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RECURSION RELATIONS AND SIMPLIFIED EXPRESSIONS FOR THE 6-j SYMBOLS

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Abstract

It is shown that the 144 symmetry properties of the 6-j symbols which were discovered by Regge can be displayed with a 4 x 4 array which is invariant to the ordering of three of the rows and all columns. The elements of the array are linear combinations of the quantum numbers. In addition, it is shown that a large set of three-term recursion relations exist. The recursion relations are used to deduce simplified expressions for the 6-j symbols in special cases.

The Wigner 6-j symbol,[1] which is related to the Racah W-coefficient[2] by

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{a+b+e+d}\, W(abed, cf), \tag{1}$$

can be represented by the notation

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{bmatrix} a+b-c & a+e-f & b+d-f & -c+d+e \\ a-e+f & a-b+c & c+d-e & -b+d+f \\ b-d+f & c-d+e & -a+b+c & -a+e+f \\ c+d+e+1 & b+d+f+1 & a+e+f+1 & a+b+c+1 \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}, \tag{2}$$

where every element B_{ij} is a positive integer or zero.

The advantage of the square array notation is that it efficiently displays the 144 symmetry properties of the 6-j symbol that were discovered by Regge[3] since it is invariant to the ordering of the four columns and to the ordering of the first three rows. This notation is similar to that used by Regge[4] to display the symmetry properties of the 3-j symbols. The invariance of the square array can be proved with the help of Racah's formula for the W-coefficient[2] modified according to Eq. 1 and expressed in terms of the elements B_{ij}:

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1. E. P. Wigner, "On the Matrices Which Reduce the Kronecker Products of Representations of Simple Reducible Groups," unpublished.
2. G. Racah, Phys. Rev. 62, 438 (1942); Phys. Rev. 63, 367 (1943).
3. T. Regge, Nuovo cimento 11, 116 (1959).
4. T. Regge, Nuovo cimento 10, 544 (1958).

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} = (-1)^{B_{41}+1} \left[\frac{B_{22}!\ B_{23}!\ B_{24}!\ B_{32}!\ B_{33}!\ B_{34}!\ B_{41}!}{B_{11}!\ B_{12}!\ B_{13}!\ B_{14}!\ B_{21}!\ B_{31}!\ B_{42}!\ B_{43}!\ B_{44}!} \right]^{\frac{1}{2}}$$

$$\times \quad \left\{ \begin{matrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{matrix} \right\}, \tag{3}$$

where

$$\left\{ \ \right\} = \sum_k \frac{(-1)^{k+B_{11}}\ B_{11}!\ B_{12}!\ B_{13}!\ B_{14}!\ B_{21}!\ B_{31}!\ (B_{11}+B_{41}-k)!}{k!\,(B_{11}-k)!\ (B_{12}-k)!\ (B_{13}-k)!\ (B_{14}-k)!\ (B_{21}-B_{11}+k)!\ (B_{31}-B_{11}+k)!\ B_{41}!} \tag{3a}$$

With the aid of the new notation it is possible to display (144)[3] three-term recursion relations by means of the single expression

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} = - \left[\frac{B_{12}\ B_{13}\ B_{14}\ B_{41}}{B_{11}\ B_{42}\ B_{43}\ B_{44}} \right]^{\frac{1}{2}} \begin{bmatrix} B_{11}-1 & B_{12}-1 & B_{13}-1 & B_{14}-1 \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41}-1 & B_{42}-1 & B_{43}-1 & B_{44}-1 \end{bmatrix}$$

$$- \left[\frac{(B_{11}+B_{41})^2\ B_{21}\ B_{31}}{B_{11}\ B_{42}\ B_{43}\ B_{44}} \right]^{\frac{1}{2}} \begin{bmatrix} B_{11}-1 & B_{12} & B_{13} & B_{14} \\ B_{21}-1 & B_{22} & B_{23} & B_{24} \\ B_{31}-1 & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42}-1 & B_{43}-1 & B_{44}-1 \end{bmatrix}, \tag{4}$$

which may be verified with Eq. 3.

