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THE REACTOR TRANSIENTS ACCOMPANYING PULSED SOURCE OPERATION AND STEP CHANGES OF REACTIVITY

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ABSTRACT

The method of over-all eigenfunctions developed by Foderaro and Garabedian is used to derive a method for calculating the ascending and descending reactor transients due to the pulsed operation of a neutron source and the transient resulting from a step change of reactivity. The method is applicable to multiregion reactors of arbitrary composition including reflected cores. The two-group approximation is employed including epithermal fission, and the solution is obtained for one effective group of delayed neutron precursors.

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THE REACTOR TRANSIENTS ACCOMPANYING PULSED SOURCE OPERATION AND STEP CHANGES OF REACTIVITY

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Introduction

The use of a pulsed neutron source for measuring the multiplication factor in subcritical systems is presently being planned for significant portions of the experimental program accompanying the initial startup of the Experimental Gas-Cooled Reactor (EGCR). In addition, certain rod calibration data will be obtained by measuring the period of the transient produced by control rod withdrawal. The formalism presented here was developed to provide a means for calculating with some precision the nature of the transient accompanying these operations in order that important questions concerning feasibility could be answered.

In the pulsed neutron experiments these questions relate primarily to intensity (detector count rate) and the effect of a very thick reflector, or more specifically

1) intensity as related to pulse length for the prompt neutron decay constants to be encountered,

2) intensity as related to the delay time necessary to obtain the asymptotic transient,

3) effect of the thick reflector, if any, on the nature of the asymptotic transient.

In the case of control rod induced transients the principal question relates to the magnitude of the flux increase which occurs before the asymptotic transient is obtained.

Scope of the Development

The treatment of the reactor transient problems which is given here is based on the method of over-all eigenfunctions developed by Foderaro

and Garabedian.[1] This method is unique in that (1) the criticality determinants which arise have an order which is independent of the number of groups and regions, and (2) no explicit formulation of the boundary condition at interfaces is necessary.

The discussion given below extends the treatment given in Ref. 1 in several respects, including (1) introduction of an extraneous source and the transients associated with pulsing the source, and (2) inclusion of epithermal fission and epithermal absorption.

As in Ref. 1, the treatment is given for two groups of neutrons and one group of delayed neutron precursors. The notation and general line of development parallels that in Ref. 1 whenever possible.

Derivation of the General Time Dependence

For two neutron groups we write the diffusion equations as

$$\nabla\cdot(D_s\nabla\phi_s) - \Sigma_s\phi_s + p\Sigma_f\phi_f - \frac{1}{v_s}\frac{\partial\phi_s}{\partial t} = 0 \tag{1}$$

$$\nabla\cdot(D_f\nabla\phi_f) - \Sigma_f\phi_f + (1-\beta)\left[\frac{k_s}{p}\Sigma_s\phi_s + k_f\Sigma_f\phi_f\right] + \lambda C + S - \frac{1}{v_f}\frac{\partial\phi_f}{\partial t} = 0 \tag{2}$$

and the equation for the delayed neutron precursors is

$$\frac{\partial C}{\partial t} + \lambda C - \beta\left[\frac{k_s}{p}\Sigma_s\phi_s + k_f\Sigma_f\phi_f\right] = 0 \quad . \tag{3}$$

The subscripts s and f refer to the slow (thermal) and fast (epithermal) energy groups, β is the effective delayed neutron fraction, C and λ are the concentration and decay constant for the delayed neutron precursor, p is the probability in an infinite core that a fission source neutron will reach the slow group, k_s and k_f are the source production

[1] A. Foderaro and H. L. Garabedian, "Two-Group Reactor Kinetics," Nucl. Sci. Eng., 14(1): 22-29 (September 1962).

rates in the two groups per initial source neutron in an infinite reactor, and v_s and v_f are the average neutron velocities in the two groups. The term S represents the spatial dependence of the extraneous source.

We first eliminate the delayed neutron precursors from Eqs. (2) and (3). To do this we differentiate Eq. (2) with respect to time and subtract Eq. (3). We assume that $\partial S/\partial t = 0$. The result is then combined with Eq. (2) to yield

$$\begin{aligned}\frac{\partial}{\partial t}\left[\nabla\cdot(D_f\nabla\phi_f)\right] - \Sigma_f\frac{\partial\phi_f}{\partial t} + (1-\beta)\left[\frac{k_s}{p}\Sigma_s\frac{\partial\phi_s}{\partial t} + k_f\Sigma_f\frac{\partial\phi_f}{\partial t}\right] \\ + \lambda\left\{\nabla\cdot(D_f\nabla\phi_f) - \Sigma_f\phi_f + \left[\frac{k_s}{p}\Sigma_s\phi_s + k_f\Sigma_f\phi_f\right]\right. \\ \left. + S - \frac{1}{v_f}\frac{\partial\phi_f}{\partial t}\right\} - \frac{1}{v_f}\frac{\partial^2\phi_f}{\partial t^2} = 0 \quad .\end{aligned} \tag{4}$$

