \underset{\sim}{k}_a - (m_A/m_B)\,\underset{\sim}{k}_b$ as z-axis (as under option KB4 = 1) the amplitudes β^m_R are obtained from the β^m referred to $\underset{\sim}{k}_a$ by a

rotation $(0, -\theta_R, 0)$, where θ_R is the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{q}$.

$$\beta_R^0 = \tfrac{1}{2}(3\cos^2\theta_R - 1)\,\beta^0 + \sqrt{\tfrac{3}{2}}\sin 2\theta_R\,\beta^1 + \sqrt{\tfrac{3}{2}}\sin^2\theta_R\,\beta^2 ,$$

$$\beta_R^1 = -\sqrt{\tfrac{3}{8}}\sin 2\theta_R\,\beta^0 + \cos 2\theta_R\,\beta^1 + \tfrac{1}{2}\sin 2\theta_R\,\beta^2 ,$$

$$\beta_R^2 = \sqrt{\tfrac{3}{8}}\sin^2\theta_R\,\beta^0 - \tfrac{1}{2}\sin 2\theta_R\,\beta^1 + (1 - \tfrac{1}{2}\sin^2\theta_R)\beta^2$$

For the $\hat{d}_{KQ}$ this corresponds to using $\underset{\sim}{q}$ instead of $\underset{\sim}{k}_a$ as $\hat{x}$-axis, (i.e. a rotation of $(0, 0, -\theta_R)$, so the $|d_{KQ}|$ (and hence α, β, γ) are unchanged, but the $\phi_{i,R}$ are now measured from $\underset{\sim}{q}$ (right-hand rotation about $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ being positive) $\phi_{i,R} = \phi_i + \theta_R$, and in the plane wave limit the $\phi_{i,R} = 0$.

<u>(iV.) Transfer of $\ell = 3$ and 4</u>

The only quantities computed are the cross-section

$$\sigma(\theta) = N\left[\,|\beta^0|^2 + 2\sum_{m=1}^{\ell}|\beta^m|^2\right],$$

and the polarization

$$\pi(\theta) = \frac{2N}{\ell\sigma}\sum_{m=0}^{\ell=1}\sqrt{(\ell+m+1)(\ell-m)}\;\mathrm{Im}(\beta^m\beta^{m+1*}),$$

which always vanishes, $\pi = 0$, at $\theta = 0^\circ$ or 180°.

Appendix III

Tables of Γ Coefficients

The Γ coefficient is defined by eq. (16),

$$\Gamma^{\ell m}_{LL'} = i^{L'-L-\ell}(2L+1)\sqrt{\frac{(L-m)!}{(L+m)!}}\,\langle L\ell 00|L'0\rangle\langle L\ell m,-m|L'0\rangle$$

We give here algebraic expressions for $\ell = 0$ through 4.

(i) $\ell = 0$

L' = L only : $\boxed{\Gamma^{00}_{LL} = (2L+1)}$

(ii) $\ell = 1$

L' = L $\pm$ 1 only :

$\Gamma^{11}_{L,L+1} = \Gamma^{11}_{L,L-1} = \sqrt{\frac{1}{2}}$		
$\Gamma^{10}_{L,L+1} = L+1$		
$\Gamma^{10}_{L,L-1} = -L$		

(iii) $\ell = 2$

$\Gamma^{2m}_{LL'} =$

L' \ m	0	1	2
L+2	$\frac{3(L+1)(L+2)}{2(2L+3)}$	$\sqrt{\frac{3}{2}}\,\frac{L+2}{2L+3}$	$\sqrt{\frac{3}{2}}\,\frac{1}{2(2L+3)}$
L	$-\frac{L(L+1)(2L+1)}{(2L-1)(2L+3)}$	$\sqrt{\frac{3}{2}}\,\frac{2L+1}{(2L-1)(2L+3)}$	$\sqrt{\frac{3}{2}}\,\frac{2L+1}{(2L-1)(2L+3)}$
L-2	$\frac{3L(L-1)}{2(2L-1)}$	$-\sqrt{\frac{3}{2}}\,\frac{L-1}{2L-1}$	$\sqrt{\frac{3}{2}}\,\frac{1}{2(2L-1)}$

