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STUDIES OF THE ION-CYCLOTRON RESONANCE INSTABILITY

G. K. Soper

(Thesis)

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THERMONUCLEAR DIVISION

STUDIES OF THE ION-CYCLOTRON RESONANCE INSTABILITY

G. K. Soper

Submitted as a dissertation to the Graduate Council of The University of Tennessee in partial fulfillment of the requirements for the degree of Doctor of Philosophy

OCTOBER 1964

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PREFACE

This report is based upon a thesis submitted to the University of Tennessee in partial fulfillment of the requirements for the doctoral degree. The report describes research carried out in the Thermonuclear Division at the Oak Ridge National Laboratory.

The research was supported by the Oak Ridge Graduate Fellowship Program of the Oak Ridge Institute of Nuclear Studies and was directed by a committee appointed by Dean H. A. Smith of the University of Tennessee Graduate School which was composed of Dr. E. G. Harris and Dr. H. C. Schweinler of the Department of Physics at the University of Tennessee, Dr. G. W. Hoffman of the Electrical Engineering Department at the University of Tennessee, Dr. R. L. Becker of the Physics Division, and Dr. G. E. Guest of the Thermonuclear Division of the Oak Ridge National Laboratory.

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CHAPTER I

INTRODUCTION

I. BACKGROUND OF THE PROBLEM AND A GENERAL REVIEW OF THE LITERATURE

The study of high temperature gaseous matter in a highly ionized state has become known as the field of plasma physics. Such a state exists in nature, for example, in the gases of interstellar space, the ionized gases on the surface of the sun and the ionosphere. Man-made plasmas include such well-known devices as thyratron tubes, mercury-vapor rectifiers and plasma-filled waveguides. A plasma surrounds a space vehicle during re-entry (causing the so-called communications blackout), occurs in explosions and lightning bolts, and has been proposed for rocket propulsion motors in interplanetary travel. Recently, a great deal of attention has been focused on plasma physics through the attempts to harness the energy available from a thermonuclear reaction.[1]

A number of experimental devices, such as the DCX's[2] and OGRA[3] injection machines, for example, have been built in an effort to produce

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[1] D. J. Rose and M. Clark, Jr., Plasmas and Controlled Fusion (M.I.T. Press and John Wiley and Sons, New York, 1961), pp. 1-12; Proc. 2nd U. N. Conf. on Peaceful Uses of Atomic Energy, Vols. 31 and 32, United Nations, Geneva (1958); A. Bishop, Project Sherwood (Addison-Wesley, Reading, Mass., 1958), pp. 1-14; and Proc. Conf. on Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Salzburg, 1961), to name but a few.

[2] C. F. Barnett, P. R. Bell, J. S. Luce, E. D. Shipley, and A. Simon, Proc. 2nd U. N. Conf. on Peaceful Uses of Atomic Energy, Vol. 31, United Nations, Geneva (1958), p. 298.

[3] L. A. Artsimovich, Proc. 2nd U. N. Conf. on

this reaction, and from these experiments come many theoretical problems. Of these problems, the stable confinement of this plasma for a sufficiently long time that fusion can occur has turned out to be one of the most perplexing.[4] Since fusion temperatures are in excess of 10,000,000°K, "containment and insulation" of the plasma using ordinary laboratory vessels is impossible. However, at these energies a "magnetic bottle" can be used where the interaction of the fully ionized gas with an electromagnetic field is so adjusted that the electromagnetic force on each particle is always directed away from the walls. It has become evident, however, that instabilities, that is, small disturbances which grow rapidly in either space or time in the plasma, contribute to the mechanisms which eventually disrupt the confined system and terminate the reaction. Impurity radiation is an example of a major non-instability loss mechanism in controlled thermonuclear processes.[5]

The hydromagnetic[6] instabilities (interchange, kink, sausage, for example) which involve the plasma as a whole; plasma oscillations

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Peaceful Uses of Atomic Energy, Vol. 31, United Nations, Geneva (1958), p. 6.

[4] A. S. Bishop, Physics Today 17, No. 3, 19 (1964). This article also contains an excellent description of the principles of the different thermonuclear machines, their advantages and limitations.

[5] R. F. Post, J. Nucl. Energy, Pt. C 3, 273 (1961); S. Cuperman, F. Englemann, and J. Oxenius, Phys. Fluids 6, 108 (1963); and Phys. Fluids 7, 428 (1964).

[6] M. N. Rosenbluth and C. L. Longmire, Annals of Physics 1, 120 (1957); W. B. Thompson, lecture, Plasma Physics Summer School of the Danish Atomic Energy Commission, Risö, Denmark (August 1960), p. 273. (Unpublished); and Masatomo Sato, "Stability of Plasma (I)," IPPJ-7 (March 1963). This review article contains many references to magnetohydrodynamic instability research.

or "micro-instabilities"[7] which involve local turbulence or oscillations of rapid growth rates (compared with hydromagnetic instabilities); and another oscillatory instability characteristic of a weakly inhomogeneous fully ionized plasma confined by a strong magnetic field called the "universal" or drift-wave instability[8] are examples of disrupting wave-type disturbances which have been predicted theoretically and whose behavior has been studied in great detail. Of course, other instabilities exist, and furthermore, it seems evident that more instabilities remain to be discovered.

To study the hydromagnetic instabilities one introduces the assumptions that the velocity distribution of the particles is nearly isotropic and that the system is adiabatic.[9] Using these simplifying postulates, the hydrodynamic equations of mass, momentum, and energy conservation can be used. Also, one must include in the force term the magnetic stress, $\frac{1}{c}\vec{V} \times \vec{B}$, then employ the Maxwell equations to determine the fields. The hydromagnetic instabilities predicted on the basis of this theory are of low frequency (much less than the cyclotron frequency of the positive ions[10]) and a relatively slow growth rate and lead to

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[7] N. Rostocker, "Plasma Stability," GA-2617 (December 14, 1961); and M. N. Rosenbluth, lecture, Plasma Physics Summer School of the Danish Atomic Energy Commission, Risö, Denmark (August 1960), p. 201. (Unpublished).

[8] H. Lashinsky, Phys. Rev. Letters 12, 121 (1964); and N. Krall and M. N. Rosenbluth, Phys. Fluids 6, 254 (1963), for example.

[9] S. Glasstone and R. H. Lovberg, Controlled Thermonuclear Reactions (D. Van Nostrand Co., Inc., Princeton, New Jersey, 1960), p. 487, for example.

[10] L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, New York, 1962), p. 51.

the expulsion of a confined plasma by gross drift motions. Stabilization of a number of these disturbances, however, is theoretically possible.[11]

A large class of instabilities, however, exist which are usually highly localized and represent charged particles interacting through collective fields, not necessarily at low frequencies. These have been dubbed the "micro-instabilities."[12] Theoretically predicted micro-instabilities include: 1) the two-stream instability which is described as simply being produced by two interpenetrating streams of particles,[13] 2) the electrostatic cyclotron resonance instability[14] which describes resonant coupling between plasma oscillations and particle rotation in a magnetic field, and 3) the pinch-type instabilities[15] associated with

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[11] The reader will find in the literature many papers devoted to this subject. As an example, M. N. Rosenbluth, Proc. 3rd International Conference on Ionized Gases, Venice (1957), p. 903. In addition to the bibliography of Sato (footnote 6), W. B. Thompson, An Introduction to Plasma Physics (Pergamon Press, Oxford, 1962), pp. 97-138, is an excellent reference to magnetohydrodynamic stability.

[12] As with the hydromagnetic instabilities, most books on the subject of plasma physics devote a great deal of their space discussing these phenomena and usually the original works are well referenced. An excellent review article describing many Russian contributions to these instabilities is A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Soviet Physics Uspekhi 4, 332 (1961).

[13] There has been a great deal of research on this instability mode; for example, in the theoretical research on beam-plasma interactions. The following reference is an excellent review article on the subject and is well referenced. Ya. B. Fainberg, J. Nucl. Energy, Pt. C 3, 203 (1962). Specifically see O. Buneman, Phys. Rev. Letters 1, 8 (1958); and Phys. Rev. 115, 503 (1959).

[14] The pioneering work was done by E. G. Harris, "Unstable Plasma Oscillations in a Magnetic Field," ORNL-2728 (June 5, 1959); and Phys. Rev. Letters 2, 34 (1959).

[15] H. P. Furth, Phys. Fluids 6, 48 (1963). This reference gives a very thorough cataloging of the pinch-type instabilities.

self-focusing of charged particle streams. One could continue this list to include the "universal" instabilities, mentioned briefly above, and the mirror hydromagnetic and slow-growth-rate Alfvén-wave instabilities described by Post,[16] to name but a few.

These plasma oscillations, often characterized by large growth rates and short wavelengths, by their nature do not lead to gross drift motions but "act to destroy effective confinement through causing enhanced diffusion across the magnetic field or through causing excessively rapid energy exchange among components of the plasma."[17]

These instabilities manifest themselves in the details of the velocity space distribution function. The "origin" of the micro-instability is a departure from isotropic conditions (anisotropy), a property which afflicts magnetically confined systems.[18] It seems plausible then that these instabilities should be difficult to stabilize since a rearrangement of the distribution function would be called for.

It is the purpose of this dissertation to study in some detail one particular micro-instability, the ion-cyclotron resonance instability,[19] to survey what has been done to date, to discuss its role in the thermonuclear machines, to offer new calculations regarding finite-

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[16] R. F. Post, "Some Observations on Plasma Instabilities in the Mirror Machine," Plasma Hydromagnetics, D. Bershader, editor (Sixth Lockheed Symposium on Magnetohydrodynamics, Stanford University Press, Stanford, California, 1962), p. 16.

[17] Ibid., p. 17.

[18] Ibid., p. 16.

[19] E. G. Harris, J. Nucl. Energy, Pt. C 2, 138 (1961).

geometry aspects of the theory and stable-unstable operating regimes, and finally to suggest further calculations to be attempted.

II. BEGINNING CONSIDERATIONS

Plasma oscillations as a new and separate phenomenon were first studied by Langmuir and Tonks[20] in 1929. They observed that collective effects disrupted an electron beam in a plasma in a distance much shorter than could be explained by collisions. They interpreted this disrupting mechanism as plasma-electron oscillations at the plasma frequency. In 1938 A. A. Vlasov[21] treated a system of charged particles by means of a modified Boltzmann equation. Vlasov showed that for a plasma the electron oscillation frequency could be much larger than the frequency of collision between charged particles and neutral gas, and consequently, one could neglect all "collision" interactions. This approach is valid for low density and high temperatures when the thermal energy is much greater than the mean potential energy between neighboring particles. The self-consistent electromagnetic fields were calculated using the distribution functions as sources in the Maxwell equations, and a dispersion relation calculated for both longitudinal and transverse waves.

However, L. Landau[22] in 1946, using Vlasov's basic equation, found that Vlasov's results were incorrect and furthermore found that in the rigorous sense no proper dispersion relation exists for waves

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[20] I. Langmuir and L. Tonks, Phys. Rev. 33, 195, 990 (1929).

[21] A. A. Vlasov, J. Exp. Theor. Phys. (U.S.S.R.) 8, 291 (1938). See also USAEC Translation 2729.

[22] L. D. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

in a plasma. He showed that for a given wave vector $\vec{k}$, arbitrary values of ω, the frequency of a disturbance in the plasma, are possible. Landau's correct initial-value treatment led to qualitatively new results, namely a damping mechanism which could absorb energy from oscillations in a collisionless plasma. Van Kampen[23] solved the problem in a strikingly different manner. Landau's treatment did not answer the question of divergence of a singular integral in the stationary wave method. Van Kampen showed that it is possible to construct a complete set of stationary plane wave solutions of arbitrary wave number and arbitrary frequency. Although each "Van Kampen mode" alone does not exhibit damping, an arbitrary initial disturbance will produce a spectrum of modes and the gross perturbation will decay with Landau damping behavior.[24]

The problems considered above are based on an infinite, homogeneous model in which the time-dependent parts of the dependent variables are assumed small compared with the time-independent parts. A perturbation technique is employed in which products of perturbation terms are neglected, the equations linearized, and the Fourier components of the dependent variables are derived. In the work that follows, a similar model will be used except in the consideration of finite-geometry cases where slight modifications will be in order. Also, the

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[23] N. G. Van Kampen, Physica 21, 949 (1955).

[24] T. H. Stix, The Theory of Plasma Waves (McGraw-Hill Book Company, New York, 1962), p. 148. For rigorous mathematical extensions and discussions of the Landau and Van Kampen formulations, see K. M. Case, Annals of Physics 7, 349 (1959); G. Bacus, J. Math. Phys. 1, 178 (1960); and J. N. Hayes, Phys. Fluids 4, 1387 (1961).

perturbed part of the magnetic field is neglected so that the electric field is a curl-free vector limiting then the study to longitudinal oscillations. Drummond, Rosenbluth, and Johnson[25] claim to have shown that this is a valid assumption if the ratio of the plasma pressure to magnetic field energy density is small.

III. MATERIAL IN SUCCEEDING CHAPTERS

In Chapter II a review of the literature on the ion-cyclotron resonance instability research will be given. Included in this resumé will be a discussion of finite-geometry effects and the attempt by this author to describe a set of vacuum-plasma boundary conditions.

Chapter III begins with a discussion of the Vlasov equations. Then the dispersion relation for four separate models is derived. These models are: (A) The infinite plasma model which as considered by Harris[26] is extended to include the dispersion relation using a two-temperature distribution function. The groundwork will be laid here for the evaluation of this dispersion relation with the aid of a high-speed digital computer. (B) The Harris model limited in z (where z is the direction of the external magnetic field) in which the infinite model is replaced by a plasma slab bounded at $z = \pm a$. The particles are considered to have no velocity spread in the z-direction but to have a MB* spread in the direction perpendicular to z. The singular character of

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[25] W. E. Drummond, M. N. Rosenbluth, and M. L. Johnson, Bull. Am. Phys. Soc., Series II, 6, No. 2, 185 (1961).

[26] Harris, loc. cit.

*Maxwell-Boltzmann.

the differential equations involved is discussed and compared with a similar problem in the literature. (C) The Burt-Harris[27] model where a cylindrical-shaped plasma is considered in which the ions move in Larmor orbits in the ion-cyclotron frequency. This model is used as an approximation to a weak mirror thermonuclear device. The dispersion relation is studied in the limit of infinite length, finite radius and visa-versa. Finally, (D) the Shima[28] model in which a finite problem of one or two ion orbits across is approximated by an exactly solvable problem, again an infinite plasma but one in which the Maxwellian ion distribution rotates as a whole at the ion-cyclotron frequency. It is shown how these results are incorporated into the computer-code of model (A).

In Chapter IV are presented the details of the computer calculations of models (A) and (D). For various temperature parameters, regions of stability and instability are determined and a perturbation technique used to determine growth rates. The computer work was done on the Control Data Corporation 1604-A Machine.

In Chapter V the conclusions are drawn and suggestions for further research presented, particularly in connection with further exhaustive computer calculations emphasizing DCX parameters.

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[27] P. B. Burt and E. G. Harris, Phys. Fluids 4, 1412 (1961); and P. B. Burt, disseration, University of Tennessee, 1961.

[28] Y. Shima, Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1962, ORNL-3392, p. 61.

CHAPTER II

PREVIOUS WORK

I. LITERATURE SURVEY

The ion-cyclotron resonance instability theory had its inception in attempts to explain oscillatory phenomena evident in many plasma-type experiments. One such early experiment was that by Alfvén[1] and his co-workers concerning electron phenomena in trochotron tubes.* At low electron density the motion follows classical behavior. At higher densities, however, the electron's energy distribution is so altered that an appreciable part of the current is collected by electrodes negative with respect to the filament. Simultaneously, strong beam noise was detected. Malmfors[2] suggested that the noise was due to collective behavior of the electrons in the beam and predicted unstable oscillations with the frequency within small intervals of multiples of the gyromagnetic (cyclotron) frequency.**

Gross[3] in a later paper reported an error in Malmfors' work. Taking into account random thermal motion, he shows for a peaked velocity

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[1] Alfvén et al., Theory and Applications of Trochotrons, Kungl. Tekniska Högskolans Handlingar, NR 22 (1948).

*So named because electrons emitted from a hot cathode in a tube with crossed electric and magnetic fields travel to the anode in trochoidal paths.

[2] K. G. Malmfors, Arkiv för Fysik 1, 569 (1950).

**For very high electron beam density Malmfors' theory predicted unstable oscillations at all frequencies.

[3] E. P. Gross, Phys. Rev. 82, 232 (1951).

distribution the existence of gaps in the spectrum at frequencies which are approximate multiples of the cyclotron frequency. Gross demonstrated that when the effect of the thermal motion is small, no instability could exist. Furthermore, he postulated that the same type of calculations should hold whatever the effect of the thermal motion. While Gross's approach was a kinetic theory treatment, Sen,[4] looking for a connection between solar "enhanced radiation" and plasma oscillations, followed Landau's prescription and found that the system was unstable in frequency bands around multiples of the gyrofrequency. Upon further investigation of Gross's dispersion relation, Sen showed numerically that unstable oscillations did exist (using parameters suited to solar data, i.e., far from Gross's original approximation).

These then were some of the first few attempts to give a possible explanation to some of the experimental phenomena occurring in electrical discharges in the laboratory (and in nature, too) on the basis of unstable plasma oscillations. Next came the thermonuclear machines and with them a seemingly unending source of confinement disrupting mechanisms to be studied theoretically. Quite early it became all too obvious that violent localized disturbances on the microscopic scale were contributing to the unstable configurations. Colgate and Furth,[5] writing on the stabilized pinch and controlled fusion power, revealed that although the pinch could be "stabilized," small-scale turbulence exists within the gross configuration. There was evidence of electrons interacting with electrostatic

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[4] H. K. Sen, Phys. Rev. 88, 816 (1952).

[5] S. A. Colgate and H. P. Furth, Science 128, 337 (1958).

plasma waves enhancing the resistivity effect at high power levels discouraging the use of a stabilized pinch as a thermonuclear plasma container. Cooperative phenomena in the B-1 Stellarator discharges were reported by Bernstein et al.[6] These data indicated the existence of unexplained collective processes. X-rays were produced by "runaway" electrons striking the walls; furthermore, the discharge current decayed in abrupt steps correlated in time with the bursts of X-rays. There was even evidence that the runaway electrons themselves affected the development of the discharge.

In an effort to give a possible theoretical explanation to phenomena of this nature Harris,[7] expanding on the work of Malmfors, Gross, and Sen, developed a theory in which the anisotropies of the distribution function were considered to give rise to these instabilities. The remainder of the discussion in this dissertation will be based on Harris's basic works.

Harris in his first paper* derives the general dispersion relation for longitudinal waves based on the Vlasov set of equations following fairly closely the work of Bernstein.[8] Many simplifying assumptions are

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[6] W. Bernstein et al., Phys. Fluids 1, 420 (1958).

[7] E. G. Harris, "Unstable Plasma Oscillations in a Magnetic Field," ORNL-2728 (June 5, 1959); Phys. Rev. Letters 2, 34 (1959); and J. Nucl. Energy, Pt. C 2, 138 (1961).

*Harris's papers will be referred to as I, II, and III as they appear in footnote 7. Paper II is a condensed version of I; however, the theory's applicability to DCX and OGRA is discussed. Paper III includes all the cases of I, and also transverse oscillations are considered (i.e., the full set of Maxwell's equations are used in the formulation of the problem).

[8] I. B. Bernstein, Phys. Rev. 109, 10 (1958).

made and will be discussed in some detail in Chapter III of this dissertation. The distribution function used by Malmfors, Gross, and Sen, in which all electrons and ions move with the same speed perpendicular to the field, is employed by Harris:

$$f_{oj} = \frac{1}{2\pi}\,\delta(v_z)\,\frac{\delta(v_\perp - v_j)}{v_\perp} \tag{1}$$

where $\delta(x)$ is the Dirac δ-function,[9] and the symbols $\perp$ and z represent directions with respect to the external magnetic field, and j represents either species of the plasma. If the ion motions are neglected (i.e., let $m_i = \infty$) and set $k_z = 0$, where k_z is the component of the wave vector along the magnetic field, one finds

$$1 = -\frac{\omega_{pe}^2}{\omega_{ce}^2}\sum_{n=-\infty}^{\infty}\frac{n\omega_{ce}}{(\omega + n\omega_{ce})}\,\frac{1}{\lambda}\,\frac{d}{d\lambda}\,J_n^2(\lambda) \tag{2}$$

where

$$\omega_{pe}^2 = \frac{4\pi N^o e^2}{m_e} \tag{3}$$

is called the plasma frequency,

$$\omega_{ce}^2 = \left(\frac{eB_o}{m_e c}\right)^2 \tag{4}$$

is called the cyclotron frequency, and

$$\lambda = \frac{k_\perp v_e}{\omega_{ce}}\ .$$

It was in using this dispersion relation that Gross showed no unstable modes of oscillation for $\lambda \ll 1$, that Sen showed that instabilities could

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[9]See, for example, P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill Book Company, New York, 1953), p. 122.

exist for $\lambda \ll 1$ and finally that Harris, using a Nyquist diagram technique,[10] was able to show that instabilities set in when $\lambda > 1.84$. It was shown next for $k_\perp = 0$ that the plasma was stable, but when neither $k_\perp$ nor k_z was zero, there could be instabilities for $\lambda < 1.84$. In order to consider the effect of a spread of velocities on stability the following zero-order anisotropic distribution function was used:

$$f_{oj} = \frac{1}{2\pi} \frac{2}{\alpha_{\perp j}^2} \frac{\alpha_{zj}}{\pi} \frac{e^{-v_\perp^2/\alpha_{\perp j}^2}}{v_z^2 + \alpha_{zj}^2} \tag{6}$$

where $\alpha_\perp$ and α_z measure the spread in velocity in the directions perpendicular and parallel to the field. Using this distribution function and assuming again no motion of the ions, it is shown that for waves propagating parallel to the magnetic field the oscillations are damped at the plasma frequency (Landau damping); and furthermore that there is neither instability nor Landau damping for $k_z = 0$. However, with no restrictions on $\vec{k}$, taking into account the motion of the ions but letting the temperature along the field be zero (i.e., let $\alpha_z = 0$), one finds a very fundamental relationship. It is shown that a sufficient condition for unstable oscillations in a plasma is:

$$\omega_{pe}^2 > \ell^2 \omega_{ci}^2 , \ \ell = \pm 1 , \ \pm 2 , \ \ldots . \tag{7}$$

where ω_{pe} is defined by Equation (3) and ω_{ci} is the ion-cyclotron frequency given by Equation (4) with m_e replaced by m_i.

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[10] H. M. James, N. B. Nichols, R. S. Philips, Theory of Servomechanism (MIT Radiation Laboratory Series, McGraw-Hill, New York, 1947), p. 70; and P. Penrose, Phys. Fluids 3, 258 (1960).

A small scale trochotron-type experiment was tried by Harris and his students[11] in an attempt to verify Equation (7). Powers found, however, that other phenomena occurred long before the critical current required for instability based on the theory of Harris could be reached.

Equation (7) is the now well-known result of Harris. Since any confinement of a plasma by a magnetic field necessarily produces anisotropic velocity distributions, Harris thought mirror-type thermonuclear machines might breed these instabilities readily. It now seems quite probable that many thermonuclear devices do indeed exhibit this "Harris instability." In what follows is a brief description of a few experimental devices* and the results they yield.

II. EXPERIMENTAL OBSERVATIONS

The DCX plasma is formed by injecting 600-keV molecular hydrogen ions into a magnetic mirror geometry.[12] The molecular ions dissociate either by colliding with neutrals or by Lorentz breakup and are trapped in the magnetic field, which is perpendicular to the injection plane. By this means a high concentration of energetic ions are accumulated in a charged ring and held until they have randomized into a high temperature plasma.

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[11] P. A. Thompson, Bull. Am. Phys. Soc., Series II, 6, 196 (1960); and W. L. Powers, Master's thesis, University of Tennessee, 1963.

*Included will be the molecular-ion injection type machines DCX-I and II and the Russian's OGRA and the neutral injection machines ALICE and Phoenix.

[12] See, for example, The ORNL Thermonuclear Program Jan. 15, 1958, ORNL-2457, p. 7ff. Also the Thermonuclear Division of ORNL publishes bi-annually a progress report in which

It has become evident that a density-limiting mechanism is present in DCX-I of the micro-instability type.[13] Strong proton cyclotron radiation with harmonics through the seventh was observed with each even harmonic weaker than the following odd harmonic. It was seen that the amplitude of these signals was dependent on the input current and magnetic field. The resonance character of these collective oscillations was given the following interpretation. The charge waves following the rotating ions close on themselves. After many passes in this ring, only those waves rotating in phase survive, forming a circular array of charged "bunches."[14] These bunches manifest themselves as oscillations at multiples of the ion-cyclotron frequency. That these protons in the trapped ring might be "bunching," could possibly explain space and energy diffusion in DCX through a resonance instability of ion-cyclotron motion with electron plasma oscillations.[15] A thesis problem by Burt[16] using an idealized

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the detailed description of the thermonuclear program is included.

[13] T. K. Fowler, Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1963, ORNL-3564, p. 81. An argument against micro-instabilities in DCX devices is also given in this section.

[14] This discussion of charge "bunching" in DCX follows very closely the Section 8.1.2 of Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1963, ORNL-3564. Also see J. D. Jackson, J. Nucl. Energy, Pt. C 1, 171 (1961); and Thermonuclear Div. Semiann. Progr. Rept. Apr. 30, 1963, ORNL-3472, p. 75.

[15] "Negative-Mass" type instabilities have also been suggested as ion-bunching mechanisms. See C. E. Nielsen, A. M. Sessler, and K. R. Symon, CERN Conf. Report, 1959, Geneva, European Organization for Nuclear Research, p. 239.

[16] P. Burt, dissertation, University of Tennessee, 1961; and P. Burt and E. G. Harris, Phys. Fluids 4, 1412 (1961).

DCX geometry model, showed that the ions tended to "bunch," giving rise to large electric fields within the plasma.

The DCX-II[17] device has recently begun operation in which the density buildup is accomplished by multiple passes of an injected molecular ion beam through an arc discharge. This machine attains ion-beam intensities seven to eight times greater than DCX-I, and with a more efficient dissociation of the incident beam. As with its earlier counterpart, DCX-II also emits a considerable amount of radio-frequency energy consisting principally of the harmonics of the molecular-and-atomic-ion-cyclotron frequencies[18] (18.6 Mc). Some "white" noise extending above 100 Mc is also present but is not explained.

Recently, however, it has been shown that these Harris instabilities may be suppressed somewhat by heating the electrons in the trapped plasma. Harris[19] has suggested that in an idealized representation of the DCX-I plasma, passing an electron beam through the circulating ion beam stabilizes the lowest frequency modes, but further calculations indicated that this was not promising. It is concluded that stability occurs because energy is drained out of the machine by convection at a more rapid rate than it is produced. However, new unstable modes at

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[17] P. R. Bell et al., Thermonuclear Div. Semiann. Progr. Rept. Apr. 30, 1964, ORNL-3315, pp. 15-20; Nucl. Fusion: 1962 Suppl., Pt. 1, 251-258; and ORNL CF-60-1-73 (Mar. 4, 1960).

[18] Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1962, ORNL-3392, pp. 17-22.

[19] E. G. Harris, "Stabilization of a Plasma by Convection." (Privately circulated note); and Thermonuclear Div. Semiann. Progr. Rept. Apr. 30, 1963, ORNL-3472, pp. 79-81.

higher frequencies are introduced. In addition, a sufficient temperature spread in ions and electrons should also limit the Harris-type instability.*

The Russian contributions to plasma physics and the controlled thermonuclear processes has been quite impressive. They have put into operation a large magnetic mirror system called OGRA** very similar to DCX-II except for the injection energy (here 200 keV) and method of break-up (on the background gas).[20] Measurements indicate that both an ion-cyclotron and low-frequency flute instability are present. The plasma is characterized by a large positive space potential creating a strong radial electric field. As the plasma rotates, spikes are formed at its surface, thereby spewing the electrons and ions across the magnetic field. Also, a significant portion of the electrons acquire enough energy to escape through the mirrors along the magnetic field lines. The signals received by the external detectors indicate low-frequency oscillations that coincide in frequency with frequency of the rotation of the plasma plus intense coherent radiation at the ion-cyclotron frequency and overtones. Experiments on injected currents agree qualitatively with theoretical predictions of Harris. However, when a positive potential is impressed on end grids, located beyond the mirrors outside the plasma, the oscillation amplitude of the cyclotron frequencies decreases. This

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[20] The discussion of the OGRA device was taken from G. F. Bogdanov, Kurchatov Institute of Atomic Energy, Moscow, 1962 No. 23/236 reprint, Translated by J. Lewin of the Thermonuclear Division, ORNL, October 31, 1962.

* This fact will be clearly seen in Chapter III using Model (A) and also the computer results presented in Chapter IV.

** After L. A. Artsimovich and I. N. Golovin.

partial stabilizing effect is possibly explained by the decrease in electron density and an increase in the electron temperature.* Again, the computer work in Chapter IV will bear out this fact. It is concluded that the interaction of the electrons' oscillations and the ion-cyclotron waves is the mechanism by which the electrons acquire energy large enough to pass through the end mirrors in the presence of the high plasma potential.

The neutral injection machines, ALICE[21] at Lawrence Radiation Laboratory and the Phoenix[22] at Culham Laboratories in England, have recently begun operation, and experimental results indicate that the Harris instability is here also. In ALICE a 20 keV beam of excited neutral atoms enters a discharge chamber and these atoms are ionized by Lorentz breakup or by collision with background gas resulting in the buildup of a high energy plasma. Phoenix is different only in the fact that the primary beam is 60 keV neutral hydrogen atoms and is injected into the mirror magnetic field perpendicular to the field lines in the median plane. The experimental results from Phoenix indicate that when the electron plasma frequency is greater than the ion

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*It was conjectured that an increase in the electron temperature was due to an increase in the time spent in the interior of the plasma by the electrons as a result of the elimination of the flute instability by these end grids.

[21] Controlled Thermonuclear Research Semiannual Report June, 1962, UCRL-10294, pp. 15-25; and UCRL-10852, pp. 19-35.

[22] L. G. Kuo et al., "Experimental and Theoretical Studies of Instabilities in a High Energy Neutral Injection Mirror Machine," CLM-P32. (November, 1963), p. 305. (Submitted for publication to Phys. Fluids). See also, D. R. Sweetman, Nuclear Fusion, 1962 Supplement, Part 1, 279 (1962).

cyclotron frequency, discontinuous oscillation at the ion-cyclotron frequency (~ 60 mc/s at 40 kg), continuous oscillation at the ion-cyclotron frequency, and strong low frequency (~ 100 kc/s) oscillation occur. These instabilities promote strong scattering of the ions out of the transverse direction but cannot be observed to cause actual loss of plasma. This leads to an apparent "self-quenching" of the oscillation by changing the anisotropy of the ion-velocity distribution enough to terminate or weaken the instability. There were also observed oscillations at one-half the ion-cyclotron frequency--a fact that will be discussed in connection with results from the computer analysis of Harris's dispersion relations.

It is apparently obvious that the experimental observations presented above are consistent with the electrostatic ion-cyclotron resonance instability predicted by Harris. The presence of this instability would be expected to increase the energy transferred between ions and electrons and to enhance other transport coefficients in the plasma.[23]

III. FURTHER THEORETICAL DEVELOPMENTS OF THE HARRIS INSTABILITY

In order to extend the original theory, Harris[24] suggested that if there were a spread in the cyclotron frequencies due to a non-uniform

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[23] R. F. Post, "Some Observations on Plasma Instabilities in the Mirror Machine," Plasma Hydromagnetics, Daniel Bershader, editor (Sixth Lockheed Symposium on Magnetohydrodynamics, Stanford University Press, Stanford, California, 1962), p. 21.

[24] E. G. Harris, "The Effect of a Spread in Frequencies on the Stability of a Plasma," ORNL CF-62-8-37 (Aug. 21, 1962); also see Thermonuclear Div. Semiann. Progr.

magnetic field, the coherence of the oscillations may be destroyed, since many of the particles will be out of resonance. It was found (using DCX-I parameters) that there was only a narrow band of densities for which the plasma goes unstable. Next, the effect of non-zero α_z on the dispersion relation was investigated[25] assuming propagation of waves almost parallel to the magnetic field. It was found, using the Burt-Harris model, that the solution of the dispersion relation could be attained without the aid of a computer. The results are presented in a parameterized form as a graph on which are mapped stable and unstable regions for given values of the temperatures occurring in the plasma. Thus for a given value of the density and magnetic field, one could predict stable or unstable operating conditions as a function of plasma anisotropies (temperatures).