We next expand the fluxes and the source in terms of the complete set of functions formed by the solutions of Helmholtz's equation

$$\nabla^2 W_\nu(\underline{r}) + B_\nu^2 W_\nu(\underline{r}) = 0$$

which satisfy the condition $W_\nu(\underline{r}) = 0$ at the outer boundary of the reactor.

$$\begin{aligned}\phi_f(\underline{r},t) &= \sum_\nu A_\nu^f W_\nu(\underline{r})\, T_\nu^f(t) \\ \phi_s(\underline{r},t) &= \sum_\nu A_\nu^s W_\nu(\underline{r})\, T_\nu^s(t) \\ S(\underline{r}) &= \sum_\nu S_\nu W_\nu(\underline{r})\end{aligned} \tag{5}$$

Note that complete generality is maintained at this point and no assumption regarding separability of the time dependence is made.

We substitute Eqs. (5) into Eqs. (1) and (4), multiply by $W_\mu(\underline{r})$, and integrate over the reactor volume. From Eq. (1) we obtain

$$\sum_\nu \int d\underline{r}\, W_\mu \nabla\cdot(D_s \nabla A^s_\nu W_\nu T^s_\nu) - \sum_\nu \int d\underline{r}\, \Sigma_s W_\mu A^s_\nu W_\nu T^s_\nu$$
$$+ \sum_\nu \int d\underline{r}\, \nu\Sigma_f A^f_\nu W_\mu W_\nu T^f_\nu - \frac{1}{v_s}\frac{\partial}{\partial t}\sum_\nu \int d\underline{r}\, A^s_\nu W_\mu W_\nu T^s_\nu = 0 \,. \tag{6}$$

We make use of the relationship that

$$W_\mu \nabla\cdot(D_s \nabla W_\nu) = (\nabla D_s)\cdot W_\mu \nabla W_\nu + D_s W_\mu \nabla^2 W_\nu$$
$$= (\nabla D_s)\cdot W_\mu \nabla W_\nu - D_s W_\mu B^2_\nu W_\nu \tag{7}$$

and define

$$R^s_{\mu\nu} = -\int d\underline{r}(\nabla D_s)\cdot W_\mu \nabla W_\nu + B^2_\nu \int d\underline{r}\, D_s W_\mu W_\nu$$
$$+ \int d\underline{r}\, \Sigma_s W_\mu W_\nu \tag{8}$$

and

$$Q^f_{\mu\nu} = \int d\underline{r}\, \nu\Sigma_f W_\mu W_\nu \,^{*} \tag{9}$$

We may then write Eq. (6) as

$$-\sum_\nu R^s_{\mu\nu} A^s_\nu T^s_\nu + \sum_\nu Q^f_{\mu\nu} A^f_\nu T^f_\nu - \frac{1}{v_s}\sum_\nu A^s_\nu \delta_{\mu\nu} \dot{T}^s_\nu = 0 \tag{10}$$

where the dot refers to a time derivative and we have made use of the orthogonality of the functions $W_\nu(\underline{r})$.

In performing these same operations on Eq. (4) we define the following quantities.

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*See the Appendix for the analytical procedures used to evaluate these and similar integrals.

$$R^f_{\mu\nu} = -\int d\underline{r}\, \nabla D_f \cdot w_\mu \nabla w_\nu + B_\nu^2 \int d\underline{r}\, D_f\, w_\mu\, w_\nu + \int d\underline{r}\, \Sigma_f\, w_\mu\, w_\nu \tag{11}$$

$$V^f_{\mu\nu} = \int d\underline{r}\, k_f\, \Sigma_f\, w_\mu\, w_\nu \tag{12}$$

$$V^s_{\mu\nu} = \int d\underline{r}\, \frac{k_s}{p}\, \Sigma_s\, w_\mu\, w_\nu \tag{13}$$

and Eq. (4) then becomes

$$\begin{aligned} &-\sum_\nu R^f_{\mu\nu} A^f_\nu \dot{T}^f_\nu + (1-\beta)\left[\sum_\nu V^s_{\mu\nu} A^s_\nu \dot{T}^s_\nu + \sum_\nu V^f_{\mu\nu} A^f_\nu \dot{T}^f_\nu\right] \\ &+\lambda\Big\{-\sum_\nu R^f_{\mu\nu} A^f_\nu T^f_\nu + \left[\sum_\nu V^s_{\mu\nu} A^s_\nu T^s_\nu + \sum_\nu V^f_{\mu\nu} A^f_\nu T^f_\nu\right] \\ &+\sum_\nu \delta_{\mu\nu} S_\nu - \frac{1}{v_f}\sum_\nu \delta_{\mu\nu} A^f_\nu \dot{T}^f_\nu\Big\} - \frac{1}{v_f}\sum_\nu \delta_{\mu\nu} A^f_\nu \ddot{T}^f_\nu = 0. \end{aligned} \tag{14}$$