(iv) $\ell = 3$

$$\Gamma^{3m}_{LL'} =$$

L' \ m	0	1	2	3
$L+3$	$\frac{5}{2}\,\frac{(L+1)(L+2)(L+3)}{(2L+3)(2L+5)}$	$\frac{5\sqrt{3}}{4}\,\frac{(L+2)(L+3)}{(2L+3)(2L+5)}$	$\sqrt{\frac{15}{8}}\,\frac{L+3}{(2L+3)(2L+5)}$	$\frac{\sqrt{5}}{4}\,\frac{1}{(2L+3)(2L+5)}$
$L+1$	$-\frac{3}{2}\,\frac{L(L+1)(L+2)}{(2L-1)(2L+5)}$	$-\frac{\sqrt{3}}{4}\,\frac{(L+2)(L-5)}{(2L-1)(2L+5)}$	$\sqrt{\frac{15}{8}}\,\frac{L+4}{(2L-1)(2L+5)}$	$\frac{3\sqrt{5}}{4}\,\frac{1}{(2L-1)(2L+5)}$
$L-1$	$\frac{3}{2}\,\frac{L(L-1)(L+1)}{(2L-3)(2L+3)}$	$-\frac{\sqrt{3}}{4}\,\frac{(L-1)(L+6)}{(2L-3)(2L+3)}$	$-\sqrt{\frac{15}{8}}\,\frac{L-3}{(2L-3)(2L+3)}$	$\frac{3\sqrt{5}}{4}\,\frac{1}{(2L-3)(2L+3)}$
$L-3$	$-\frac{5}{2}\,\frac{L(L-1)(L-2)}{(2L-1)(2L-3)}$	$\frac{5\sqrt{3}}{4}\,\frac{(L-1)(L-2)}{(2L-1)(2L-3)}$	$-\sqrt{\frac{15}{8}}\,\frac{L-2}{(2L-1)(2L-3)}$	$\frac{\sqrt{5}}{4}\,\frac{1}{(2L-1)(2L-3)}$