The effect of non-zero ion and electron temperatures on the general dispersion relation* has been investigated by a number of authors. Sagdeev et al.[26] showed that if the longitudinal and transverse energies are approximately equal, it was reasonable to "assume" that the plasma was stable. Analyzing the longitudinal part of the conductivity tensor,

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Rept. Oct. 31, 1962, ORNL-3392, pp. 56-60.

[25] E. G. Harris, "The Effect of Finite Ion and Electron Temperature on the Ion-Cyclotron Resonance Instability," CLM-R32 (October, 1963).

*That is, the dispersion relation derived from a bi-Maxwellian distribution function. This derivation will be given in the discussion of Model (A) of Chapter III.

[26] R. Z. Sagdeev et al., Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Vol. 31, United Nations, Geneva (1958), p. 15; and A. A. Vedenov, E. P. Velichov, and R. Z. Sagdeev, Soviet Physics Uspekhi 4, 332 (1961).

Drummond, Rosenbluth, and Johnson[27] found that $T_\perp/T_z$ * of the ions must be greater than eight or nine for instability to occur, where

$$\alpha_{\perp j}^{2} = \frac{2\kappa T_{\perp j}}{m_j} \tag{8}$$

and

$$\alpha_{zj}^{2} = \frac{2\kappa T_{zj}}{m_j} \tag{9}$$

and

$\kappa \equiv$ Boltzmann's Constant.

Ozawa, Kaji, and Kito,[28] using a complex potential technique, derive the stability criterion for longitudinal waves in a magnetic field. For a one species plasma with a bi-Maxwellian velocity distribution

$$f_o = \frac{1}{\pi^{3/2}\alpha_\perp^{2}\alpha_z} e^{-v_\perp^{2}/\alpha_\perp^{2} - v_z^{2}/\alpha_z^{2}} \tag{10}$$

they show stability when $T_\perp/T_z < 2$ and $\omega_p/\omega_c < 0.6$. These results will be compared with work from Chapter IV of this dissertation. A. V. Timofeev[29] attributes the buildup of ion acoustic vibrations in an anisotropic, nonisothermal plasma as a result of cyclotron resonance. He showed that for an instability to occur one must have $T_{zi}/T_{\perp i} < 0.5$ and

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[27] W. E. Drummond, M. N. Rosenbluth, and M. L. Johnson, Bull. Am. Phys. Soc., Series II, 6, No. 2, 185 (1961).

* The symbols $T_\perp$ and T_z are the familiar representation of temperatures perpendicular and parallel to the magnetic field. A more physically correct statement would be the velocity spread perpendicular and parallel to the magnetic field.

[28] Y. Ozawa, I. Kaji, and M. Kito, J. Nucl. Energy, Pt. C 4, 271 (1962).

[29] A. V. Timofeev, Soviet Physics JETP 12, 281 (1961).

$T_{zi}/T_{ze} \lesssim 0.2$. Again comparison with results of this dissertation is possible. Kahn,[30] making calculations on the same lines as Harris but assuming instead that the electrons have zero velocity and that the ions are bi-Maxwellian, proves a sufficient condition of instability to be $\omega_p > \omega_c$ and $T_\perp = 2T_z$. In work closely resembling Model (A) of Chapter III, Dnestrovsky, Kostomarov, and Pistunovich[31] found analytically that the ion anisotropy $T_{\perp i}/T_{zi}$ must be greater than two and the ratio of the electron plasma frequency to the ion-cyclotron frequency must be greater than one half for the plasma to exhibit unstable behavior. They showed that the largest region of instability corresponds to long wave length disturbances propagated almost perpendicular to the magnetic field. The waves satisfying

$$\frac{k_z}{k}\frac{\omega_{pe}}{\omega_{ci}} \approx n ,$$

n being some integer, are shown to possess the largest growth rate. The effect of non-zero electron temperature on the stability of the system is indicated. The above work was carried out using asymptotic expansions of the dispersion relation. Also Hall and Heckrotte[32] prove a necessary condition for stability similar to those above. Hall[33] looks also at

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[30] F. D. Kahn, report to be published.

[31] Yu. N. Dnestrovsky, D. P. Kostomarov, and V. I. Pistunovich, Nuclear Fusion 3, 30 (1963).

[32] L. S. Hall and W. Heckrotte, "Stability of Longitudinal Oscillations in a Uniform Magnetized Plasma with Anisotropic Velocity Distribution," UCRL-7627 (Dec. 9, 1963); and T. Kamash and W. Heckrotte, Phys. Rev. 131, 2129 (1963).

[33] L. S. Hall, W. Heckrotte, and T. Kamash, "Ion-Cyclotron Electrostatic Instabilities." Paper Presented at Sherwood Plasma Physics Theory Meeting, Gatlinburg, Tennessee, May 7-8, 1964.

ion-ion and electron-electron type interactions and derives instability criteria for these cases.

IV. FINITE GEOMETRY CONSIDERATIONS

In most of the above calculations the major simplifying assumption was that of a plasma of infinite extent. In an effort to determine the effect of finite boundaries Burt[34] derived boundary conditions for a cylindrical shell of ions rotating at the ion-cyclotron frequency. It was shown that space charge piling up on the boundaries led to discontinuities in the electric field. Assuming that one could integrate the potential across this discontinuity, a complicated dispersion relation was found which after various simplifications was found not to be very sensitive to the boundaries. Soper[35] found it more appropriate to consider each boundary separately, calculate the surface charge that migrates to the boundary upon perturbation of the plasma and derive the boundary conditions without recourse to step function distributions, etc. Again it was found that the boundary conditions entered only in a secondary manner. New "modes" depending on the boundaries were found to be present, however. An extension of these boundary conditions to the present work was attempted but was abandoned in favor of a more rigorous method.* Following an article by Frieman *et al.*,[36] who show in a case similar to

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[34] Burt, *loc. cit.*

[35] G. K. Soper, Master's thesis, University of Tennessee, 1962.

*See Model (B), Chapter III for this discussion.

[36] E. A. Frieman *et al.*, Phys. Fluids 5, 196 (1962).

Model (B) that the differential equation governing the electric potential must be singular somewhere in the interface between plasma and vacuum, the connection formulas are written down in the limiting case of a sharp-beam boundary. Following this prescription, and using the Burt-Harris model, a finite geometry problem approximating a mirror machine which has both strong and weak mirrors is solved. Finally, Shima[37] has extended Harris's original problem to include a rotating beam with finite temperatures. It is shown that the dispersion relation is identical with that of Harris (paper III) except that the ions are Doppler shifted due to the rotations. The solution of this dispersion relation with the aid of a computer is included in Chapter IV.

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[37] Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1962, ORNL-3392, p. 61.

CHAPTER III

SOLUTION OF THE PROBLEM

Although each of the four models discussed in the introduction will be taken up separately, the basic plasma equations, common to all four models, will first be presented and discussed in some detail. A common technique for the solution of plasma oscillation problems is based on a perturbation technique in which the basic equations are linearized, Fourier and Laplace analyzed, and a dispersion equation obtained. The most general theoretical approach to microscopic plasma oscillation problems is through the Boltzmann and Maxwell equations. These equations connect the self-consistent electric and magnetic fields with the particle distribution functions. The Boltzmann equation without collision terms for the electrons and ions plus Maxwell's equations are called the Vlasov equations.

I. THE VLASOV EQUATIONS

Due to the long range of the electric and magnetic forces between particles in a plasma, each particle is almost constantly interacting with a large number of other particles. One assumes that each particle of the plasma moves in an average electric and magnetic field produced by all the other particles of the plasma and by external fields. So if one considers the plasma as a continuous medium to which electrodynamics applies, the description of the plasma is given by:

$$\frac{\partial f_j}{\partial t} + \vec{v} \cdot \nabla f_j + \frac{e_j}{m_j} (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \nabla_v f_j = (\frac{df}{dt})_{coll} = 0 \qquad (1)$$

and

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{2a}$$

$$\nabla \cdot \vec{B} = 0 \tag{2b}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2c}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \tag{2d}$$

and where ∇_v represents the gradient operator in velocity space.* Equation (1) describes the rate of change of the distribution function f_j for each species of the plasma and, of course, Equations (2a-d) are Maxwell's equations. These equations are made self-consistent with the definition of the charge and current densities:

$$\rho = \sum_j e_j \int f_j d^3v \tag{2e}$$

$$\vec{J} = \sum_j e_j \int \vec{v} f_j d^3v \ . \tag{2f}$$

It is these groups of equations which are referred to as the Vlasov equations.[1] The unrationalized set of Gaussian units is used throughout.

$$^*\nabla_v \equiv \frac{\partial}{\partial \vec{v}} = \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} + \frac{\partial}{\partial v_z}\hat{z} \ .$$

[1]A. A. Vlasov, J. Exep. Theor. Phys. (USSR) 8, 291 (1938). See also USAEC translation--2729; and E. G. Harris, "On a Self-Consistent Field Method for a Completely Ionized Gas," NRL-4944 (May 17, 1957). The rigorous molecular foundation of the Vlasov equation was first investigated by N. N. Bogoliubov, Problems of a Dynamical Theory in Statistical Physics, Moscow, 1946; translated by E. K. Gora, in Studies in Statistical Mechanics, Vol. 1, edited by J. deBoer and G. E. Uhlenbeck, Interscience, New York, 1962; N. Rostocker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960) derive the Vlasov equation in the fluid limit, i.e., $e \to 0$, $m \to 0$, $c \to \infty$ such that e/m and ec remain constant; and finally see R. Balescu, Statistical Mechanics of Charged

The Boltzmann distribution function, i.e., $f_j(\vec{x}, \vec{v}, t)d^3xd^3v$, describes the probability of finding a particle of species j* within the volume element d^3x about x and within d^3v, the element of velocity space in time t. The electric and magnetic fields are considered to be the sum of the average internal and external fields. Also, no distinction need be made between $\vec{B}$ and $\vec{H}$, and $\vec{E}$ and $\vec{D}$ since all currents and charge densities in the medium are treated explicitly.[2]

It should be noticed that the collision term in the Boltzmann equation has been set equal to zero. For this reason only oscillatory phenomena and stability questions can be treated using the Vlasov equations. Problems concerning relaxation of long-lived plasma or "resistive" instabilities, for example, cannot be solved using this assumption because of the lack of dissipative effects. It is shown in Appendix A that the interaction between individual particles can indeed be neglected.

Equations (1) and (2a-d) are non-linear since $\vec{E}$ and $\vec{B}$ depend upon f_j, making an exact solution of these equations quite difficult. However, a solution of the Vlasov equations is made tractable by considering systems which depart only slightly from an equilibrium form. One linearizes the equations by writing

$$f_j = f_{oj}(\vec{v}) + f_{ij}(\vec{x}, \vec{v}, t) \tag{3}$$

Particles (Interscience Publishers, New York, 1963), pp. 65-69.

*The symbol j represents either ions or electrons.

[2]A. Simon, An Introduction to Thermonuclear Research (Pergamon Press, New York, 1959), p. 138; lecture, Plasma Physics Summer School of the Danish Atomic Energy Commission, Risö, Denmark (August, 1960), p. 64.

$$\vec{E} = \vec{E}_o + \vec{E}_1(\vec{x}, t) \tag{4}$$

$$\vec{B} = \vec{B}_o + \vec{B}_1(\vec{x}, t) \tag{5}$$

where it is assumed that:

(a) all perturbation terms are much smaller than the equilibrium quantities, e.g., $f_1 \ll f_o$, and furthermore that all products of perturbation quantities may be neglected, (b) $\vec{E}_o \equiv 0$, there is no steady state electric field,

$$\text{(c)} \quad \vec{B}_o = B_o \hat{z} , \tag{6}$$

and finally

$$\text{(d)} \quad \vec{B}_1 = 0 \tag{7}$$

which implies

$$\text{(e)} \quad \vec{E}_1 = - \nabla\Phi(\vec{x}, t) \tag{8}$$

In this approximation the first of the Vlasov equations is linear and dropping the subscript j reduces to:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 + \frac{e}{mc} (\vec{v} \times \vec{B}_o) \cdot \nabla_v f_1 = \frac{e}{m} \nabla\Phi \cdot \nabla_v f_o \tag{9}$$

where the electric potential satisfies the Poisson equation:

$$\nabla^2\Phi = -4\pi \sum_j e_j \int f_1(\vec{x}, \vec{v}, t) d^3v . \tag{10}$$

It is not correct in general to determine the electric field from the distribution function by means of the Poisson equation in a static magnetic field problem. Instead, the full set of Maxwell equations should be used resulting in a coupling of longitudinal and transverse motions. However, it can be shown[3] that this approximation is justified

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[3] V. A. Bailey, Phys. Rev. 83, 439 (1951); and I. B. Bernstein and R. M. Kulsrud, Phys. Fluids 3, 937 (1960).

for the case when $\omega_p \ll kc$ and $\omega_c \ll kc$. This is known as the electrostatic or longitudinal wave approximation and designates Coulomb interactions as most responsible for plasma instabilities. A solution of the zero-order Vlasov equation:

$$(\vec{v} \times \vec{B}_o) \cdot \nabla_v f_o = 0 \tag{11}$$

where

$$\vec{B}_o = B_o \hat{z}$$

is

$$f_o(\vec{v}) = f_o(\vec{v}_\perp, v_z) \tag{12}$$

where

$$v_\perp^2 = v_x^2 + v_y^2 . \tag{13}$$

Although the derivation of the dispersion relation for each model is based on Equations (9) and (10), the general discussion will be terminated and the individual nature of the solutions presented.

II. THE INFINITE PLASMA MODEL

The derivation of the dispersion relation for the infinite plasma starting with Equation (9) has been given by a number of authors. The following development will follow that of Harris (paper I), but omitting much of the detail.

Since Equation (9) is linear, one can begin by Fourier and Laplace transforming the space and time dependent quantities:

$$f_{1j}(\vec{k}, \vec{v}, t) = \int e^{-i\vec{k}\cdot\vec{x}} f_{1j}(\vec{x}, \vec{v}, t) d^3x \tag{14}$$

and

$$f_{1j}(\vec{k}, \vec{v}, \omega) = \int_0^\infty e^{i\omega t} f_{1j}(\vec{k}, \vec{v}, t) dt \tag{15}$$

and similar equations for $\Phi(x, t)$. If $\mathrm{Im}\omega > \gamma$ and f_{1j} and Φ are assumed

to be of order $e^{\gamma t}$ for $t \geq 0$, then Equation (9) can be transformed to give:

$$i(\vec{k} \cdot \vec{v} - \omega)f_1 - \omega_c\left(v_y \frac{\partial f_1}{\partial v_x} - v_x \frac{\partial f_1}{\partial v_y}\right) = \frac{ie\Phi}{m} \vec{k} \cdot \nabla_v f_o - g(\vec{k}, \vec{v}) \tag{16}$$

where it has been assumed that

$$\vec{B}_o = -B_o \hat{z}$$

and where

$$g(\vec{k}, \vec{v}) \equiv f_1(\vec{k}, \vec{v}, t = 0) = \int e^{-i\vec{k}\cdot\vec{x}} f_1(\vec{x}, \vec{v}, 0)d^3x \tag{17}$$

is the Fourier transform of the initial perturbation and where

$$\omega_c = \frac{eB_o}{mc} \tag{18}$$

is the cyclotron frequency of the species.*

It is convenient to introduce cylindrical coordinates $v_\perp$, ϕ, and v_z in velocity space. Thus

$$\vec{v} = \hat{x}v_\perp \cos \phi + \hat{y}v_\perp \sin \phi + \hat{z}v_z \tag{19}$$

$$\vec{k} = \hat{x}k_\perp + \hat{z}k_z \tag{20}$$

and Equation (16) becomes

$$\frac{\partial f_1}{\partial \phi} + \frac{i}{\omega_c}(-\omega + k_\perp v_\perp \cos \phi + k_z v_z)f_1 = \frac{ie\Phi}{m\omega_c}\left(k_\perp \cos \phi \frac{\partial f_o}{\partial v} + k_z \frac{\partial f_o}{\partial v_z}\right) - \frac{g}{\omega_c} \tag{21}$$

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*The particle designation has been omitted from the above equations. The following convention is used: for the electrons $\omega_{ce} = -eB_o/m_e c$; for the ions $W_{ci} = eB_o/m_i c$.

whose solutions is:[4]

$$f_1(\vec{k}, \vec{v}, \omega) = \int_{\pm\infty}^{\phi} d\phi' \, G(\phi, \phi') \, \Phi(\vec{k}, \omega) \left[\frac{ie}{m\omega_c} \, k_\perp \cos\phi' \frac{\partial f_o}{\partial v_\perp} + k_z \frac{\partial f_o}{\partial v_z} \right]$$
$$+ \int_{\pm\infty}^{\phi} d\phi' \, G(\phi, \phi') \, g(\vec{k}, \vec{v}) \, , \tag{22}$$

where

$$G = \exp\left\{ - \left[\frac{-i\omega + ik_z v_z}{\omega_c} (\phi - \phi') + \frac{ik_\perp v_\perp}{\omega_c} (\sin\phi - \sin\phi') \right] \right\} . \tag{23}$$

The sign in the lower limit of Equation (22) is chosen so that $G(\phi, \phi')$ vanishes at the lower limit when ω has a positive imaginary part. Transforming Poisson's equation

$$k^2 \Phi(\vec{k}, \omega) = 4\pi \sum e \int f_1(\vec{k}, \vec{v}, \omega) d^3v \tag{24}$$

substituting Equation (22) and solving for Φ, one finds (restoring the particle designation) that:

$$\Phi(\vec{k}, \omega) = \frac{\sum_j \frac{4\pi e_j^2}{k^2 \omega c_j} \int S_j d^3v}{1 - D(\vec{k}, \omega)} \tag{25}$$

where S_j is the last term on the right-hand side of Equation (22),

$$D(\vec{k}, \omega) = \sum_j \frac{i\omega p_j^2}{k^2 \omega c_j} \int d^3v \left[k_\perp \frac{\partial f_o}{\partial v_\perp} \int_{\pm\infty}^{\phi} G(\phi, \phi') \cos\phi' \, d\phi' \right.$$
$$\left. + k_z \frac{\partial f_o}{\partial v_z} \int_{\pm\infty}^{\phi} G(\phi, \phi') \, d\phi' \right], \tag{26}$$

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[4] I. B. Bernstein, Phys. Rev. 109, 10 (1958).

$$\omega_{pj}^{2} = \frac{4\pi n_{oj} e^{2}}{m_j} \tag{27}$$

is the plasma frequency of species j, and where n_{oj}, the equilibrium density per unit volume, was added as a separate factor so that f_{oj} is normalized to unity.

The solution to the initial value problem is found by inversion of the Laplace and Fourier transforms, viz.,

$$\Phi(\vec{x}, t) = \int_{-\infty + i\sigma}^{+\infty + i\sigma} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \; e^{i(\vec{k}\cdot\vec{x} - \omega t)} \; \Phi(\vec{k}, \omega) \tag{28}$$

where the above integral is to be carried out in the complex ω plane above all singularities of $\Phi(\vec{k}, \omega)$. Following the arguments of Landau,[5] one assumes that $g_j(\vec{k}, \vec{v})$ are entire functions of the complex variable v. Upon invoking analytic continuation of the integrals in Equation (25), one can show for sufficiently well-behaved zero-order distribution functions that Equation (25) represents the ratio of two entire functions and that the only singularities of $\Phi(\vec{k}, \omega)$ are the zeros of the denominator of Equation (25). The time dependence of $\Phi(\vec{x}, t)$ is completely determined by these zeros.

The dispersion relation, i.e., the relation connecting ω and $\vec{k}$, is then written

$$D(\vec{k}, \omega) = 1 \; , \tag{29}$$

the solution of which in general determines the behavior of the plasma. For example, if there exist solutions to Equation (29) for which the imaginary part of the frequency, ω, has a positive real value, then plasma

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[5] L. D. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

oscillations exist whose amplitudes increase exponentially with time. It is this growth from arbitrarily small disturbances which is termed an instability.

Using Equation (26) in (29) and the identity[6]

$$e^{ix\sin(\phi - \theta)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in(\phi - \theta)} \tag{30}$$

where J_n is the Bessel function of order n, one can reduce the dispersion relation to the following form

$$1 = \sum_j \frac{2\pi\omega_{pj}^2}{k^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_\perp dv_\perp \left\{ -\frac{\omega_{cj}}{v_\perp} \frac{\partial f_{oj}}{\partial v_\perp} \frac{nJ_n^2(k_\perp v_\perp/\omega_{cj})}{\Omega_{zj}} \right.$$

$$\left. + k_z \frac{\partial f_{oj}}{\partial v_z} \frac{J_n^2(k_\perp v_\perp/\omega_{cj})}{\Omega_{zj}} \right\} . \tag{31}$$

where

$$\Omega_{zj} \equiv -\omega + k_z v_z - n\omega_{cj} . \tag{32}$$

In order to compute the effect of finite temperature distribution functions, the integrals in Equation (31) are evaluated using the following bi-Maxwellian equilibrium distribution:*

$$f_{oj} = \frac{1}{\pi^{3/2}\, \alpha_{\perp j}^2\, \alpha_{zj}} e^{-v_\perp^2/\alpha_{\perp j}^2 - v_z^2/\alpha_{zj}^2} \tag{33}$$

where $\alpha_{\perp j}^2 = 2\kappa T_{\perp j}/m_j$ and $\alpha_{zj}^2 = 2\kappa T_{zj}/m_j$ measure the spread in velocities perpendicular and parallel to the field.

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[6] G. N. Watson, Theory of Bessel Functions (Cambridge University Press, Cambridge, 1945), p. 14.

*Harris used a Maxwellian in the

The solution of Equation (31) with the bi-Maxwellian is straightforward and gives:

$$1 = \sum_j \left(\frac{\omega_{pj}}{k_z \alpha_{zj}}\right)^2 \sum_{n=-\infty}^{\infty} C_{|n|} \left\{ \left(\frac{k_z}{k}\right)^2 Z'\left(\frac{\omega - n\omega_{cj}}{k_z \alpha_{zj}}\right) - \frac{n}{\lambda_j\left(\frac{\omega_{cj}}{k_z \alpha_{zj}}\right)} \left(\frac{k_\perp}{k}\right)^2 Z\left(\frac{\omega - n\omega_{cj}}{k_z \alpha_{zj}}\right) \right\}$$

where

$$C_{|n|} = e^{-\lambda_j} I_{|n|}(\lambda_j) \ , \ I_{|n|}(\lambda_j) = \text{Imaginary Bessel function}, \tag{35}$$

$$\lambda_j = \frac{k_\perp^2 \alpha_{\perp j}^2}{2\omega_{cj}^2} , \tag{36}$$

and where

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{du e^{-u^2}}{u - \zeta} \tag{37}$$

is the plasma dispersion function of Fried and Conte.[7] Equation (37) is defined by $\mathrm{Im}\zeta > 0$ and as the analyticic continuation of this for $\mathrm{Im}\zeta \leq 0$.

In general the solution of Equation (34) must be obtained numerically. In Chapter IV simplifying approximations will be made and the solution discussed.

III. THE HARRIS MODEL LIMITED IN z

Suppose instead of an infinite plasma, one considered a plasma of finite extent. Assume a model in which there exists a plasma slab, bounded

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perpendicular direction and Lorentzian in z, later letting the temperature in the z direction go to zero.

[7]B. D. Fried and S. D. Conte, The Plasma Dispersion Function (Academic Press, New York, 1961), p. 1; and D. Pines and J. R. Schrieffer, Phys. Rev. 124, 1387 (1961).

at $z = \pm a$, and infinite in the x and y directions. Furthermore, suppose the particles have a temperature spread in the perpendicular direction and are cold* in the z direction, or:

$$f_{oj}(\vec{v}, z) = \frac{n_o(z)}{\pi\alpha_{\perp j}^2}\,\delta(v_z)\, e^{-v_\perp^2/\alpha_{\perp j}^2} \tag{38}$$

where the exact dependence of the density on position has been left arbitrary. Earlier treatments[8] of a similar finite geometry problem considered only infinitely sharp plasma-vacuum density distribution. In other words, inside the plasma the unperturbed density was given as a constant and outside, the density was exactly zero.

The form of the distribution function suggests the particles' behavior to be described by the Vlasov equation in the perpendicular direction and by the hydrodynamic equations in the z-direction. The velocity dependence in the z-direction can be eliminated by taking moments of the Vlasov equation with respect to v_z. Letting $\vec{B}_o = 0$ as a further simplification, integrating Equation (1) over dv_z, one has (dropping the label j):

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} f dv_z + \nabla \cdot \int_{-\infty}^{\infty} \vec{v}\, f dv_z + \frac{e}{m}\vec{E} \cdot \int_{-\infty}^{\infty} \nabla_v f dv_z = 0 \,. \tag{39}$$

Introducing the notations of perpendicular and parallel gradient operators:

$$\nabla = \nabla_\perp + \frac{\partial}{\partial z}\hat{z} \quad \text{where} \quad \nabla_\perp = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \tag{40}$$

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*That is, the particles have no velocity spread in the z-direction.

[8] G. K. Soper, Master's thesis, University of Tennessee, 1962.

and similar equations for the velocity gradients, Equation (39) is:

$$\frac{\partial F}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp F + \frac{\partial}{\partial z} \int_{-\infty}^{\infty} v_z f dv_z + \frac{e}{m} \vec{E}_\perp \cdot \nabla_{v\perp} F$$

$$+ \frac{e}{m} E_z \int_{-\infty}^{\infty} \frac{\partial f}{\partial v_z} dv_z = 0 \tag{41}$$

where

$$F(\vec{x}, \vec{v}_\perp, t) \equiv \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t) dv_z \ . \tag{42}$$

Defining

$$H(\vec{x}, \vec{v}_\perp, t) \equiv \int_{-\infty}^{\infty} v_z f(\vec{x}, \vec{v}, t) dv_z \tag{43}$$

and linearizing, one finds that:

$$\frac{\partial F_1}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp F_1 + \frac{\partial H_1}{\partial z} + \frac{e}{m} \vec{E}_{1\perp} \cdot \nabla_{v\perp} F_o = 0 \ . \tag{44}$$

The next step in this hierarchy is to multiply the Vlasov equation by v_z and integrate over dv_z,

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} v_z f dv_z + \int_{-\infty}^{\infty} \vec{v} \cdot \nabla f v_z dv_z + \frac{e}{m} \int_{-\infty}^{\infty} v_z \vec{E} \cdot \nabla_v f dv_z = 0$$

and upon integrating the last term by parts, the above equation gives:

$$\frac{\partial H}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp H + \int_{-\infty}^{\infty} v_z^2 \frac{\partial f}{\partial v_z} dv_z + \frac{e}{m} \vec{E}_\perp \cdot \nabla_{v\perp} H - \frac{e}{m} E_z F = 0 \ . \tag{45}$$

It is easy to see that following this procedure, one could get as many moment equations as wished, each coupled to the other. One would have, say, N coupled equations with N + 1 unknowns. To terminate this hierarchy, one rewrites Equation (45) to read:

$$\frac{\partial H}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp H + \frac{\partial}{\partial z} \int_{-\infty}^{\infty} (v_z - \langle V_z \rangle)^2 f dv_z + 2 \frac{\partial}{\partial z} \int_{-\infty}^{\infty} v_z \langle V_z \rangle f dv_z$$

$$- \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \langle V_z \rangle \langle V_z \rangle f dv_z + \frac{e}{m} \vec{E}_\perp \cdot \nabla_{v\perp} H - \frac{e}{m} E_z F = 0 \tag{46}$$

where

$$\langle V_z(\vec{x}, \vec{v}_\perp, t) \rangle \equiv \frac{\int_{-\infty}^{\infty} v_z f dv_z}{\int_{-\infty}^{\infty} f dv_z} = \frac{H}{F} . \tag{47}$$

Defining a scalar pressure

$$P \equiv m \int_{-\infty}^{\infty} (v_z - \langle V_z \rangle)^2 f dv_z$$

in analogy with the general pressure tensor,[9] the middle three terms of Equation (46) become

$$.. \frac{1}{m} \frac{\partial}{\partial z} P + 2 \frac{\partial}{\partial z} \langle V_z \rangle H - \frac{\partial}{\partial z} \langle V_z \rangle \langle V_z \rangle F \ldots$$

or from Equation (47)

$$.... \frac{\partial}{\partial z} \left(\frac{1}{m} P + H \langle V_z \rangle\right) ....$$

Thus Equation (46) is reduced to

$$\frac{\partial H}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp H + \frac{\partial}{\partial z} \left(\frac{1}{m} P + H \langle V_z \rangle\right) + \frac{e}{m} \vec{E}_\perp \cdot \nabla_{v\perp} H - \frac{e}{m} E_z F . \tag{48}$$

The above equation is linearized by writing

$$\frac{1}{m} P_o = \int_{-\infty}^{\infty} v_z^2 f_o dv_z - 2 \int_{-\infty}^{\infty} v_z \langle V_z \rangle f_o dv_z + \int_{-\infty}^{\infty} \langle V_z \rangle^2 f_o dv_z \tag{49}$$

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[9] I. B. Bernstein and S. K. Trehan, Nuclear Fusion 1, 1 (1960); and L. Spitzer, Physics of Fully Ionized Gases,

which is zero since f_o is proportional to $\delta(v_z)$, and since

$$H_o = \int_{-\infty}^{\infty} v_z f_o dv_z = 0 ,$$

therefore

$$\frac{\partial H_1}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp H_1 + \frac{1}{m}\frac{\partial P_1}{\partial z} - \frac{e}{m} E_{1z} F_o = 0 . \tag{50}$$

As was stated earlier, the particles are assumed to obey the hydrodynamic equations in the z-direction. From the adiabatic relation and the linearized equations of hydromagnetics, one finds:[10]

$$P_1 \sim P_o \nabla \cdot \vec{\xi} \tag{51}$$

where $\vec{\xi}$ denotes the displacement of an element of the fluid. Since $P_o = 0$, then $P_1 \approx 0$ and one finds that

$$\frac{\partial H_1}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp H_1 = \frac{e}{m} E_{1z} F_o \tag{52}$$

thus terminating the hierarchy. The solution of Equation (52) when substituted into Equation (44) will determine the source function for Maxwell's equations.

It shall now be assumed that all first order quantities have their space and time dependence given by a factor:

$$\sim g(z)\, e^{i\vec{k}_\perp \cdot \vec{x}_\perp - i\omega t} \tag{53}$$

(Interscience, New York, 1962), p. 24.

[10] S. Chandrasekhar, Plasma Physics (The University of Chicago Press, Chicago, 1960), pp. 99-102.

The dispersion relation obtained in this way is the same as is obtained from the complete initial value treatment following Landau's prescription for evaluating the singular integrals. A growing wave corresponds to a solution of the dispersion relation for which $\mathrm{Im}\,\omega > 0$.