We wish now to find the solutions to Eqs. (10) and (14). For the transient following insertion of the source we seek solutions of the form:

$$T(t) \sim 1 - e^{\omega t}. \tag{15}$$

With this form, the transient decays to a steady state and is therefore consistent with the known conditions for the solution. This is equivalent to assuming a form $T(t) = A + Be^{\omega t}$, and finding A and B from the initial and asymptotic conditions on the flux. If the solution in Eq. (15) is substituted into Eq. (14), then we obtain upon rearrangment of terms

$$\left\{ \sum_{\nu} \left[(\omega+\lambda)\left(R^{f}_{\mu\nu} + \frac{\omega}{v_f}\delta_{\mu\nu} \right) - (\omega - \beta\omega + \lambda) V^{f}_{\mu\nu} \right] A^{f}_{\nu} \right.$$

$$\left. - \sum_{\nu} (\omega - \beta\omega + \lambda) V^{s}_{\mu\nu} A^{s}_{\nu} \right\} e^{\omega t} = -\lambda \left\{ \sum_{\nu} \delta_{\mu\nu} S_{\nu} \right. \tag{16}$$

$$\left. + \sum_{\nu} V^{s}_{\mu\nu} A^{s}_{\nu} + \sum_{\nu} \left[V^{f}_{\mu\nu} - R^{f}_{\mu\nu} \right] A^{f}_{\nu} \right\} .$$

In Eq. (16) the left-hand side is a function of time and the right-hand side is independent of time. We must, therefore, have

LHS = Const. = RHS

and to find the value of the constant we need only to examine the origin of the right-hand side. It is not difficult to see that these terms arise from the constant term in Eq. (15) and represent the asymptotic solution for the fluxes. In fact, if the time independent solution to Eqs. (1) and (2) were sought we would obtain, simply, RHS = 0 for Eq. (16). The left-hand side of Eq. (16) may thus be set equal to zero.

We turn next to Eq. (10) for the thermal neutron amplitudes. Using the assumed solution from Eq. (15) we obtain

$$\left\{ \sum_{\nu} \left[R^{s}_{\mu\nu} + \frac{\omega}{v_s}\delta_{\mu\nu} \right] A^{s}_{\nu} - \sum_{\nu} Q^{f}_{\mu\nu} A^{f}_{\nu} \right\} e^{\omega t}$$
$$= \sum_{\nu} R^{s}_{\mu\nu} A^{s}_{\nu} - \sum_{\nu} Q^{f}_{\mu\nu} A^{f}_{\nu} \tag{17}$$

By the same reasoning as above we may determine of Eq. (17) that the left-hand side equals zero. We obtain, then, from Eqs. (16) and (17) two sets of linear homogeneous equations for the amplitudes A^{s}_{ν} and A^{f}_{ν}. It will be profitable at this point to adopt a matrix notation to simplify the algebra on these equations and we note, for illustration, that if A and B are matrices, then the matrix elements of their product are

$$\{AB\}_i = \sum_j A_{ij} B_j \ .$$

From the left-hand sides of Eqs. (16) and (17) we obtain

$$\left\{ (\omega+\lambda)\left[R_f + \frac{\omega}{v_f}\delta\right] - (\omega-\beta\omega+\lambda) V_f \right\} A_f - (\omega-\beta\omega+\lambda) V_s A_s = 0 \tag{18}$$

and

$$\left[R_s + \frac{\omega}{v_s}\delta \right] A_s - Q_f A_f = 0 \tag{19}$$

where R_f, R_s, V_f, V_s, Q_f, and δ are square matrices and A_s and A_f are column matrices.

If we define

$$\eta = (\omega-\beta\omega+\lambda) \tag{20}$$

$$M_f = R_f + \frac{\omega}{v_f}\delta \qquad M_s = R_s + \frac{\omega}{v_s}\delta \tag{21}$$

then Eqs. (18) and (19) become

$$\left[(\omega+\lambda) M_f + \eta V_f\right] A_f - \eta V_s A_s = 0 \qquad Q_f A_f - M_s A_s = 0 \tag{22}$$

and by eliminating A_f we obtain

$$\left\{ \left[(\omega+\lambda) M_f + \eta V_f\right] Q_f^{-1} M_s - \eta V_s \right\} A_s = 0 \ . \tag{23}$$