(v) $\lambda = 4$

$$\Gamma^{4m}_{LL'} =$$

$L' \backslash m$	0	1	2	3	4
$L+4$	$\frac{35}{8}\frac{(L+1)(L+2)(L+3)(L+4)}{(2L+3)(2L+5)(2L+7)}$	$\frac{7\sqrt{5}}{4}\frac{(L+2)(L+3)(L+4)}{(2L+3)(2L+5)(2L+7)}$	$\frac{7}{4}\sqrt{\frac{5}{2}}\frac{(L+3)(L+4)}{(2L+3)(2L+5)(2L+7)}$	$\frac{\sqrt{35}}{4}\frac{L+4}{(2L+3)(2L+5)(2L+7)}$	$\frac{1}{8}\sqrt{\frac{35}{2}}\frac{1}{(2L+3)(2L+5)(2L+7)}$
$L+2$	$-\frac{5}{2}\frac{L(L+1)(L+2)(L+3)}{(2L-1)(2L+3)(2L+7)}$	$-\frac{\sqrt{5}}{4}\frac{(L+2)(L+3)(2L-7)}{(2L-1)(2L+3)(2L+7)}$	$\sqrt{\frac{5}{2}}\frac{(L+3)(L+7)}{(2L-1)(2L+3)(2L-1)}$	$\frac{\sqrt{35}}{4}\frac{2L+9}{(2L-1)(2L+3)(2L+7)}$	$\frac{1}{2}\sqrt{\frac{35}{2}}\frac{1}{(2L-1)(2L+3)(2L+7)}$
L	$\frac{9}{4}\frac{(L-1)L(L+1)(L+2)(2L+1)}{(2L-3)(2L-1)(2L+3)(2L+5)}$	$-\frac{9\sqrt{5}}{4}\frac{(L-1)(L+2)(2L+1)}{(2L-3)(2L-1)(2L+3)(2L+5)}$	$\frac{3}{2}\sqrt{\frac{5}{2}}\frac{(2L+1)(9-L-L^2)}{(2L-3)(2L-1)(2L+3)(2L+5)}$	$\frac{9\sqrt{35}}{4}\frac{2L+1}{(2L-3)(2L-1)(2L+3)(2L+5)}$	$\frac{3}{4}\sqrt{\frac{35}{2}}\frac{2L+1}{(2L-3)(2L-1)(2L+3)(2L+5)}$
$L-2$	$-\frac{5}{2}\frac{(L-2)(L-1)L(L+1)}{(2L-5)(2L-1)(2L+3)}$	$\frac{\sqrt{5}}{4}\frac{(L-2)(L-1)(2L+9)}{(2L-5)(2L-1)(2L+3)}$	$\sqrt{\frac{5}{2}}\frac{(L-2)(L-6)}{(2L-5)(2L-1)(2L+3)}$	$-\frac{\sqrt{35}}{4}\frac{2L-7}{(2L-5)(2L-1)(2L+3)}$	$\frac{1}{2}\sqrt{\frac{35}{2}}\frac{1}{(2L-5)(2L-1)(2L+3)}$
$L-4$	$\frac{35}{8}\frac{(L-3)(L-2)(L-1)L}{(2L-5)(2L-3)(2L-1)}$	$-\frac{7\sqrt{5}}{4}\frac{(L-3)(L-2)(L-1)}{(2L-5)(2L-3)(2L-1)}$	$\frac{7}{4}\sqrt{\frac{5}{2}}\frac{(L-3)(L-2)}{(2L-5)(2L-3)(2L-1)}$	$-\frac{\sqrt{35}}{4}\frac{L-3}{(2L-5)(2L-3)(2L-1)}$	$\frac{1}{8}\sqrt{\frac{35}{2}}\frac{1}{(2L-5)(2L-3)(2L-1)}$

Appendix IV

Plane-wave limits

For convenience in comparing the results of distorted-wave calculations with those obtained with the earlier plane-wave theories[1][2] we give the plane-wave limits of the formulae discussed in Part I. The plane-wave limit means putting $\phi^{(\pm)}(\underline{k},\underline{r}) = e^{i\underline{k}\cdot\underline{r}}$ in the expressions (6), (9), etc., for the amplitudes, and corresponds to neglecting the scattered waves (or, equivalently, neglecting the distorting optical and Coulomb potentials in eq. (12)). Of course, these results can be obtained from SALLY by putting $U_i = \eta_i = 0$ in the input, but the partial wave expansions (11) used there are unnecessary. Inserting plane waves in eq. (9) leads to

$$\begin{aligned}\sqrt{2\ell+1}\; B_{\ell m} &= \frac{m_B}{m_A}\,\frac{1}{k_f^2}\,\sqrt{2\ell+1}\;\beta^{\ell m} \\ &= \frac{i^{-\ell}}{\sqrt{4\pi}}\int d\underline{r}_{fB}\int d\underline{r}_{aA}\; e^{-i\underline{k}_f\cdot\underline{r}_{fB}}\, f_{\ell m}(\underline{r}_{fB},\underline{r}_{aA})\, e^{i\underline{k}_a\cdot\underline{r}_{aA}}\end{aligned} \qquad \cdots (A.11)$$