From Equations (52) and (53) one finds:

$$H_1 = \frac{\frac{e}{m} E_{1z} F_o}{i\Omega_k} , \tag{54}$$

$$\Omega_k \equiv \vec{k}_\perp \cdot \vec{v}_\perp - \omega \tag{55}$$

and similarly for Equation (44)

$$i\Omega_k F_1 + \frac{\partial H_1}{\partial z} + \frac{e}{m} \vec{E}_{1\perp} \cdot \nabla_{v\perp} F_o = 0 . \tag{56}$$

Using

$$\vec{E}_1 = -i\vec{k}_\perp \Phi - \frac{\partial \Phi}{\partial z} \hat{z} \tag{57}$$

and substituting Equation (54) into Equation (56), one finds (restoring the j label again) that:

$$F_{1j} = - \frac{e_j}{m_j \Omega_k^2} \frac{\partial}{\partial z} \left(F_{oj} \frac{\partial \Phi}{\partial z}\right) + \frac{e_j}{m_j \Omega_k} \Phi \vec{k} \cdot \nabla_{\perp v} F_{oj} \tag{58}$$

where

$$F_{oj}(\vec{v}_\perp, v_z, z) = n_{oj}(z)\, \overline{F}_{oj}(\vec{v}_\perp, v_z) \tag{59}$$

and therefore

$$\frac{4\pi e_j^2}{m_j} n_{oj}(z) = \omega_{pj}^2(z) . \tag{60}$$

From Poisson's equation

$$\nabla \cdot \vec{E}_1 = 4\pi \sum_j e_j \int f_{1j} d^3v = 4\pi \sum_j e_j \iint F_{1j} d^2v_\perp$$

or upon substituting Equations (57) and (58), dropping bars on

$\overline{F}_{oj}(\vec{v}_\perp, v_z)$:

$$\frac{d^2\Phi}{dz^2} + k_\perp^2\Phi = \sum_j \left[- \iint_{-\infty}^{\infty} \frac{F_{oj} \frac{d}{dz}\left(\omega_{pj}^2(z)\frac{d\Phi}{dz}\right) d^2v_\perp}{\Omega_k^2} \right.$$

$$\left. + \omega_{pj}^2(z)\Phi\vec{k}_\perp \cdot \iint_{-\infty}^{\infty} \frac{\nabla_{v\perp}F_{oj}d^2v_\perp}{\Omega_k} \cdot \right] \tag{61}$$

Integrating the first term on the right hand side by parts and simplifying, gives:

$$\frac{d^2\Phi}{dz^2} - k_\perp^2\Phi = \frac{1}{k_\perp^2} \sum_j \vec{k}_\perp \cdot \iint_{-\infty}^{\infty} \frac{\nabla_{v\perp}F_{oj}d^2v_\perp}{\Omega_k} \left[\frac{d}{dz}\left(\omega_{pj}^2(z)\frac{d\Phi}{dz}\right) - \omega_{pj}^2k_\perp^2\Phi\right]$$

which can be written:

$$\frac{d}{dz}\left[(1 - G(z))\frac{d\Phi}{dz}\right] - k_\perp^2\Phi\,(1 - G(z)) = 0 \tag{62}$$

where

$$G = \sum_j \frac{\vec{k}_\perp}{k_\perp^2} \cdot \iint_{-\infty}^{\infty} \frac{\nabla_{v\perp}F_{oj}d^2v_\perp}{\Omega_k}\,\omega_{pj}^2\,(z)$$

or

$$G = \frac{1}{k_\perp^2}\left(\vec{k}_\perp \cdot \iint_{-\infty}^{\infty} \frac{\omega_{pe}^2\nabla_{v\perp}F_{oe} + \omega_{pi}^2\nabla_{v\perp}F_{oi}}{\Omega_k}\,d^2v_\perp\right). \tag{63}$$

Since

$$\omega_{pi}^2 = \frac{m_e}{m_i}\omega_{pe}^2 = \delta\omega_{pe}^2\ , \tag{64}$$

then

$$G = \frac{\omega_{pe}^2(z)}{k_\perp^2}\left(\vec{k}_\perp \cdot \iint_{-\infty}^{\infty} \frac{\nabla_{v\perp}F_{oe} + \delta\nabla_{v\perp}F_{oi}}{\Omega_k}\,d^2v_\perp\right)$$

and letting $m_i = \infty$ $(\delta \to 0)$

$$G(z) = \frac{\omega_{pe}^{2}(z)}{k_{\perp}^{2}} \left(\vec{k}_{\perp} \cdot \iint_{-\infty}^{\infty} \frac{\nabla_{v\perp} F_{oe} d^{2}v_{\perp}}{\Omega_{k}} \right) . \tag{65}$$

So Equation (62) can be written

$$\frac{d}{dz} \left[(1 - G(z)) \frac{d\Phi}{dz} \right] - k_{\perp}^{2} (1 - G(z)) \Phi = 0 .$$

Let

$$0 \leq \Delta(z) = \frac{\omega_{pe}^{2}(z)}{\omega_{po}^{2}} \leq 1 \tag{66}$$

and assume that $\omega_{pe}(z)$ has its maximum at $z = 0$ and falls monotonically to zero at $z = \pm \infty$. Further assume that

$$\vec{k}_{\perp} = k_{\perp} \hat{x} \tag{67}$$

and use

$$F_{o} = \int_{-\infty}^{\infty} f_{o} dv_{z} = \frac{1}{\pi \alpha_{\perp}^{2}} e^{-v_{\perp}^{2}/\alpha_{\perp}^{2}} \tag{68}$$

to finally show that Equation (62) can be written as:

$$\frac{d}{dz} \left[(1 - \eta\Delta(z)) \frac{d\Phi}{dz} \right] - k_{\perp}^{2} (1 - \eta\Delta(x)) \Phi = 0 \tag{69}$$

where

$$\eta = \frac{\omega_{po}^{2}}{k_{\perp}^{2} \alpha_{\perp}^{2}} Z' \left(\frac{\omega}{k_{\perp} \alpha_{\perp}} \right) \tag{70}$$

where $Z(\zeta)$ is the Fried-Conte plasma dispersion function as defined by Equation (37). The prime means differentiation with respect to the argument.

It will be shown that Equation (69) has no solutions that are free of singularities. The proof follows the work of Frieman _et al._,[11] whose

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[11] E. A. Frieman _et al._, Phys. Fluids _5_, 196 (1962).

paper on the streaming instabilities of a finite beam passing through a cold plasma is based on an equation very similar to Equation (69) above. Assume that Φ and Φ' are finite everywhere. Multiply Equation (69) by Φ^* and integrate by parts to get:

$$\int_{-\infty}^{\infty} dz(1 - \eta\Delta(z)) \left(\left|\frac{d\Phi}{dz}\right|^2 + k_\perp^2 \, |\Phi|^2 \right) = 0 \tag{71}$$

where the surface terms vanish since one assumes regularity at $z = \pm\infty$. Taking the imaginary part of Equation (71) shows that η must be real, and since the sign of the integral must change somewhere, then η must be greater than unity. So if the solutions to Equation (69) are to be regular everywhere, then

$$\eta \text{ real}, \quad \eta > 1 \, . \tag{72}$$

Therefore, somewhere in the plasma, at $z = z_o$, say, the coefficient $1 - \eta\Delta(z)$ of the second derivative of Equation (69) will vanish, and at that point this differential equation will have a regular singular point.[12] Let

$$\psi = \frac{1}{\Phi} \frac{d\Phi}{dz} \tag{73}$$

so that Equation (69) can be written as a Riccati equation:

$$\frac{d}{dz} (1 - \eta\Delta(z))\psi = (1 - \eta\Delta(z))(k_\perp^2 - \psi^2) \tag{74}$$

and since Φ is regular at infinity, then clearly as $z \to +\infty$, $\psi \to -k_\perp < 0$; as $z \to -\infty$, $\psi \to +k_\perp > 0$. Since Equation (69) is invariant under reflection in z, $\Phi(-z)$ is also a solution as is any linear combination of solutions. Then it is clear that in constructing solutions to Equation (69),

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[12] E. D. Rainville, Elementary Differential Equations (The Macmillan New York, 2nd ed., 1958), pp. 276-7.

one can choose even or odd solutions. From Equation (73) it is seen that ψ must be an odd function of z. Considering then the behavior for $z > 0$, it is seen from the asymptotic behavior of ψ that the only way $\psi(z)$ can become positive is if it vanishes at some z_1. Note that even if $\psi(z)$ can change sign by becoming infinite at $z = z_2$, then $\Phi(z_2) = 0$. Since $\Phi(z) \sim e^{-k_\perp z}$ as $z \to \infty$, there must be a finite $z_1 > z_2$ at which $\Phi'(z_1) = 0$ and hence $\psi(z_1) = 0$. But at any z_1 for which ψ vanishes one has from Equation (74):

$$(1 - \eta\Delta(z_1))\, \psi'(z_1) = (1 - \eta\Delta(z_1))\, k_\perp^2 \tag{75}$$

If $z_1 \neq z_o$, then $\psi'(z_1) = k_\perp^2 > 0$ and hence $\psi(z)$ cannot <u>decrease</u> through zero. Since $\psi < 0$ at infinity, it must be negative for all $z > z_o$. On the other hand, at $z = z_o \neq 0$ or ∞ and if $\Phi(z_o) \neq \infty$, then from Equation (69) $\Phi'(z_o) = 0$ and from Equation (74)

$$\psi(z_o) = 0 \tag{76}$$

unless $\Phi(z_o) = 0$. But if $\Phi = 0$ at $z = z_o$, then since $\Phi' = 0$ here also, there must exist a $z_3 > z_o$ for which $\Phi'(z_3) = 0$ and $\Phi(z_3) \neq 0$. This implies that $\psi(z_3) = 0$. But we have shown above that $\psi < 0$ for $z > z_o$. Thus $\Phi(z_o)$ cannot vanish. If for any $z_o < z < \infty$, $\psi'(z)$ can become negative, then since $1 - \eta\Delta(z) > 0$ for $z > z_o$, one can infer from Equation (74) that $\psi^2 > k_\perp^2$, that is

$$|\psi| > k_\perp \ . \tag{77}$$

From the parity arguments it has been shown Φ can be chosen to be either even or odd. As one moves infinitesimally away from the origin toward positive z, Φ and Φ' have the same sign which implies $\psi > 0$ for $z = 0^+$. It is seen that if there exists a z_4, $0 < z_4 < z_o$ such that $\psi(z_4) = 0$

then $\psi'(z_4) = k_\perp^2 > 0$. Hence $\psi > 0$ for $0 < z < z_o$. The coefficient $1 - \eta\Delta(z)$ is less than zero for $0 < z < z_o$ and if $\psi'(z)$ can become negative, then again from Equation (74) it is inferred that $|\psi| > k_\perp$. Thus it is concluded that about $z = z_o$, ψ is bounded above and below by the value $k_\perp$. Hence if Φ is to be regular at $z = +\infty$, then it cannot be regular at $z = z_o$. Similar arguments hold for $z = -z_o$.*

It has been seen above that the electric potential has a singularity which depends upon η. It was shown[13] that the η's formed a continuum and that by constructing wave-packet solutions the singular character of the solution was "smeared out" thus enabling them to derive a connection formula for the solution across the singularity in the beam. This connection formula turns out to be for the slab geometry simply:

$$(1 - \eta\Delta)\frac{d\Phi}{dz} \text{ is continuous across } z = z_o \tag{78}$$

which is just

$$D_n = K_{33}E_n \text{ is continuous} \tag{79}$$

[14] —

familiar from electrostatics, where

$$K_{33} = 1 - \eta\Delta \ . \tag{80}$$

However, one does not have Φ continuous across the singularity.

In the special case of a sharp boundary between plasma and vacuum, Frieman *et al.*[15] argue that one or more of the original assumptions, such

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*The author acknowledges helpful discussions with Dr. R. L. Becker and Dr. G. E. Guest.

[13] *Ibid.*

[14] A. B Kitsenko and K. N. Stepanov, J. Tech. Phys. 6, 127 (1961). This reference gives the components of the dielectric tensor in terms of integrals over the equilibrium distribution function.

[15] Frieman *et al.*, *loc. cit.*

as zero parallel temperature, small fields, etc., breaks down within the slab edge, so that Φ is no longer singular and one does have the boundary condition:

$$\Phi \text{ is continuous} . \tag{81}$$

With this additional boundary condition the eigenfrequencies are effectively discrete.[16] The sharp edge case will be used throughout this dissertation, again referring the reader to the original paper of Frieman et al. for details of the "weak-beam" situation. Appendix B gives a brief outline of the above considerations using the initial value treatment.

Letting ω_{pe} be constant inside the slab and drop rapidly to zero within a narrow edge at $z = \pm z_o$, Equation (69) then becomes:

$$(1 - \eta)\left(\frac{d^2\Phi}{dz^2} - k_\perp{}^2\Phi\right) = 0 \quad \text{for } |z| < a . \tag{82}$$

Outside the plasma:

$$\frac{d^2\Phi}{dz^2} - k_\perp{}^2\Phi = 0 . \tag{83}$$

The solutions of the above equations are:

$$\Phi = a_1 e^{-k_\perp z} \qquad z < a \tag{84}$$

$$\Phi = a_2 e^{-k_\perp z} + a_3 e^{k_\perp z} \qquad |z| < a \tag{85}$$

$$\Phi = a_4 e^{k_\perp z} \qquad z < -a . \tag{86}$$

One also recovers the infinite plasma solution

$$\eta = 1 \tag{87}$$

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[16] Equations (79) and (81) constitute the plasma-vacuum or free boundary conditions as considered by W. P. Allis, S. J. Buchsbaum, and A. Bers, Waves in Anisotropic Plasma (M.I.T. Press, Cambridge, Mass., 1963), pp. 132-ff.

from Equation (82). Equation (87) indicates an arbitrary choise of $\Phi(z)$ inside the plasma slab. Thus, since the connection formula (78) can be written

$$\left.\frac{d\Phi}{dz}\right|_{a-\epsilon} - \left.\frac{d\Phi}{dz}\right|_{a+\epsilon} = \eta \left.\frac{d\Phi}{dz}\right|_{a-\epsilon}$$

and since $\eta = 1$, then

$$\left.\frac{d\Phi}{dz}\right|_{a+\epsilon} = 0 \ ,$$

that is, just outside the plasma and similarly for $z = -a$. Then since $\Phi = 0$ at $z = \pm\infty$, $\Phi(z)$ must be zero everywhere outside the plasma slab. This also indicates that the potential in the interior must vanish at $z = \pm a$ since from Equation (81) the potential must be continuous across the interface. Since the solutions vanish at $z = \pm a$, it is sufficient to take $k_z = \frac{n\pi}{2a}$, $n = 0, \pm 1, \pm 2, \ldots$ as the z component of the wave vector.*

Using Equations (84), (85), and (86) with Equations (79), (80), and (81), gives a set of four homogeneous, algebraic equations with constant coefficients. For a non-trivial solution to exist the determinant of the coefficients must vanish, giving:

$$\eta = \frac{2}{1 \pm e^{-2k_\perp a}} \ . \tag{88}$$

One should notice that Equations (87) and (88) satisfy the original conditions that η must be real and greater than or equal to unity. One can find the dependence of ω on $k_\perp$, a, $\omega_{pe}{}^2$, and $\alpha_\perp$ from Equations (88) and (70).

The effect of the boundary conditions can be seen from the limiting values of Equation (88). In the limit

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*The author is grateful to Dr. T. K. Fowler for pointing out this fact.

$$k_\perp a \to \infty \tag{89}$$

then

$$\eta \to 2 \tag{90}$$

indicating that for a bounded plasma the boundary effects have a role even in the case of infinite separation. In the limit

$$k_\perp a \to 0 \tag{91}$$

then for the positive root of Equation (88)

$$\eta \to 1 \tag{92}$$

and for the negative root

$$\eta \to \infty \, . \tag{93}$$

In general, using Equation (70) and the expansion formulas found in Fried and Conte, the dispersion relation can be solved for any value of $k_\perp a$.

However, from Equation (93) one infers for the negative root of Equation (88) that the plasma is always stable. One can conclude that the boundary effects although present do not limit the behavior of the plasma in any major respect in comparison with the infinite plasma behavior. This is in general agreement with the paper of Frieman et al.[17] who conclude that the worst instabilities of a finite beam are essentially the same as those for an infinite beam.

If one includes a static magnetic field in the z-direction, the solution of the Vlasov equations, although more involved, goes through as before, and one finally finds the differential equation for the transformed electric potential $\Phi(\vec{k}_\perp, \omega, z)$ to be:*

$$\frac{d}{dz}\left[(1 - \zeta\Delta(z))\frac{d\Phi}{dz}\right] - k_\perp^2(1 - \xi\Delta(z))\Phi = 0 \tag{94}$$

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[17] Frieman et al., loc. cit.

*The reader is referred to Appendix C for the details of the derivation of Equation (94).

where

$$\zeta \equiv \sum_j \omega_{poj}^2 \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_j)e^{-\lambda_j}}{(\omega - n\omega_{cj})^2} , \tag{95}$$

$$\xi \equiv \sum_j \frac{\omega_{poj}^2}{\lambda_j \omega_{cj}} \sum_{n=-\infty}^{\infty} \frac{nI_n(\lambda_j)e^{-\lambda_j}}{(\omega - n\omega_{cj})} , \tag{96}$$

and

$$\lambda_j \equiv \frac{k_\perp^2 \alpha_{\perp j}^2}{2\omega_{cj}^2} . \tag{97}$$

Multiplying by Φ^*, assuming regularity at $z = \infty$, integrating by parts, and taking the imaginary part, gives:

$$\mathrm{Im}\left\{\langle k_z^2\rangle \zeta + \langle k_\perp^2\rangle \xi \right\} = 0 \tag{98}$$

where

$$\langle k_z^2\rangle \equiv \int dz \left|\frac{d\Phi}{dz}\right|^2 \Delta \tag{99}$$

and

$$\langle k_\perp^2\rangle \equiv \int dz\, k_\perp^2 |\Phi|^2 \Delta . \tag{100}$$

Equation (98) is very similar to the infinite dispersion relation for $\vec{B} = B_o\hat{z}$:[18]

$$1 = \left(\frac{k_z}{k}\right)^2 \zeta + \left(\frac{k_\perp}{k}\right)^2 \xi . \tag{101}$$

Equation (94) is singular when the coefficient, $1 - \zeta\Delta(z)$, of the second derivative vanishes for some $z = +z_o$. Since $\Delta \leq 1$, then ζ must be greater than or equal to unity for this condition to be satisfied. However, unlike Equation (69) there are no restrictive conditions on ζ other than that

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[18] E. G. Harris, "Unstable Plasma Oscillations in a Magnetic Field," ORNL-2728 (June 5, 1959).

given by Equation (98). Thus one has for the sharp-edge interface:

$$\frac{d^2\Phi}{dz^2} - m^2 \Phi = 0 \qquad |z| < a ,$$

and

$$\frac{d^2\Phi}{dz^2} - k_\perp{}^2 \Phi = 0 \qquad |z| > a \tag{103}$$

where

$$m^2 \equiv k_\perp{}^2 \left(\frac{1 - \xi}{1 - \zeta}\right) . \tag{104}$$

For an infinite plasma the potential varies as

$$\Phi \sim e^{ik_z z} \tag{105}$$

and Equation (102) becomes:

$$(-k_z{}^2 - m^2)\Phi = 0 \tag{106}$$

or

$$m^2 = -k_z{}^2 . \tag{107}$$

Using Equation (104), (95), and (96), Equation (107) can be written

$$Y(\omega) = 1 = \sum_j \sum_{n=-\infty}^{\infty} \omega_{poj}{}^2 \; e^{-\lambda_j} I_n(\lambda_j) \left\{ \left(\frac{k_\perp}{k}\right)^2 \frac{n}{\lambda_j \omega_{cj}(\omega - n\omega_{cj})} + \left(\frac{k_z}{k}\right)^2 \frac{1}{(\omega - n\omega_{cj})^2} \right\} \tag{108}$$

which is just Equation (49) of Harris's[19] original work.

In order to make the solution of the non-zero magnetic field case a little easier, assume only a one plasma-vacuum interface at $z = 0$. Also first assume that $m^2 < 0$ and let

$$m^2 \equiv -k_z{}^2 . \tag{109}$$

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[19] Ibid.

Then the solution to Equation (102) is

$$\Phi = a_1 e^{ik_z z} + a_2 e^{-ik_z z} \tag{110}$$

and the solution to Equation (103) is

$$\Phi = a_3 e^{-k_\perp z} . \tag{111}$$

Putting Equations (110) and (111) into the boundary conditions given by Equations (79) and (81), where

$$K_{33} = 1 - \zeta\Delta \tag{112}$$

gives the following conditions:

$$a_1 + a_2 = a_3 \tag{113}$$

and

$$a_3 = - \frac{ik_z}{k_\perp} (1 - \zeta)(a_1 - a_2) . \tag{114}$$

Eliminating a_3 and solving for a_1/a_2 gives:

$$\frac{a_1}{a_2} = - \frac{\left[1 - \frac{ik_z}{k_\perp} (1 - \zeta)\right]}{\left[1 + \frac{ik_z}{k_\perp} (1 - \zeta)\right]} . \tag{115}$$

This case corresponds to waves approaching the boundary and being perfectly reflected.* To solve the problem one must choose ω arbitrarily and calculate k_z from:

$$-k_z{}^2 = k_\perp{}^2 \left(\frac{1 - \xi(\omega)}{1 - \zeta(\omega)}\right) . \tag{116}$$

If one does not make the approximation of a single interface, but considers the original the original slab geometry, the solution of the

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*Notice $|a_1|/|a_2| = 1$, as would be expected since longitudinal waves cannot propagate in the vacuum region.

problem results in the evaluation of the familiar secular determinant which finally reduces to:

$$\sin 2k_z a = \frac{2(1 - \zeta)k_z k_\perp}{k_\perp^2 + (1 - \zeta)^2 k_z^2} , \tag{117}$$

where as before $m^2 \equiv -k_z^2$. Using Equation (109), Equation (104) can be written:

$$Y(\omega, k_z) = 1 = \sum_j \omega_{poj}^2 \sum_{n=-\infty}^{\infty} e^{-\lambda_j} I_n(\lambda_j) \left\{ \frac{k_\perp^2}{k_z^2 + k_\perp^2} \frac{n}{\lambda_j \omega_{cj}(\omega - n\omega_{cj})} + \frac{k_z^2}{k_z^2 + k_\perp^2} \frac{1}{(\omega - n\omega_{cj})^2} \right\} \tag{118}$$

Given a value of the slab width, a, one could in principle solve Equations (117) and (118) simultaneously for ω and k_z. Obviously, however, this would require solving a very involved trancendental equation. Instead, choose k_z arbitrarily, then determine ω from the solution of Equation (118). Now k_z, ω and hence ζ are known. Finally, use Equation (117) to find a, the length of the machine for which the calculation applies. As a simple example let

$$k_z \gg k_\perp . \tag{119}$$

Then Equation (118) is almost independent of k_z, and its solution determines ω. From Equation (117)

$$\sin 2k_z a = \frac{2(1 - \zeta)k_z k_\perp}{k_\perp^2 + (1 - \zeta)^2 k_z^2} \approx \frac{2k_\perp}{(1 - \zeta)k_z} \approx 0$$

if $1 - \zeta$ does not become zero for the above value of ω.

Thus

$$2k_z a = n\pi$$

or

$$k_z = \frac{n\pi}{2a} . \tag{120}$$

Now assume $m^2 > 0$ and go back to the single interface approximation. From the solution of the boundary value problem one finds:

$$m = \frac{k_\perp}{\zeta - 1} . \tag{121}$$

Using Equation (121) in Equation (104) gives:

$$Y(\omega) = 1 = \sum_j \omega_{poj}^2 \sum_{n=-\infty}^{\infty} e^{-\lambda_j} I_n(\lambda_j) \left\{ \frac{1}{1 - (1 - \zeta)^2} \frac{1}{(\omega - n\omega_{cj})^2} - \frac{(1 - \zeta)^2}{1 - (1 - \zeta)^2} \frac{n}{\lambda_j \omega_{cj}(\omega - n\omega_{cj})} \right\} . \tag{122}$$

If one examines $Y(\omega)$ in the neighborhood of $\omega_{ce} \gg \omega \sim \ell\omega_{ci}$, only the term with $n = \ell$ in the above sum is important. From Equation (95) one sees that:

$$\zeta \to \infty \quad \text{as } \omega \to \ell\omega_{ci} \tag{123}$$

and

$$\frac{(1 - \zeta)^2}{1 - (1 - \zeta)^2} \to -1 \tag{124}$$

and

$$\frac{1}{1 - (1 - \zeta)^2} \to -\frac{1}{\zeta^2} . \tag{125}$$

Therefore, Equation (122) can be reduced to the following approximate expression:

$$1 = Y(\omega) \approx \frac{4\ell^2 \omega_{pi}^2 C_\ell}{\lambda_i(\omega^2 - \ell^2\omega_{ci}^2)} \tag{126}$$

which indicates that ω^2 is always positive and hence ω has no imaginary part which means stability.

For the case

$$m = 0 \tag{127}$$

Equation (104) turns out to be the result Harris[20] finds for the infinite plasma case with $k_z = 0$. It was shown[21] that there was neither instability nor Landau damping for waves propagating perpendicular to the magnetic field.

Boundary value problems similar to the Harris bounded model have been studied by Allis, Buchsbaum, and Bers[22] who devote the second part of their book to the analysis of plasma waveguides and associated boundary-value problems.

IV. THE BURT-HARRIS MODEL

This problem is based on the finite cylindrical shell of cold plasma model of Burt and Harris[23] in which the initial ion density moves in concentric Larmor orbits with velocities:

$$\vec{V}_{oi} = r\omega_{ci}\hat{\phi} \tag{128}$$

and where the electrons have zero velocity:

$$\vec{V}_{oe} = 0 \ . \tag{129}$$

This model would be applicable to a thermonuclear device with very weak mirrors in which $v_{zj}^{o} \approx 0$. The equations governing the particles for the

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[20] Ibid.

[21] I. B. Bernstein, loc. cit.

[22] Allis, Buchsbaum, and Bers, loc. cit.

[23] P. Burt, dissertation, University of Tennessee, 1961; and P. Burt and E. G. Harris, Phys. Fluids 4, 1412 (1961).

cold plasma are

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{v}_j) = 0 \tag{130}$$

which is the equation of continuity and

$$\frac{\partial \vec{v}_j}{\partial t} + \vec{v}_j \cdot \nabla \vec{v}_j = \left(\frac{e}{m}\right)_j \vec{E} + \left(\frac{e}{m}\right)_j \frac{\vec{v}_j \times \vec{B}}{c} \tag{131}$$

which is the equation of motion for the particles where the subscript j represents the species of the plasma.

The equations are linearized just as before by assuming

$$\vec{E} = \vec{E}_1 \ , \tag{132}$$

$$\vec{B} = -B_o \hat{z} \ , \tag{133}$$

$$n_j = n_{oj}(r, z) + n_{1j}(r, \phi, z, t) \ , \tag{134}$$

and

$$\vec{v}_j = \vec{V}_{oj} + \vec{u}_j(r, \phi, z, t) \tag{135}$$

where products of perturbation quantities will be neglected, and where the r and z dependence of the equilibrium density has been left arbitrary. Substituting Equations (132) through (135) into Equation (131), gives (for the ions:*

$$\frac{\partial \vec{u}_i}{\partial t} + \omega_{ci} \frac{\partial \vec{u}_i}{\partial \phi} + \omega_{ci}(u_{ri}\hat{\phi} - u_{\phi i}\hat{r}) = \frac{e}{m_i} \vec{E} \ . \tag{136}$$

Assuming that all perturbation quantites vary as

$$\sim g(r,z)\ e^{i\omega t - i\ell\phi} \tag{137}$$

one can write Equation (136) as

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*The fact that $\partial\hat{r}/\partial\phi = \hat{\phi}$ and $\partial\hat{\phi}/\partial\phi = -\hat{r}$ has been used in this derivation.

$$\vec{u}_i = \frac{e}{m_i} \overleftrightarrow{\sigma}_i \cdot \vec{E} \tag{138}$$

where the tensor $\overleftrightarrow{\sigma}_i$ is defined by

$$\overleftrightarrow{\sigma}_i \equiv \begin{bmatrix} \dfrac{i\Omega_\ell}{\omega_{ci}^2 - \Omega_\ell^2} & \dfrac{\omega_{ci}}{\omega_{ci}^2 - \Omega_\ell^2} & 0 \\ -\dfrac{\omega_{ci}}{\omega_{ci}^2 - \Omega_\ell^2} & \dfrac{i\Omega_\ell}{\omega_{ci}^2 - \Omega_\ell^2} & 0 \\ 0 & 0 & \dfrac{1}{i\Omega_\ell} \end{bmatrix}, \tag{139}$$

and where

$$\Omega_\ell \equiv \omega - \ell\omega_{ci} \ . \tag{140}$$

Similarly, for the electrons one gets:

$$\vec{u}_e = -\frac{e}{m_e} \overleftrightarrow{\sigma}_e \cdot \vec{E} \tag{141}$$

where

$$\overleftrightarrow{\sigma}_e \equiv \begin{bmatrix} \dfrac{i\omega}{\omega_{ce}^2 - \omega^2} & \dfrac{\omega_{ce}}{\omega_{ce}^2 - \omega^2} & 0 \\ -\dfrac{\omega_{ce}}{\omega_{ce}^2 - \omega^2} & \dfrac{i\omega}{\omega_{ce}^2 - \omega^2} & 0 \\ 0 & 0 & \dfrac{1}{i\omega} \end{bmatrix}$$

and

$$\omega_{ce} = -\frac{eB_o}{m_e c} \ . \tag{143}$$

Using Equation (137) in (130) and substituting the perturbed density into Poisson's equation which is:

$$\nabla \cdot \vec{E}_1 = 4\pi e \left(n_{1i} - n_{1e}\right) , \tag{144}$$

one gets

$$\nabla \cdot \left[\left(\overleftrightarrow{1} + \frac{\omega_{pi}^{2}(r,\ z)}{i\Omega_{\ell}} \overleftrightarrow{\sigma}_i + \frac{\omega_{pe}^{2}(r,\ z)}{i\omega} \overleftrightarrow{\sigma}_e \right) \cdot \vec{E} \right] = 0 \ . \qquad (145)$$

If one neglects the perturbed part of the magnetic field so that

$$\vec{E}_{\perp} = -\nabla\Phi \ , \qquad (146)$$

then

$$\nabla \cdot \left[\overleftrightarrow{K} \cdot \nabla\Phi \right] = 0 \qquad (147)$$

is the differential equation for the electric potential of the Burt-Harris plasma, with

$$\overleftrightarrow{K} \equiv \overleftrightarrow{1} + \frac{\omega_{pi}^{2}(r,\ z)}{i\Omega_{\ell}} \overleftrightarrow{\sigma}_i + \frac{\omega_{pe}^{2}(r,\ z)}{i\omega} \overleftrightarrow{\sigma}_e \qquad (148)$$

defined as the dielectric tensor of the plasma.

Dividing Equation (147) into its perpendicular and parallel parts, results in:

$$\nabla_{\perp} \cdot \left[K_{11} \nabla_{\perp}\Phi \right] - \frac{i\ell}{r} \Phi \frac{\partial K_{12}}{\partial r} + \frac{\partial}{\partial r} \left(K_{33} \frac{\partial\Phi}{\partial z} \right) = 0 \ . \qquad (149)$$

If one considers only the region of ω for which

$$\omega_{pi} \ll \omega \approx \ell\omega_{ci} \approx \omega_{pe} \ll |\omega_{ce}| ,$$

then the dielectric tensor becomes:

$$\overleftrightarrow{K} = \begin{bmatrix} 1 - \dfrac{\omega_{pi}^{2}}{\Omega_{\ell}^{2} - \omega_{ci}^{2}} & -\dfrac{\omega_{ci}\omega_{pi}^{2}}{i\Omega_{\ell}(\Omega_{\ell}^{2} - \omega_{ci}^{2})} & 0 \\ \dfrac{\omega_{ci}\omega_{pi}^{2}}{i\Omega_{\ell}(\Omega_{\ell}^{2} - \omega_{ci}^{2})} & 1 - \dfrac{\omega_{pi}^{2}}{\Omega_{\ell}^{2} - \omega_{ci}^{2}} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_{pi}^{2}}{\Omega_{\ell}^{2}} - \dfrac{\omega_{pe}^{2}}{\omega^{2}} \end{bmatrix} \qquad (150)$$

and Equation (149) reads:

$$\nabla_\perp \cdot \left(1 - \frac{\omega_{pi}^2}{\Omega_\ell^2 - \omega_{ci}^2}\right) \nabla_\perp \Phi + \frac{\ell}{r} \Phi \frac{\omega_{ci}}{\Omega_\ell(\Omega_\ell^2 - \omega_{ci}^2)} \frac{\partial \omega_{pi}^2}{\partial r}$$
$$+ \frac{\partial}{\partial z}\left[\left(1 - \frac{\omega_{pi}^2}{\Omega_\ell^2} - \frac{\omega_{pe}^2}{\omega^2}\right)\frac{\partial \Phi}{\partial z}\right] = 0 . \tag{151}$$

Concentrating on the root near $\omega \sim \ell\omega_{ci}$, one has

$$1 - \frac{\omega_{pi}^2}{\Omega_\ell^2 - \omega_{ci}^2} \simeq 1 + \frac{\omega_{ci}^2}{\omega_{ci}^2} \sim 1$$

and if

$$\omega_{pi}^2 = \omega_{pi}^2(z) ,$$

then

$$\frac{\partial \omega_{pi}^2}{\partial r} = 0$$

and if

$$\omega_{pi}^2 = \omega_{pi}^2(r) ,$$

then assume

$$\frac{1}{\Omega_\ell \omega_{ci}} \frac{\partial \omega_{pi}^2}{\partial r} \ll 1$$

so that the differential equation becomes

$$\nabla_\perp^2 \Phi + \frac{\partial}{\partial z}\left[(1 - \eta\Delta(r, z)) \frac{\partial \Phi}{\partial z}\right] = 0 , \tag{152}$$

where the $\Delta(r, z)$ is defined so that it is less than or equal to unity but greater than zero as r and z vary, and where

$$\eta \equiv \frac{\omega_{poi}^2}{\Omega_\ell^2} + \frac{\omega_{poe}^2}{\omega^2} \tag{153}$$

is not a function of r or z.