Up to this point we have assumed that Eq. (5) represented unrestricted expansions for the fluxes and source. We now assume that

these sums extend only up to ν = m and neglect all higher terms. In this case, Eq. (23) represents m homogeneous equations for the matrix elements A_ν^s, and in order to possess a solution other than $A_\nu^s = 0$ we must have zero for the determinant of the matrix in brackets. With a certain amount of algebra we can express this condition as

$$\left| A\omega^3 + B\omega^2 + C\omega + D \right| = 0 \tag{24}$$

where the matrices A, B, C, and D are given by

$$A = \frac{Q_f^{-1}}{v_f v_s} \tag{25}$$

$$B = \frac{Q_f^{-1} R_s}{v_f} + \frac{R_f Q_f^{-1}}{v_s} + \frac{\lambda Q_f^{-1}}{v_f v_s} + \frac{(1-\beta) V_f Q_f^{-1}}{v_s} \tag{26}$$

$$\begin{aligned} C = {} & R_f Q_f^{-1} R_s + \frac{\lambda}{v_s} R_f Q_f^{-1} + \frac{\lambda}{v_f} Q_f^{-1} R_s \\ & + (1-\beta) V_f Q_f^{-1} R_s + \frac{\lambda}{v_s} V_f Q_f^{-1} - (1-\beta) V_s \end{aligned} \tag{27}$$

$$D = \lambda \left[R_f Q_f^{-1} R_s + V_f Q_f^{-1} R_s - V_s \right] . \tag{28}$$

Since the matrices A, B, C, and D are of order m, it is clear that the determinant in Eq. (24) is of order 3m in ω and there will, therefore, be 3m roots. These roots are the values of ω for which the determinant equals zero and are the 3m values of ω for which solutions of the form in Eq. (15) exist for Eqs. (10) and (14). The most general solutions of Eqs. (10) and (14) may thus be written to this order as

$$A_\nu^s T_\nu^s(t) = \sum_{\alpha=1}^{3m} A_{\nu\alpha}^s \left(1 - e^{\omega_\alpha t}\right) \tag{29}$$

and

$$A_\nu^f T_\nu^f(t) = \sum_{\alpha=1}^{3m} A_{\nu\alpha}^f \left(1 - e^{\omega_\alpha t}\right) \tag{30}$$

and the solutions of Eqs. (1) and (2) are given entirely if we have the $6m^2$ coefficients $A_{\nu\alpha}^s$ and $A_{\nu\alpha}^f$ and the $3m$ roots ω_α.

In order to find these coefficients we write Eq. (19) as

$$A_f = \left\{ Q_f^{-1} R_s + \frac{\omega}{v_s} Q_f^{-1} \right\} A_s \tag{31}$$

and Eq. (23) as

$$\left\{ A\omega^3 + B\omega^2 + C\omega + D \right\} A_s = 0 \ . \tag{32}$$

For each of the $3m$ roots ω_α, Eq. (32) represents m homogeneous equations for $A_{\nu\alpha}^s$ and may be used to find $m - 1$ values for the $A_{\nu\alpha}^s$ in terms of one of them, say $A_{1\alpha}^s$. We may express this result as

$$A_{\nu\alpha}^s = \rho_{\nu\alpha}^s A_{1\alpha}^s \tag{33}$$

where the $3m^2$ constants $\rho_{\nu\alpha}^s$ may be found from Eq. (32) (note that $\rho_{1\alpha}^s = 1$).

Now Eq. (31) is the transformation equation which transforms the $A_{\nu\alpha}^s$ into the $A_{\nu\alpha}^f$. That is, if

$$Y(\omega_\alpha) \equiv Q_f^{-1} R_s + \frac{\omega_\alpha}{v_s} Q_f^{-1} \tag{34}$$

and

$$\rho_{\nu\alpha}^f \equiv \sum_{\mu=1}^{m} \left\{ Y(\omega_\alpha) \right\}_{\nu\mu} \rho_{\mu\alpha}^s \tag{35}$$

then

$$A^{f}_{\nu\alpha} = \rho^{f}_{\nu\alpha} A^{s}_{1\alpha} \ . \tag{36}$$

We have now expressed all of the $6m^2$ coefficients $A^{f}_{\nu\alpha}$ and $A^{s}_{\nu\alpha}$ in terms of the 3m constants $A^{s}_{1\alpha}$, and the problem is now reduced to finding these constants. These remaining constants may be obtained from the initial and asymptotic conditions for each case of interest as shown below.

Subcritical Reactor With Pulsed Source

At this point we consider the ascending and subsequently the descending transients due to a pulsed source when the reactivity is independent of time. We have shown that the RHS of Eqs. (16) and (17) may be set equal to zero, and if we express these for the present problem we obtain

$$\left\{R_f - V_f\right\} A_f + V_s A_s = -S \tag{37}$$

$$Q_f A_f - R_s A_s = 0 \tag{38}$$

from which we may write

$$\left\{\left[V_f - R_f\right] Q_f^{-1} R_s - V_s\right\} A_s = S \ . \tag{39}$$

Equation (39) is inhomogeneous and may be used to find the m components of A_s, and Eq. (38) may be used to find the m components of A_f. That is,

$$Q_f^{-1} R_s A_s = A_f \ . \tag{40}$$

These components will simply be the coefficients A^{s}_{ν} and A^{f}_{ν} which are the expansion coefficients for the flux in the asymptotic steady-state case with the source present.