In terms of Wigner's 6-j symbols, Eq. 4 becomes

$$(a+b+d+e+2)\left[(a-e+f+1)(b-d+f+1)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$+\left[(a+e-f)(b+d-f)(-c+d+e)(c+d+e+1)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f+\frac{1}{2} \end{Bmatrix}$$

$$+\left[(a+b-c+1)(b+d+f+2)(a+e+f+2)(a+b+c+2)\right]^{\frac{1}{2}}\begin{Bmatrix} a+\frac{1}{2} & b+\frac{1}{2} & c \\ d & e & f+\frac{1}{2} \end{Bmatrix} = 0. \qquad (4a)$$

By combining sets of two equations of the type given in Eq. 4 it is possible to obtain only four independent relations which are not related through the symmetry properties. Examples of the four relations are

$$\left[(-c+d+e)(a-e+f+1)(b-d+f+1)(c+d+e+1)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$+\left[\frac{(-c+d+e)(c+d+e+1)-(a+b-c+1)(a+b+c+2)}{(a+b+d+e+2)}\right]\left[(a+e-f)(b+d+f)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f+\frac{1}{2} \end{Bmatrix}$$

$$-\left[(a+b-c+1)(-b+d+f)(-a+e+f)(a+b+c+2)\right]^{\frac{1}{2}}\begin{Bmatrix} a+\frac{1}{2} & b+\frac{1}{2} & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f \end{Bmatrix} = 0, \qquad (5)$$

$$(-a+c-d+f)\left[(a-e+f+1)(c+d+e+1)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$+\left[(b-d+f+1)(a+e-f)(b+d-f)(-c+d+e)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f+\frac{1}{2} \end{Bmatrix}$$

$$-\left[(a+b-c+1)(-a+b+c)(c-d+e)(-a+e+f)\right]^{\frac{1}{2}}\begin{Bmatrix} a+\frac{1}{2} & b & c-\frac{1}{2} \\ d & e-\frac{1}{2} & f \end{Bmatrix} = 0, \qquad (5a)$$

$$\left[\frac{(a+b-c)(a+b+c+1) - (-c+d+e)(c+d+e+1)}{(a+b+d+e+1)}\right]\left[(b+d+f+1)(a+e+f+1)\right]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$+ \left[(a-e+f)(b-d+f)(a+b-c)(a+b+c+1)\right]^{\frac{1}{2}} \begin{Bmatrix} a-\frac{1}{2} & b-\frac{1}{2} & c \\ d & e & f-\frac{1}{2} \end{Bmatrix}$$

$$- \left[(-a+e+f)(-b+d+f)(-c+d+e)(c+d+e+1)\right]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f-\frac{1}{2} \end{Bmatrix} = 0, \qquad (5b)$$

and

$$(a-c+d-f)\left[(c+d+e+1)(b+d+f+1)(a+e+f+1)(a+b+c+1)\right]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$(b+c+e+f+1)\left[(a+b-c)(a+e-f)(b+d-f)(-c+d+e)\right]^{\frac{1}{2}} \begin{Bmatrix} a-\frac{1}{2} & b-\frac{1}{2} & c \\ d-\frac{1}{2} & e-\frac{1}{2} & f \end{Bmatrix}$$

$$- (a+b+d+e+1)\left[(b-d+f)(c-d+e)(-a+b+c)(-a+e+f)\right]^{\frac{1}{2}} \begin{Bmatrix} a & b-\frac{1}{2} & c-\frac{1}{2} \\ d & e-\frac{1}{2} & f-\frac{1}{2} \end{Bmatrix} = 0. \qquad (5c)$$

With the equations that have been presented it is possible to deduce some simplified expressions for the 6-j symbols in special cases in a manner similar to that described previously for the 3-j symbols.[5] It should be noted, for example, that the sum given in Eq. 3a is always an integer or zero since each term in the sum is an integer. Hence, when $B_{11} = 0$, the summation index can only assume the value zero and the sum is equal to unity. This fact leads to the simplified expression

$$\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} = (-1)^{B_{41}+1}\left[\frac{B_{22}!\,B_{23}!\,B_{24}!\,B_{32}!\,B_{33}!\,B_{34}!\,B_{41}!}{B_{12}!\,B_{13}!\,B_{14}!\,B_{21}!\,B_{31}!\,B_{42}!\,B_{43}!\,B_{44}!}\right]^{\frac{1}{2}}, \qquad (6)$$