Suppose that the plasma is infinite in the r direction and that

$$\Delta = \Delta(z) \tag{154}$$

then assuming the potential to be given by:

$$\Phi = e^{i\omega t}\ \ell^{-i\ell\phi} J_\ell\ (k_\perp r) F(z) \tag{155}$$

so that the r and ϕ dependence of $\Phi(r, \phi, z)$ transform like:

$$\int_0^\infty J_m(k_\perp r)\ \frac{\partial}{\partial r}\ (r\ \frac{\partial \Phi}{\partial r})\ dr\ -\ \ell^2 \int_0^\infty \frac{1}{r}\ J_m(k_\perp r)\ \Phi dr$$

$$= -k_\perp{}^2 \int_0^\infty J_m(k_\perp r)\Phi(r) r dr + (m^2 - \ell^2) \int_0^\infty \frac{1}{r}\ J_m(k_\perp r)\Phi(r) dr$$

$$= -k_\perp{}^2\ \Phi \text{ when } m = \ell$$

and Equation (152) becomes

$$\frac{d}{dz}\left[(1 - \eta\Delta)\ \frac{dF}{dz}\right] - k_\perp{}^2 F = 0\ . \tag{156}$$

Applying the same analysis as in the Harris bounded model, one can show that η must be real and

$$n > 1\ . \tag{157}$$

Equation (156) is very similar to Equation (69) of Section III and one can prove similarly that if the potential is to be regular at $z = \pm\infty$, then the solutions must be singular somehwere in the plasma edge. However, assuming a sharp boundary so that the eigenfrequencies are discrete, one can write the equations for the potential as

$$\nabla \cdot (\overleftrightarrow{K} \cdot \nabla\Phi) = 0 \qquad \text{for } |z| < a \tag{158}$$

and

$$\nabla^2\Phi = 0 \qquad \text{for } |z| < a\ , \tag{159}$$

and taking $\Delta = 1$ inside the plasma and falling rapidly to zero at $z = \pm a$, then the boundary conditions can be written as

$$(1 - \eta) \frac{d\Phi}{dz}\Big|_{z = \pm a \mp \epsilon} = \frac{d\Phi}{dz}\Big|_{z = \pm a \pm \epsilon} \tag{160}$$

and

$$\Phi\Big|_{z = \pm a \mp \epsilon} = \Phi\Big|_{z = \pm a \pm \epsilon} . \tag{161}$$

Equation (158) becomes, using Equation (154),

$$\frac{d^2F}{dz^2} = - \left(\frac{k_\perp{}^2}{\eta - 1}\right) F , \quad |z| < a$$

whose solution is

$$F \sim \sin k_z z$$

where

$$k_z{}^2 \equiv \frac{k_\perp{}^2}{\eta - 1} , \tag{162}$$

and for Equation (159) the equation is:

$$\frac{d^2F}{dz^2} = k_\perp{}^2 F \qquad |z| > a$$

and whose solution is

$$F \sim e^{-k_\perp z} \quad \text{for} \quad z > a$$

and

$$F \sim e^{k_\perp z} \quad \text{for} \quad z < -a .$$

Substituting these forms into the boundary conditions, gives a secular determinant, the solution of which is:

$$\tan 2k_z a = \frac{2\kappa}{1 - \kappa^2} \tag{163}$$

and where

$$\kappa \equiv \frac{k_\perp}{k_z} . \tag{164}$$

Using the tangent double-angle formula, the dispersion relation (163) can be written:

$$\frac{2 \tan k_z a}{1 - \tan^2 k_z a} = \frac{2\kappa}{1 - \kappa^2} \tag{165}$$

which means that

$$\tan k_z a = \left\{ \begin{matrix} \kappa \\ -\frac{1}{\kappa} \end{matrix} \right\} . \tag{166}$$

It can be shown that the only solutions to the above equation are those for real k_z.[24] This is reasonable since it has been shown that $\eta \geq 1$ and real, and k_z and η are related through Equation (162). One can plot Equation (166) to find the solutions as seen in Figure 1. In principle, the roots of Equation (166) can be found and the dispersion relation solved. Particularly for large $k_z a$ the solutions to Equation (163) can be written:

$$\kappa \approx (n + \frac{1}{2})\, \pi \tag{167}$$

and

$$\kappa \approx n\pi \ . \tag{168}$$

Equation (167) and (168) are very similar to the odd and even modes derived by Dawson and Oberman[25] for a similar geometry but where the particles are constrained to move along the field line. However, in general the roots must be found numerically, and calling these roots κ_n, the dispersion relation becomes:

$$\eta_n = 1 + \kappa_n \tag{169}$$

η being given by Equation (153).

Now if one assumes z to be the infinite coordinate, the differential equation is written (see Equation (152)):

$$\nabla_\perp^2 \Phi - k_z^2 (1 - \eta \Delta(r))\, \Phi = 0 \tag{170}$$

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[24] Soper, loc. cit.

[25] J. Dawson and C. Oberman, Phys. Fluids 2, 103 (1959).

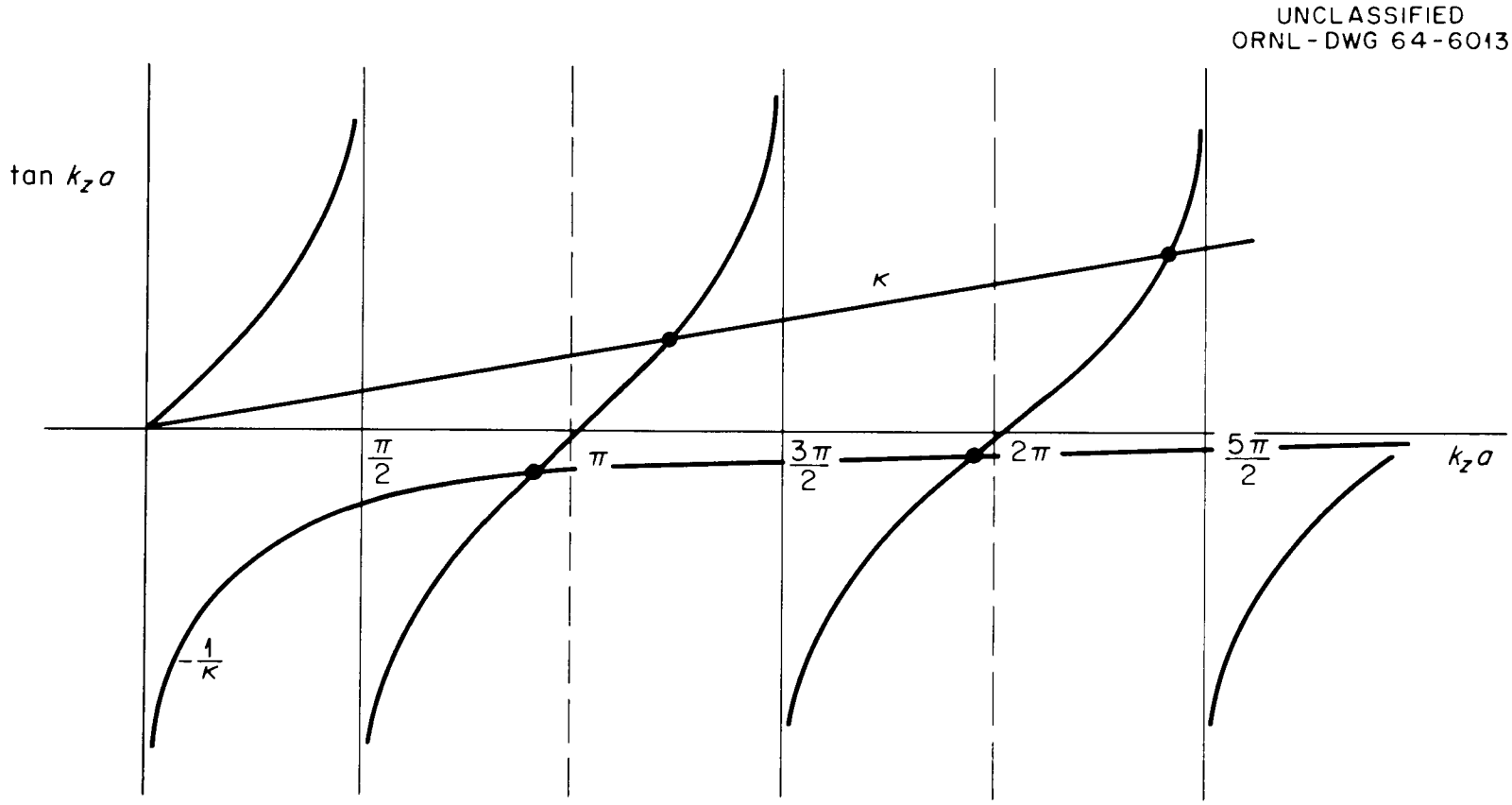

Figure 1. Graphical representation of the Burt-Harris dispersion relation.

where the z dependence has been given by a factor $e^{ik_z z}$. One can see that Equation (170) is singular nowhere, the eigenfrequencies are discrete, and in principle Equation (170) can be solved for any r dependence on Δ. However, for simplicity only the sharp boundary case is considered here. With $\Delta = 1$ for $r < R$, R being the radius of the plasma cylinder, the differential equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{\ell^2}{r^2}\Phi - k_z^{\,2}(\eta - 1)\Phi = 0$$

whose solution is:

$$\Phi = a_1 J_\ell\,(k_\perp r) + a_2 Y_\ell\,(k_\perp r)$$

with

$$k_\perp^{\,2} \equiv k_z^{\,2}\,(\eta - 1)\ . \tag{171}$$

But $a_2 = 0$ since $Y_\ell(0) = \infty$, so

$$\Phi(r) = a_1 J_\ell\,(k_\perp r) \qquad r < R\ . \tag{172}$$

Outside where $\Delta = 0$ the differential equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial f}\right) - \frac{\ell^2}{r^2}\Phi - k_z^{\,2}\Phi = 0$$

whose solution is

$$\Phi(r) = a_3 I_\ell\,(k_z r) + a_4 K_\ell\,(k_z r)\ .$$

But $a_3 = 0$ since $I_\ell(\infty) = \infty$, and so one has for $r > R$

$$\Phi(r) = a_4 K_\ell\,(k_z r)\ .$$

Applying the familiar boundary conditions and solving the resulting secular determinant gives,

$$k_\perp(1 - \eta)\,\frac{J_\ell'(k_\perp R)}{J_\ell(k_\perp R)} = k_z\,\frac{K_\ell'(k_z R)}{K_\ell(k_z R)} \tag{174}$$

where the prime means differentiation with respect to the argument. This equation determines a discrete set of $k_\perp$ for a given k_z and this used in Equation (171)

$$\eta = 1 + \frac{k_\perp^2}{k_z^2} \tag{175}$$

gives discrete η. Equation (175) is essentially Equation (120) of Frieman et al.,[26] with $k_\perp/k_z$ of Equation (171) above replacing their ξ^2. These authors find, as mentioned before, that the worst instabilities of a finite beam are essentially the same as those of an infinite beam.

V. THE SHIMA MODEL*

For the infinite plasma with bi-Maxwellian distribution described in Section II, the many ion orbits overlap. There was no average velocity and no average current. However, for machines such as the DCX's and OGRA in which the plasma is of finite size--about one or two ion orbits across--the infinite plasma approximation is no longer justified. Although there is a uniform field, there will be a strong diamagnetic current because of the finite size of the plasma. Shima[27] has shown that this situation can be approximated by another exactly solvable problem, again an infinite plasma, but one in which the bi-Maxwellian ion distribution rotates as a whole at the ion-cyclotron frequency. Also, the ions are allowed to have different temperatures parallel and perpendicular to the external magnetic field as seen in the reference frame rotating with the ions.

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[26] Frieman et al., loc. cit.

*This discussion is based on remarks by T. K. Fowler and Y. Shima.

[27] Y. Shima, Thermonuclear Div. Semiann. Progr. Rept. Oct. 31, 1962, ORNL-3392, p. 61.

So for Shima's case the equilibrium distribution function for the electrons is of the form $f_{oe}(v_\perp, v_z)$ and for the ions is of the form $f_{oi}(v_\perp', v_z)$ where $v_\perp'$ is the velocity magnitude in the x-y plane of reference rotating at the ion-cyclotron frequency and in the same sense as the ions. (See Figure 2.) The rotating distribution function $f_{oi}(v_\perp', v_z)$, except for boundary effects, describes a spatially uniform beam with finite temperature in a uniform field. The Burt-Harris problem is a special case of Shima's model in that Burt and Harris make the approximation of zero temperature in the rotating frame of reference.

Analyzing the problem in cylindrical coordinates in order to couple the rotating ions to the fixed electrons, Shima found in the extreme* case of ions rotating at the ion-cyclotron frequency, that the dispersion relation is just exactly as that for the infinite plasma model except that the ion terms of the species sum have been Doppler shifted by $\ell\omega_{ci}$, where ℓ is an integer denoting the azimuthal mode number. Thus for numerical work one need only add the term $\ell\omega_{ci}/k_z\alpha_{zi}$ to the argument of the Fried-Conte plasma dispersion function in the ion terms in order to take over the infinite plasma formalism.

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*The word extreme is used since in most machines the frequency of rotation of the ions is somewhat less than ω_{ci}.

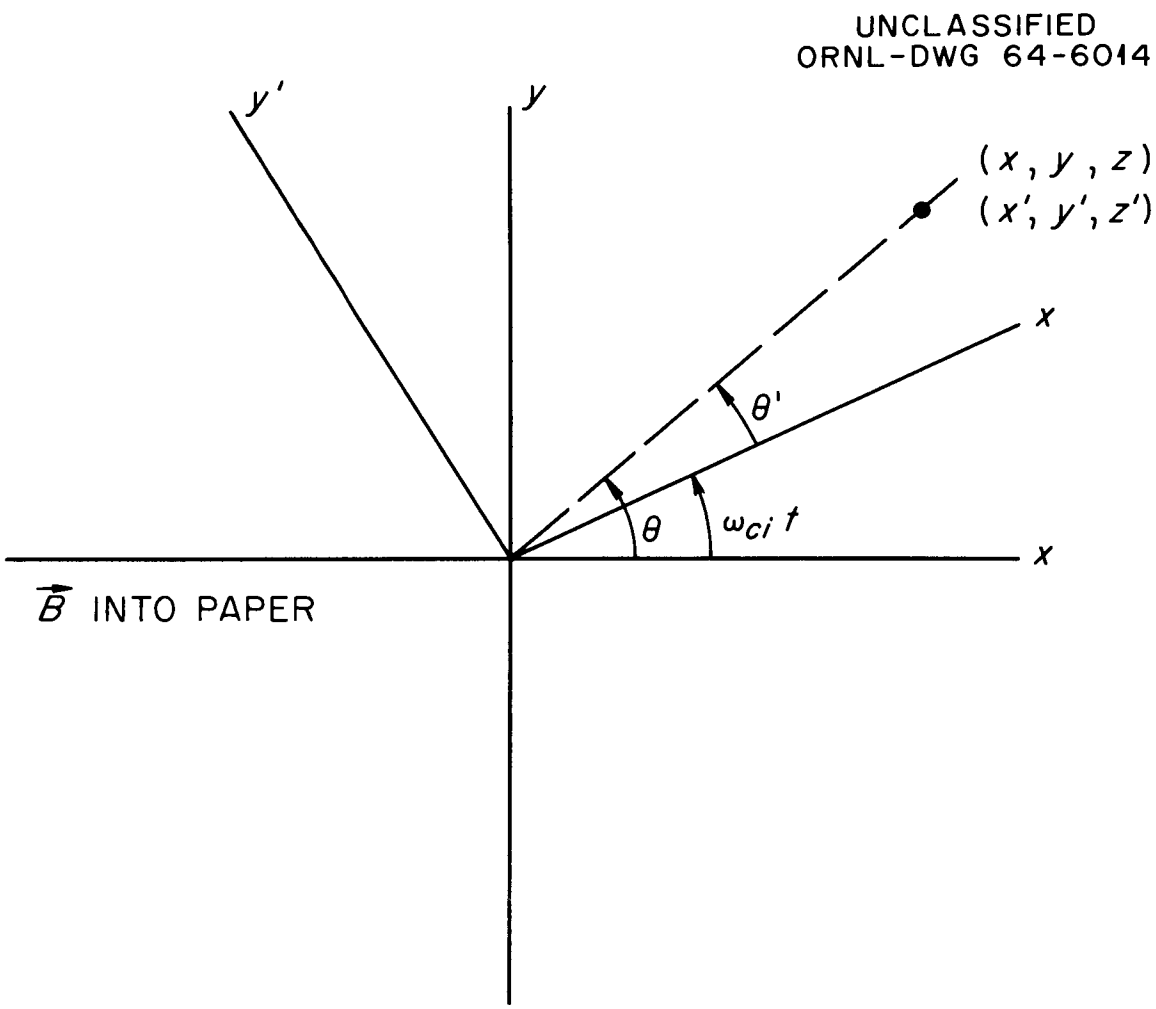

Figure 2. Coordinate system for the Shima model.

CHAPTER IV

THE NUMERICAL SOLUTION OF THE INFINITE PLASMA DISPERSION RELATION

The solution of the infinite plasma dispersion relation for ions and electrons with bi-Maxwellian velocity distributions has been investigated previously in an attempt to extend Harris's original work.[1] The work included in this chapter has been done with parameters and simplifying assumptions which approximate the thermonuclear machines mentioned in Chapter II. Comparison with previous results show qualitatively similar results. However, the numerical solution of the Shima Model indicates that one of the anisotropic modes of the infinite problem interacts strongly with the diamagnetic current and becomes essentially the instability found by Burt and Harris.[2]

The numerical work was done with the aid of the Control Data Corporation 1604-A computer system. The numerical codes were written in Fortran 62 language and the total computer time for each run was usually less than three minutes. A typical code is included at the end of this chapter.

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[1] A. V. Timofeev, Soviet Physics JETP 12, 281 (1961); Yu. N. Dnestrovsky, D. P. Kostomarov, and V. I. Pistunovich, Nuclear Fusion 3, 30 (1963); Y. Ozawa, I. Kaji, and M. Kito, J. Nucl. Energy, Pt. C 4, 271 (1962); and Laurence S. Hall and W. Heckrotte, "Stability of Longitudinal Oscillations in a Uniform Magnetized Plasma with Anisotropic Velocity Distribution," UCRL-7627 (December 9, 1963).

[2] P. Burt and E. G. Harris, Phys. Fluids 4, 1412 (1961); and E. G. Harris, "The Effect of Finite Ion and Electron Temperatures on the Ion-Cyclotron Resonance Instability," CLM-R32 (October, 1963).

I. REDUCTION OF THE DISPERSION RELATION

The dispersion relation to be solved was derived in Equation (34) of Chapter III:

$$1 = \sum_j \left(\frac{\omega_{pj}}{k_z \alpha_{zj}}\right)^2 \sum_{n=-\infty}^{\infty} C_{|n|} \left\{ \left(\frac{k_z}{k}\right)^2 Z'\left(\frac{\omega - n\omega_{cj}}{k_z \alpha_{zj}}\right) - \left(\frac{k_\perp}{k}\right)^2 \frac{n}{\lambda_j\left(\frac{\omega_{cj}}{k_z \alpha_{zj}}\right)} Z\left(\frac{\omega - n\omega_{cj}}{k_z \alpha_{zj}}\right) \right\} \tag{1}$$

where the quantities above have been defined by Equations (34) through (36) of Chapter III. Performing the sum over the species of the plasma and writing the dispersion relation in dimensionless form, the above equation becomes:

$$\begin{aligned} 1 = {} & \delta \mathfrak{z}_p^2 \sum_{n=-\infty}^{\infty} C_{|n|}(\lambda_i) \left\{ \left(\frac{k_z}{k}\right)^2 Z'(\mathfrak{z} - n\mathfrak{z}_c) - \left(\frac{k_\perp}{k}\right)^2 \frac{nZ(\mathfrak{z} - n\mathfrak{z}_c)}{\lambda_i \mathfrak{z}_c} \right\} \\ & + \frac{\delta}{\theta} \mathfrak{z}_p^2 \sum_{n=-\infty}^{\infty} C_{|n|}(\lambda_e) \left\{ \left(\frac{k_z}{k}\right)^2 Z'(\delta^{1/2}\theta^{-1/2}\mathfrak{z} - n\delta^{-1/2}\theta^{-1/2}\mathfrak{z}_c) \right. \\ & \left. - \left(\frac{k_\perp}{k}\right)^2 \frac{\delta^{1/2}\theta^{-1/2} n}{\mathfrak{z}_c \lambda_e} Z(\delta^{1/2}\theta^{-1/2}\mathfrak{z} - n\delta^{-1/2}\theta^{-1/2}\mathfrak{z}_c) \right\} \end{aligned} \tag{2}$$

where λ_j is defined by Equation (36) of Chapter III

and

$$\mathfrak{z} = \frac{\omega}{k_z \alpha_{zi}}, \tag{3}$$

$$\mathfrak{z}_c = \frac{\omega_{ci}}{k_z \alpha_{zi}}, \tag{4}$$

$$\mathfrak{z}_p = \frac{\omega_{pe}}{k_z \alpha_{zi}}, \tag{5}$$

$$\delta = \frac{m_e}{m_i} , \tag{6}$$

and

$$\theta = \frac{T_{ze}}{T_{zi}} .^{*} \tag{7}$$

But λ_e can be expressed in terms of the gyro-radius of the electrons:

$$\lambda_e = \frac{1}{2} r_{Le}^{2} k_{\perp}^{2} \tag{8}$$

where

$$r_{Le} = \alpha_{\perp e}/\omega_{ce} \tag{9}$$

is the Larmor radius of the electrons, and if the temperature of the electrons is less than or of the same order of magnitude as the temperature of the ions, then

$$r_{Le} \ll r_{Li}$$

and one has

$$\lambda_e \ll \lambda_i . \tag{10}$$

The waves considered in this problem are ones for which

$$k_{\perp} r_{Li} \sim 1 \tag{11}$$

and thus

$$\lambda_e \ll 1 . \tag{12}$$

In view of this approximation only the n = 0 term of the electron contribution is important, since from Equation (35) of Chapter III:

$$C_{|n|} = e^{-\lambda_e} I_{|n|}(\lambda_e) \approx 0 \quad \text{for } n \neq 0$$

and furthermore, $C_o = e^{-\lambda_e} I_o(\lambda_e) \approx 1$ and the dispersion relation can now be written:

---

*This term will be referred to as the electron temperature term.

$$\frac{1}{\delta\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2} = \frac{1}{\theta} Z'(\delta^{1/2}\,\theta^{-1/2}\mathcal{Z}) + \sum_{n=-\infty}^{\infty} C_{|n|}(\lambda_i)\left[Z'(\mathcal{Z} - n\mathcal{Z}_c) - 2n\mathcal{Z}_c T Z(\mathcal{Z} - n\mathcal{Z}_c)\right] \tag{13}$$

and where

$$T \equiv \left(\frac{\alpha_{zi}}{\alpha_{\perp i}}\right)^2 = \frac{T_{zi}}{T_{\perp i}}\,^{*} \tag{14}$$

is the term which measures the degree of anisotropy of the ions. The first term of the right hand side of Equation (13) is the electron term, the second the ion term. For the Shima Model all one need do is replace the variable $\mathcal{Z}$ by $\mathcal{Z} + \ell\mathcal{Z}_c$ in the ion term <u>only</u>.

In principle, Equation (13) can be solved to find:

$$\mathcal{Z} = \mathcal{Z}\left(\frac{k_z}{k}\mathcal{Z}_p, \mathcal{Z}_c, T, \theta, \lambda, \delta, \ell\right). \tag{15}$$

Let

$$\mathcal{Z} = x + iy \tag{16}$$

where

$$x = x\left(\frac{k_z}{k}\mathcal{Z}_p, \mathcal{Z}_c, T, \theta, \lambda, \delta, \ell\right) \tag{17}$$

and

$$y = y\left(\frac{k_z}{k}\mathcal{Z}_p, \mathcal{Z}_c, T, \theta, \lambda, \delta, \ell\right) \tag{18}$$

and for a given ℓ, δ, θ, T and λ, one can draw curves of constant x and y in the $(k_z/k)\mathcal{Z}_p - \mathcal{Z}_c$ plane. The curve y = 0 is particularly important since along it $\mathcal{Z}$ is real and y must change sign as the curve is crossed. This curve then divides the stable and unstable regions of the $(k_z/k)\mathcal{Z}_p - \mathcal{Z}_c$ plane. Since the time dependence was assumed to be of the form $e^{-i\omega t}$, then substituting Equation (16), one sees that

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*This term appears from the $(1/\lambda_i)(k_\perp/k)^2(n/\mathcal{Z}_c)$ term of Equation (2), using Equation (36) of Chapter III and Equation (4) of this chapter.

$$e^{-i\omega t} = e^{-i(x + iy)tk_z\alpha_{zi}} = e^{-ixtk_z\alpha_{zi}}\, e^{ytk_z\alpha_{zi}}$$

so since the sign of the imaginary part of the frequency changes from negative to positive as one crosses the $y = 0$ curve, one has passed from a stable operating region to an unstable region. It will be shown later how to calculate the growth rates in these unstable regions.

The procedure for the computer calculation can be described as follows:

For $y = 0$, take the imaginary part of Equation (13)*

$$S(x) = 0 = \frac{1}{\theta}\,\mathrm{Im}Z'(\delta^{1/2}\,\theta^{-1/2}x) + \sum_{n=-\infty}^{\infty} C_{|n|}\Big\{\mathrm{Im}Z'(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c) - 2n\mathfrak{z}_c T\,\mathrm{Im}Z(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c)\Big\} \tag{19}$$

and the real part

$$\frac{1}{\delta\left(\frac{k_z}{k}\mathfrak{z}_p\right)^2} = \frac{1}{\theta}\,\mathrm{Re}Z'(\delta^{1/2}\,\theta^{-1/2}x) + \sum_{n=-\infty}^{\infty} C_{|n|}\Big\{\mathrm{Re}Z'(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c) - 2n\mathfrak{z}_c T\,\mathrm{Re}Z(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c)\Big\}. \tag{20}$$

Choose λ, θ, T, δ, and ℓ, then for a given value of $\mathfrak{z}_c$ (some value of the magnetic field) plot Equation (19) and find its roots; that is, find the values of x for which $S(x) = 0$. Use these roots in Equation (20) to find $(k_z/k)\mathfrak{z}_p$. This then locates points in the $(k_z/k)\mathfrak{z}_p$ - $\mathfrak{z}_c$ plane. One then repeats the procedure for another value of $\mathfrak{z}_c$ until the map is complete.

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*For completeness the factor $\ell\mathfrak{z}_c$ is included in the ion term.

In order to make use of such a stability-instability plot, one must first choose the parameters for which the most unstable plasma exists, that is, parameters for which the unstable region of the $(k_z/k)\mathcal{Z}_p - \mathcal{Z}_c$ plane is largest. Then one can vary these parameters to find their effect on the stability of the plasma system. For the present general discussion the value of ℓ will be taken to be zero. Its very important effect on the stability boundary will be discussed later.

II. THE $\ell = 0$ CASE

The most unstable plasma exists when the electron temperature term (θ) and the ion anisotropy term (T) are zero and the value of λ_i is approximately three.* Figure 3 shows a plot of S(x) vs x for various values of λ_i. It can be seen that these curves are not very sensitive to changes in λ_i near $\lambda_i \approx 3$. It can be shown using Equation (36) of Chapter III and Equations (4) and (14) of this chapter that

$$\lambda_i = \frac{k_\perp^2 \alpha_{\perp i}^2}{2\omega_{ci}^2} = \frac{1}{2}\left(\frac{k_\perp}{k_z}\right)^2 \frac{1}{T\mathcal{Z}_c^2} \tag{21}$$

So as T and $\mathcal{Z}_c$ vary, then λ_i should change also. However, λ_i is held constant by varying $k_\perp/k_z$ appropriately. The solution of the problem was found to be

$$\mathcal{Z} = \mathcal{Z}\left(\frac{k_z}{k}\mathcal{Z}_p, \mathcal{Z}_c, \theta, T, \lambda_i, \delta, \ell\right);$$

it could just as well have been written

$$\mathcal{Z} = \mathcal{Z}\left(\frac{k_z}{k}\mathcal{Z}_p, \mathcal{Z}_c, \theta, T, \frac{k_z}{k_\perp}, \delta, \ell\right). \tag{22}$$

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*λ_i was chosen to be three, since this made the first few $e^{-\lambda_i} I_{|n|}(\lambda_i)$ near their maximum values.

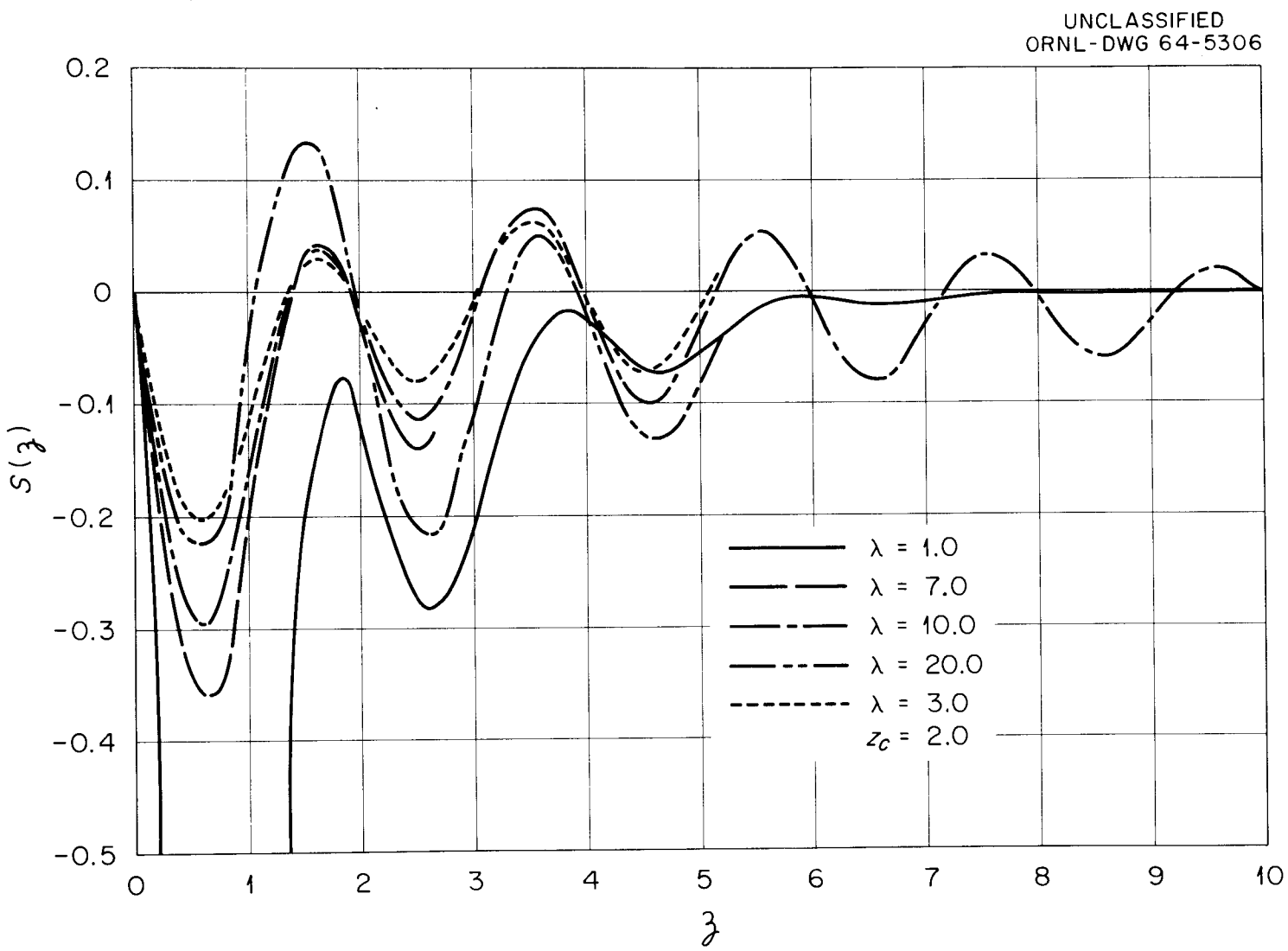

Figure 3. The effect of λ_i on the roots of Equation (19).