Since we now have A^{s}_{ν} and A^{f}_{ν} we evaluate Eqs. (29) and (30) for the asymptotic ($t = \infty$) case and may then write

$$\sum_{\alpha} A^{s}_{\nu\alpha} = A^{s}_{\nu} \tag{41}$$

$$\sum_{\alpha} A^{f}_{\nu\alpha} = A^{f}_{\nu} \, . \tag{42}$$

Consider next the initial condition when the source is turned on. We have then $\phi_s = \phi_f = C = 0$ and Eq. (2) reduces to

$$\frac{1}{v_f} \frac{\partial \phi_f}{\partial t} \Big|_{t=0+} = S \, , \tag{43}$$

where the expression for ϕ_f is given by

$$\phi_f(\underline{r},t) = \sum_{\nu} W_{\nu}(\underline{r}) \sum_{\alpha} A^{f}_{\nu\alpha} \left(1 - e^{\omega_\alpha t}\right) . \tag{44}$$

If we substitute Eq. (44) into Eq. (43), multiply by W_μ and integrate over the reactor volume we obtain

$$\sum_{\alpha} A^{f}_{\mu\alpha} \omega_\alpha = - v_f S_\mu \, . \tag{45}$$

With the 3m Eqs. (41), (42), and (45) we have sufficient relations to find the 3m coefficients $A^{s}_{1\alpha}$. These relations are

$$\sum_{\alpha} \rho^{s}_{\nu\alpha} A^{s}_{1\alpha} = A^{s}_{\nu} \tag{46}$$

$$\sum_{\alpha} \rho^{f}_{\nu\alpha} A^{s}_{1\alpha} = A^{f}_{\nu} \tag{47}$$

$$\sum_{\alpha} \rho^{f}_{\nu\alpha} \omega_\alpha A^{s}_{1\alpha} = - v_f S_\nu \, . \tag{48}$$

There are 3m equations above, and these may be used directly to solve for the constants $A^{s}_{1\alpha}$. All $6m^2$ of the constants $A^{s}_{\nu\alpha}$ and $A^{f}_{\nu\alpha}$ are now

available and, together with the 3m values ω_α, may be used to construct the ascending transient due to the pulsed source. The descending transient is discussed in the next section which covers the general case, in which the initial conditions are specified in a source-free system.

Source-Free Reactor With Specified Initial Conditions – General Treatment

Even if no source is present the decay constants ω_α will still be determined by Eq. (24), since no specific effect of the source is present in that relationship. The time dependence in Eq. (15) has been used to obtain this relationship, but any function of the form $A + B\, e^{\omega t}$ would have given the same determinantal equation. The only differences would have been in the right-hand sides of Eqs. (16) and (17). Indeed, if we had chosen the form $e^{\omega t}$ these terms on the right-hand side would have been absent. In addition to the equation determining the ω_α, we will also have the same relations given in Eqs. (33) and (36) which generate all the expansion coefficients from the 3m constants $B^s_{1\alpha}$. (As will be seen directly, it is convenient to reserve the symbol A for the coefficients during the ascending transient with the source present.) The problem then reduces again to writing 3m conditions involving the $B^s_{1\alpha}$.

The general form of the fluxes for this case is

$$\phi_s(\underline{r},t) = \sum_\nu W_\nu(\underline{r}) \sum_\alpha B^s_{\nu\alpha}\, e^{\omega_\alpha t} \tag{49}$$

$$\phi_f(\underline{r},t) = \sum_\nu W_\nu(\underline{r}) \sum_\alpha B^f_{\nu\alpha}\, e^{\omega_\alpha t} \,. \tag{50}$$

If we know both ϕ_s and ϕ_f at t = 0, then we may expand them in terms of the $W_\nu(\underline{r})$ and obtain

$$\phi_s(\underline{r},0) = \sum_\nu B^s_\nu W_\nu(\underline{r}) \tag{51}$$

and

$$\phi_f(\underline{r},0) = \sum_\nu B_\nu^f W_\nu(\underline{r}) \tag{52}$$

where the B_ν^f and B_ν^s may be found, numerically if necessary. With these quantities known, we may write

$$B_\nu^s = \sum_\alpha B_{\nu\alpha}^s = \sum_\alpha \rho_{\nu\alpha}^s B_{1\alpha}^s \tag{53}$$

$$B_\nu^f = \sum_\alpha B_{\nu\alpha}^f = \sum_\alpha \rho_{\nu\alpha}^f B_{1\alpha}^s \quad . \tag{54}$$

These equations are 2m relations for the $B_{1\alpha}^s$, and the remaining equations necessary may be obtained from the distribution of delayed neutron precursors at t = 0. We expand this distribution in terms of the functions $W_\nu(\underline{r})$ and obtain

$$C(\underline{r},0) = \sum_\nu C_\nu W_\nu(\underline{r}) \tag{55}$$

where the C_ν may be found, numerically if necessary.