The zero-range approximation (10) gives immediately

$$\begin{aligned}B_{\ell m} &= \frac{1}{\sqrt{(2\ell+1)4\pi}}\int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}}\; F_\ell(r)\,\left[i^\ell Y_\ell^{\,m}(\hat{r})\right]^* \\ &= \sqrt{\frac{4\pi}{2\ell+1}}\; Y_\ell^{\,m*}(\hat{q})\int j_\ell(qr)\,F_\ell(r)\,r^2\,dr \qquad \cdots (A.12) \\ &= \sqrt{\frac{4\pi}{2\ell+1}}\; Y_\ell^{\,m*}(\hat{q})\; I_\ell\,, \quad \text{say,}\end{aligned}$$

where $\underset{\sim}{q} = \underset{\sim}{k}_a - {}^{m_A}/_{m_B}\, \underset{\sim}{k}_b$. If $\underset{\sim}{q}$ is chosen as z axis, this becomes

$$B_{\ell m} = I_\ell \; \delta_{m,0} \; ; \; I_\ell = \int j_\ell(qr)\, F_\ell(r)\, r^2 dr \qquad \text{--(A.13)}$$

so that only transfers with zero component of ℓ along $\underset{\sim}{q}$ are allowed. A popular way to evaluate this integral has been to apply a lower cut-off at some interaction radius R (of the same order as the nuclear radius), usually on the grounds that strong absorption makes negligible the contributions from the nuclear interior, although this is inconsistent with the use of plane waves,

$$I_\ell \approx I_\ell(R) = \int_R^\infty j_\ell(qr)\, F_\ell(r)\, r^2 dr \qquad \text{---(A.14)}$$

Now, for example, in stripping reactions we may assume the wave-function of the captured particle is

$$F_\ell(r) = u_\ell(r) = a_\ell\, h_\ell^{(1)}(i\kappa r)\,, \qquad \text{for } r \geqslant R$$

with $a_\ell = u_\ell(R)/h_\ell^{(1)}(i\kappa r)$, and $\hbar^2\kappa^2/2\mu$ the binding energy, which should be a good approximation for the capture of <u>neutral</u> particles provided R is somewhat larger than the nuclear radius. (Options 0, 7, 12, 13 in SALLY give this form for $r \geqslant R_N$ for neutral particles). The integral (A14) is then readily evaluated in Wronskian form

$$I_\ell(R) = -\frac{a_\ell R^2}{q^2+\kappa^2}\, W\left[j_\ell(qR),\, h_\ell^{(1)}(i\kappa R)\right]$$

$$= -\frac{R^2 u_\ell(R)}{q^2+\kappa^2}\, \frac{dh_\ell^{(1)}(i\kappa R)/dR}{h_\ell^{(1)}(i\kappa R)} \left[j_\ell(qR) - \frac{h_\ell^{(1)}(i\kappa R)}{dh_\ell^{(1)}(i\kappa R)/dR}\, dj_\ell(qR)/dR \right] \text{--(A.15)}$$

$$= -\frac{i\kappa R^2 u_\ell(R)}{q^2+\kappa^2}\, \frac{h_{\ell-1}^{(1)}(i\kappa R)}{h_\ell^{(1)}(i\kappa R)} \left[j_\ell(qR) + \frac{iq}{\kappa}\, \frac{h_\ell^{(1)}(i\kappa R)}{h_{\ell-1}^{(1)}(i\kappa R)}\, j_{\ell-1}(qR) \right] \text{--(A.16)}$$

This is the expression given by Butler[1][2] for the amplitude. Values of $R^3 |I_\ell(R)/u_\ell(R)|^2$ have been tabulated by Lubitz.[29] A more severe ("surface") approximation is sometimes made by replacing the integral (A.14) by the value of its integrand at the interaction radius, so $I_\ell \propto j_\ell(qR)$. Expression (A.15) only leads to this form if the log derivative of $u_\ell(r)$ at r = R is large compared to q, for then the factor in brackets may be written

$$j_\ell(qR) + \delta R \frac{d}{dR}[j_\ell(qR)] \approx j_\ell(qR')$$

where

$$R' = R + \delta R , \qquad \delta R = - \frac{h_\ell^{(1)}(iKR)}{d h_\ell^{(1)}(iKR)/dR}$$

In general then this surface approximation will require a larger interaction radius than the more complete (A.15).