The most unstable choice of $k_\perp/k_z$ is the one which makes $\lambda_i \approx 3$. Therefore, for a given θ and T, the $\mathrm{Im}\,\mathcal{Z} = 0$ curve in the $(k_z/k)\mathcal{Z}_p - \mathcal{Z}_c$ plane traces out the most unstable region if $k_\perp/k_z$ is continually adjusted to keep $\lambda_i \approx 3$. If $k_\perp/k_z$ is varied in any other way, a smaller region of instability would result.

For real values of the argument the real and imaginary part of the FC* functions have the following properties:

$$\mathrm{Re}Z(x) = -\mathrm{Re}Z(-x) \ , \tag{23}$$

$$\mathrm{Im}Z(x) = \mathrm{Im}Z(-x) \ , \tag{24}$$

and

$$Z'(x) = -2 - 2xZ(x) \ . \tag{25}$$

Thus, when looking for the solution of Equation (19), one need only consider $x = \mathrm{Re}\,\mathcal{Z} > 0$. One should notice, however, that this is true <u>only</u> for $\ell = 0$. For the above parameters, the behavior of the imaginary part of the solution is given in Figure 4 for a few values of $\mathcal{Z}_c$.

It was found that for $\mathcal{Z}_c \gg 1$, the roots of the imaginary equation were approximately (taking T = 0, first):

$$x = (j - \tfrac{1}{2})\,\mathcal{Z}_c \tag{26}$$

and

$$x = j\mathcal{Z}_c \tag{27}$$

where

$$j = 1, 2, 3, \ldots . \tag{28}$$

and substituting these values into the real part of the dispersion relation gives:**

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*Fried-Conte plasma dispersion function.

**With $\theta = 0$, the electron term of the imaginary part of the dispersion relation was zero. For the real

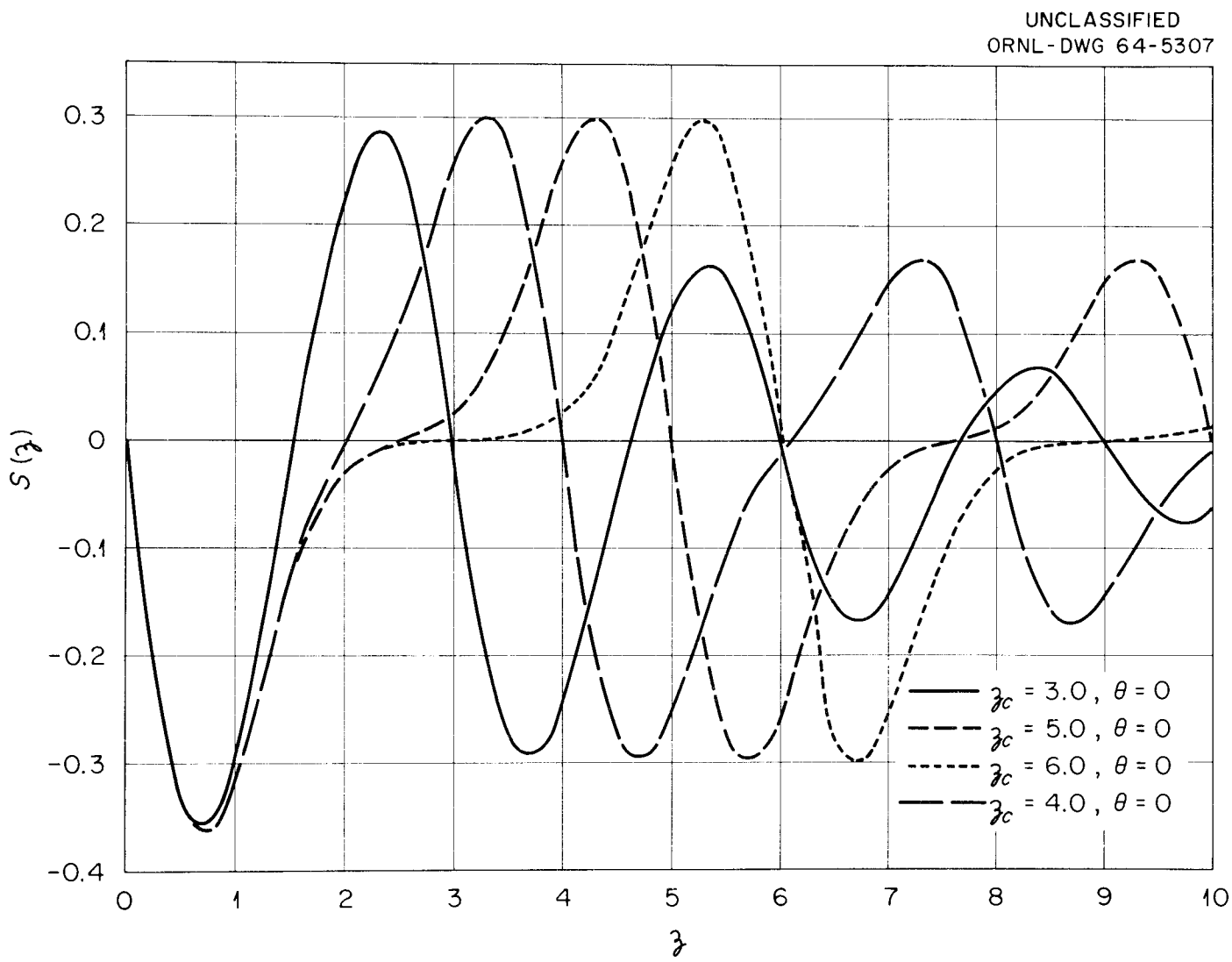

Figure 4. The effect of ζ_c on the roots of Equation (19).

$$\frac{1}{\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2} = \frac{1}{\left(j - \frac{1}{2}\right)^2 \mathcal{Z}_c^{\,2}} + \delta \sum_{n=-10}^{10} C_{|n|}\ \mathrm{ReZ}'\left[\left(j - n - \tfrac{1}{2}\right)\mathcal{Z}_c\right]. \qquad (29)^*$$

The second term on the right hand side is negligibly small; so

$$\mathcal{Z}_c = \frac{1}{(j - \frac{1}{2})}\left(\frac{k_z}{k}\mathcal{Z}_p\right). \qquad (30)$$

For the second set of roots

$$\frac{1}{\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2} = \frac{1}{j^2 \mathcal{Z}_c^{\,2}} + \delta \sum_{n=-10}^{10} C_{|n|}\ \mathrm{ReZ}'\left[(j - n)\mathcal{Z}_c\right] \qquad (31)$$

where the second term on the right hand side becomes

$$\delta C_\ell \mathrm{ReZ}'(0) = -2\delta C_\ell\ . \qquad (32)$$

Thus

$$\mathcal{Z}_c = \frac{1}{j}\ \frac{(k_z/k)\,\mathcal{Z}_p}{\sqrt{1 + 2\delta C_\ell\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2}}$$

and approaches

$$\mathcal{Z}_c = 1/\sqrt{2j^2 C_\ell} \qquad (34)$$

as $(k_z/k)\mathcal{Z}_p$ approaches infinity. The behavior of these curves in the $(k_z/k)\mathcal{Z}_p$ plane is shown in Figure 5 with the regions of stable and unstable operating regions marked.

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part the electron term reduces to $1/x^2$.

*The infinite sum was replaced by a sum from $n = -10$ to $n = +10$, since $C_{|n|}$ (3) became negligibly small for $n > 10$. For large values of $\mathcal{Z}_c$ only two terms of the sum were significant, $n = j - 1$ and $n = j$, the remaining terms, exponentially small.

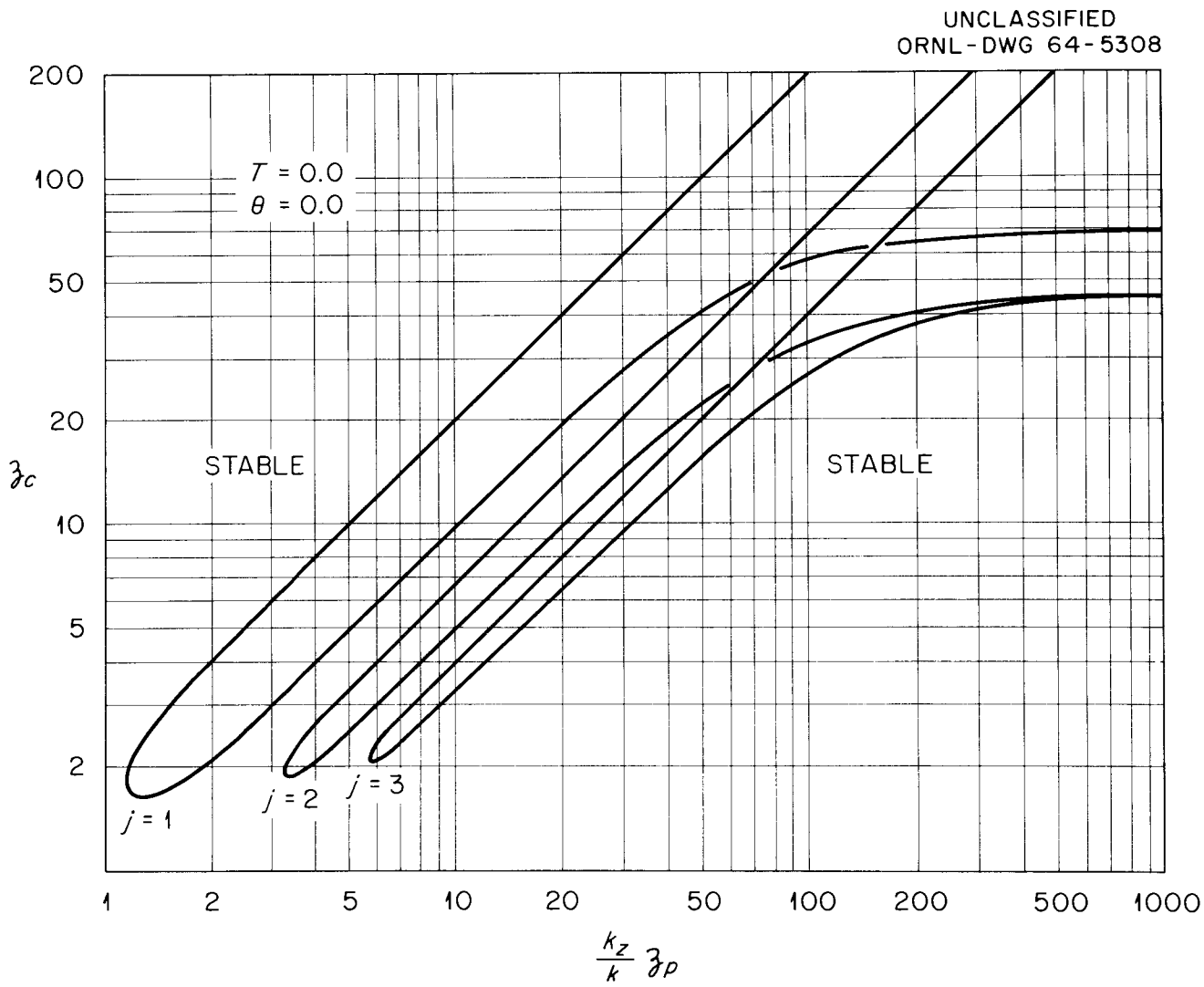

Figure 5. Regions of stability-instability for cold electron plasma and T = 0.

Giving the ion anisotropy term, T, a non-zero value increases the stability of the plasma as seen from Figure 6. The root which was located at $j\mathcal{Z}_c$ will be shifted to $j\mathcal{Z}_c(1 - T)$. The root located halfway between $(j - 1)\mathcal{Z}_c$ and $j\mathcal{Z}_c$ is not shifted. So, for $\mathcal{Z}_c \gg 1$, the solutions to the imaginary part are:

$$x = (j - \tfrac{1}{2})\mathcal{Z}_c \ , \tag{35}$$

and

$$x = j\mathcal{Z}_c(1 - T) \ . \tag{36}$$

It was found that the unstable region would vanish* if

$$(j - \tfrac{1}{2})\mathcal{Z}_c > j\mathcal{Z}_c(1 - T)$$

or if

$$T > \frac{1}{2j} \tag{37}$$

a relation found analytically by Dnestrovsky et al.[3] for j = 1. Comparison with the work of Ozawa, Kaji, and Kito.[4] who solve the dispersion relation for a one species plasma, in a different manner from that in this dissertation, is accomplished by redoing this problem using $\lambda_i = 0.1$ and T = 0.1 and neglecting the electron term. Figure 7 of this work is identical to Figure 6 of their paper.

Heating the electrons, that is, giving θ a non-zero value, also improves the stability. For T = 0, and non-zero θ, the imaginary part of the dispersion relation, is:

$$S(x) = 0 = \frac{1}{\theta}\,\mathrm{Im}Z'(\delta^{1/2}\,\theta^{-1/2}x) + \sum_{n=-10}^{10} C_{|n|}\,\mathrm{Im}Z'(x - n\mathcal{Z}_c) \ , \tag{38}$$

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*That is, there were no roots of Equation (19).

[3]Yu. Dnestrovsky et al., loc. cit.

[4]Ozawa et al., loc. cit.

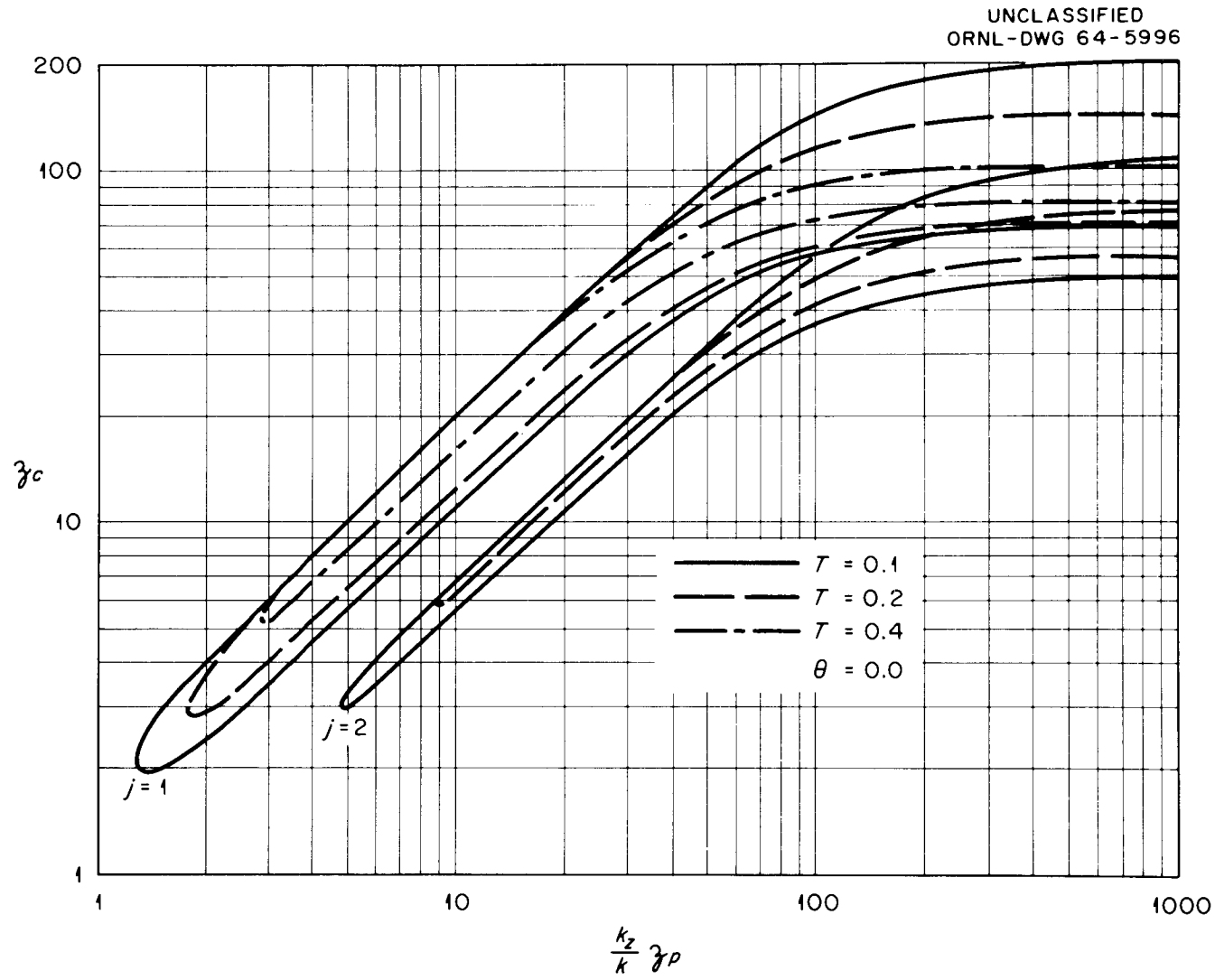

Figure 6. The effect of ion anisotropy on the stability of a plasma.

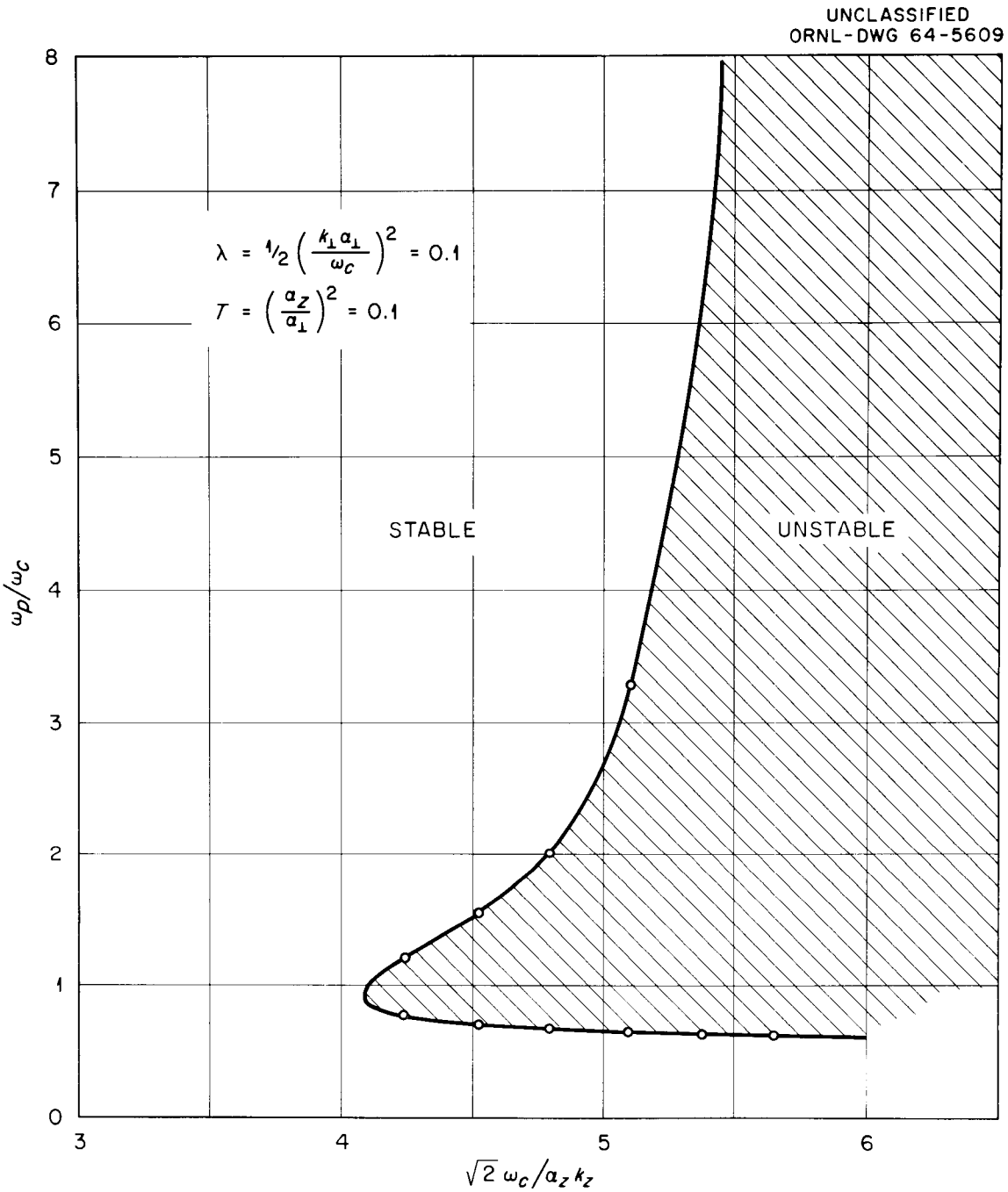

Figure 7. Comparison of regions of stability with the work of Ozawa, Kaji, and Kito.

and for θ less than about five the amplitude of the electron contribution is greater than that of the ion contribution, as seen in Figure 8. For sufficiently large $\mathcal{Z}_c$ there will be a solution of the imaginary part of the dispersion relation. The $0 < \theta \lesssim 5$ curves then are displaced upward and the "half-integral" roots are displaced toward the integral roots as seen for the $\theta = 0.01$ curve in Figure 9. For θ greater than about five, the amplitude of the electron contributions is less than the ion contribution (see Figure 10) and the imaginary part of the dispersion relation has a solution for all values of $\mathcal{Z}_c$,* except near $\mathcal{Z}_c = 1$ where the $n = 0$ ion contribution is important. Figures 11 and 12 are plots for θ about five and larger. With both θ and T non-zero the imaginary part of the dispersion relation takes the form:**

$$S(x) = 0 = \frac{1}{\theta}\,\mathrm{Im}Z'(\delta^{1/2}\,\theta^{-1/2}x) + C_1\left\{Z'(x - \mathcal{Z}_c) - 2\mathcal{Z}_c T Z(x - \mathcal{Z}_c)\right\}. \tag{39}$$

One root is about at $x = \mathcal{Z}_c(1 - T)$ and the other is found from (assuming T is small):

$$\frac{1}{\theta}\,\mathrm{Im}Z'(\delta^{1/2}\,\theta^{-1/2}x) + C_1\mathrm{Im}Z'(x - \mathcal{Z}_c) = 0\ , \tag{40}$$

whose solution is approximately

$$x \simeq \frac{\mathcal{Z}_c}{1 + \delta^{1/2}\,\theta^{-1/2}}\ . \tag{41}$$

One sees that qualitatively the instability vanishes when

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*However, some of the solutions to the imaginary part when substituted into the real equation yield negative values for $(k_z/k\,\mathcal{Z}_p)^2$.

**This form was suggested by E. G. Harris after conversation with M. N. Rosenbluth.

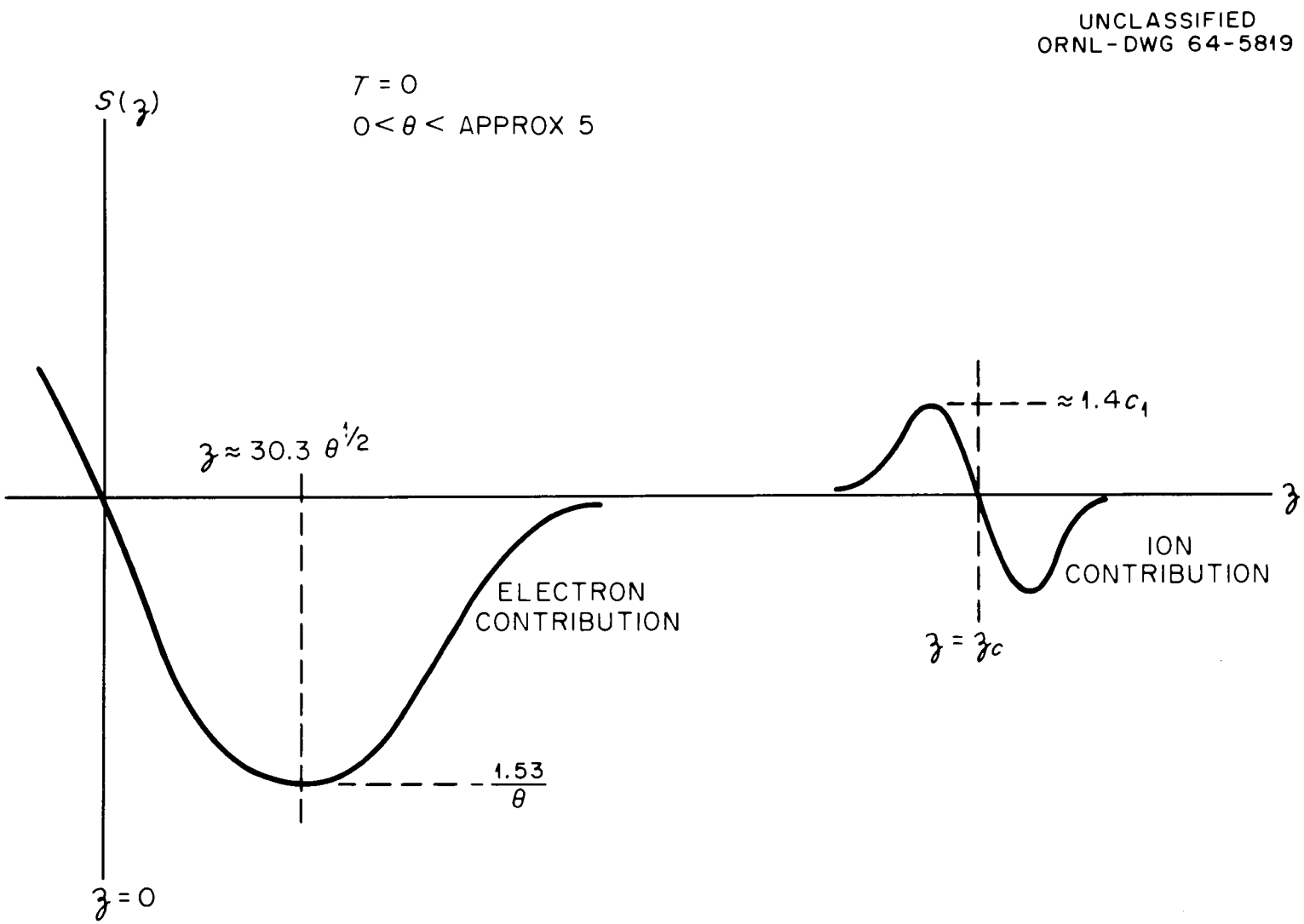

Figure 8. Comparison of electron and ion contributions to $S(\zeta)$ for $\theta < 5$.

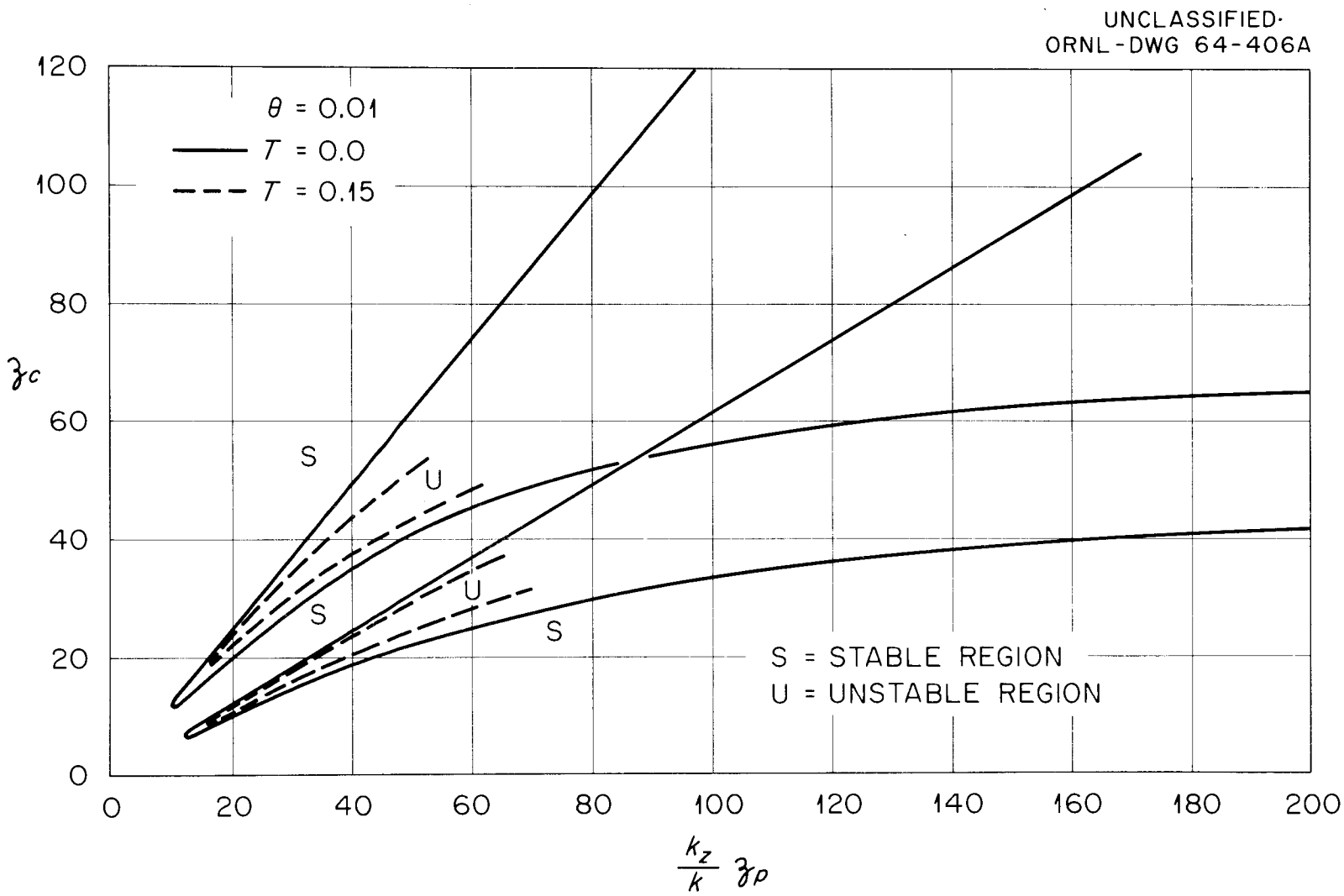

Figure 9. Regions of stability-instability for an electron temperature ratio less than five.
than five.