We now look at Eq. (2) for the source-free case at t = 0. We substitute Eqs. (52) and (55) into this equation, multiply each term by $W_\mu(\underline{r})$, and integrate over the entire reactor volume. By now this operation is familiar and only the result will be given.

$$\begin{aligned} &\sum_\nu \left\{ R_{\mu\nu}^f - (1-\beta) V_{\mu\nu}^f \right\} \sum_\alpha B_{\nu\alpha}^f - (1-\beta) \sum_\nu V_{\mu\nu}^s \sum_\alpha B_{\nu\alpha}^s \\ &+ \frac{1}{v_f} \sum_\nu \delta_{\mu\nu} \sum_\alpha B_{\nu\alpha}^f \omega_\alpha - \lambda \sum_\nu \delta_{\mu\nu} C_\nu = 0 \end{aligned} \tag{56}$$

If we now make use of Eqs. (33) and (36) written for the $B_{\nu\alpha}^f$ and $B_{\nu\alpha}^s$ and define the following matrix:

$$\{X\}_{\mu\alpha} = \sum_{\nu} \Big\{ \big[R^{f}_{\mu\nu} - (1-\beta) V^{f}_{\mu\nu} \big] \rho^{f}_{\nu\alpha} - (1-\beta) V^{s}_{\mu\nu} \rho^{s}_{\nu\alpha} + \frac{1}{v_f} \delta_{\mu\nu} \rho^{f}_{\nu\alpha} \omega_{\alpha} \Big\} \tag{57}$$

then in terms of X we may write Eq. (56) as

$$\sum_{\alpha} X_{\mu\alpha} B^{s}_{1\alpha} = \lambda c_{\mu} . \tag{58}$$

Equations (53), (54), and (58) are now 3m equations for the 3m constants $B^{s}_{1\alpha}$ which complete the general solution for this case.

Specific problems may easily be expressed in the form given here and two such cases will be considered next. These are the descending transient which occurs after the source pulse is turned off, and the ascending or descending transient which occurs when reactivity is respectively added or subtracted stepwise while the reactor is in a just critical steady-state condition.

Descending Transient After Removal of Source

For this case we assume that the source pulse has remained on for a time t_o and that the problem of the ascending transient has been solved to obtain the ω_α, $A^{s}_{\nu\alpha}$, and $A^{f}_{\nu\alpha}$. We then know $\phi_s(\underline{r},t_o)$ and $\phi_f(\underline{r},t_o)$ in terms of their expansions, Eqs. (29) and (30), and may substitute these into Eq. (3) which describes the time dependence of the delayed neutron precursor. If we do this Eq. (3) may then be solved, since its form is simply that of the linear first order differential equation whose solution is well known. That is, if

$$\frac{d}{dt} y(t) + P(t) y(t) = Q(t)$$

then

$$y e^{\int P dt} = \int e^{\int P dt} Q \, dt + \text{Const.}$$

If we use $C(\underline{r},0) = 0$ as the initial condition, then the solution to Eq. (3) becomes

$$\lambda C(\underline{r},t_o) = \beta \frac{k_s}{\beta} \Sigma_s \sum_\nu W_\nu \sum_\alpha A^s_{\nu\alpha} \left\{ 1 - \frac{\lambda}{\omega_\alpha+\lambda} e^{\omega_\alpha t_o} - \left(1 - \frac{\lambda}{\omega_\alpha+\lambda}\right) e^{-\lambda t_o} \right\} + \beta k_f \Sigma_f \sum_\nu W_\nu \sum_\alpha A^f_{\nu\alpha} \left\{ 1 - \frac{\lambda}{\omega_\alpha+\lambda} e^{\omega_\alpha t_o} - \left(1 - \frac{\lambda}{\omega_\alpha+\lambda}\right) e^{-\lambda t_o} \right\} . \tag{59}$$

We now multiply each term of Eq. (59) by $W_\mu(\underline{r})$ and integrate over the reactor volume. If we define

$$f_\alpha(t_o) = 1 - \frac{\lambda}{\omega_\alpha+\lambda} e^{\omega_\alpha t_o} - \left(1 - \frac{\lambda}{\omega_\alpha+\lambda}\right) e^{-\lambda t_o} \tag{60}$$

then this operation gives us

$$\lambda C_\mu = \beta \left\{ \sum_\nu V^s_{\mu\nu} \sum_\alpha A^s_{\nu\alpha} f_\alpha(t_o) + \sum_\nu V^f_{\mu\nu} \sum_\alpha A^f_{\nu\alpha} f_\alpha(t_o) \right\} . \tag{61}$$