Finite-range effects on stripping reactions in the plane wave limit are easily seen. Using eqs. (55) and (59),

$$f_{\ell m}(\underline{r}_{tB}, \underline{r}_{aA}) = V(r_{tx})\, \phi_a(r_{tx})\, u_\ell(r_{xA})\, Y_\ell^{m*}(\hat{r}_{xA})\, J/D_0 .$$

Inserting this in eq. (A.11), we soon find

$$B_{\ell m} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^{m*}(\hat{q})\, I_\ell\, G(K)/D_0$$

where

$$G(K) = \int d\underline{r}\, V(r)\, \phi_a(r)\, e^{i\underline{K}\cdot\underline{r}}$$

and

$$\underline{K} = \frac{m_t}{m_a}\underline{k}_a - \underline{k}_t , \qquad \underline{q} = \underline{k}_a - \frac{m_A}{m_B}\underline{k}_t ,$$

and I_ℓ is the same as for the zero-range approximation. Thus the only effect of finite range is to modulate the amplitude by the fourier transform G(K)/Do, as discussed on page 27.

With $\underset{\sim}{k}_a$ as z-axis and $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ as y-axis, it follows from (A.12) that the plane-wave amplitudes (as calculated by SALLY) are

$$\beta^{\ell m} = \beta^{\ell}_{PW} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\; P^{m}_{\ell}(\theta_R), \qquad m \geq 0$$
$$= (-)^m \beta^{\ell -m}$$

where $\beta^{\ell}_{PW} = \frac{m_A}{m_B} k_b^2 I_\ell$ is real,

and θ_R is the angle between $\underset{\sim}{k}_a$ and $\underset{\sim}{q}$. All the m-dependence is thrown into the Y_ℓ^m. Referred to $\underset{\sim}{q}$ as z-axis (option KB4 = 1 in SALLY), only $\beta^{\ell 0} = \beta^{\ell}_{PW}$ is non-vanishing.

An immediate consequence of the reality of the $\beta^{\ell m}$ is that the polarization π vanishes (although the tensor polarization moments t_{2q} in general will not be zero). It also follows that when referred to $\underset{\sim}{q}$ as z-axis the d_{KQ} vanish unless Q = 0 and K is even, and

$$d_{K0} = 1, \qquad K \text{ even}, \quad \text{for all } \ell.$$

Referred to $\underset{\sim}{k}_a \times \underset{\sim}{k}_b$ as $\hat{z}$ axis, $\underset{\sim}{q}$ as $\hat{x}$-axis we now have $\hat{\beta}^{\ell m} = (-)^m \hat{\beta}^{\ell -m}$ with plane waves, so that $\hat{\rho}_{KQ}$ is real and vanishes if K is odd. We soon find for the $\hat{d}_{KQ}$ that

$$\left.\begin{array}{lll} \hat{d}_{20} = -\frac{1}{2}, & \hat{d}_{22} = \sqrt{\frac{3}{8}}, & \\ \hat{d}_{40} = \frac{3}{8}, & \hat{d}_{42} = -\sqrt{\frac{5}{32}}, & \hat{d}_{44} = \sqrt{\frac{35}{128}} \end{array}\right\} \text{ for all } \ell.$$

Since they are real, their phases $\phi_i = 0$. When measured from $\underset{\sim}{k}_a$ as $\hat{x}$-axis the $|\hat{d}_{KQ}|$ are unchanged but we then have $\phi_i = -\theta_R$.

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(15) In general $u_\ell(r)$ will also depend on j (and s if x is composite) but this dependence will generally be weak and is usually neglected.

(16) Exchange may arise in two ways, (i) from exchange forces, (ii) from (anti-) symmetry requirements. Here we specifically neglect exchange integrals for which the space-coordinates of the outgoing particle differ from those of the incident particle. An exception occurs when the two-body interaction has zero-range, for then this distinction between exchange and direct terms disappears. The transition amplitudes for (p,n) or (n,p) reactions may have two basic contributions (see Bloom et al., Phys. Rev. Lett. 3, 98 (1959)). One arises from charge-exchange without space exchange, thus having the same form as the inelastic scattering described in Section 5.3.3, while the other involves exchange of space-coordinates and has the form of a knock-on term as in Section 5.4.

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