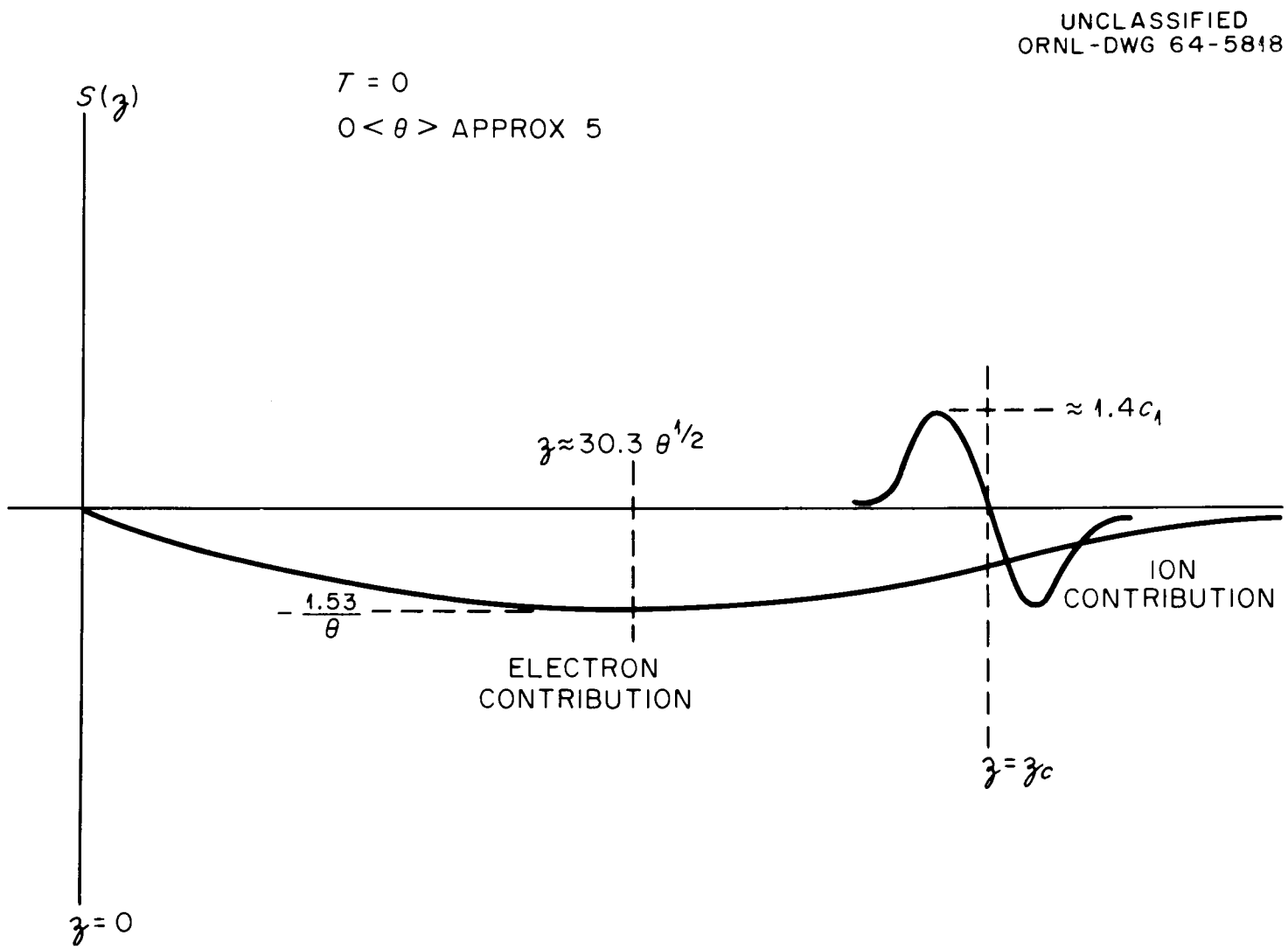

Figure 10. Comparison of electron and ion contributions to $S(z)$ for $\theta \gtrsim 5$.

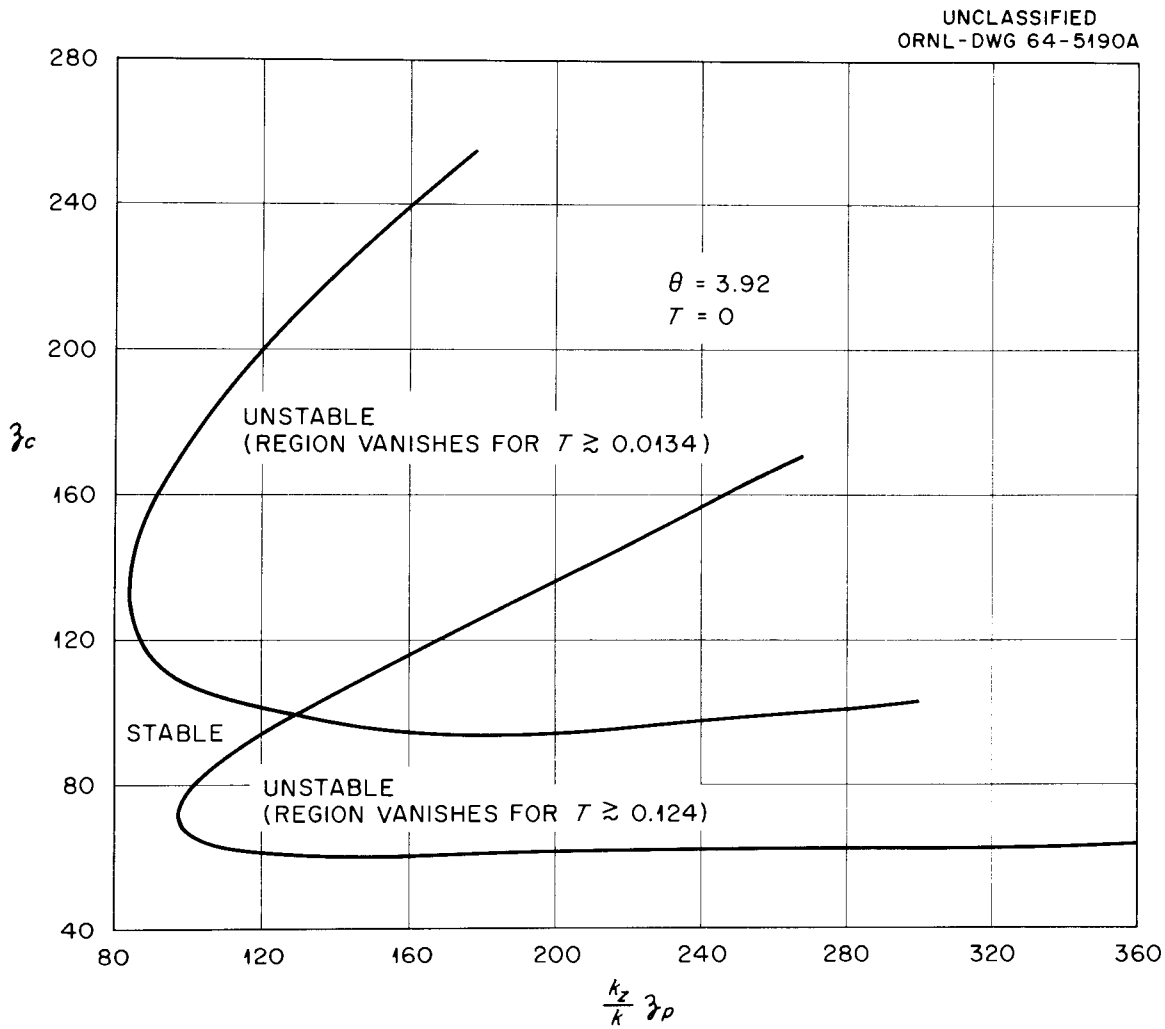

Figure 11. Regions of stability-instability for an electron temperature ratio in the vicinity of five.

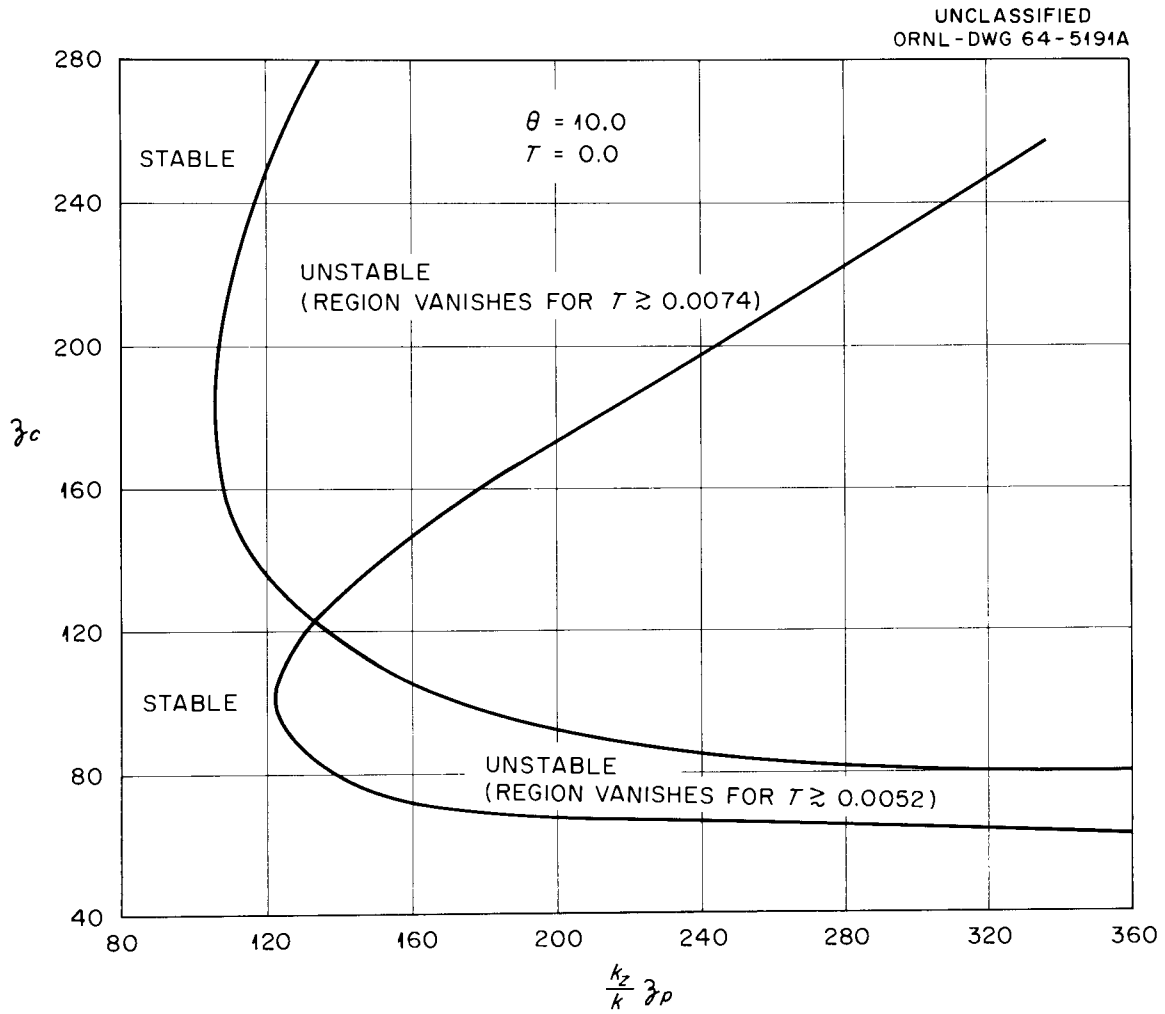

Figure 12. Regions of stability-instability for an electron temperature ratio greater than five.

$$\frac{\mathcal{Z}_c}{1 + \delta^{1/2}\,\theta^{-1/2})} > \mathcal{Z}_c(1 - T)$$

or

$$T > \frac{\delta^{1/2}\,\theta^{-1/2}}{1 + \delta^{1/2}\,\theta^{-1/2}}\,, \quad j = 1\,, \tag{42}$$

again a result found quantitatively by Dnestrovsky et al.[5] Thus the plasma is stable if condition (42) is satisfied or if Equation (37) holds, whichever is smaller. A plot similar to the one drawn by Timofeev[6] is shown in Figure 13. However, Timofeev's curve did not include the portion for $\theta \lesssim 5$, thus indicating stability for $\theta < 5$ at all values of T.

For $\ell = 0$, one can draw the operating regimes for ALICE and Phoenix and the $\ell = 0$ modes of the DCX's and OGRA. These regimes are presented in Figure 14. The boundaries of the operating regions are determined by the maximum and minimum values of k_z. For $k_z = k$, the maximum value of k_z, one can draw a boundary whose slope is ω_{pe}/ω_{ci}. Since

$$\omega_{pe} \simeq 5 \times 10^4 \sqrt{n_o} \tag{43}$$

and considering typical ion cyclotron frequencies of 10^8 sec^{-1}, one has:

$$\text{slope} \simeq 5 \times 10^{-4} \sqrt{n_o}\,. \tag{44}$$

For ALICE, DCX-I, and Phoenix, the densities are between 10^7 and 10^8 cm^{-3}, and the slope as a lower bound on the operating region can be drawn as shown by the long dashed lines on Figure 14. For OGRA, $n_o \sim 10^9$ cm^{-3} and for DCX-II, densities as high as 10^{10} cm^{-3} have been reached. The minimum value of k_z is determined by the maximum wave length that can

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[5] Yu. Dnestrovsky et al., loc. cit.

[6] Timofeev, loc. cit.

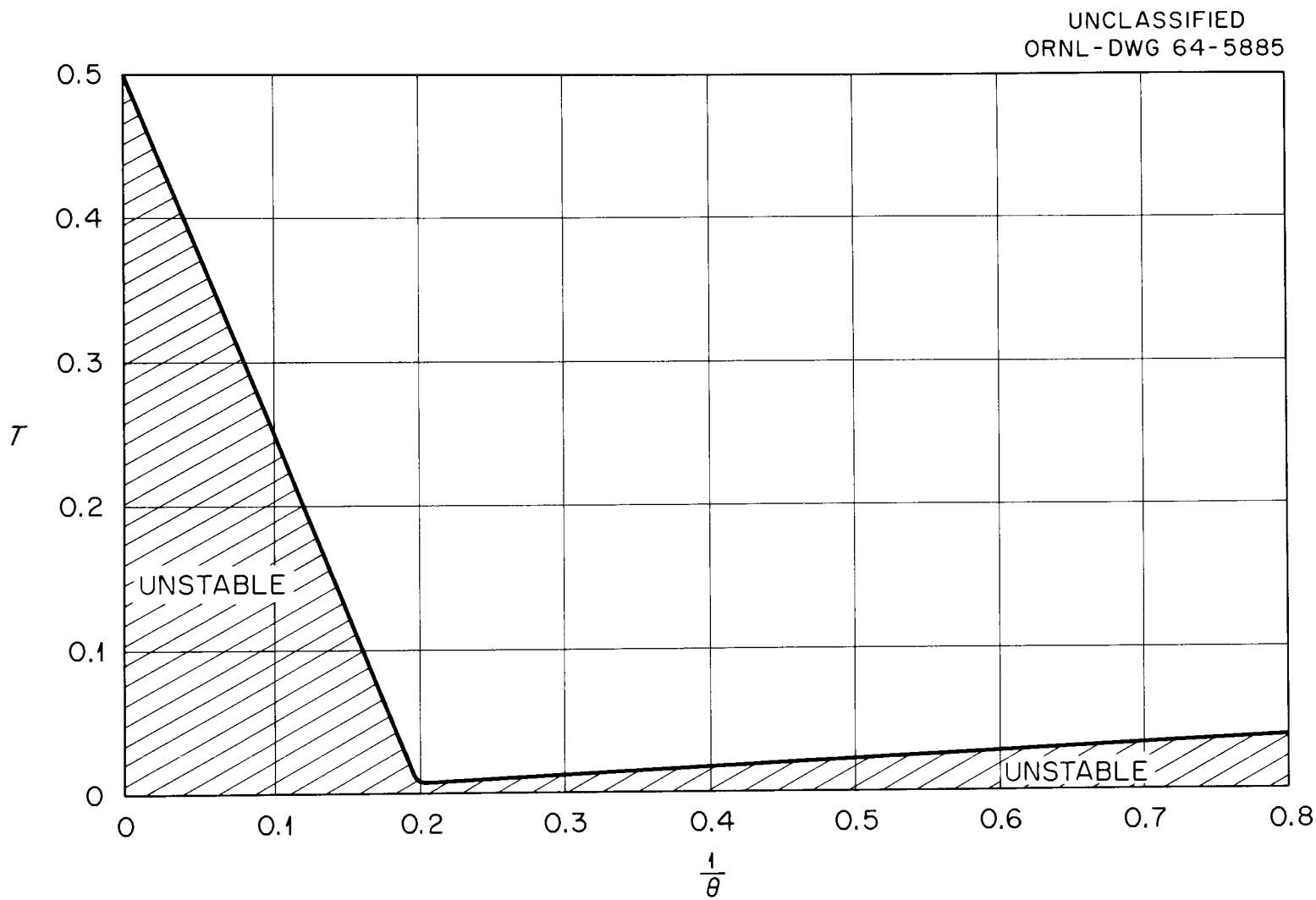

Figure 13. Regions of stability-instability as a function of ion anisotropy and electron temperature.

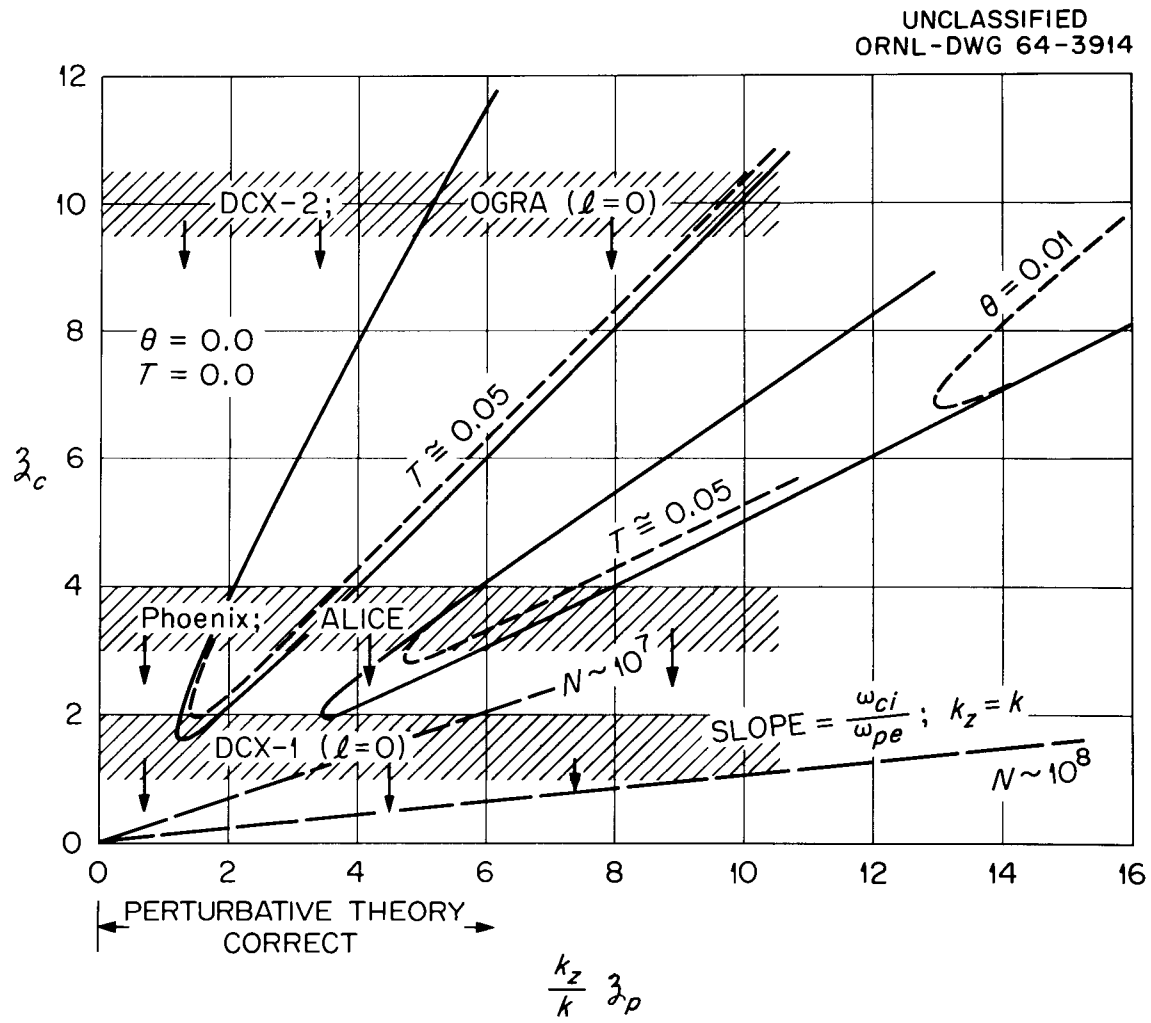

Figure 14. Bounds on the regions of applicability for the four thermonuclear machines studied.

exist in the machine, which is approximately the length of the machine. For ALICE, Phoenix, and DCX-I, this is about 10 cm; for DCX-II, 1 meter; for the OGRA, up to 10 meters. Taking α_{zi} to be about 10^8 cm/sec, one can write

$$\mathcal{Z}_c = \frac{\omega_{ci}}{k_z \alpha_{zi}} \sim \frac{10^8}{\frac{2\pi}{L}\, 10^8} = \frac{L}{2\pi} . \tag{45}$$

This bound is represented by the crosshatching in Figure 14, this bound being flexible because of the uncertainty in the length L. The DCX-I region is lower than Phoenix or ALICE because of the higher injection energy and therefore larger α_{zi}. The effect of ion-anisotropy and electron temperature can clearly be seen from this figure. A small amount of electron temperature, theoretically, can stabilize the lower operating regions with respect to the Harris instability and may eliminate some of the low harmonics in the larger injection machines. Growth rates have been determined for the unstable regions using a perturbation technique. This aspect will be discussed later.

III. THE $\ell \neq 0$ CASE

For DCX-I and II and OGRA the Larmor radius of the ions is comparable to the machine radius and the infinite plasma model is a poor approximation. However, the approximation introduced by Shima, representative of plasmas about one Larmor diameter across, can also be solved numerically. It has been shown that the dispersion relation for this case is:

$$\frac{1}{\delta\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2} = \frac{1}{\theta} Z'(\delta^{1/2}\theta^{-1/2}x) + \sum_{n=-\infty}^{\infty} C_{|n|}\left[Z'(x + \ell\mathcal{Z}_c - n\mathcal{Z}_c) - 2n\mathcal{Z}_c TZ(x + \ell\mathcal{Z}_c - n\mathcal{Z}_c)\right] . \tag{46}$$

The general procedure outlined for the infinite plasma solution can be followed for this case. However, due to the ion rotation, the imaginary part of the dispersion relation is no longer symmetric with respect to $\mathrm{Re}\,\mathcal{Z} = x$. In fact, it was found that for each value of the azimuthal mode number ℓ, that anisotropy mode with angular velocity ($= \mathrm{Re}\,\omega/\ell$) most nearly coinciding with that of the ions changes character and becomes essentially the instability found by Burt-Harris[7] of zero-temperature ions rotating through electrons.

Shima found it more convenient to analyze the problem using cylindrical waves, so perturbations were assumed to go as:

$$\sim e^{-i\omega t - i\ell\theta} J_\ell(k_\perp r)\, e^{ik_z z} . \tag{47}$$

Since

$$e^{-i\omega t - i\ell\theta} = e^{\mathrm{Im}\,\omega t}\, e^{-i\mathrm{Re}\,\omega t - i\ell\theta} , \tag{48}$$

and if $\mathrm{Re}\,\omega < 0$, then this represents a wave moving in the positive θ direction with angular velocity $|\mathrm{Re}\,\omega|/\ell$. Since the ions rotate in the positive θ direction (remember it was assumed that $\vec{B} = -B_o\hat{z}$), it is those waves with $\mathrm{Re}\,\omega < 0$ which interact most strongly with the ions. In order to display these results, it is convenient to take $(k_z/k)\mathcal{Z}_p$ negative for $x = \mathrm{Re}\,\mathcal{Z}$ negative, always with the understanding that $(k_z/k)\mathcal{Z}_p$ is really positive.

Figure 15 shows the results of these calculations for $\theta = 0$, $T = 0$, and $\ell = 1$. For the portion $\mathrm{Re}\,\omega < 0$, one sees regions unique in the analysis thus far. First of all, one finds a boundary ($\mathrm{Im}\,\mathcal{Z} = 0$) extending into the origin similar to the Burt-Harris mode (see Figure 16).

---

[7]Burt and Harris, <u>loc</u>. <u>cit</u>.

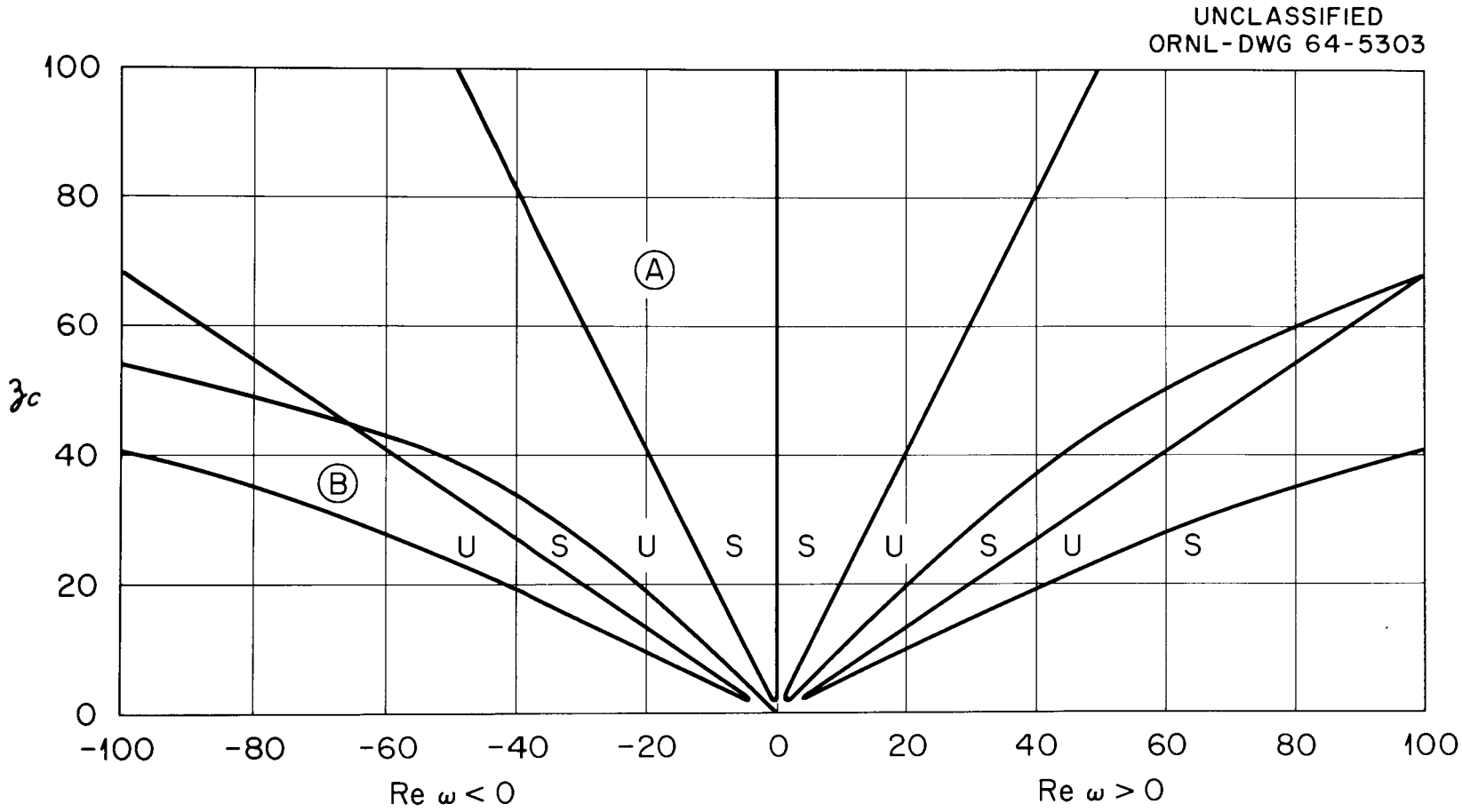

Figure 15. Shima model results for T = 0, θ , 0, and ℓ = 1.

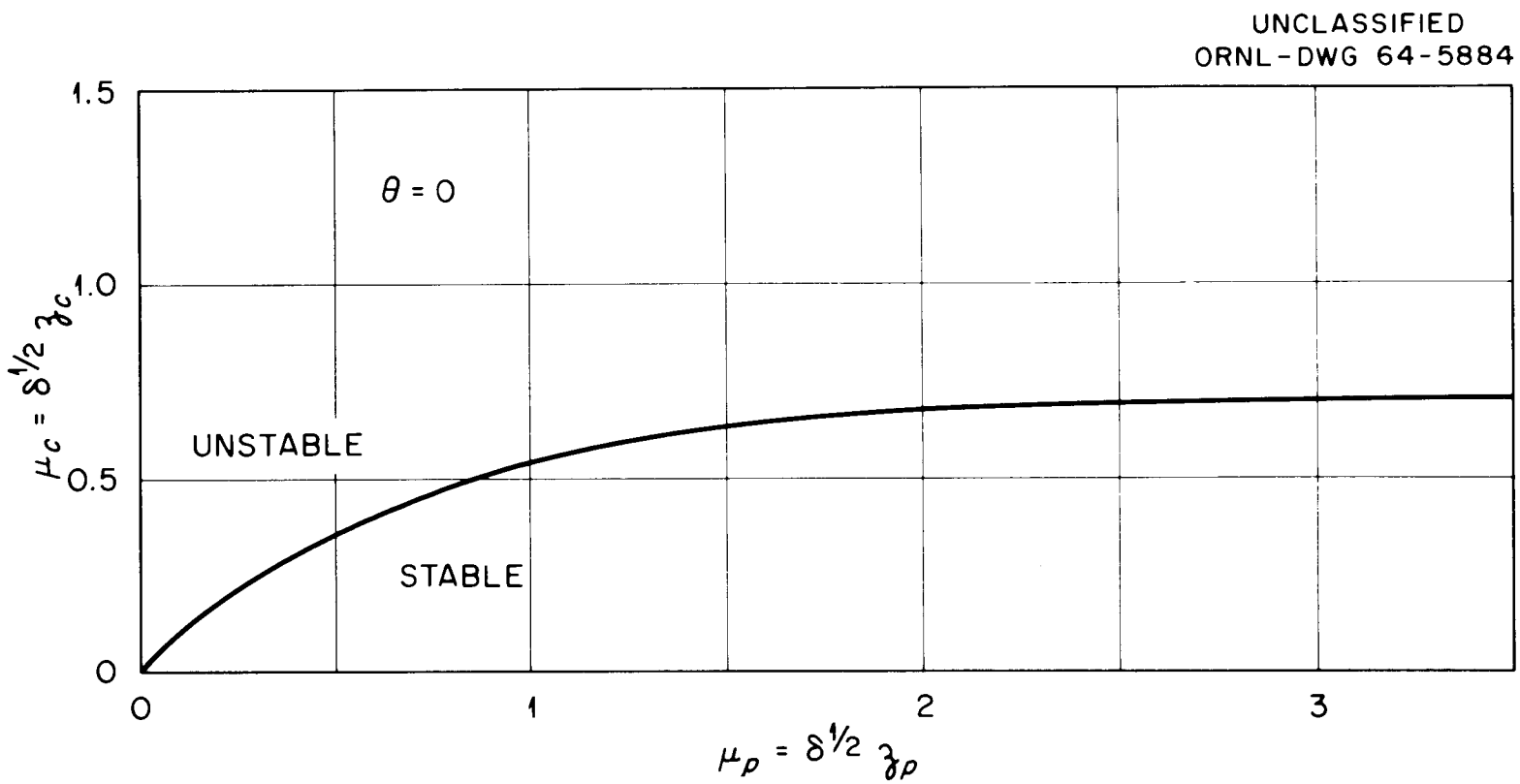

Figure 16. The Burt-Harris mode.

Furthermore, for this case, one finds "more stability" than is found in the Burt-Harris problem (as seen by region A in Figure 15). Also the anisotropy modes, regions B,* contribute "spikes" of unstable regions. Figures 17, 18, 19, and 20 indicate how these modes are affected by the ion anisotropy. Increasing T removes the anisotropic modes but leaves the Burt-Harris mode effectively unchanged. However, giving the electrons a finite temperature removes the Burt-Harris instability and moves the unstable regions to higher $\mathcal{Z}_c$ values, as was found in the infinite plasma solution. Figures 21 and 22 are examples of this fact.

The effect of ℓ can be seen by comparing Figure 23 with Figure 17. For low values of $\mathcal{Z}_c$ increasing ℓ increases the unstable regions for low density and decreases the stable region for larger density. Figure 24 is a plot of the operating regimes for the machines applicable to the Shima approximation. The dashed line on the Burt-Harris modes indicates the effect of heating the electrons.

In order to complete the computer calculations one must be able to determine the growth rates for the unstable regions. To do this exactly is a formidable problem, indeed, and for the purposes of this dissertation a perturbation technique** was used. Using Equation (46) and for simplicity letting T = 0, one gets:

$$\frac{1}{\delta\left(\frac{k_z}{k}\mathcal{Z}_p\right)^2} = F(\mathcal{Z}) = \frac{1}{\theta} Z'(\delta^{1/2}\,\theta^{-1/2}) + \sum_{n=-\infty}^{\infty} C_{|n|}\, Z'(\mathcal{Z} + \ell\mathcal{Z}_c - n\mathcal{Z}_c) \qquad (49)$$

---

*There are further anisotropy modes, as seen in Figure 17. For T = 0, in theory there are an infinite number of the anisotropy modes each with decreasing unstable area as the mode number j increases.