These constants are the expansion coefficients for the distribution of delayed neutron precursors at t_o, and, as this is the initial time for the descending transient, are the constants which appear in Eq. (58). The constants B^f_ν and B^s_ν for this case are also required and may be obtained from the expansions of the fluxes. That is,

$$\sum_{\nu} W_{\nu}(\underline{r})\, B_{\nu}^{f} = \sum_{\nu} W_{\nu}(\underline{r}) \sum_{\alpha} A_{\nu\alpha}^{f} \left(1 - e^{\omega_{\alpha} t_{0}}\right) \tag{62}$$

$$\sum_{\nu} W_{\nu}(\underline{r})\, B_{\nu}^{s} = \sum_{\nu} W_{\nu}(\underline{r}) \sum_{\alpha} A_{\nu\alpha}^{s} \left(1 - e^{\omega_{\alpha} t_{0}}\right) \tag{63}$$

from which, if we multiply by $W_{\mu}(\underline{r})$ and integrate over the reactor volume, we obtain

$$B_{\nu}^{f} = \sum_{\alpha} A_{\nu\alpha}^{f} \left(1 - e^{\omega_{\alpha} t_{0}}\right) \tag{64}$$

$$B_{\nu}^{s} = \sum_{\alpha} A_{\nu\alpha}^{s} \left(1 - e^{\omega_{\alpha} t_{0}}\right) . \tag{65}$$

Since we now have B_{ν}^{f}, B_{ν}^{s}, and C_{ν}, we may find all of the $B_{\mu\alpha}^{f}$ and $B_{\mu\alpha}^{s}$, and will then have the complete solution for the descending transient.

Transient Following a Step Addition or Subtraction of Reactivity

Consider next the case of step changes in reactivity. In order to obtain the 3m constants B_{ν}^{s}, B_{ν}^{f}, and C_{ν}, we again make use of initial conditions for the fluxes and delayed neutron precursors; the condition here being one of time independence.

The equations determining the time independent solution for the flux may be written down from Eqs. (10) and (14). If the time derivatives are set to zero then we obtain for the source-free case

$$-\sum_{\nu} R_{\mu\nu}^{f} B_{\nu}^{f} + \sum_{\nu} V_{\mu\nu}^{s} B_{\nu}^{s} + \sum_{\nu} V_{\mu\nu}^{f} B_{\nu}^{f} = 0 \tag{66}$$

and

$$-\sum_{\nu} R^{s}_{\mu\nu} B^{s}_{\nu} + \sum_{\nu} Q^{f}_{\mu\nu} B^{f}_{\nu} = 0 \tag{67}$$

or, in matrix notation

$$\left\{ V_f - R_f \right\} B_f + V_s B_s = 0 \tag{68}$$

$$Q_f B_f - R_s B_s = 0 \tag{69}$$

from which we obtain

$$\left\{ \left[V_f - R_f \right] Q_f^{-1} R_s + V_s \right\} B_s = 0 \, . \tag{70}$$

Equation (70) will have non-trivial solutions for B_s only if the determinant of the matrix in brackets equals zero. That is, we require that

$$\left| \left[V_f - R_f \right] Q_f^{-1} R_s + V_s \right| = 0 . \tag{71}$$

This determinantal equation is, in fact, the criticality condition for the reactor and represents a restriction on the values of k_s and k_f which appear as factors in the matrix elements of V_f and V_s.

Once the criticality condition is fulfilled, then Eq. (70) may be used to find m − 1 of the B^s_ν in terms of one of them, say B^s_1. We then write

$$B_f = Q_f^{-1} R_s B_s \tag{72}$$

from Eq. (69) and can find m values for the B^f_ν from those for B^s_ν. The remaining constant, B^s_1, is determined by the power level of the reactor.

The constants C_ν are determined by the condition that the concentration of delayed neutron precursors is independent of time. We impose this condition on Eq. (3), substitute the expansions for the fluxes,

multiply by $W_\mu(\underline{r})$, and integrate over the reactor volume. The result is

$$\lambda C_\mu = \beta \left\{ \sum_\nu V^s_{\mu\nu} B^s_\nu + \sum_\nu V^f_{\mu\nu} B^f_\nu \right\}. \tag{73}$$

Now since the B^s_ν and B^f_ν have been obtained already, we may calculate all of the λC_μ, and all of the coefficients $B^s_{\nu\alpha}$ and $B^f_{\nu\alpha}$ will then be accessible.

Systematic Procedure for Solving Specific Problems

The stepwise procedure which should be employed in solving specific problems may be illustrated as follows for the ascending transient due to the source.

1. Calculate the matrix elements of the numerous matrices which appear in the treatment, e. g., $R^f_{\mu\nu}$, $R^s_{\mu\nu}$, $V^s_{\mu\nu}$, etc.

2. Form the determinantal equation, Eq. (24), and find its 3m roots ω_α.

3. Write $A^s_{\nu\alpha} = \rho^s_{\nu\alpha} A^s_{1\alpha}$, $\rho^s_{1\alpha} = 1$ and use m − 1 of the relations in Eq. (32) to obtain the remaining $\rho^s_{\nu\alpha}$ for each root ω_α.

4. Use Eq. (35) to find $3m^2$ values of $\rho^f_{\nu\alpha}$ from the values of $\rho^s_{\nu\alpha}$.

5. Find the 3m constants $A^s_{1\alpha}$ from the selected set of 3m equations derived from the appropriate initial or asymptotic conditions.