5. C. D. Zerby and R. R. Coveyou, Some Simplified Expressions for the 3-j Symbols, ORNL-TM-514 (to be published).

which applies to the case in which any B_{ij} for $i \leq 3$ is equal to zero since that element can be transposed to the B_{11} position using the symmetry properties. Thus a simplified equation for the 6-j symbol is obtain whenever $j_k = j_\ell + j_m$ in any set of quantum numbers that form a triangle relation in the 6-j symbol.

If the minimum of the B_{ij} is greater than zero, then a form of the recursion relation given in Eq. 4 can be used to relate the symbol to symbols with the value of the element reduced by one. In this way the symbol can finally be reduced to the form given in Eq. 6. Alternatively, the element with the minimum value can be transposed to the B_{11} position and Eq. 4 can be used in the form it is presented. In the latter case there is an advantage to using the recursion relation for the symbol defined in Eq. 3a since it is always an integer. The substitution of Eq. 3 into Eq. 4 yields

$$\begin{Bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{Bmatrix} = \begin{bmatrix} B_{12} & B_{13} & B_{14} \end{bmatrix} \begin{Bmatrix} B_{11}-1 & B_{12}-1 & B_{13}-1 & B_{14}-1 \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41}-1 & B_{42}-1 & B_{43}-1 & B_{44}-1 \end{Bmatrix}$$

$$- \begin{bmatrix} B_{21} & B_{31} & (B_{11}+B_{41}) \end{bmatrix} \begin{Bmatrix} B_{11}-1 & B_{12} & B_{13} & B_{14} \\ B_{21}-1 & B_{22} & B_{23} & B_{24} \\ B_{31}-1 & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42}-1 & B_{43}-1 & B_{44}-1 \end{Bmatrix} . \quad (7)$$

From Eqs. 3 and 7 and the fact that the symbol with braces is equal to unity when the element in the B_{11} position is equal to zero, one can quickly deduce that

$$\begin{bmatrix} 1 & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} = (-1)^{B_{41}+1} \left[\frac{B_{22}!\ B_{23}!\ B_{24}!\ B_{32}!\ B_{33}!\ B_{34}!\ B_{41}!}{B_{12}!\ B_{13}!\ B_{14}!\ B_{21}!\ B_{31}!\ B_{42}!\ B_{43}!\ B_{44}!} \right]^{\frac{1}{2}}$$

$$\times \quad \left\{ \left[B_{12}\ B_{13}\ B_{14} \right] - \left[B_{21}\ B_{31}\ (B_{41}+1) \right] \right\} \qquad (8)$$

$$\begin{bmatrix} 2 & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} = (-1)^{B_{41}+1} \left[\frac{B_{22}!\ B_{23}!\ B_{24}!\ B_{32}!\ B_{33}!\ B_{34}!\ B_{41}!}{2\ B_{12}!\ B_{13}!\ B_{14}!\ B_{21}!\ B_{31}!\ B_{42}!\ B_{43}!\ B_{44}!} \right]^{\frac{1}{2}}$$

$$\times \left\{ \left[B_{12}\ B_{13}\ B_{14} \right] \left\{ (B_{12}-1)(B_{13}-1)(B_{14}-1) - B_{21}\ B_{31}\ B_{41} \right\} \right.$$

$$\left. - \left[B_{21}\ B_{31}\ (B_{41}+2) \right] \left\{ B_{12}\ B_{13}\ B_{14} - \left[(B_{21}-1)(B_{31}-1)(B_{41}+1) \right] \right\} \right\}, \qquad (9)$$

etc. Equation 8 provides expressions for the symbols when $j_k = j_\ell + j_m - 1$ in any of the four triangle relations and Eq. 9 provides similar expressions for $j_k = j_\ell + j_m - 2$.

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