**This method was suggested by T. K. Fowler and E. G. Harris.

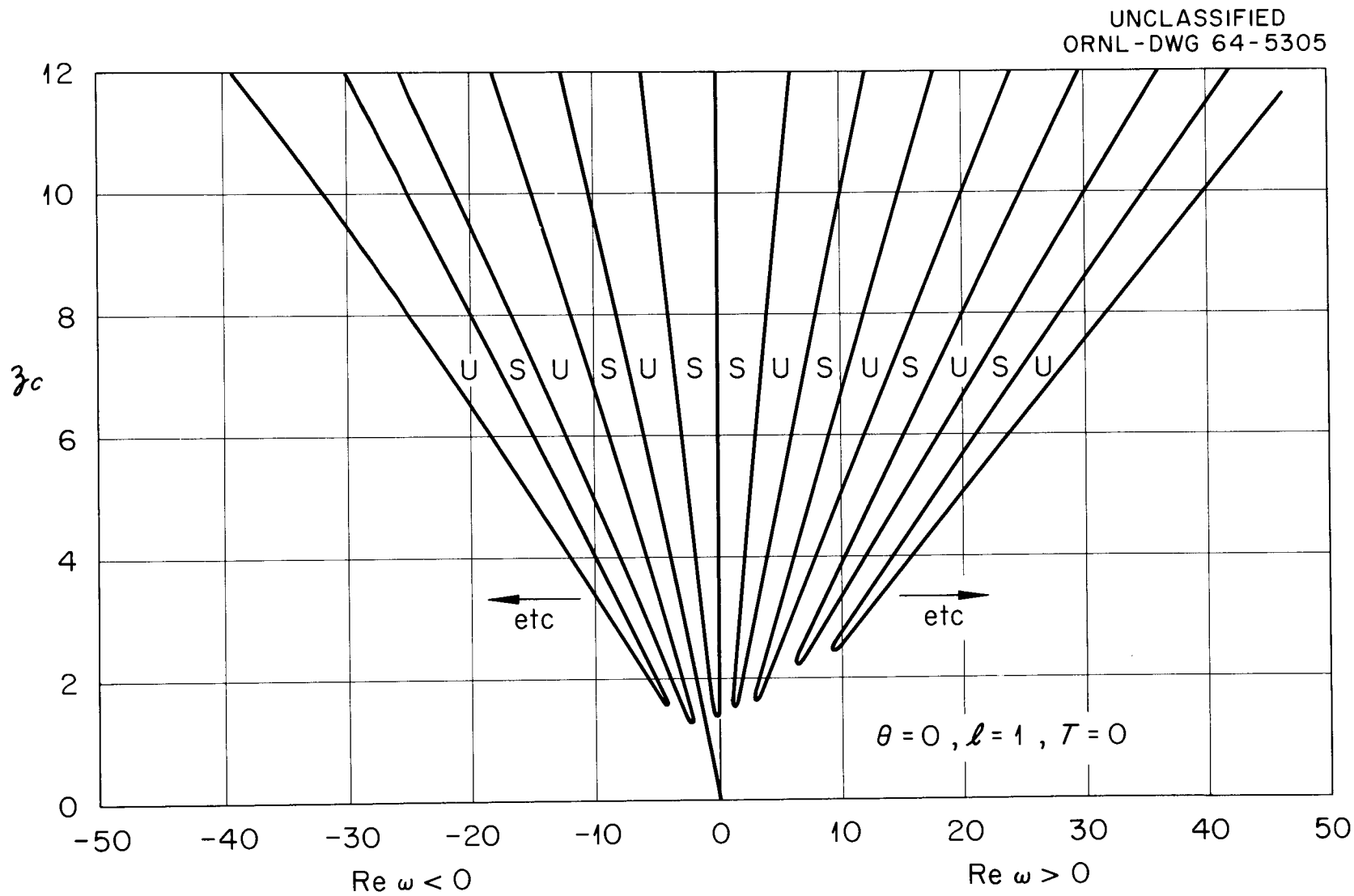

Figure 17. The anisotropy modes and the Burt-Harris mode of the Shima model for T = 0.

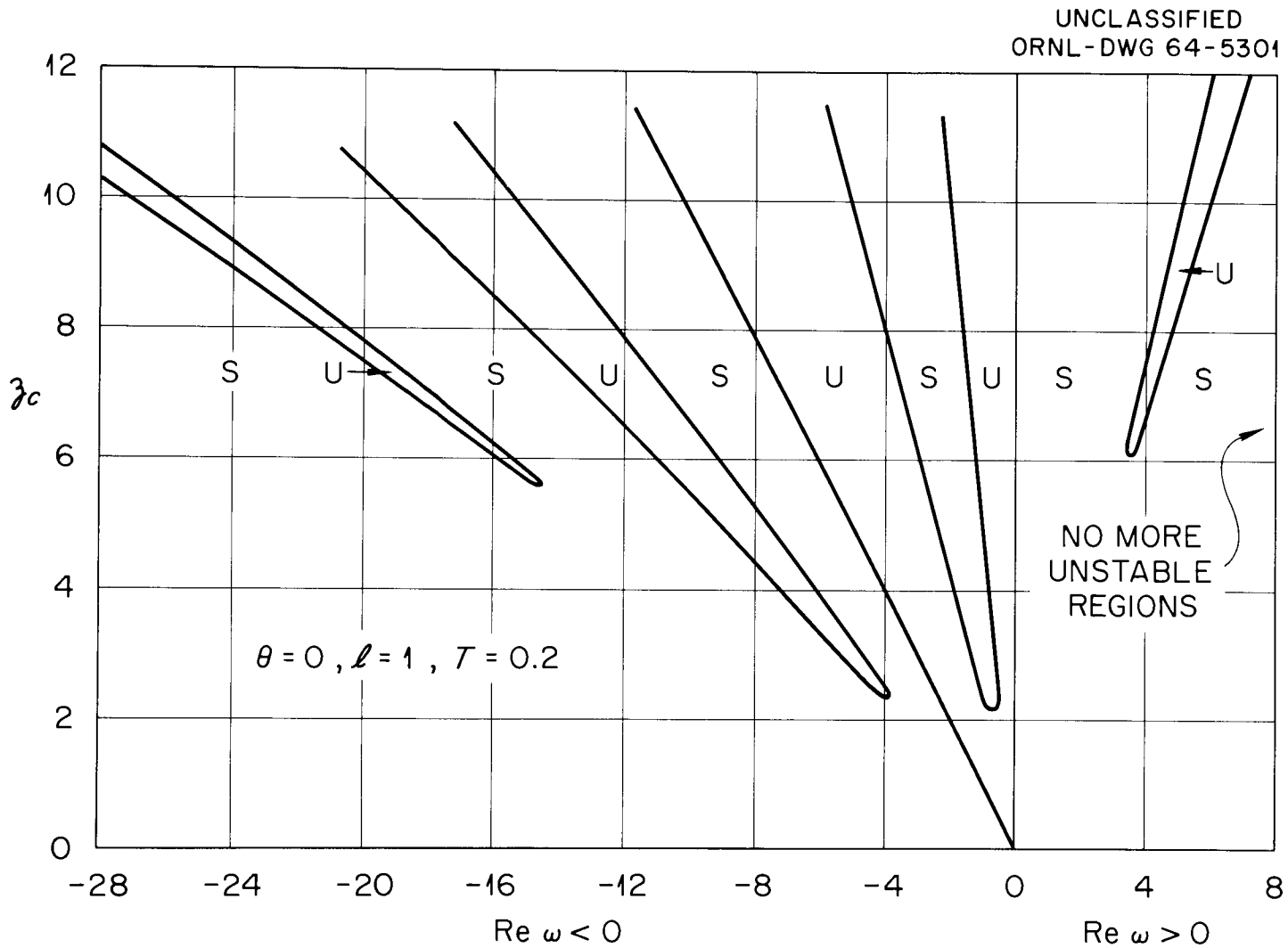

Figure 18. The effect of increased ion anisotropy on the stability regions of the Shima model for T = 0.2.

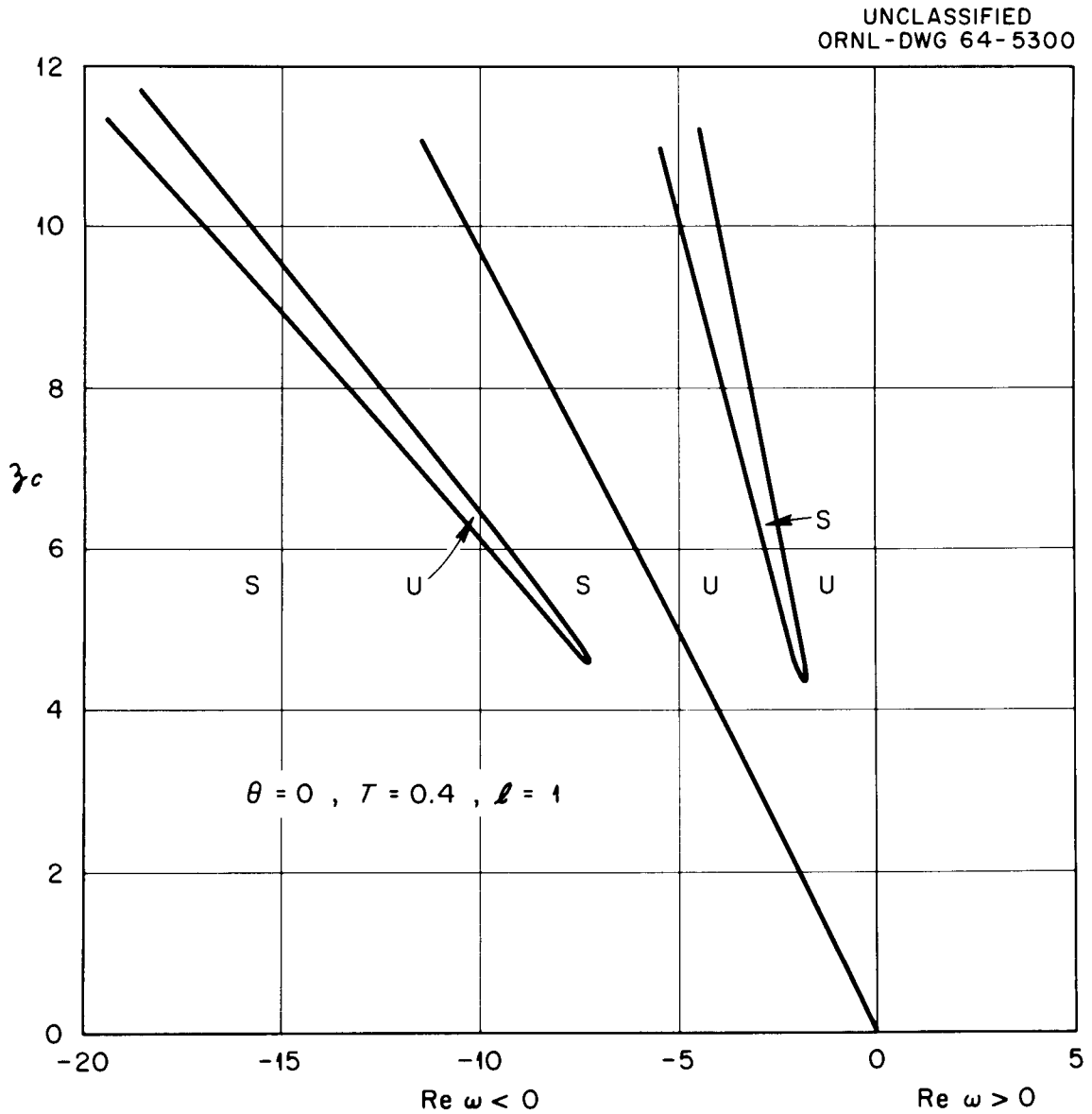

Figure 19. The effect of increased ion anisotropy on the stability regions of the Shima model for T = 0.4.

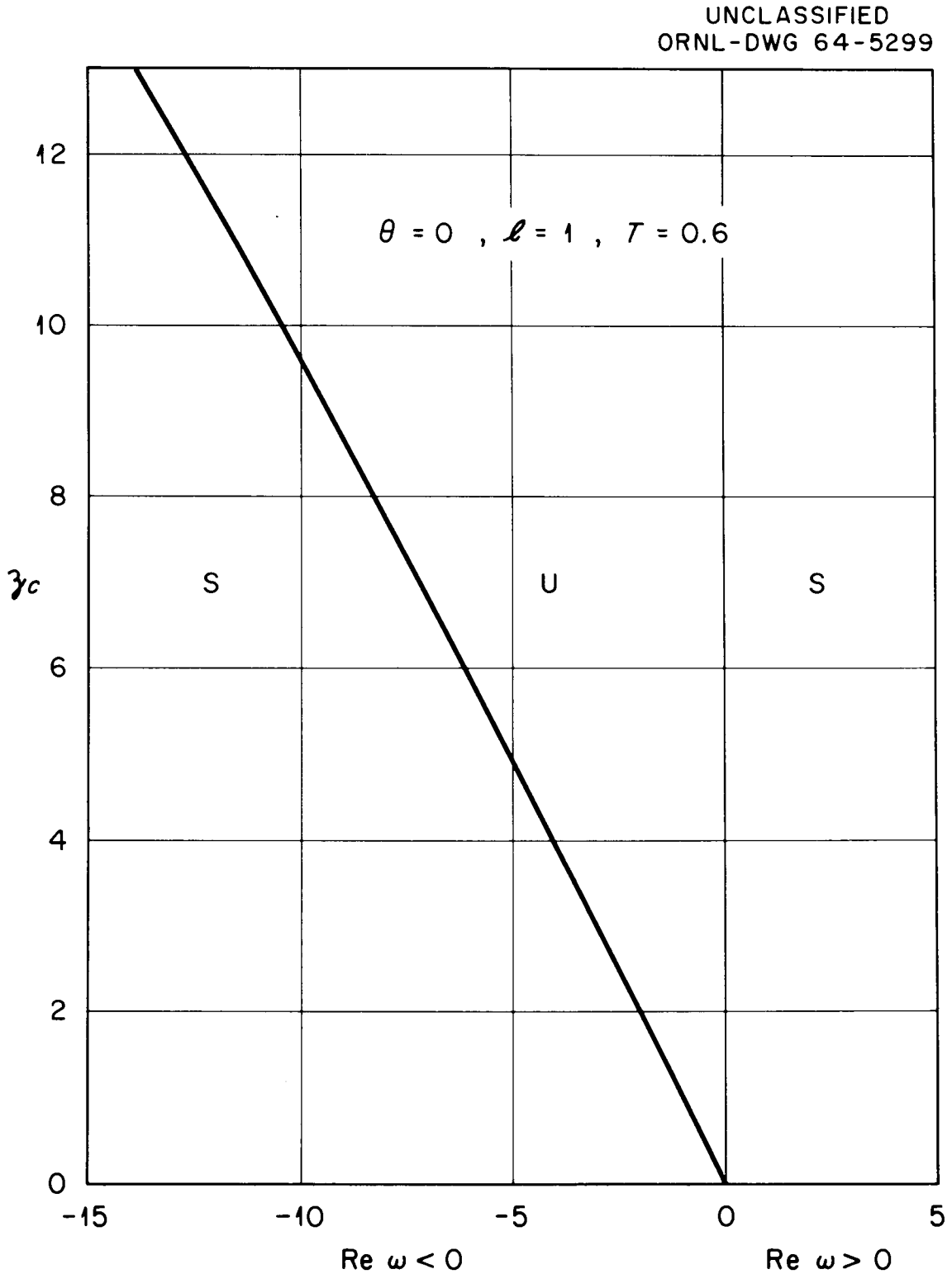

Figure 20. The complete removal of the anisotropy modes by increasing T beyond 0.5.

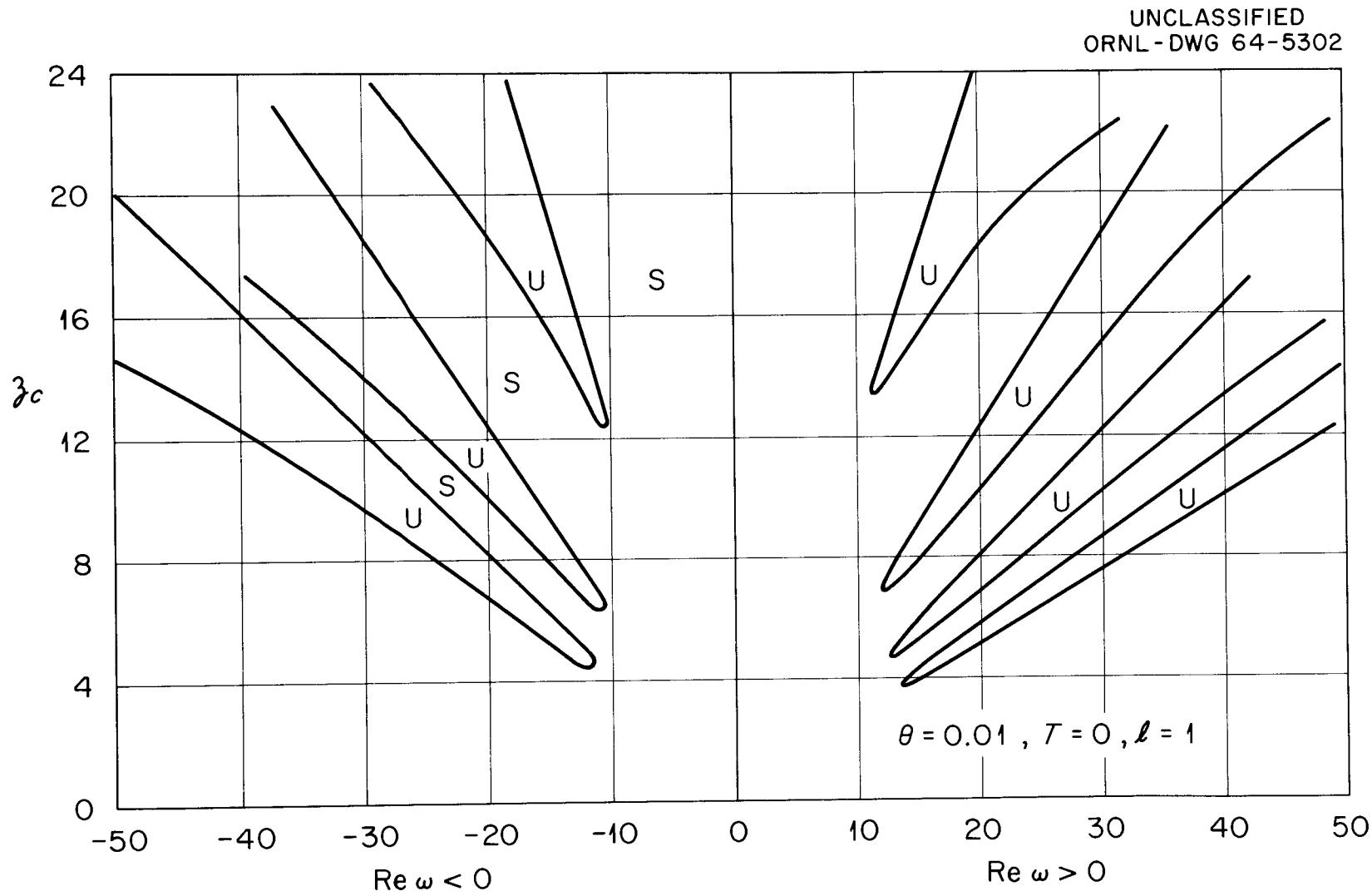

Figure 21. Removal of the Burt-Harris mode by increasing the electron temperature.

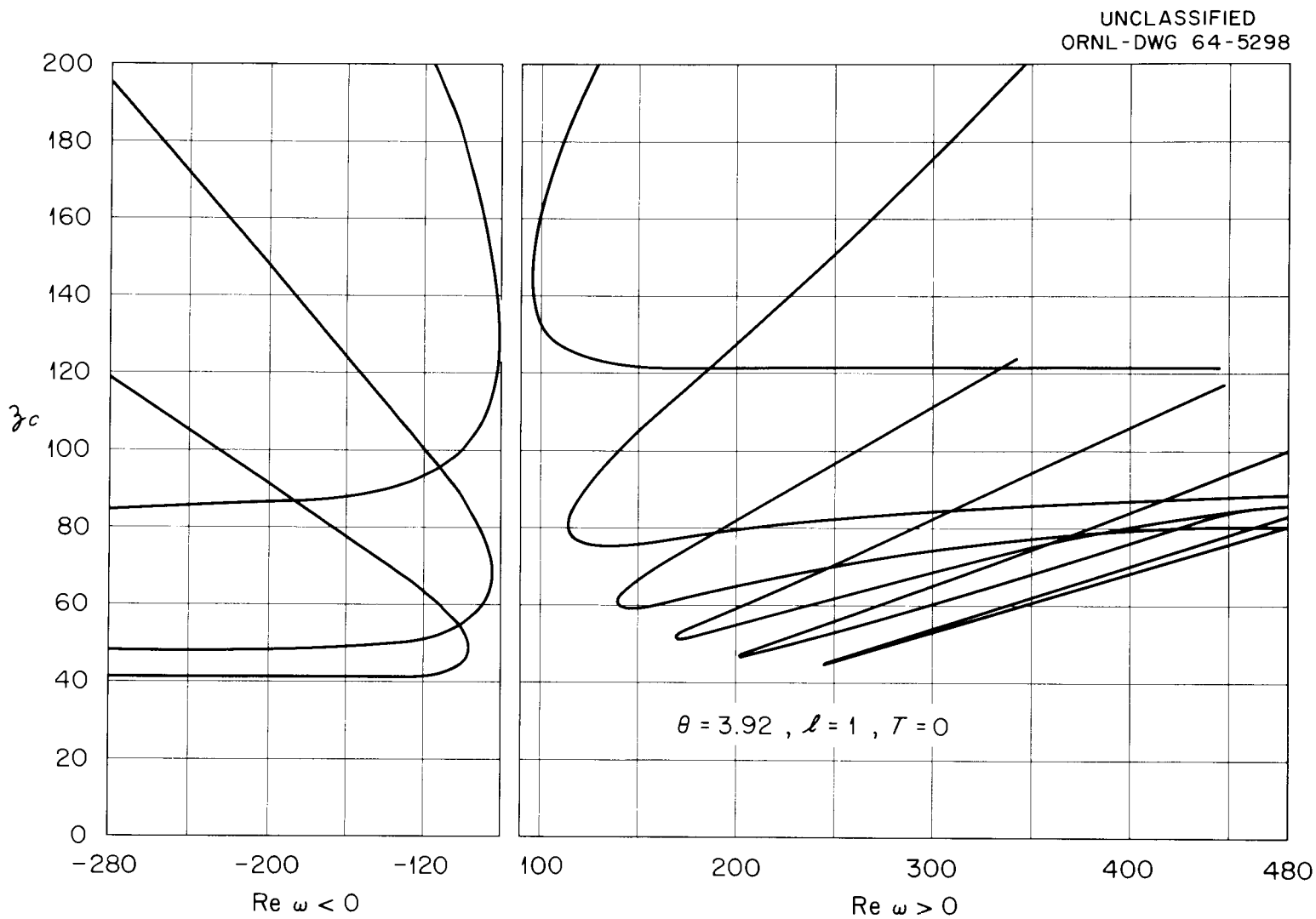

Figure 22. Removal of the Burt-Harris mode by increasing the electron temperature.

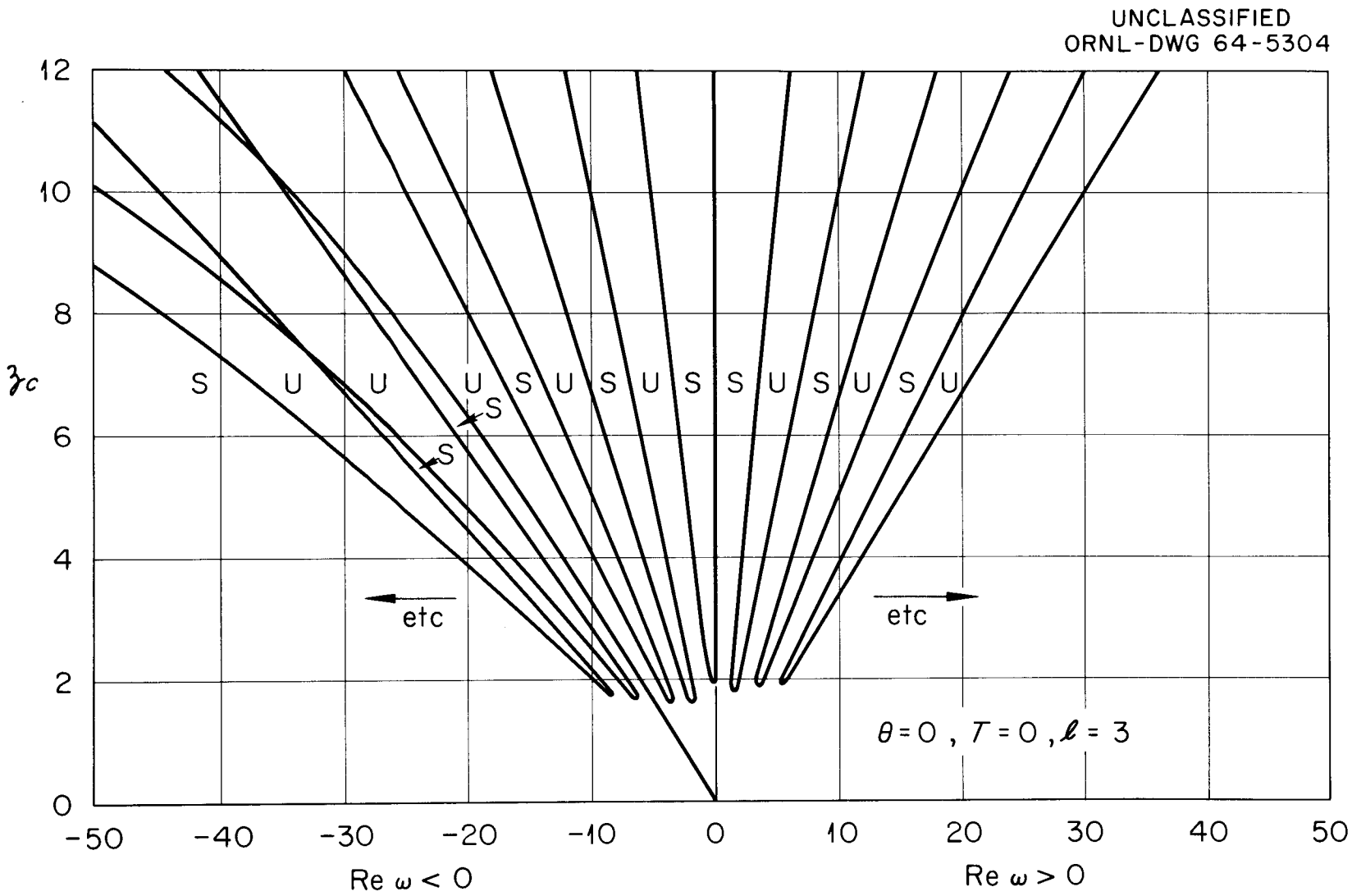

Figure 23. The effect of the azimuthal mode number, ℓ, on the stability plot.

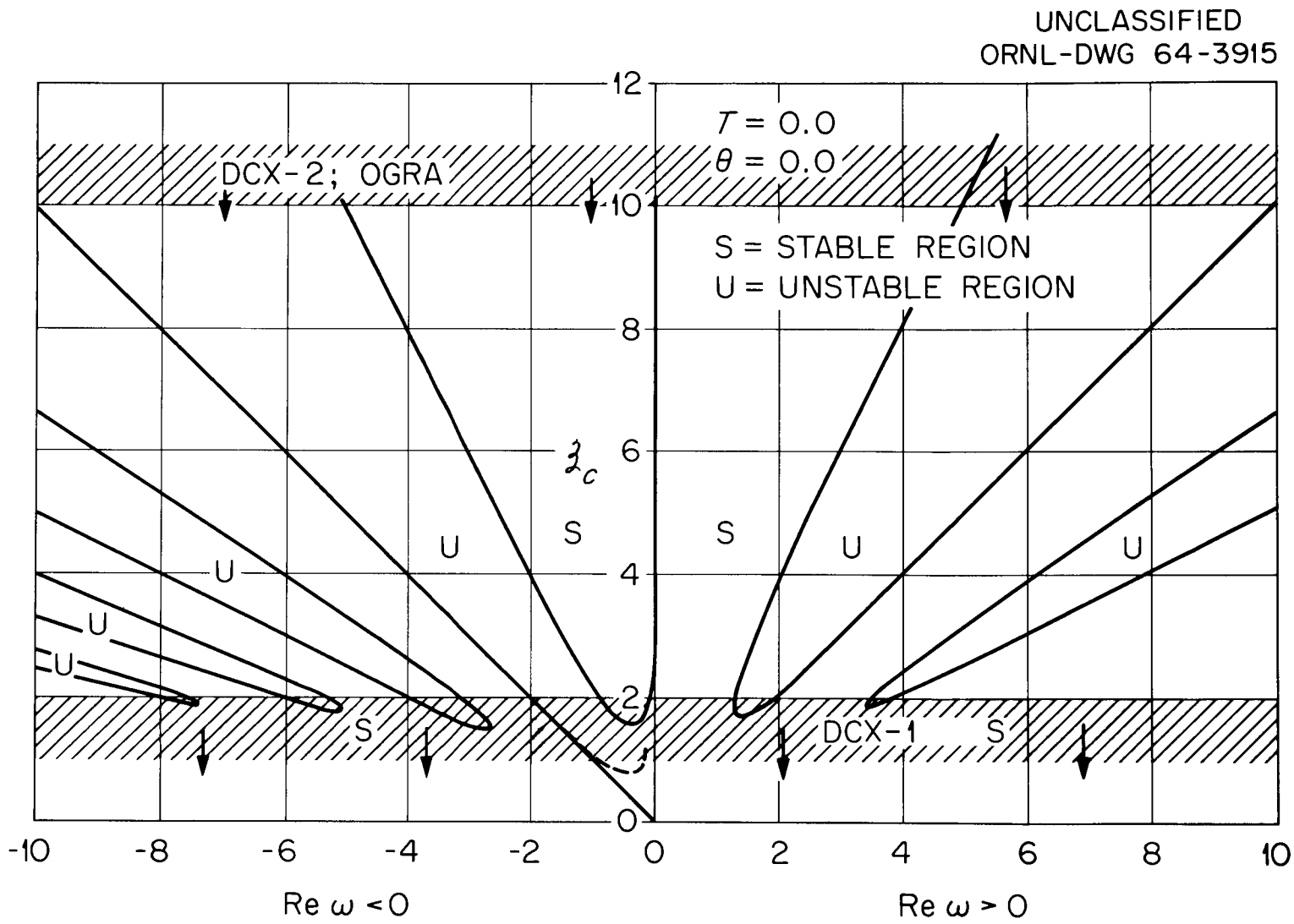

Figure 24. Operating regimes for the machines applicable to the Shima approximation.

where

$$F(\mathfrak{z}) = R(\mathfrak{z}) + iI(\mathfrak{z}) \ , \quad \mathfrak{z} = x + iy \tag{50}$$

and where

$R(\mathfrak{z})$ is real for real $\mathfrak{z}$

and

$I(\mathfrak{z})$ is real for real $\mathfrak{z}$.

The boundary between stable and unstable regions is given by:

$$I(x) = 0 \ . \tag{51}$$

Expanding about small values of y, one finds:

$$F(x + iy) \simeq F(x) + iyF'(x)$$

or

$$F(x + iy) = R(x) + iI(x) + iy\left[R'(x) + iI'(x)\right] \tag{52}$$

$$= \frac{1}{\delta\left(\frac{k_z}{k}\mathfrak{z}_p\right)^2} \text{ which is real.}$$

Therefore, the imaginary part of Equation (52) reads:

$$0 = I(x) + yR'(x)$$

or

$$y = \mathrm{Im}\,\mathfrak{z} = -\frac{I(x)}{R'(x)} \tag{53}$$

where

$$x = \mathrm{Re}\,\mathfrak{z} . \tag{54}$$

For $\theta = 0$, one can write

$$R(x) = \frac{1}{\delta x^2} + \sum_{n=-\infty}^{\infty} C_{|n|}\ \mathrm{Re}Z'(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c) \tag{55}$$

so

$$R'(x) = -\frac{2}{\delta x^3} + \sum_{n=-\infty}^{\infty} C_{|n|}\ \mathrm{Re}Z''(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c) \tag{56}$$

and

$$I(x) = \sum_{n=-\infty}^{\infty} C_{|n|}\ \mathrm{Im}Z'(x + \ell\mathfrak{z}_c - n\mathfrak{z}_c) \ . \tag{57}$$

Using Equations (57) and (56) in Equation (53) determines the growth

rate for the unstable modes. Again the use of a high-speed computer system makes the analysis of equation (53) quite rapid. A sample program for this calculation is found at the end of the chapter.