6. Calculate the $6m^2$ coefficients $A^s_{\nu\alpha}$ and $A^f_{\nu\alpha}$ using Eqs. (33) and (36).

7. Evaluate $\phi_s(\underline{r},t)$ and $\phi_f(\underline{r},t)$ at the desired positions and times.

Appendix 1

Evaluation of Core Integrals

The formulae for calculating the matrix elements involving integrals over the reactor volume have been written down by Foderaro and Garabedian[2] and are included here for the sake of completeness.

Cylindrical Geometry

$$W_\nu(\underline{r}) = \frac{1}{R\sqrt{\pi}} \frac{J_0\left(\frac{z_\nu r}{R}\right)}{J_1(z_\nu)} \tag{A1}$$

where Z_ν is the ν-th zero of $J_0(x)$ and R is the outer boundary of the reactor. For $\mu \neq \nu$ we have

$$\int_{r_1}^{r_2} W_\mu(r) W_\nu(r) 2\pi r\, dr = \frac{2}{R J_1(z_\mu) J_1(z_\nu)(z_\mu^2 - z_\nu^2)} \Bigg\{ r_2 \cdot \left[z_\mu J_1\left(\frac{z_\mu r_2}{R}\right) J_0\left(\frac{z_\nu r_2}{R}\right) - z_\nu J_0\left(\frac{z_\mu r_2}{R}\right) J_1\left(\frac{z_\nu r_2}{R}\right) \right] - r_1 \left[z_\mu J_1\left(\frac{z_\mu r_1}{R}\right) J_0\left(\frac{z_\nu r_1}{R}\right) - z_\nu J_0\left(\frac{z_\mu r_1}{R}\right) J_1\left(\frac{z_\nu r_1}{R}\right) \right] \Bigg\} \tag{A2}$$

where r_1 and r_2 bound regions of uniform composition.

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[2] A. Foderaro and H. L. Garabedian, "A New Method for the Solution of Group Diffusion Equations," Nucl. Sci. Eng., 8(1): 44-52 (July 1960).

For $\mu = \nu$ we have

$$\int_{r_1}^{r_2} W_\nu(r)\, W_\nu(r)\, 2\pi r\, dr = \frac{1}{R^2 J_1^2(z_\nu)} \left\{ r_2^2 \left[J_0^2\left(\frac{z_\nu r_2}{R}\right) + J_1^2\left(\frac{z_\nu r_2}{R}\right)\right] - r_1^2 \left[J_0^2\left(\frac{z_\nu r_1}{R}\right) + J_1^2\left(\frac{z_\nu r_1}{R}\right)\right]\right\} \tag{A3}$$

and finally

$$\int_0^R \nabla D \cdot W_\mu \nabla W_\nu\, 2\pi r\, dr = \frac{2 z_\nu}{R^3 J_1^2(z_\nu)} \sum_n \left(D_{n+1} - D_n\right) r_n J_0\left(\frac{z_\mu r_n}{R}\right) J_1\left(\frac{z_\nu r_n}{R}\right) \tag{A4}$$

where the summation over n is a summation over each internal boundary of the reactor at which D has a discontinuity.

Slab Geometry

$$W_\nu(x) = \sqrt{\frac{2}{X_0}}\, \cos\left[(2\nu - 1)\frac{\pi x}{2 X_0}\right] \tag{A5}$$

where $\nu = 1, 2, 3, \ldots$ and X_0 is the outer boundary. For $\mu \neq \nu$ we have

$$\int_{x_1}^{x_2} W_\mu W_\nu\, dx = \frac{\sin\left[(\mu-\nu)\frac{\pi x_2}{X_0}\right]}{(\mu-\nu)\pi} + \frac{\sin\left[(\mu+\nu-1)\frac{\pi x_2}{X_0}\right]}{(\mu+\nu-1)\pi} - \frac{\sin\left[(\mu-\nu)\frac{\pi x_1}{X_0}\right]}{(\mu-\nu)\pi} - \frac{\sin\left[(\mu+\nu-1)\frac{\pi x_1}{X_0}\right]}{(\mu+\nu-1)\pi} \tag{A6}$$

where x_1 and x_2 bound regions of uniform composition.

For $\mu = \nu$ we have

$$\int_{x_1}^{x_2} W_\nu W_\nu \, dx = \frac{x_2 - x_1}{x_0} + \frac{\sin\left[(2\nu-1)\frac{\pi x_2}{x_0}\right]}{(2\nu-1)\pi} - \frac{\sin\left[(2\nu-1)\frac{\pi x_1}{x_0}\right]}{(2\nu-1)\pi} \tag{A7}$$

and finally

$$\int_0^{x_0} \nabla D \cdot W_\mu \nabla W_\nu \, dx = \frac{(2\nu-1)\pi}{x_0^2} \sum_n (D_{n+1} - D_n) \sin\left[(2\nu-1)\frac{\pi x_n}{x_0}\right] \cos\left[(2\nu-1)\frac{\pi x_n}{x_0}\right] \tag{A8}$$

where the summation over n is a summation over each internal boundary at which D has a discontinuity.

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