Notice for $\delta x^2 \ll 1$ (i.e., for $x \ll \sqrt{(m_i/m_e)}$), and using $|\mathrm{Re}Z'|_{max} = 2$, that

$$R(x) \sim \frac{1}{\delta x^2} \tag{58}$$

and therefore

$$\frac{1}{\delta(\frac{k_z}{k}\mathcal{Z}_p)^2} = R(x) \simeq \frac{1}{\delta x^2}$$

which yields

$$\frac{k_z^2}{k}\mathcal{Z}_p \simeq |x| \tag{59}$$

a fact which was used in displaying the curves on the $\mathrm{Re}\omega < 0$ side of the $(k_z/k)\mathcal{Z}_p - \mathcal{Z}_c$ map. The numerical solution of Equation (53) showed that most rapidly growing waves are those for which

$$\frac{k_z}{k}\omega_{pe} \approx j\omega_{ci} ,$$

indicating the resonant character of this instability. The growth rates for the

$$" \frac{k_z}{k}\omega_{pe} \approx (j - \frac{1}{2})\omega_{ci} "$$

waves were negligibly small.

The perturbation technique is limited to values of x in Equation (53) which keep the value of y small. Equations (56) and (53) show that values of x for which

$$\frac{2}{\delta x^3} \sim \sum_{n=-\infty}^{\infty} C_{|n|} \ \mathrm{Re}Z''$$

makes y very large. It was estimated[8] that the perturbation technique was valid for values of $|x| \leq 12$.

IV. THE FORTRAN CODE

The program for the location of the points for the stability map was found to be most efficient when a separate run was made for each value of θ. Program DAMNIT was coded such that the machine, after giving it a value of θ, would pick a value of T, a value of ℓ, and a value of γ_c, and start computing S(x), the imaginary part of the dispersion relation. It was usually faster to have decks for the programs with x = Re less than or greater than zero than to make IF statements to exclude the value x = 0. The machine would compute a value for S(x), the increments in x being 0.2, until S(x) changed sign indicating a crossing of the axis or a root. Using the method of false position, the roots were located and stored and used in the calculation for the value of $(k_z/k)\gamma_p$. The results were printed out as values of $(k_z/k)\gamma_p$ for each root as a function of γ_c. The machine would then pick another value of γ_c and start the process all over. The normal computer time for a typical run was about three minutes. Program MINKOW, which computed the growth rates as a function of x, was very much the same. Sample programs are included at the end of this chapter.

---

[8] T. K. Fowler, Private communication.

```
BEGIN JOB G04  05-12-64
COOP,10601,GK SOPER
FORTRAN,1,0,56,
              PROGRAM DAMNIT
              DIMENSION C(21),ZP(10),ROOT(10)
              C(1)=0.00000097
              C(2)=0.00000659
              C(3)=0.00004051
              C(4)=0.00022265
              C(5)=0.00107956
              C(6)=0.00454090
              C(7)=0.01621591
              C(8)=0.04778332
              C(9)=0.11178255
              C(10)=0.19682671
              C(11)=0.24300035
              C(12)=0.19682671
              C(13)=0.11178255
              C(14)=0.04778332
              C(15)=0.01621591
              C(16)=0.00454090
              C(17)=0.00107956
              C(18)=0.00022265
              C(19)=0.00004051
              C(20)=0.00000659
              C(21)=0.00000097
              PI=3.5449078
              PII=0.5641896
              D=1./1836.
              SQRTD=SQRTF(1./1836.)
              Q=0.0
              T=0.0
              A1=1.
              WRITE(9,300)Q,T,A1
          300 FORMAT(1H1,2HQ=,F4.1,2X,2HT=,F4.1,2X,3HA1=,F4.1)
              WRITE(9,302)(L1,L1=1,7)
          302 FORMAT(1H0,16X,7(2HL=,I1,10X),9X,3HL=8)
              DO2J=1,15
              ZC=J
              L=1
              M1=15.*ZC-5.
              M2=30.*ZC-1.
              DO3K=M1,M2
              Y3=K
              Z=0.2*(-30.*ZC+Y3)
              IF(ABSF(Z)-10.*ZC)70,70,2
           70 S=0.0
              DO5N=1,21
              E=N-11
            5 S=S+C(N)*(Z+A1*ZC+E*ZC*(1.-T))*EXPF(-(Z+(E+A1)*ZC)**2)
              S=-S*PI
              IF(ABSF(Z+3.*ZC+1.)-1.E-5)11,11,13
           13 S2=S
              Z2=Z
              IF(ABSF(S)-1.E-300)14,14,7
            7 G=S0/S
              IF(G)10,11,11
           10 Z1=Z
              S1=S
              NCOUNT=0
           25 Z=Z1-S1*(Z1-Z0)/(S1-S0)
```

```
      S=0.0
      DO77N=1,21
      E=N-11
   77 S=S+C(N)*(Z+A1*ZC+E*ZC*(1.-T))*EXPF(-(Z+(E+A1)*ZC)**2)
      S=-S*P1
      NCOUNT=NCOUNT+1
      IF(NCOUNT-10)151,151,14
  151 IF(S)152,14,152
  152 W=S0/S
      IF(W)19,19,20
   20 IF(ABSF((Z-Z0)/Z0)-.005)14,14,201
  201 S0=S
      Z0=Z
      GO TO 25
   19 IF(ABSF((Z-Z1)/Z1)-.005)14,14,191
  191 S1=S
      Z1=Z
      GO TO 25
   14 ROOT(L)=Z
      IF(ROOT(L))35,36,35
   35 SUM=0.0
      DO67N=1,21
      K1=N-11
      B1=K1
      X2=ROOT(L)+(B1+A1)*ZC
      IF(ABSF(X2)-10.)65,65,66
   65 S5=0.0
      DO111N1=1,40
      E2=N1
  111 S5=S5+(EXPF(-(E2**2/4.-E2*X2))-EXPF(-(E2**2/4.+E2*X2)))/E2
      REZX2=-PI1*EXPF(-X2**2)*(X2+S5)
      GO TO 67
   66 REZX2=-1./X2-1./(2.*(X2**3))
   67 SUM=SUM+C(N)*REZX2*(X2-B1*ZC*T)
      REZP2=-2.*D*(1.+SUM)+1./(ROOT(L)**2)
      IF(REZP2)98,98,99
   99 ZP(L)=SQRTF(1./REZP2)
      GO TO 100
   36 ZP(L)=0.0
      GO TO 100
   98 ZP(L)=9.999
  100 L=L+1
      IF(L-8)22,22,29
   22 S=S2
      Z=Z2
   11 S0=S
      Z0=Z
    3 CONTINUE
   29 L=L-1
      IF(L)222,222,224
  222 WRITE(9,223)ZC
  223 FORMAT(1H0,6HZP(ZC=F5.1,10H)=NO ROOTS)
      GO TO 2
  224 WRITE(9,303)ZC,(ZP(L1),L1=1,L)
  303 FORMAT(1H0,6HZP(ZC=,F5.1,2H)=,8(E11.3,1X))
  225 WRITE(9,304)(ROOT(L3),L3=1,L)
  304 FORMAT(4X,10HROOT(S=0)=,8(E11.3,1X))
    2 CONTINUE
      STOP
      END DAMNIT
```

```
BEGIN JOB L10  05-04-64
COOP,10601,GK SOPER.
FORTRAN,0,0,56.
              PROGRAM MINK5W
              DIMENSION C(21),V(4)
              C(1)=0.00000097
              C(2)=0.00000259
              C(3)=0.00004051
              C(4)=0.00022265
              C(5)=0.00107956
              C(6)=0.00454090
              C(7)=0.01621591
              C(8)=0.04778332
              C(9)=0.11178255
              C(10)=0.19682671
              C(11)=0.24300035
              C(12)=0.19682671
              C(13)=0.11178255
              C(14)=0.04778332
              C(15)=0.01621591
              C(16)=0.00454090
              C(17)=0.00107956
              C(18)=0.00022265
              C(19)=0.00004051
              C(20)=0.00000259
              C(21)=0.00000097
              V(1)=2.
              V(2)=10.
              V(3)=30.
              V(4)=60.
              PI=3.5449078
              PII=.5641896
              D=1./1836.
              Q=0.
              T=0.
              AI=1.
              DO20I=1,4
              ZC=V(I)
              WRITE(9,300)Q,T,AI,ZC
          300 FORMAT(1H1,2HQ=,F3.1,2X,2HT=,F3.1,2X,2HJ=,F3.1,2X,
             13HZC=,F4.1///)
              L1=15.*ZC
              L2=45.*ZC
              DO1J=L1,L2
              A=J
              X=0.2*(-30.*ZC+A)
              IF(ABSF(X)-1.E-5)1,1,6
            6 WRITE(9,302)X
          302 FORMAT(1H0,2HX=,F5.1)
              SUM1=0.
              SUM2=0.
              SUM3=0.
              SUM4=0.
              SUM5=0.
              SUM6=0.
              DO2N=1,21
              L=N-11
              B=L
              XN=X+(B+AI)*ZC
```

```
      SUM5=SUM5-PI*C(N)*XN*EXPF(-(XN**2))
      SUM6=SUM6-PI*C(N)*(1.-2.*(XN)**2)*EXPF(-(XN**2))
      IF(ABSF(XN)-10.)3,3,4
    3 SI=0.
      DO5M=1,40
      E=M
    5 SI=SI+(EXPF(-(E**2/4.-E*XN))-EXPF(-(E**2/4.+E*XN)))/E
      REZXN=-PI1*EXPF(-(XN**2))*(XN+SI)
      SUM1=SUM1-2.*C(N)*XN*REZXN
      SUM2=SUM2-2.*C(N)*(1.-2.*(XN)**2)*REZXN
      GO TO 2
    4 R=-1./XN+1./(2.*(XN**3))
      SUM3=SUM3-2.*C(N)*R*XN
      SUM4=SUM4-2.*C(N)*(1.-2.*(XN)**2)*R
    2 CONTINUE
      SUM7=SUM1+SUM3
      SUM8=SUM2+SUM4
      QOFXN=SUM5
      QPROFXN=SUM6
      ROFXN=1./(D*(X**2))-2.+SUM7
      RPROFXN=-2./(D*(X**3))+4.*(X+A1*ZC)+SUM8
      Y=-QOFXN/RPROFXN
      REZP=ROFXN-Y*QPROFXN
      IF(REZP)90,90,91
   90 ZP=9.9999
      GO TO 92
   91 ZP=SQRTF(1./(REZP*D))
   92 WRITE(9,501)Y,ZP,QOFXN,QPROFXN,ROFXN,RPROFXN
  501 FORMAT(1H ,2HY=,E12.4,2X,5HREZP=,E12.4,2X,6HIOFXN=,E12.4,2X
     1 8HIPROFXN=,E12.4,2X,6HROFXN=,E12.4,2X,8HRPROFXN=,E12.4)
    1 CONTINUE
   20 CONTINUE
      STOP
      END MINKOW
```

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

I. FINITE PLASMA GEOMETRY

One of the main conclusions drawn from the present study was that, although a bounded plasma problem did exhibit modes of oscillation which were unique, these effects were relatively minor. However, the differences between the infinite and bounded plasma cases should be pointed out and discussed. For the plasma slab with zero magnetic field, three distinct solutions to the problem were found. One turned out to be the infinite plasma solution which allowed arbitrary $\Phi(z)$ inside the plasma slab and zero potential, hence zero field everywhere outside the plasma. It was also found that this solution of the infinite plasma mode required Φ to vanish at $z = \pm a$. The remaining two solutions were the surface charge modes which depended upon the dimensions of the plasma. These surface modes were relatively insensitive to the separation of the boundaries.

In the Harris model with a static magnetic field along z the solution of the problem was more involved. The solutions did not separate into surface and infinite plasma modes. For negative values of m^2, essentially the square of the z component of the wave vector, a relationship between $m \equiv ik_z$ and the width, a, of the plasma could be calculated. As a possible suggestion for further research, Equations (117) and (118) of Chapter III could be coded for a numerical evaluation by a computer with k_z as the variable parameter. For each value of k_z (given a value

of the particle density and magnetic field strength) one could compute the plasma width for this problem. For positive values of m^2 it was seen that the plasma was stable for frequencies in the neighborhood of the ion-cyclotron frequency. The Burt-Harris model was assumed to be a good representation of a mirror machine with very weak mirrors. The ions traveled in concentric orbits in a plane perpendicular to the magnetic field (z-axis). The problem was found to be solvable in general for finite r and z dimensions. The results obtained indicated effects due to finite boundaries but, again, effects not very different from an infinite plasma type problem.

Although these boundary effects did not manifest themselves in any major way, such as being a source for a new instability, the effects could be calculated. Also, approximations were used which would not apply in a real plasma. The most serious deviation arises in the linearization of the basic equations, since a real plasma responds in a distinctly non-linear manner. Furthermore, one should not restrict the treatment to longitudinal oscillations, for it is apparent that there must be some coupling between the longitudinal and transverse modes.

More important than the actual results of the finite cases, however, was the fact that the solution of the differential equation defining the electric potential, although in most cases singular somewhere in the plasma edge, was found to lead to the familiar electrostatic boundary conditions only in the case of a perfectly sharp interface between plasma and vacuum. This was demonstrated, using results derived in an earlier paper,[1] by

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[1] E. A. Frieman *et al.*, Phys. Fluids *5*, 196 (1962).

assuming the unperturbed or equilibrium particle density to be an arbitrary function of position. It was interesting to note that the boundary conditions derived* in previous work of this author[2] using a "cold" plasma slab model with perfectly sharp edges were consistent with the above results. However, the attempt to take over the boundary conditions for the Harris finite model with temperature effects by merely replacing the equation for the surface charge by its average over the new distribution function, led one to boundary conditions which were not consistent with the above paper--a strange result indeed. Harris[3] pointed out that instead of averaging over the velocities, one should assume that the distribution was made up from many streams, then add up all the streams, and then replace the summation by integrals to finally arrive at the correct equations.

II. DISPERSION RELATION ANALYSIS

There has been considerable interest in possible ways of eliminating the strong ion-cyclotron oscillations that are present, for example, in the ALICE, Phoenix, DCX, and OGRA machines. A possible interpretation of the mechanism for these oscillations was given by Harris,[4] who assumed that this instability originated from the anisotropy in velocity space of

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*These boundary conditions were derived by calculating the surface charge which migrates to the boundary after perturbation, and from this the discontinuity in the normal component of the electric field was derived.

[2] G. K. Soper, Master's thesis, University of Tennessee, 1961.

[3] E. G. Harris, Private communication.

[4] E. G. Harris, "Unstable Plasma Oscillations in a Magnetic Field," ORNL-2728 (June 5, 1959).

the particle distribution function. Harris's work was extended by this author[5] to include finite ion and electron temperature effects with special emphasis on the machines named above. It was found that on a plot consisting essentially of the magnetic field versus the density, regions of instability appeared which arose from the anisotropic nature of the plasma. It was found that either reducing the anisotropy of the ions, $T_{\perp i}/T_{zi}$, or increasing the electron temperature tends to stabilize these anisotropy modes. Also, even with cold electrons and $T_{\perp i} \gg T_{zi}$, reducing the magnetic field stabilizes a plasma of finite length, as was observed in the Phoenix experiment.[6] The largest growth rates occurred at frequencies very near the ion-cyclotron or multiples of the ion-cyclotron frequency. Strong oscillations were seen in Phoenix at frequencies of one-half the ion-cyclotron frequency. Although it was found in this dissertation that the plasma could become unstable for values of the frequency satisfying $1/2\ \omega_{ci} < \omega < \omega_{ci}$, the growth rate that was calculated for the wave at one-half the ion-cyclotron frequency was negligible compared to that at $\omega = \omega_{ci}$. Thus it was concluded that the observed strong oscillations at $\omega = 1/2\ \omega_{ci}$ were not explainable on the basis of the model assumed in this dissertation.

For the Shima model, representative of plasma about one Larmor diameter across, as in the DCX's and OGRA, most of the anisotropic modes

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[5]See also Y. Ozawa, I. Kaji, and M. Kito, J. Nucl. Energy, Pt. C 4, 271 (1962); and Yu. N. Dnestrovsky, D. P. Kostomarov, and V. I. Pistunovich, Nuclear Fusion 3, 30 (1963).

[6]L. G. Kuo et al., "Experimental and Theoretical Studies of Instabilities in a High Energy Neutral Injection Machines," CLM-P32 (November, 1963).

found above remained essentially unchanged. However, for each value of the azimuthal mode number, ℓ, that anisotropy mode with angular velocity which most nearly coincided with that of the ions changed character and became essentially the instability found by Burt and Harris. In this case, reduction of the ion anisotropy removed the unstable anisotropic modes but left the Burt-Harris mode unaffected. A finite electron temperature, however, had a stabilizing effect on the Burt-Harris modes.

III. SUGGESTIONS FOR FURTHER STUDY*

Dr. J. B. Taylor suggested making a complete density versus magnetic field stability map to provide a basis for deciding which machine is best. Dr. R. F. Post suggested trying the large density limit ($\omega_{pe} \gg \ell\omega_{ci}$; $\omega \sim \ell\omega_{ci}$) to get a stability map in $T_{\perp i}/T_{zi}$ vs T_{ze}/T_{zi} for unlimited k_z.

Since the bi-Maxwellian distribution function is more representative of the asymptotic state in the DCX's, one might try an ion distribution function which is approximately a delta function in $|v_\perp|$, that is more representative of the injection conditions of most machines. The growth rates calculated in this dissertation were based on a perturbation technique which was applicable only in a limited region. By major modifications of the present code, the analysis could include growth rates for all regions. Those areas of special interest would be those where the unstable regions of two or more resonances overlap.

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*These remarks are partly based on a program suggested by Dr. T. K. Fowler for further analysis of the Harris dispersion relation.

One should also redo the computer analysis for the case of a Lorentzian distribution in v_z, mainly to be able to compare the results with other authors.* Furthermore, analysis of the case where $T_e \gg T_i$ could be incorporated into the computer code by changing the original assumptions as the regions of interest. Here one would want, for example, to examine the region near the electron cyclotron frequency. This has obvious bearing on the problem of the heating a plasma by electron cyclotron resonance by R. A. Dandl at ORNL.

In conclusion, it should be noted that, as stated by N. Rostoker:[7] "In order to produce a confined plasma for interesting times, we must eliminate intolerable instabilities, tolerate instabilities that cannot be eliminated, and know which is which."

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*e.g., Laurence S. Hall at Livermore.

[7]N. Rostoker, "Plasma Stability," GA-2617 (Dec. 14, 1961).

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APPENDIX A

THE APPROXIMATION OF A COLLISION-FREE PLASMA

In the discussion of collisions between particles in a plasma, one usually speaks of an interaction as an encounter leading to a deflection of 90° or more. In order to understand the physical assumption of a collisionless plasma, it is noted that the collisions that can occur can be roughly divided into two types. The first is the "short-range encounters" in which the ordinary short-range force acting on a particle when it makes a close collision gives rise to heavy momentum changes. In a low density plasma these encounters introduce only minor corrections to the collisionless treatment. However, at higher densities this effect may destroy the collective behavior of the plasma. The second type is that of "long-range encounters" which represent many interactions of a single particle with other members of the plasma giving rise to a large angle scattering and a very small momentum transfer per individual encounter. In principle these encounters extend over the whole plasma since the range of the Coulomb force is infinite.

Assuming that the long-range encounters make a more significant contribution that collisions resulting in large momentum changes,[1] one can derive a cross section, σ_D, for large angle, distant scattering to be:[2]

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[1] S. Chapman and T. G. Cowling, Mathematical Theory of Non-Uniform Gases (Cambridge University Press, London, 1952), pp. 178-179.

[2] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957); and R. S. Cohen, L. Spitzer, Jr., and P. McR. Routly, Phys. Rev. 80, 230 (1950).

$$\sigma_D \approx \frac{2\pi e^2}{E^2} \ln\Lambda \tag{A1}$$

where E is the energy of the scattered particle and where

$$\Lambda \equiv \frac{b_{max}}{b_{min}} \tag{A2}$$

where b represents the impact parameter. The minimum value of the impact parameter is taken equal to the distance of closest approach for a scattering of 90° in a single encounter. The maximum value of the impact parameter is given by the Debye shielding distance

$$b_{max} = \lambda_D \equiv \left(\frac{\kappa T}{4\pi n_o e^2}\right)^{1/2} \tag{A3}$$

a distance beyond which the plasma is considered to be electrically neutral so that the particle under consideration is not affected by Coulomb forces. For probable conditions existing in a thermonuclear reaction $\ln\Lambda$ would be in the vicinity of twenty.[3]

The mean collision time for an ion scattered by distant encounters with other ions can be written[4]

$$t_{ii} = \frac{1}{n_i \sigma_D v_i} \tag{A4}$$

where v_i is the average ion velocity. For 300 kev protons Equation (A4), using Equation (A1), gives

$$t_{ii} \sim .25 \text{ sec} \tag{A5}$$

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[3] L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, New York, 1962), p. 128.

[4] S. Glasstone and R. H. Lovberg, Controlled Thermonuclear Reactions (D. Van Nostrand Company, Inc., Princeton, 1960), p. 95.

where a value of $n_i = n_e = n_o = 10^{15}$ particles per cm^3 was used. However, it is easy to show that the typical frequency which occurs in a plasma is:[5]

$$\omega_p = \left(\frac{4\pi n_o e^2}{m}\right)^{1/2} \tag{A6}$$

and for a plasma of electrons and protons, Equation (A6) yields:

$$\omega_p \sim 6 \times 10^{11} \text{ sec}^{-1} . \tag{A7}$$

Therefore, the average time necessary to randomize the initial distribution is very large compared with the oscillation time of the plasma, and one sees that for a relatively long time the Vlasov equations are quite useful.

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[5]For example, see J. D. Jackson, Classical Electrodynamics (John Wiley and Sons, New York, 1962), p. 337. Almost any book or article on beginning plasma physics will have the derivation of this expression.

APPENDIX B

THE INITIAL VALUE TREATMENT OF THE BOUNDED SLAB PROBLEM*

If Equation (39)** had been subjected to the formal initial value analysis prescribed by Equations (14) and (15), Equation (69) would have appeared

$$\frac{d}{dz}\left[(1 - \eta(\omega)\Delta(z))\frac{d\Phi}{dz}(\vec{k}_\perp, \omega, z)\right] - k_\perp^2(1 - \eta(\omega)\Delta(z))\Phi = S(\vec{k}_\perp, \omega, z) \tag{B1}$$

where $S(\vec{k}_\perp, \omega, z)$ represents the initial value terms. Solving Equation (B1), one can find the time dependence of the electric potential by inversion:

$$\Phi(t, z) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} d\omega e^{-i\omega t}\,\Phi(\omega, z) \tag{B2}$$

From solutions of the homogeneous equation one can construct solutions to the inhomogeneous Equation (B1).

First solve the homogeneous equation

$$\frac{d}{dz}\left[(1 - \eta_n\Delta(z))\frac{d\Phi_n}{dz}\right] - k_\perp^2(1 - \eta_n\Delta(z))\Phi_n = 0 \tag{B3}$$

which can be written

$$\hat{H}\Phi_n - \eta_n\hat{L}\Phi_n = 0 \tag{B4}$$

---

*The author is grateful to Dr. E. G. Harris for the details of this section.

**Unless noted, all equation numbers will refer to equations found in Chapter III.

where

$$\hat{H} \equiv \frac{d^2}{dz^2} - k_\perp^2 \tag{B5}$$

and

$$\hat{L} \equiv \frac{d}{dz}\,\Delta(z)\,\frac{d}{dz} - k_\perp^2\Delta(z)\ . \tag{B6}$$

Since $\hat{H}$ and $\hat{L}$ are Hermitian operators, it is easy to show that,

$$\int dz\Phi_m{}^*\,\hat{L}\Phi_n = \delta_{mn} \tag{B7}$$

and thus one can construct a solution of the homogeneous equation from a linear combination of the orthogonal functions Φ_n. Therefore,

$$\Phi(\omega,\ z) = \sum_n a_n\Phi_n\ , \tag{B8}$$

and substituting Equation (B8) into Equation (B1), one finds:

$$(\hat{H} - \eta(\omega)\hat{L})\Phi(\omega,\ z) = \sum_n a_n(\eta_n - \eta(\omega))\hat{L}\Phi_n = S\ . \tag{B9}$$

Multiplying by $\Phi_m{}^*$, integrating and using Equation (B7), one sees that

$$a_n = \frac{\int dz\Phi_n{}^*S}{\eta_n - \eta(\omega)}\ . \tag{B10}$$

Therefore, from Equations (B8) and (B10)

$$\Phi(\omega,\ z) = \sum_n \frac{\Phi_n(z)}{\eta_n - \eta(\omega)}\int dz\Phi_n{}^*S \tag{B11}$$

and finally,

$$\Phi(t,\ z) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega e^{-i\omega t}\sum_n \frac{\Phi_n(z)}{\eta_n - \eta(\omega)}\int dz\Phi_n{}^*(z)S(\omega,\ z) \tag{B12}$$

If one assumes that there are only simple poles and let ω_{nj}* be a solution of

$$\eta(\omega) = \eta_n \ , \tag{B13}$$

then one can write Equation (B12) as:

$$\Phi(t, z) = \sum_n \sum_j \Phi_n(z) e^{-i\omega_{nj} t} A_{nj} + G(t, z) \tag{B14}$$

where

$$A_{nj} = i \text{ Residue of } \frac{1}{\eta_n - \eta(\omega)} \int dz \Phi_n{}^* S \text{ at } \omega = \omega_{nj} \tag{B15}$$

and $G(t, z)$ is the contribution due to any poles of $\int dz \Phi_n{}^* S$ which is of less importance.

Therefore it is seen that the frequencies of the system are determined by the eigenvalues of the homogeneous equation. If the η's form a continuum, then

$$\Phi(\omega, z) = \int a(\eta) \Phi(\eta, z) d\eta$$

and

$$\Phi(t, z) = \sum_j \int d\eta \Phi(\eta, z) e^{-i\omega_j(\eta) t} A_j(\eta) + G(t, z) \tag{B16}$$

where the integration in Equation (B16) will smear out the singularity if the singularity is mild enough for the integral to converge.

---

*For each eigenvalue η_n there may be many roots, so distinguish each of them with a subscript j.

APPENDIX C

THE DERIVATION OF EQUATION (94)

With

$$\vec{B}_o = -B_o\hat{z} \tag{C1}$$

Equation (1)* is written:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \omega_c \left[v_y \frac{\partial f}{\partial v_x} - v_x \frac{\partial f}{\partial v_y} \right] = - \frac{e\vec{E}}{m} \cdot \nabla_v f \ . \tag{C2}$$

Introducing cylindrical coordinates and using the moment technique of Section III of Chapter III, one can show that the differential equations for the linearized form of the F and H functions defined by Equations (42) and (43) are:

$$\frac{\partial F_1}{\partial \phi} + \frac{i}{\omega_c} (k_\perp v_\perp \cos \phi - \omega) F_1 = - \frac{1}{\omega_c} \left[\frac{\partial H_1}{\partial z} + \frac{e}{m} \vec{E}_{1\perp} \cdot \nabla_{v\perp} F_o \right] \tag{C3}$$

and

$$\frac{\partial H_1}{\partial \phi} + \frac{i}{\omega_c} (k_\perp v_\perp \cos \phi - \omega) H_1 = \frac{eE_{1z}F_o}{m\omega_c} \tag{C4}$$

where the particle subscript has been suppressed for simplicity. The solution of Equation (C4) is[1]

$$H_1 = \frac{1}{\omega_c} \int_{\pm\infty}^{\phi} d\phi' G(\phi, \phi') \frac{eE_{1z}F_o}{m} \tag{C5}$$

and similarly for Equation (C5)

$$F_1 = - \frac{1}{\omega_c} \int_{\pm\infty}^{\phi} d\phi' G(\phi, \phi') \left[\frac{\partial H_1}{\partial z} + \frac{e}{m} \vec{E}_{1\perp} \cdot \nabla_{v\perp} F_o \right] \tag{C6}$$

---

*Again, all equation numbers will refer to equations from Chapter III.

[1] I. B. Bernstein, Phys. Rev. 109, 10 (1958).

where

$$G(\phi, \phi') = \exp\left\{\frac{i\omega}{\omega_c}(\phi - \phi') - \frac{ik_\perp v_\perp}{\omega_c}(\sin\phi - \sin\phi')\right\}. \quad (C7)$$

When Equation (C5) is substituted into Equation (C6), F_1 becomes:

$$F_1 = -\frac{1}{\omega_c}\int_{\pm\infty}^{\phi} d\phi' G(\phi, \phi')\frac{e}{m}E_\perp \cdot \nabla_{v\perp}F_o$$

$$-\frac{1}{\omega_c^2}\int_{\pm\infty}^{\phi} d\phi' G(\phi, \phi')\int_{\pm\infty}^{\phi'} d\phi'' G(\phi', \phi'')\frac{e}{m}\frac{\partial(F_o E_{\perp z})}{\partial z}. \quad (C8)$$

Since

$$F_o \equiv \frac{n_o(z)}{\pi\alpha_\perp^2} e^{-v_\perp^2/\alpha_\perp^2}, \quad (C9)$$

$$e^{(ik_\perp v_\perp/\omega_c)\sin\phi} = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{in\phi}, \quad (C10)$$

and

$$\cos\phi\, e^{(ik_\perp v_\perp/\omega_c \sin\phi)} = \sum_{n=-\infty}^{\infty} \frac{n}{(k_\perp v_\perp/\omega_c)} J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{in\phi}, \quad (C11)$$

$$F_1 = \frac{2eF_o\Phi k_\perp v_\perp}{m\alpha_\perp^2} e^{-(ik_\perp v_\perp/\omega_c)\sin\phi}\sum_{n=-\infty}^{\infty}\frac{n}{(k_\perp v_\perp/\omega_c)}\frac{J_n(k_\perp v_\perp/\omega_c)\, e^{in\phi}}{(\omega - n\omega_c)}$$

$$-\frac{e}{m}\frac{\partial}{\partial z}\left(F_o\frac{\partial\Phi}{\partial z}\right) e^{-(ik_\perp v_\perp/\omega_c)\sin\phi}\sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\frac{e^{in\phi}}{(\omega - n\omega_c)^2}. \quad C12)$$

Inserting Equation (C12) into Poisson's equation:

$$k_\perp^2\Phi - \frac{\partial^2\Phi}{\partial z^2} = 4\pi\sum_j e_j\int_0^\infty v_\perp dv_\perp\int_0^\infty d\phi F_{1j}(\omega, v_\perp, \phi, z) \quad (C13)$$

and using the identity found in Watson[2]

$$\int_0^\infty J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) e^{-v_\perp^2/\alpha_\perp^2} v_\perp dv_\perp = \frac{1}{2}\alpha_\perp^2 e^{-\lambda} I_{|n|}(\lambda) \tag{C14}$$

where λ is defined by Equation (97), then one finds:

$$k_\perp^2\Phi - \frac{\partial^2\Phi}{\partial z^2} = k_\perp^2\Phi \sum_j \frac{\omega_{pj}^2(z)}{\lambda_j\omega_{cj}} \sum_{n=-\infty}^{\infty} \frac{nI_{|n|}(\lambda_j)e^{-\lambda_j}}{\omega - n\omega_{cj}}$$

$$- \sum_j \frac{\partial}{\partial z}\left(\omega_{pj}^2 \frac{\partial\Phi}{\partial z}\right) \sum_{n=-\infty}^{\infty} \frac{I_{|n|}(\lambda_j)e^{-\lambda_j}}{(\omega - n\omega_{cj})^2} \tag{C15}$$

which reduces to Equation (94), where the spatial dependence has been absorbed into the factor $\Delta(z)$.

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[2] G. N. Watson, Theory of Bessel Functions (Cambridge University Press, Cambridge, 1945), p. 395.

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